Electroweak double-box integrals for Møller scattering with three Z bosons

Iris Bree | ibredeme@uni-mainz.de

based on work with **Dmytro Melnichenko** and **Stefan Weinzierl**

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JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Motivation

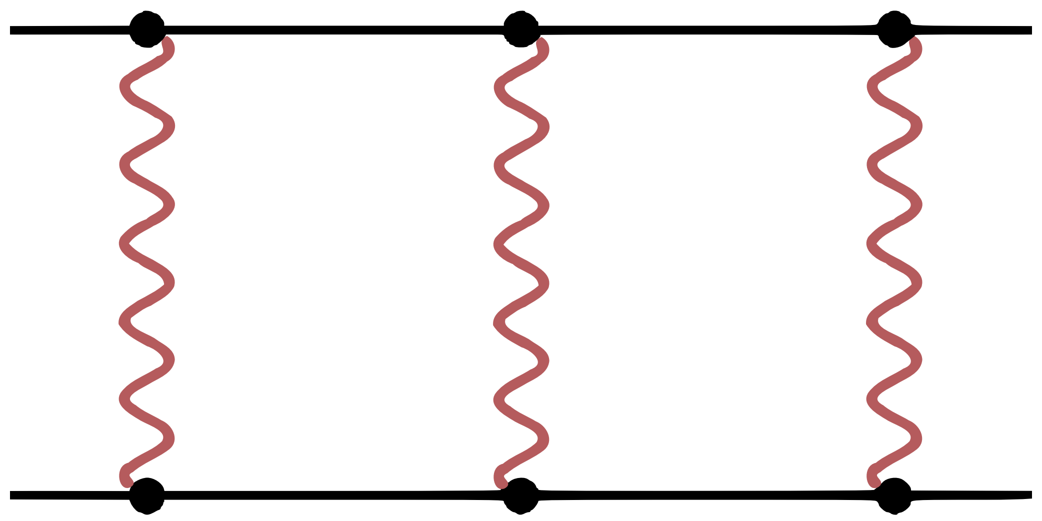
- **Møller scattering** ($e^- e^- \to e^- e^-$) provides a clean probe of the weak mixing angle at low energies

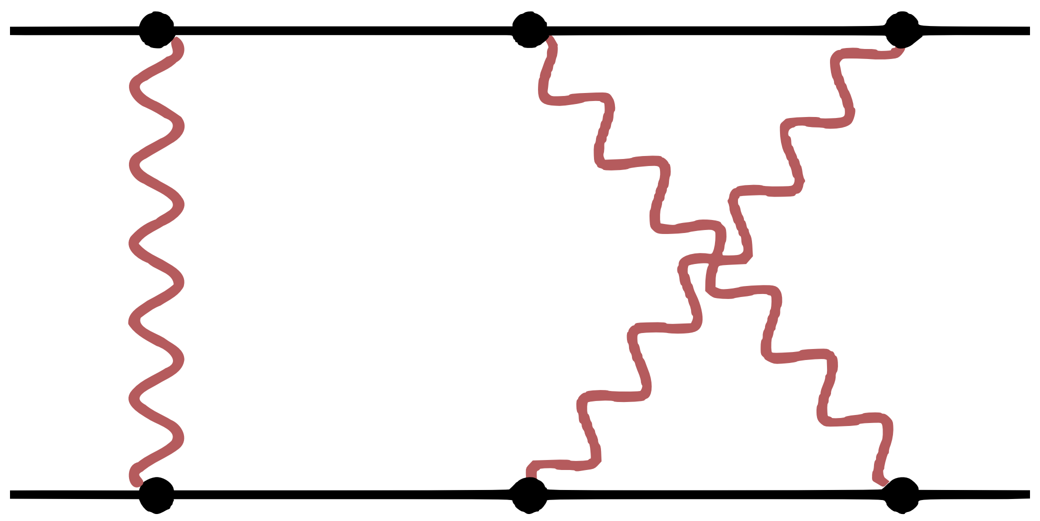
higher-order perturbative corrections needed

compute electroweak two-loop corrections

- Most complicated Feynman integrals at the two-loop level relevant to this process are **planar** and **non-planar double-box** integrals
    - 3 massless bosons — Smirnov '99, Tausk '99
    - 1 or 2 massless bosons — Schwanemann, Weinzierl '25

3 massive bosons
exchanged between fermion lines



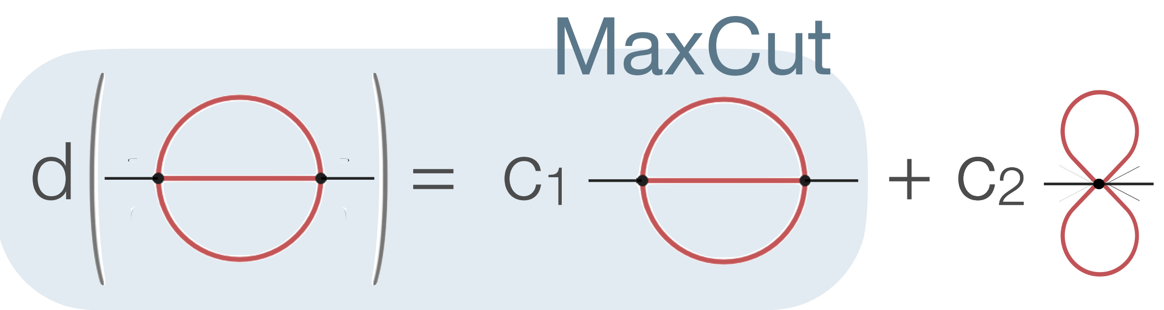


integrals with
challenging geometries

Geometry of Feynman Integrals

- Take an integral I and write it in Baikov representation: $I = \int P_\epsilon(z_1, ..., z_n)\, dz_1 ... dz_n$

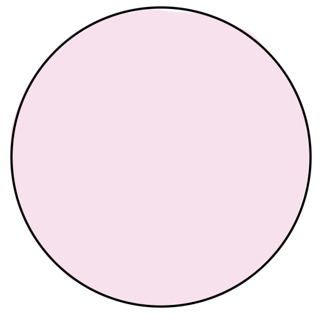
go on the maximal cut



$$I_{\text{Maxcut}} = \int U_\epsilon(z_1, ..., z_{N_B})\, dz_1 ... dz_{N_B}$$

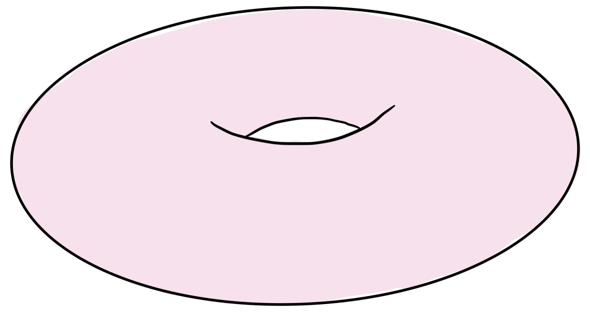
- Taking ε=0, we can observe the geometry of the integral:

$$U_{\epsilon=0} \sim \frac{1}{\sqrt{(z_1 - a)(z_1 - b)}}$$



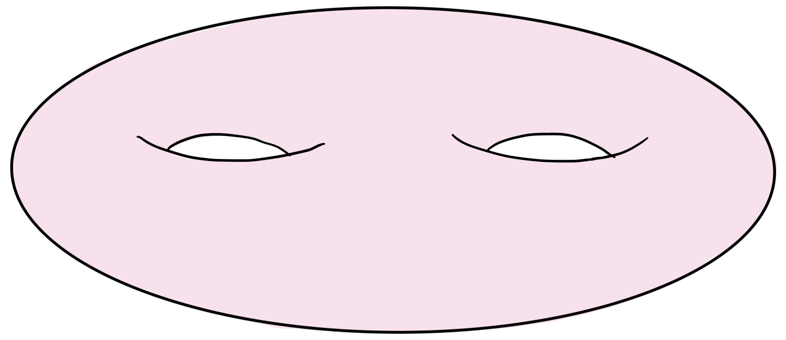
genus 0

$$U_{\epsilon=0} \sim \frac{1}{\sqrt{P_4(z_1)}}$$



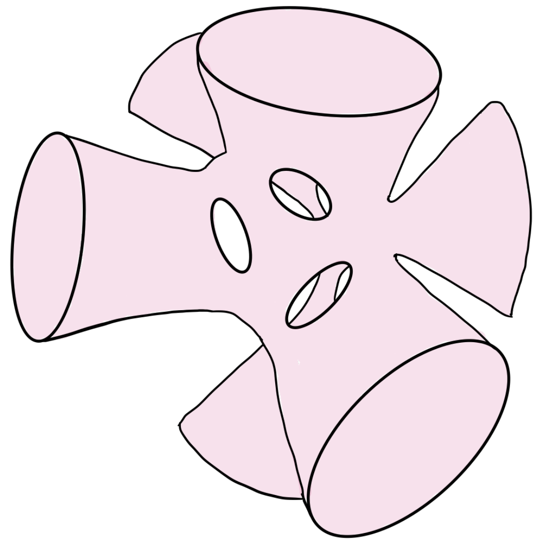
genus 1

$$U_{\epsilon=0} \sim \frac{1}{\sqrt{P_6(z_1)}}$$



genus 2

$$U_{\epsilon=0} \sim \frac{1}{\sqrt{P_6(z_1, z_2)}}$$



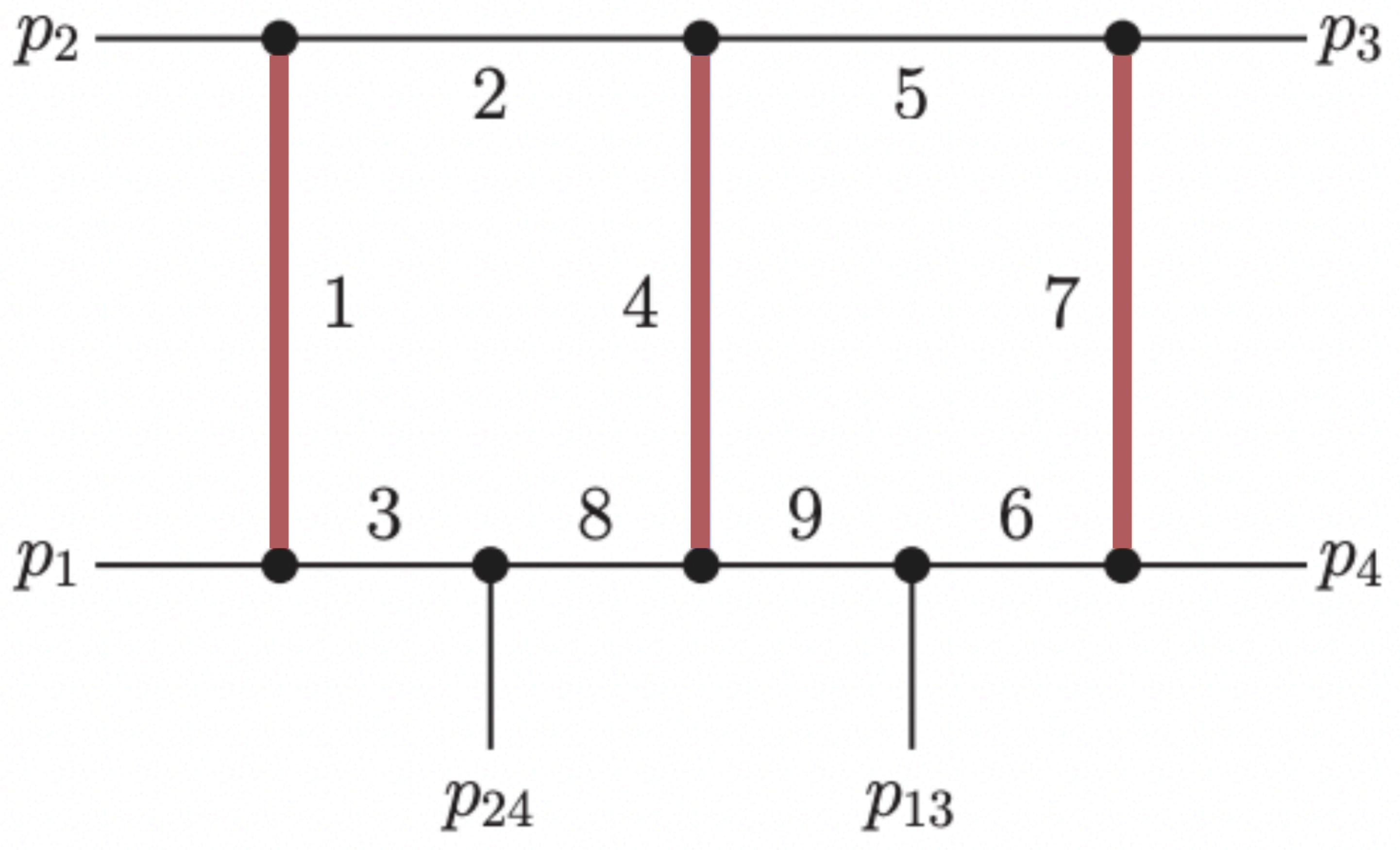
K3

...

Planar Double Box

$D = 4 - 2\varepsilon$

Scales: m^2, t, s — rescale away (s)



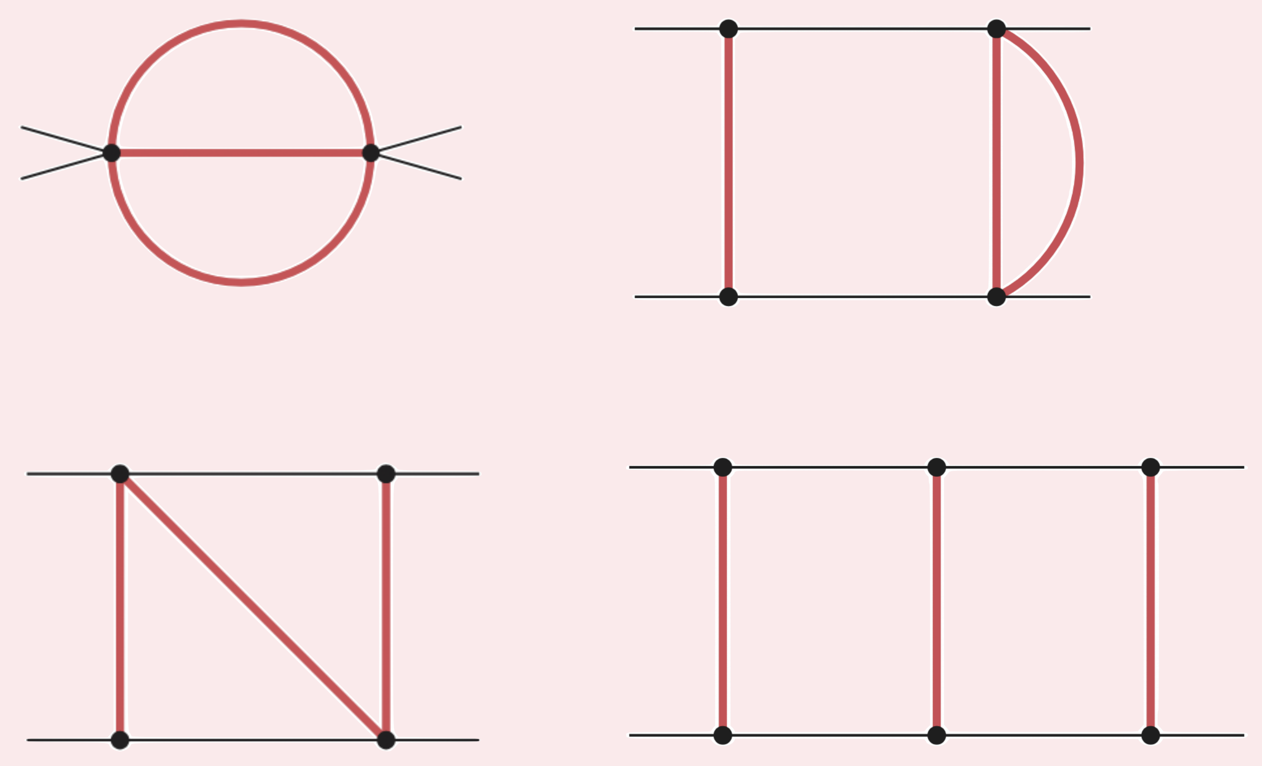
$$\sigma_1 = -(k_1 - p_1)^2 + m^2, \quad \sigma_2 = -(k_1 - p_{12})^2, \quad \sigma_3 = -k_1^2,$$
$$\sigma_4 = -(k_1 + k_2)^2 + m^2, \quad \sigma_5 = -(k_2 + p_{12})^2, \quad \sigma_6 = -k_2^2,$$
$$\sigma_7 = -(k_2 + p_{123})^2 + m^2, \quad \sigma_8 = -(k_1 - p_{13})^2, \quad \sigma_9 = -(k_2 + p_{13})^2$$

IBP reduction

- **35** master integrals - **20** sectors

Geometries:

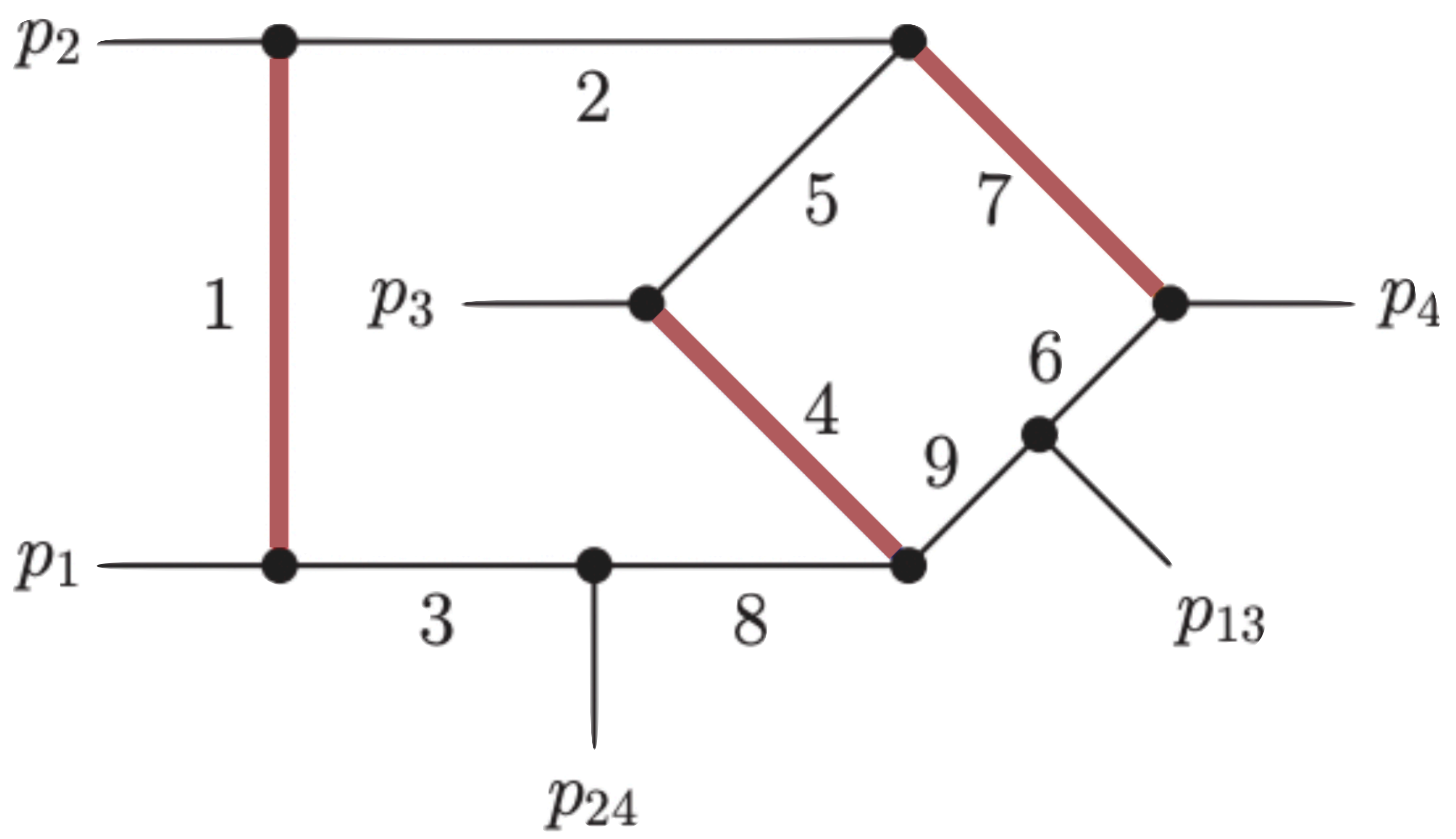
- most sectors are **genus 0**
- 4 **genus 1** sectors:



Non Planar Double Box

$D = 4 - 2\varepsilon$

Scales: m², t, s — rescale away



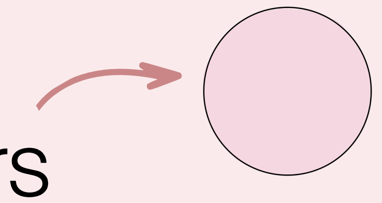
$$\sigma_1 = -(k_1 - p_1)^2 + m^2, \quad \sigma_2 = -(k_1 - p_{12})^2, \quad \sigma_3 = -k_1^2,$$
$$\sigma_4 = -(k_1 + k_2)^2 + m^2, \quad \sigma_5 = -(k_{12} + p_3)^2, \quad \sigma_6 = -k_2^2,$$
$$\sigma_7 = -(k_2 + p_{123})^2 + m^2, \quad \sigma_8 = -(k_1 - p_{13})^2, \quad \sigma_9 = -(k_2 + p_{13})^2$$

IBP reduction

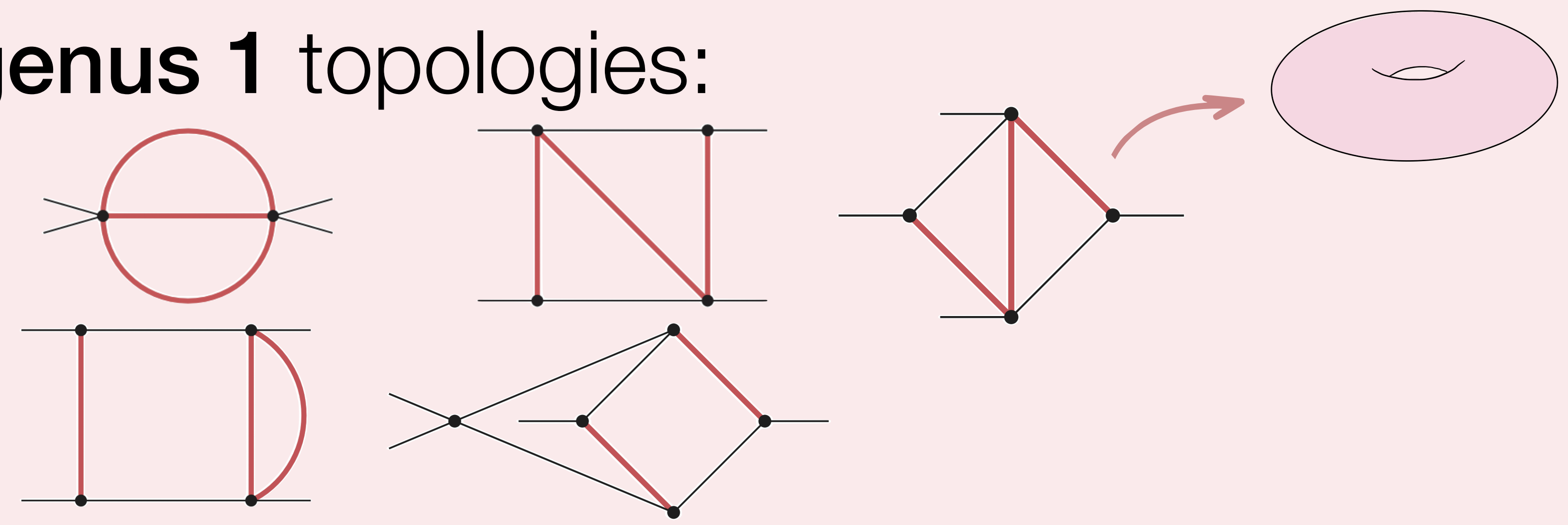
- **67** master integrals - **27** sectors

Geometries:

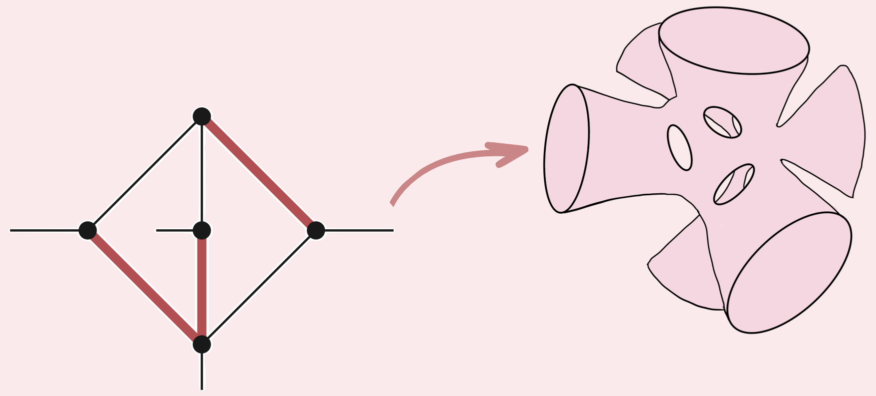
- **genus 0** sectors



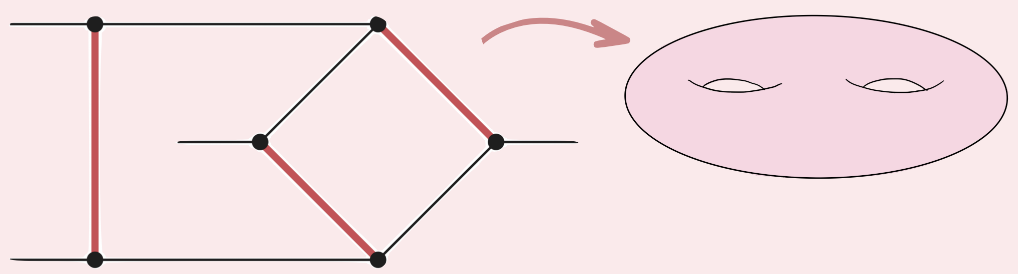
- **genus 1** topologies:



- **K3** surface



- **genus 2** curve



JG|U

Differential Equations and ε-factorization

- By applying **integration-by-parts**, one obtains a differential equation of the form

$$dI = A(\varepsilon, x)\, I$$

find transformation $I = RK$

ε-factorized DEQ

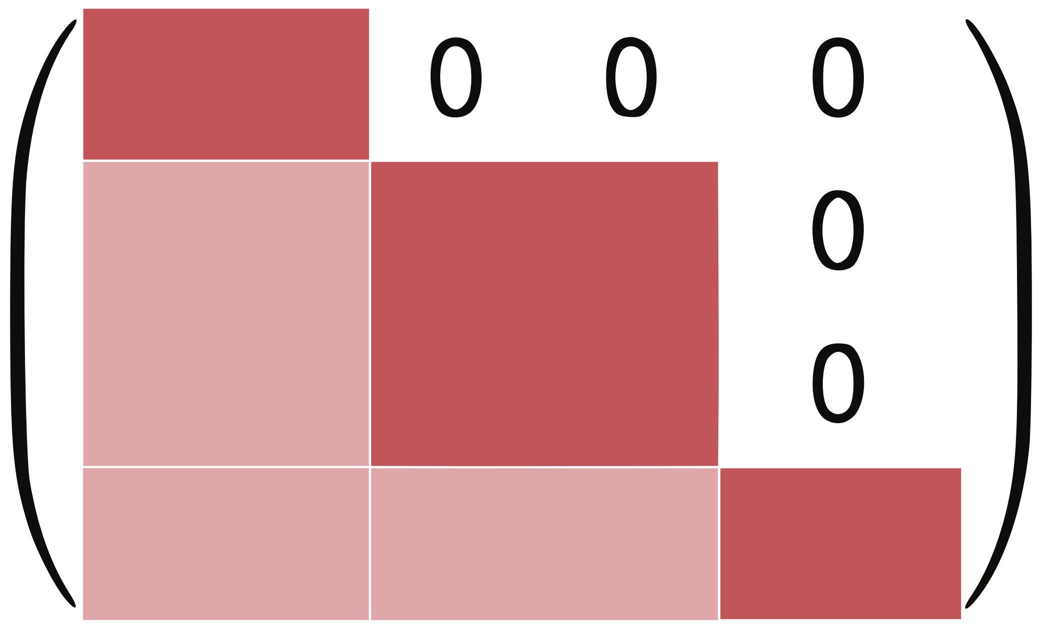
$$dK = \varepsilon \tilde{A}(x)\, K$$

solve this DEQ with appropriate boundary conditions in terms of **iterated integrals**

- If we sort the integrals from least to most complicated, the matrix A has a lower block-triangular structure:

masters fixed on the maximal cut extend naturally beyond it

$A =$



the maximal cut is the **challenging bit** (especially for "complicated" geometries)

How to ε-factorize differential equations

[arXiv:2506.09124]

A few minutes ago
D. Melnichenko

Goal: pick a set of master integrals that ε-factorizes the differential equation without explicitly exploiting the geometry

2-Step Procedure:

1. Pick a basis of **integrands** Ψ using ordering criteria based on pole order and residue.

   Translate into a basis of **integrals** J. The differential equation for J is in **Laurent polynomial form**:

$$dJ = \sum_{k=k_{\text{min}}}^{1} \varepsilon^k A^{(k)}(x)\, J$$

2. Construct a matrix R, such that the differential equation for $K = R^{-1} J$ is in **ε-factorized form**

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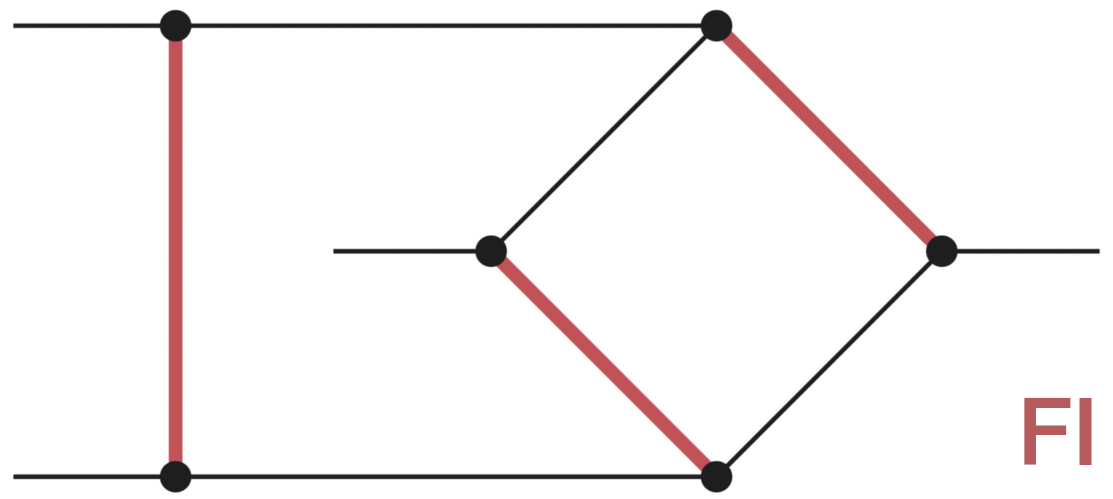
Sector 127

Non Planar Double Box



FI side: 5 MIs

- We can find a **one-dimensional** Baikov representation. The homogenised twist reads:

$$U(z) = [p_0(z)]^{4\epsilon}\,[p_1(z)]^{-\frac{1}{2}}\,[p_2(z)]^{-\frac{1}{2}}\,[p_3(z)]^{-\frac{1}{2}-\epsilon}\,[p_4(z)]^{-\frac{1}{2}-\epsilon}$$

Even

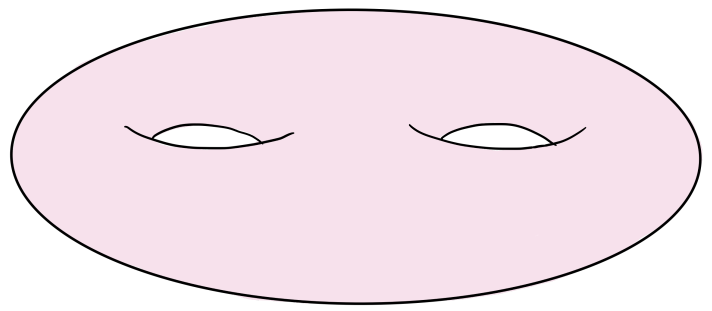
$$p_0(z) = z_0$$

Odd

$$p_1(z) = (m^2 - t)z_0 - z_1$$
$$p_2(z) = (m^2 - s - t)z_0 - z_1$$
$$p_3(z) = (9m^4 - 5m^2 s)z_0^2 - \dots$$
$$p_4(z) = 4m^4(s+t)z_0^2 + sz_1^2 - \dots$$

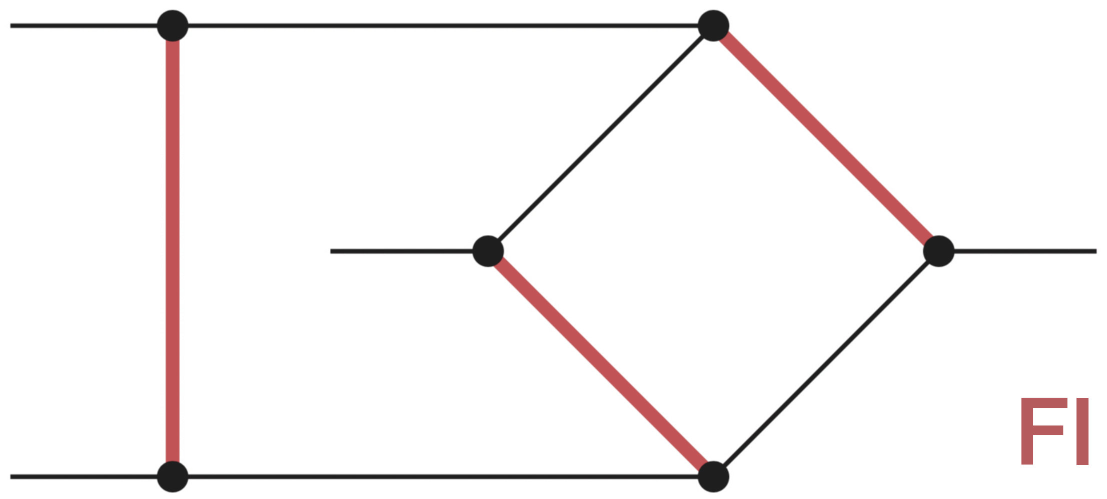
$$U_{\epsilon=0}(z_1)\big|_{z_0=1} \sim \frac{1}{\sqrt{P_6(z_1)}}$$

Genus 2



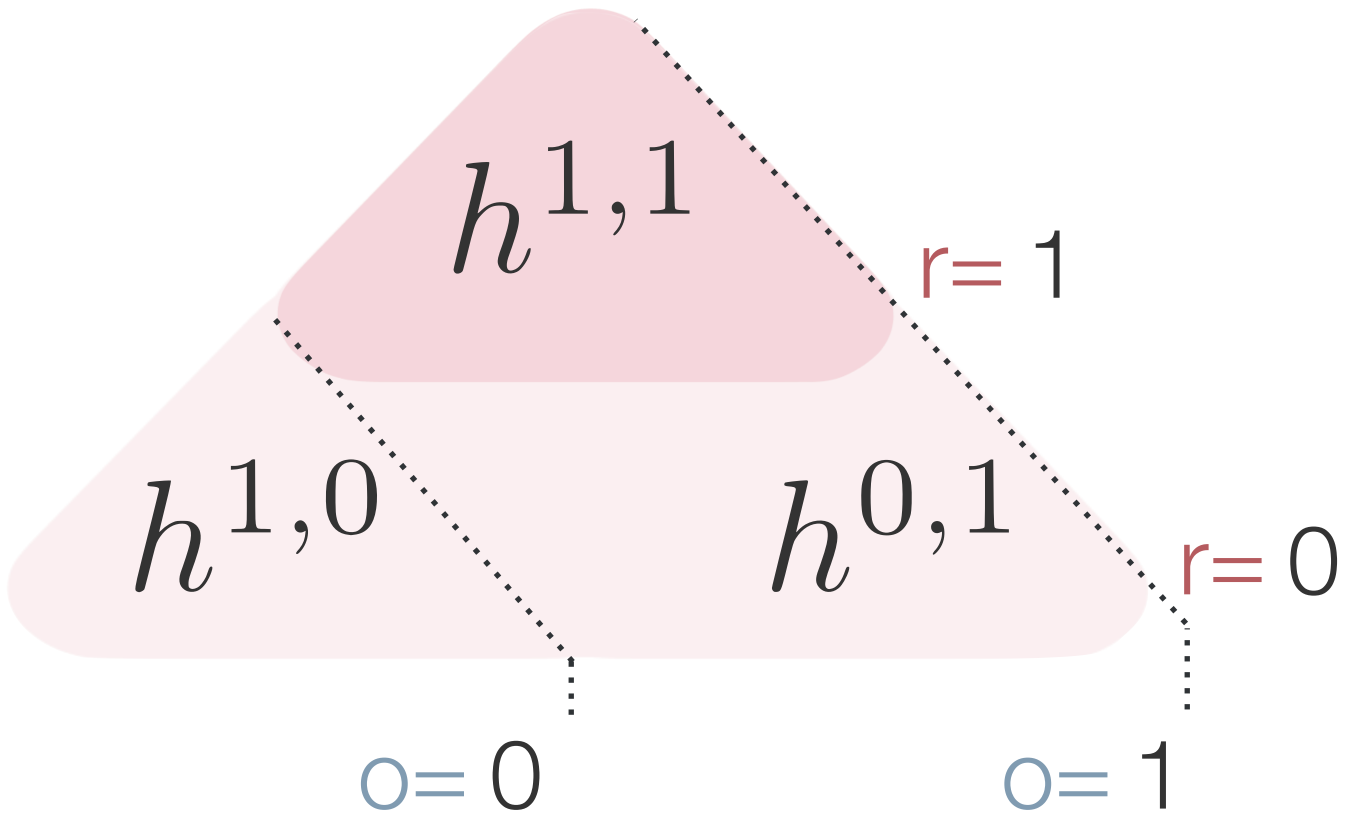
Sector 127

Non Planar Double Box



FI side: 5 MIs

- Running the full IBP reduction, we find **5** basis forms:



$$\Psi_1 = \epsilon^4\, U(z)\, z_0\, \eta$$
$$\Psi_2 = \epsilon^4\, U(z)\, z_1\, \eta$$

r= 0 o= 0 — $|\mu| = 0$

$$\Psi_3 = \epsilon^4 s\, U(z)\, \frac{z_1^2}{z_0}\, \eta$$

r= 1 o= 1

$$\Psi_4 = \epsilon^3\, U(z)\, \frac{z_0\, Q}{p_3(z)}\, \eta$$
$$\Psi_5 = \epsilon^3\, U(z)\, \frac{z_1\, Q}{p_3(z)}\, \eta$$

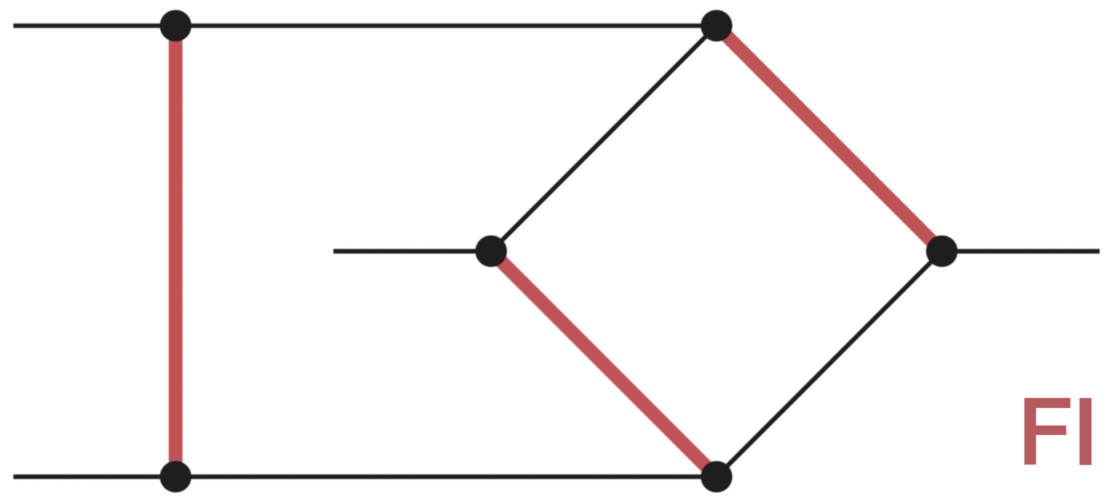
r= 0 o= 1

(Ψ_3, Ψ_4, Ψ_5): $|\mu| = 1$

$$Q = (-3m^2 + t)(z_0 + z_1)\, z_0$$

Sector 127

Non Planar Double Box



FI side: 5 MIs

- Now, we must translate the master forms into Feynman integrals.

\# master integrals   # master integrands

$$\dim V = \dim H_\omega$$

straightforward translation

H_ω

$$\Psi_1 = \epsilon^4\, U(z)\, z_0\, \eta$$

$$\Psi_2 = \epsilon^4\, U(z)\, z_1\, \eta$$

$$\Psi_3 = \epsilon^4 s\, U(z) \frac{z_1^2}{z_0}\eta$$

$$\Psi_4 = \epsilon^3\, U(z) \frac{z_0 Q}{p_3(z)}\eta$$

$$\Psi_5 = \epsilon^3\, U(z) \frac{z_1 Q}{p_3(z)}\eta$$

V

$$I^{NP}_{1\,1\,1\,1\,1\,1\,1\,0\,0\,0}$$

$$I^{NP}_{1\,1\,1\,1\,1\,1\,1\,-1\,0}$$

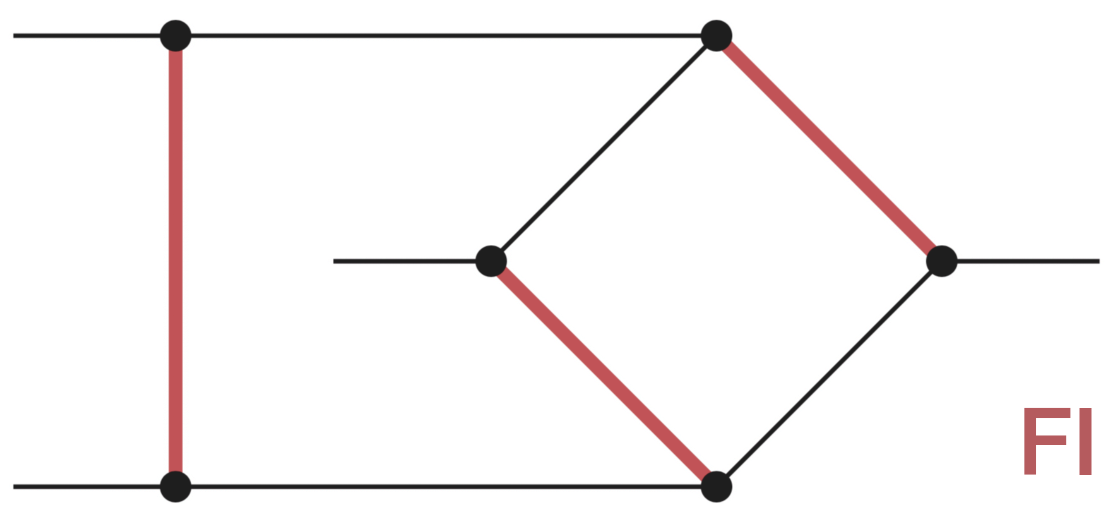
$$I^{NP}_{1\,1\,1\,1\,1\,1\,1\,-2\,0}$$

$$I^{NP}_{1\,1\,1\,1\,1\,1\,2\,0\,0}$$

$$I^{NP}_{1\,1\,1\,1\,1\,1\,2\,-1\,0}$$

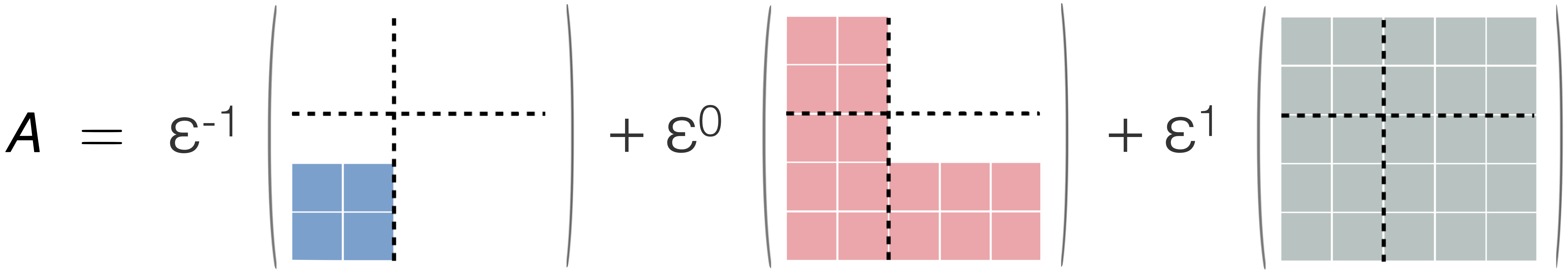
Sector 127

Non Planar Double Box

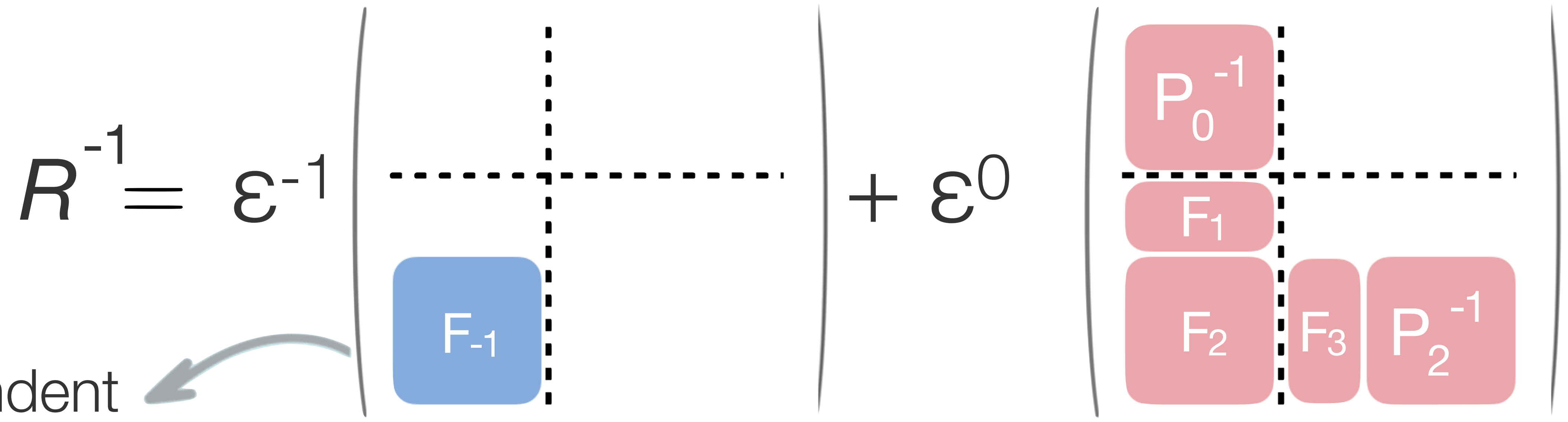


FI side: 5 MIs

- Using the basis found by the algorithm, we find the DEQ is indeed a Laurent polynomial in ε:



- The rotation to an ε-factorized DEQ can be achieved by the matrix *R*,

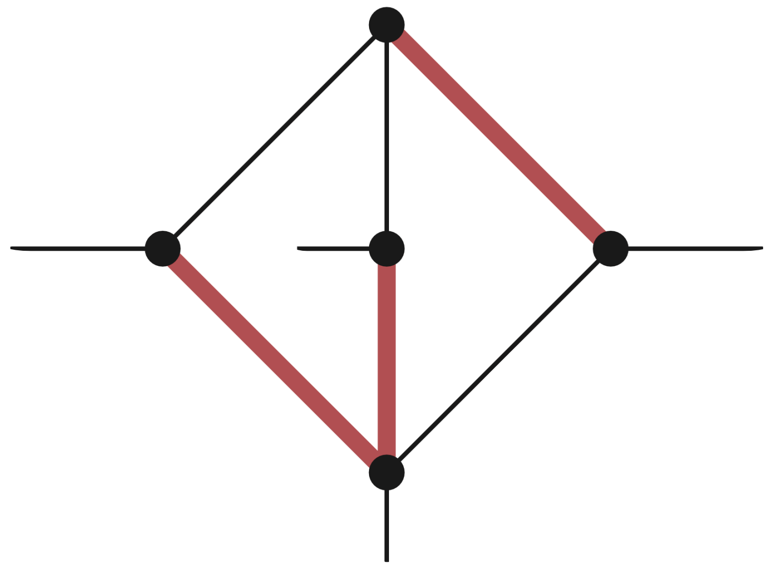


solution to ε-independent differential equations

JG|U

Sector 123

Non Planar Double Box



FI side: 8 MIs

- We can find a **two-dimensional** Baikov representation. The homogenised twist reads:

$$U(z) = [p_0(z)]^{4\epsilon}\ [p_1(z)]^{\epsilon}\ [p_2(z)]^{\epsilon}\ [p_3(z)]^{-\frac{1}{2}-\epsilon}\ [p_4(z)]^{-\frac{1}{2}-\epsilon}$$

Even

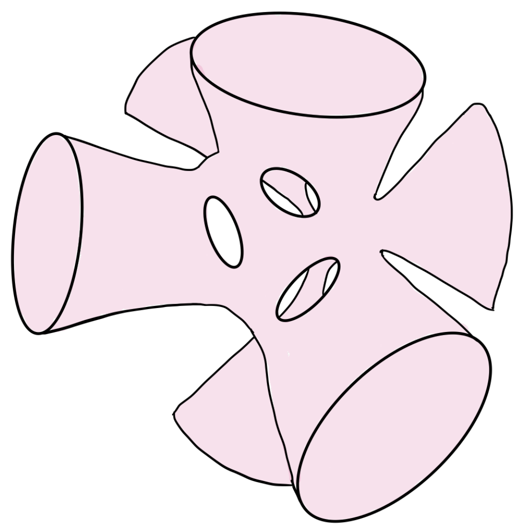
$$p_0(z) = z_0$$
$$p_1(z) = z_1$$
$$p_2(z) = z_2$$

Odd

$$p_3(z) = (m^2 s\, z_0 - t\, z_1)^2 + \ \dots$$
$$p_4(z) = m^4 z_0^2(-s\, z_0 + z_1)^2 - \ \dots$$

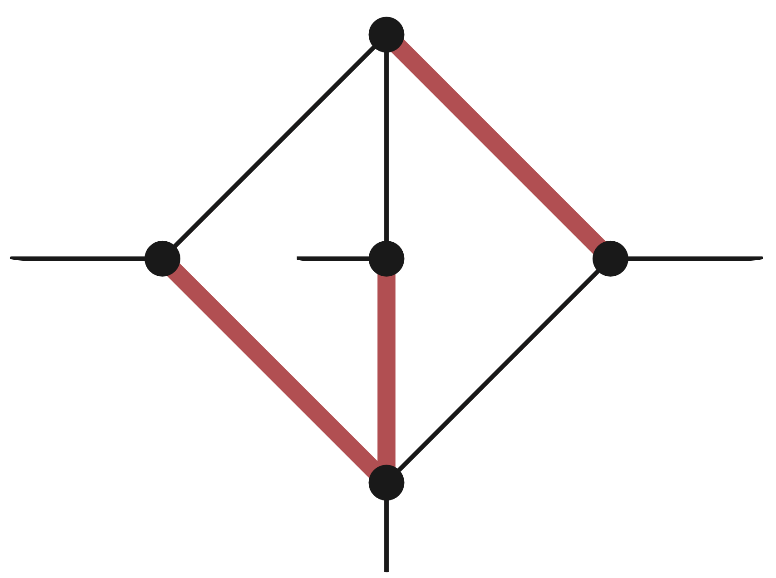
$$U_{\epsilon=0}(z_1, z_2)\big|_{z_0=1} \sim \frac{1}{\sqrt{P_6(z_1, z_2)}}$$

K3



Sector 123

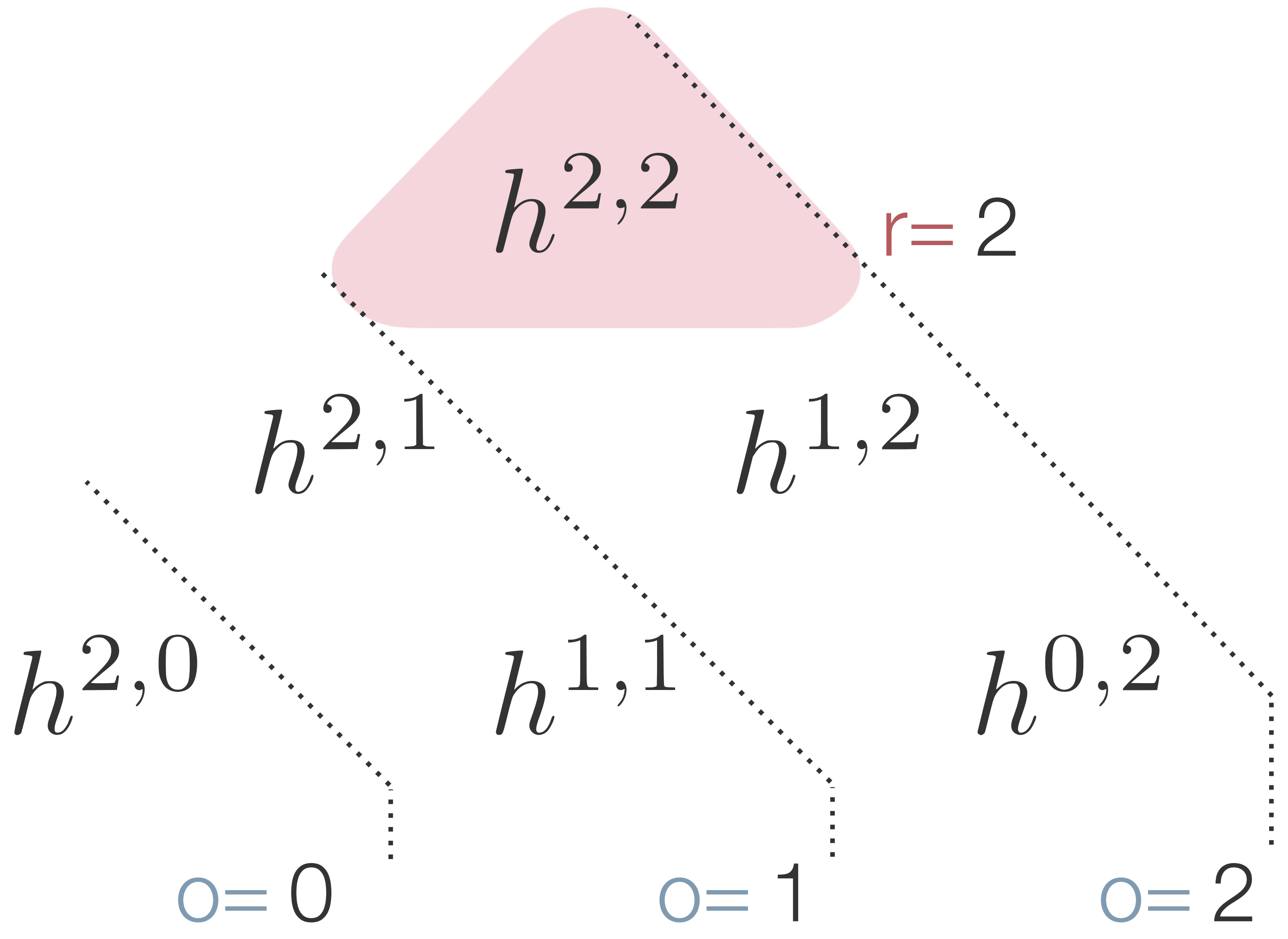
Non Planar Double Box



FI side: 8 MIs

- Running the full IBP reduction, we find **12** basis forms.

$\neq$ number Feynman Integrals



- At r= 2, we find 6 forms:

$$\Psi_2 = \epsilon^4\, U(z) \frac{z_1}{z_0} \eta$$

$$\Psi_3 = \epsilon^4\, U(z) \frac{z_2}{z_0} \eta$$

$$\Psi_4 = \epsilon^4\, U(z) \frac{z_0}{z_1} \eta$$

$$\Psi_5 = \epsilon^4\, U(z) \frac{z_2}{z_1} \eta$$

$$\Psi_6 = \epsilon^4\, U(z) \frac{z_0}{z_2} \eta$$

$$\Psi_7 = \epsilon^4\, U(z) \frac{z_1}{z_2} \eta$$

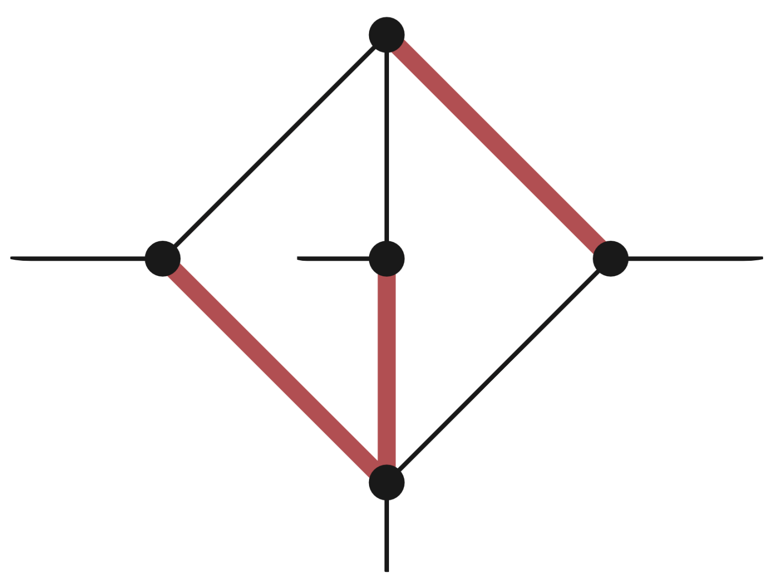
$|\mu| = 1$

Ψ_4–Ψ_7: vanish upon translation to FIs

no discrepancy in number of MIs

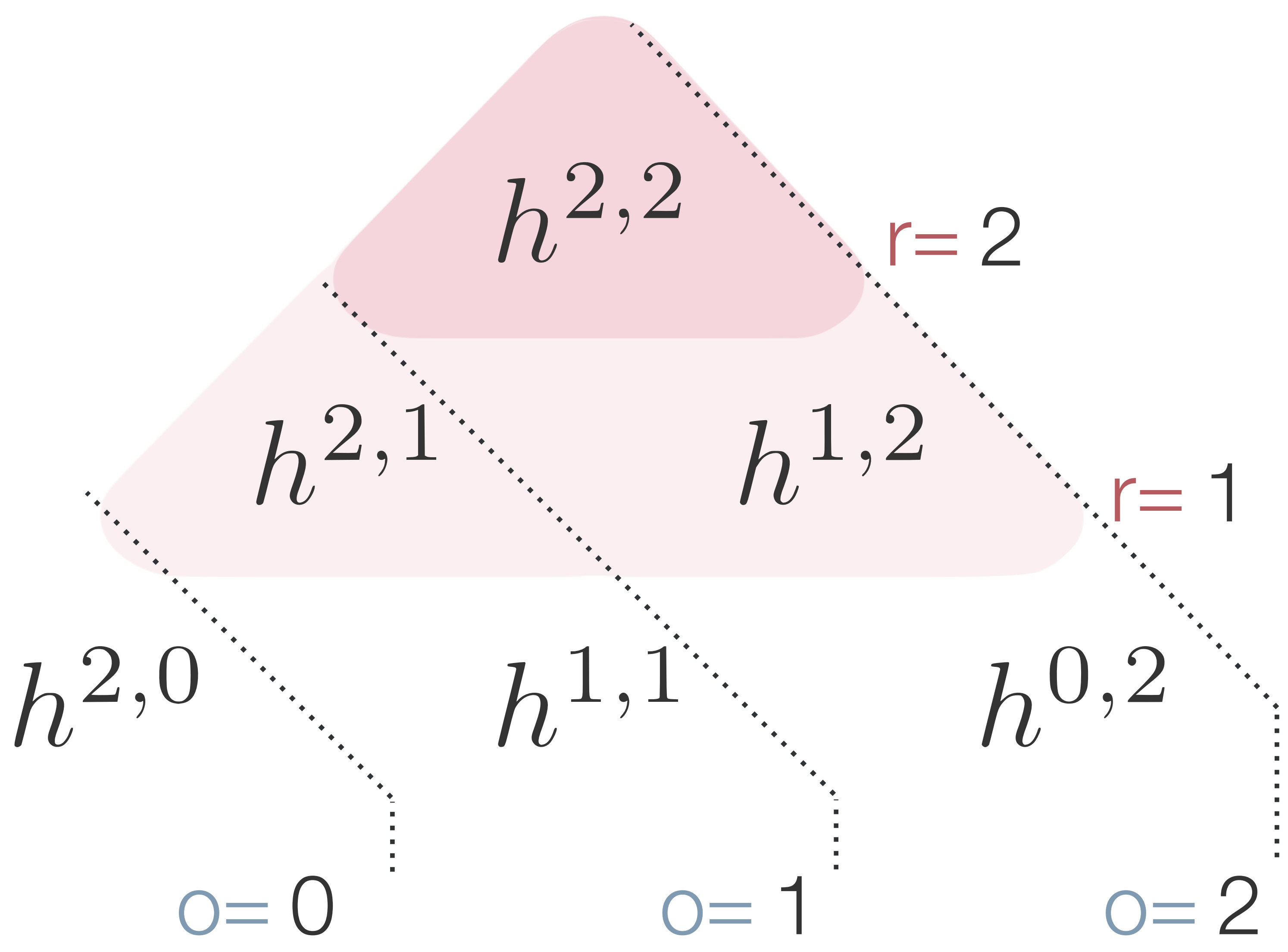
Sector 123

Non Planar Double Box



FI side: 8 MIs

- Running the full IBP reduction, we find **12** basis forms.



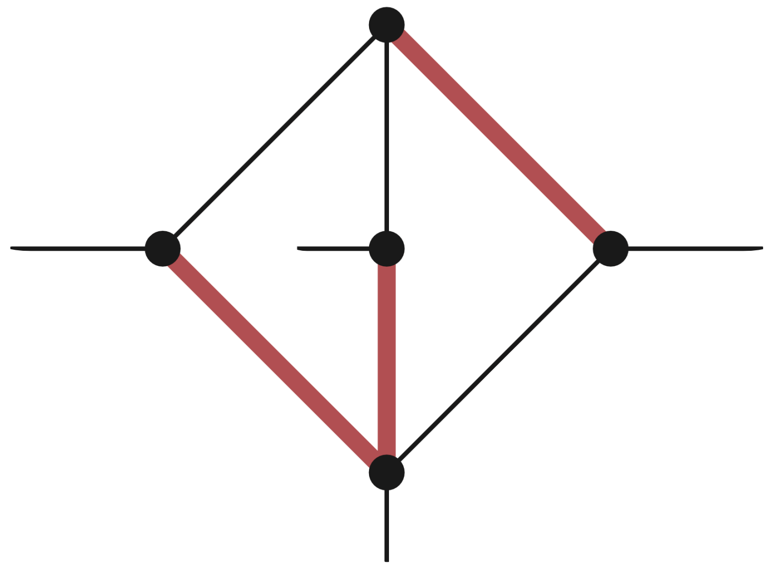
- At r= 1, we find 0 forms.

if either even polynomial is in the denominator, taking a residue will make one of the odd polynomials a **perfect square**

can take second residue

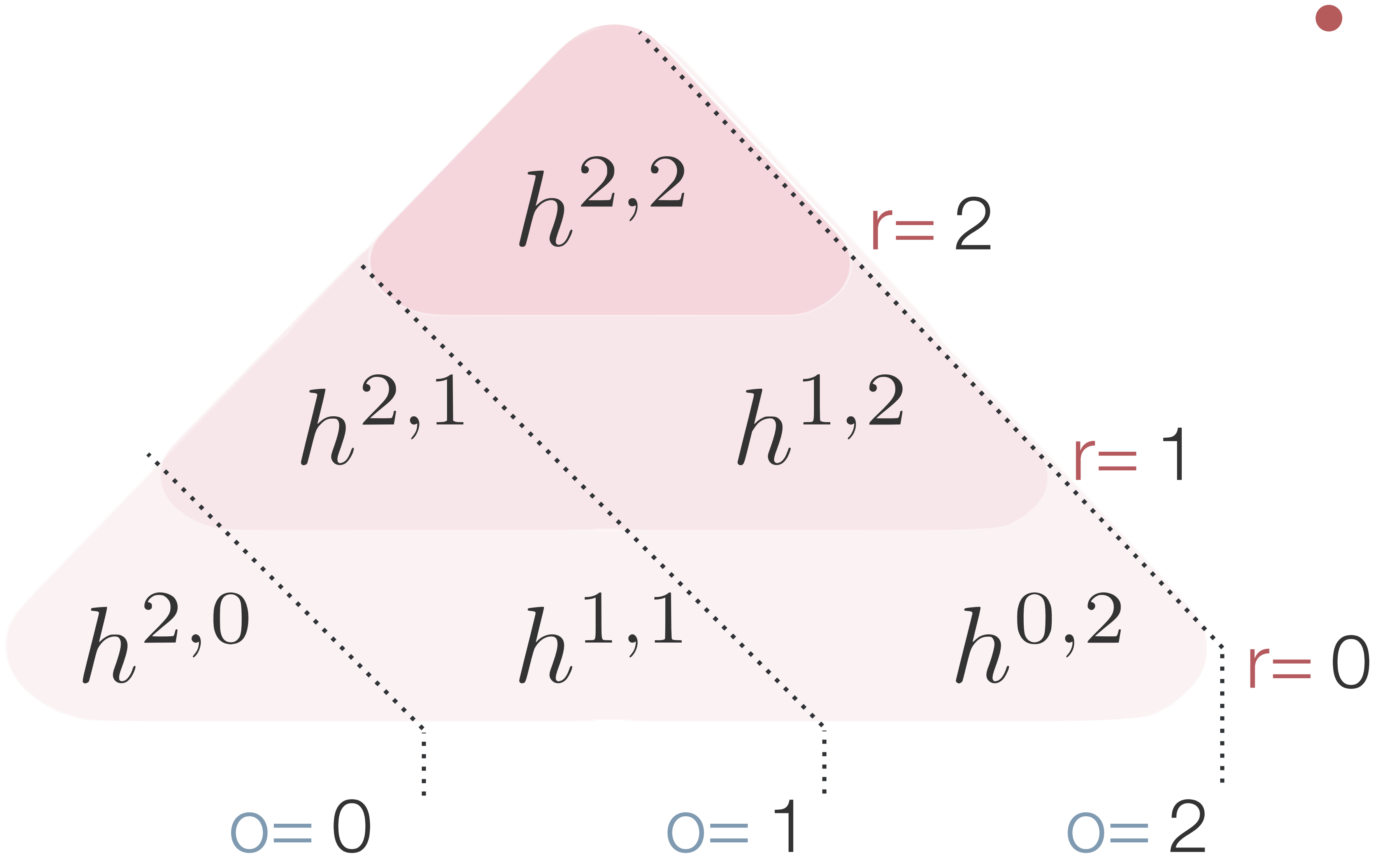
Sector 123

Non Planar Double Box



FI side: 8 MIs

- Running the full IBP reduction, we find **12** basis forms.



- At r= 0, we find, once again, 6 forms:

$$\Psi_1 = \epsilon^4\, U(z)\, \eta \qquad |\mu| = 0$$

$$\Psi_8 = \epsilon^4\, U(z) \frac{z_0^2 z_2^2}{p_4(z)} \eta$$

$$\Psi_9 = \epsilon^4\, U(z) \frac{z_0 z_2}{p_3(z)} \eta$$

$$\Psi_{10} = \epsilon^4\, U(z) \frac{z_1^2}{p_3(z)} \eta$$

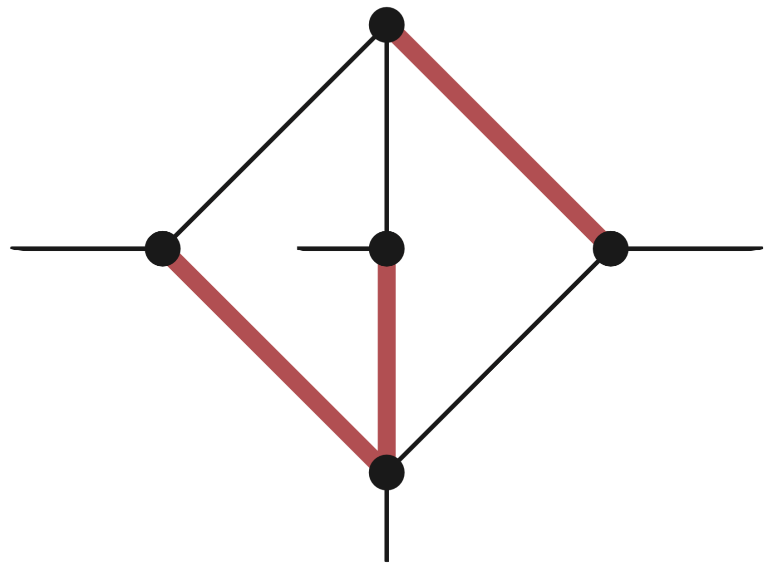
$$\Psi_{11} = \epsilon^4\, U(z) \frac{z_0 z_1}{p_3(z)} \eta$$

$$\Psi_{12} = \epsilon^4\, U(z) \frac{z_0^2}{p_3(z)} \eta$$

$\Psi_8, \ldots, \Psi_{12}$: $|\mu| = 1$

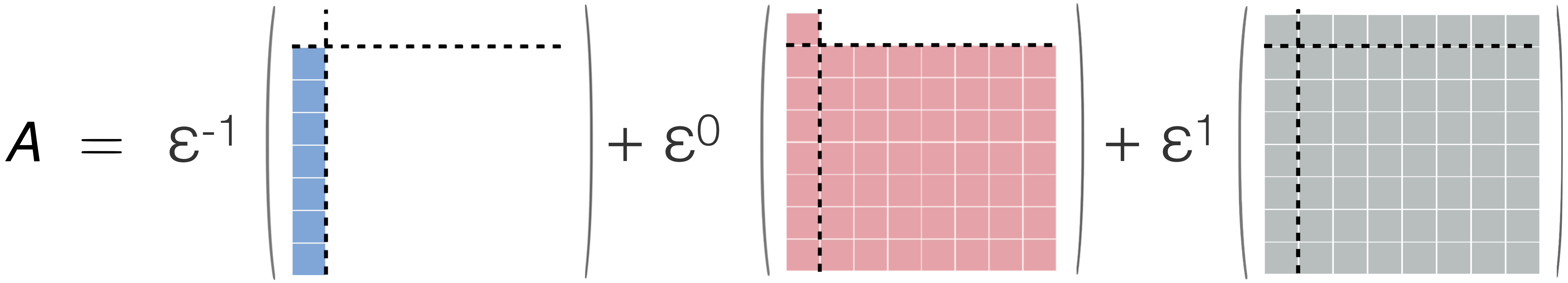
Sector 123

Non Planar Double Box

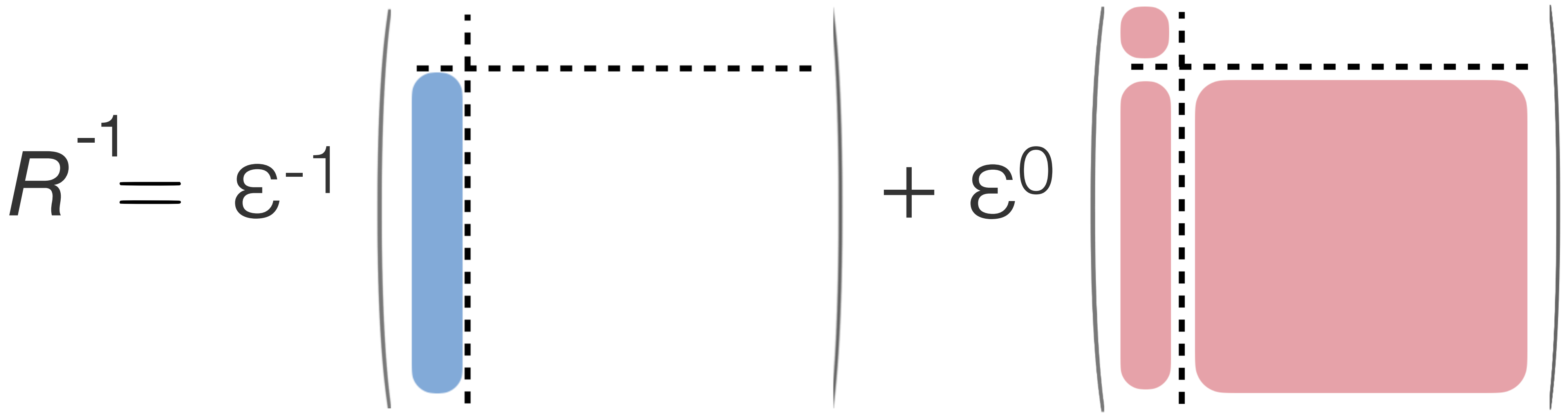


Kira: 8 MIs

- In translating the basis back into V, we must be mindful of the vanishing integrands.
- Using the basis found by the algorithm, we find the DEQ is indeed a Laurent polynomial in ε:

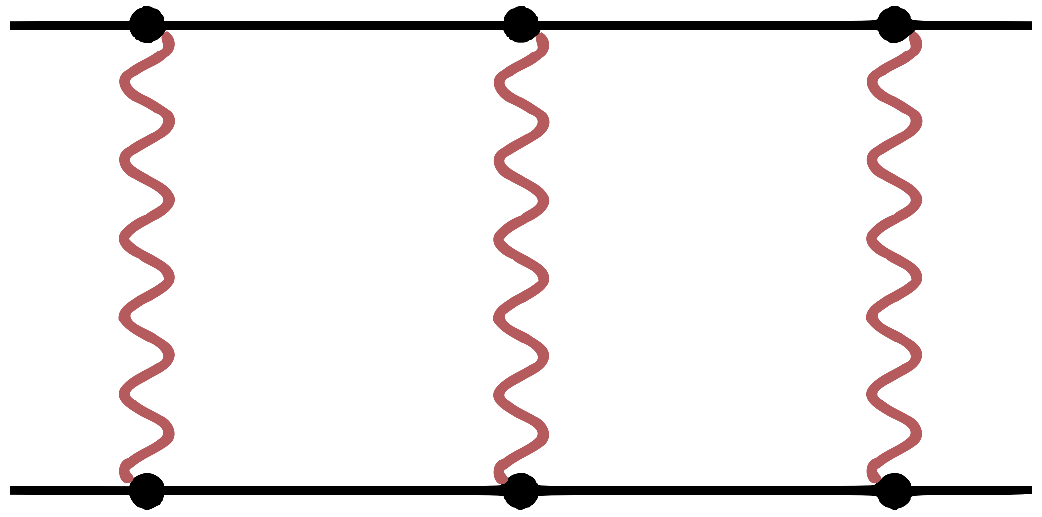


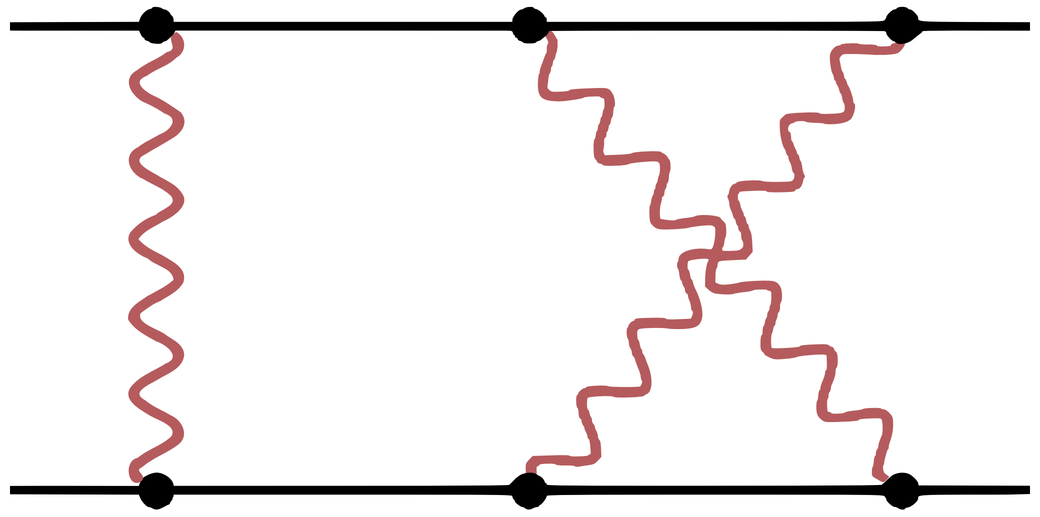
- Finally, the rotation to an ε-factorized form can be achieved by a matrix of the shape



Closing Remarks

- We considered the planar and non-planar **double-box** integrals with three massive propagators.
- The non-planar double-box is a particularly challenging integral, as it is associated to trickier geometries: **K3** and **genus 2**.
- We showed how these integrals can be computed systematically with **no prior knowledge of the geometries**, using the general algorithm of [arXiv:2506.09124] .





Thank you!