The geometric bookkeeping guide to Feynman integral reduction and ε-factorised differential equations

MPA Retreat

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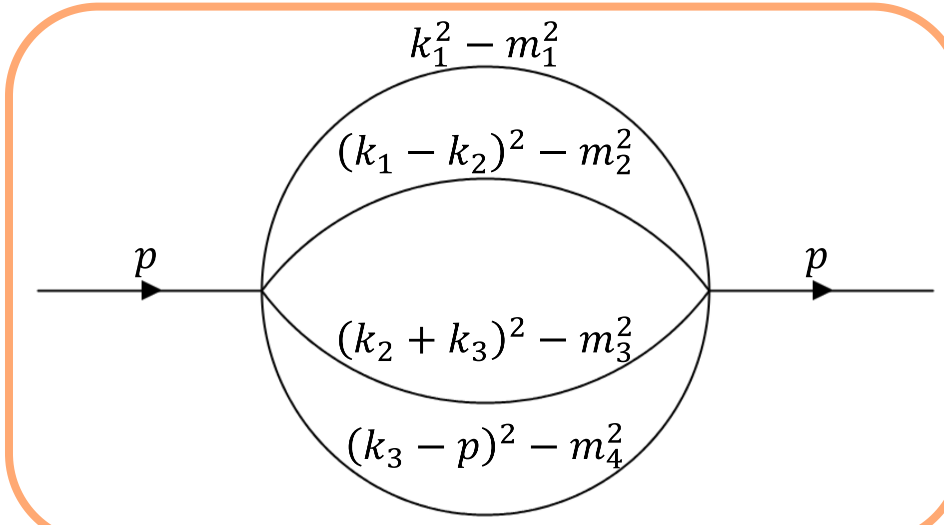
Introduction

Feynman Integrals

Feynman Diagram:



Integral Family:

$D = D_{\text{int}} - 2\varepsilon$

$$I_{n_1,\dots,n_N} \equiv e^{l\epsilon\gamma_E}\left(\mu^2\right)^{n-\frac{lD}{2}} \int \frac{d^D k_1}{(i\,\pi)^{D/2}} \cdots \frac{d^D k_l}{(i\,\pi)^{D/2}} \frac{1}{\sigma_1^{n_1}\cdot \ldots \cdot \sigma_N^{n_N}}$$

$\sigma_j = q_j^2 - m_j^2$

loop momenta

propagators

Integration-by-Parts Identities

$$I_{n_1,\dots,n_N} \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

$$\int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \frac{\partial}{\partial k_{q,\mu}} \left\{ v_\mu \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_r}} \right\} = 0 \qquad \partial_{k_q}$$

loop momentum any momentum

Laporta Algorithm

Solve ***complicated*** integrals through ***simple*** integrals

$J_1 - J_2 + 8\,n^2\,J_3 - 2\,n\,J_4 - 2\,n\,J_5 - 4\,n\,J_6 == 0$
$-8\,n\,J_3 + 2\,J_4 - 2\,J_5 + J_6 + J_7 == 0$
$J_2 - 8\,n\,J_3 - 2\,J_4 + 2\,J_5 + \left(1 + \frac{1}{\text{eps}}\right) J_8 == 0$
$-2\,n\,J_3 + J_8 - J_9 + 2\,n^2\,J_{10} - 2\,n\,J_{11} - 4\,n\,J_{12} == 0$
$-2\,J_3 - 2\,n\,J_{10} + 2\,J_{11} + J_{12} + J_{13} == 0$
$2\,J_3 + J_9 - 2\,n\,J_{10} - 2\,J_{11} + J_{14} == 0$
$J_2 - J_{15} + 2\,n^2\,J_{16} - 2\,n\,J_{17} - 2\,n\,J_{18} - 4\,n\,J_{19} == 0$
$-2\,n\,J_{16} + 2\,J_{17} - 2\,J_{18} + J_{19} + J_{20} == 0$

Gauss Elimination

Ordering?

Minimal set of ***unknown J***

$=$

Master Integrals

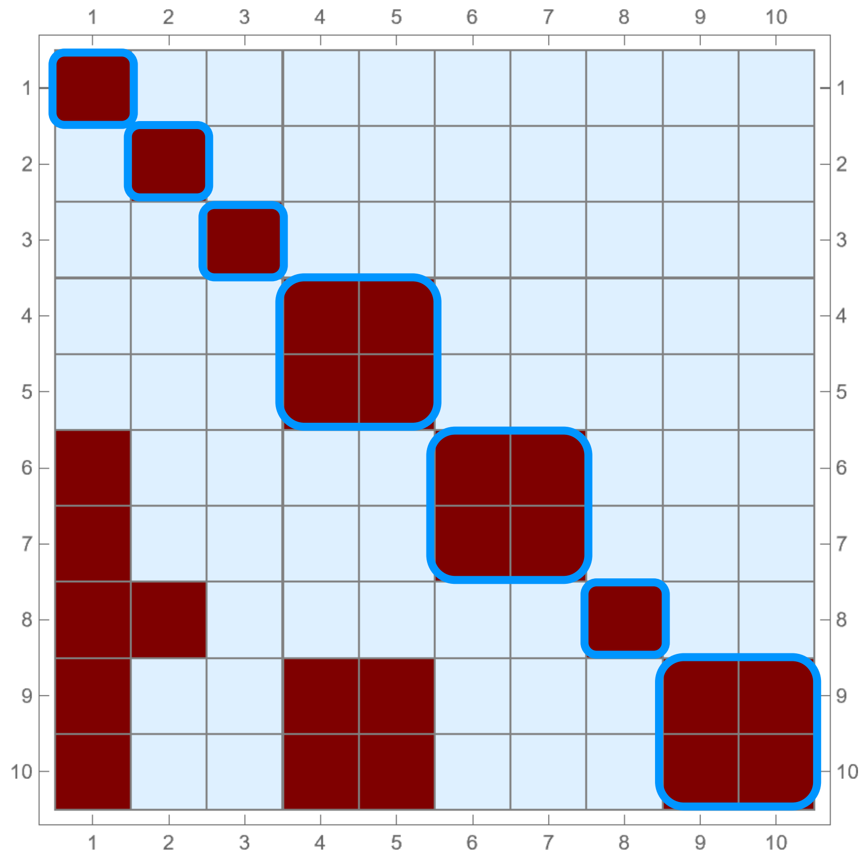
Differential Equations

$$\frac{d}{dx} I_1 = \tilde{I}_1 = c_1^1(x,\varepsilon) I_1 + \ldots + c_{N_F}^1(x,\varepsilon) I_{N_F}$$

$$\vdots \qquad \vdots \quad \text{IBPs} \qquad \vdots \quad \vdots \qquad \vdots$$

$$\frac{d}{dx} I_{N_F} = \tilde{I}_{N_F} = c_1^{N_F}(x,\varepsilon) I_1 + \ldots + c_{N_F}^{N_F}(x,\varepsilon) I_{N_F}$$

$$\boxed{\frac{d}{dx} \vec{I} = \tilde{A}(x,\varepsilon) \vec{I}}$$



Integration $\Longleftrightarrow$ **Solving DEQs**

Homogeneous Parts

ε-Factorized Form

$$R^{-1}\tilde{A}(\varepsilon,x)R - R^{-1}\frac{d}{dx}R$$

"Rotate" the basis $\vec{K} = R^{-1}(\varepsilon, x)\vec{I}$:

$$\frac{d}{dx}\vec{K} = \varepsilon \boxed{A(x)}\vec{K}$$

Ansatz $\vec{K} = \sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$:

LHS: $$\frac{d}{dx}\sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$$

RHS: $$\varepsilon A(x)\sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$$

Iterative solution:

$$O(\varepsilon^0): \frac{d}{dx}\vec{K}^{(0)} = 0$$

$$O(\varepsilon^1): \frac{d}{dx}\vec{K}^{(1)} = A(x)\vec{K}^{(0)}$$

$\vdots$

$$O(\varepsilon^n): \frac{d}{dx}\vec{K}^{(n)} = A(x)\vec{K}^{(n-1)}$$

Algorithm

Steps

1. Preliminary $\vec{J}$-basis using *clever ordering*

$$\boxed{\frac{d}{dx}\vec{J} = \sum_{k=k_{\min}}^{k=1} \varepsilon^k A^{(k)}(x)\vec{J}}$$

$$\Big\downarrow\ \vec{K} = R^{-1}(\varepsilon, x)\vec{J}$$

2. *Rotate* into ε-factorized $\vec{K}$-basis

$$\boxed{\frac{d}{dx}\vec{K} = \varepsilon A(x)\vec{K}}$$

Building Blocks

Feynman Integrand
(in Baikov representation)

$$\int \boxed{\Psi_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] = C_\varepsilon U(z) \widehat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] \eta} \propto I_{n_1,\ldots,n_N}$$

Objects

- Appropriate pre-factor C_ε
- Twist $U(z)$
- Rational polynomials P_i
- Differential form $\widehat{\Phi}$

$$U(z_0, \ldots, z_n) = \prod_{i,\mathrm{odd}} P_i^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j,\mathrm{even}} P_j^{\frac{1}{2}b_j\varepsilon}$$

$$\widehat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] = \frac{\mathrm{Q}}{\prod_i P_i^{\mu_i}}$$

$$\eta = \sum_{j=0}^{n} (-1)^j z_j dz_0 \wedge \cdots \wedge \cancel{dz_j} \wedge \cdots dz_n$$

Ordering!

$(a, r, o, |\mu|, \dots)$

Example

$$\frac{\eta}{z_1(z_2 - x)(z_1 + z_2)}$$

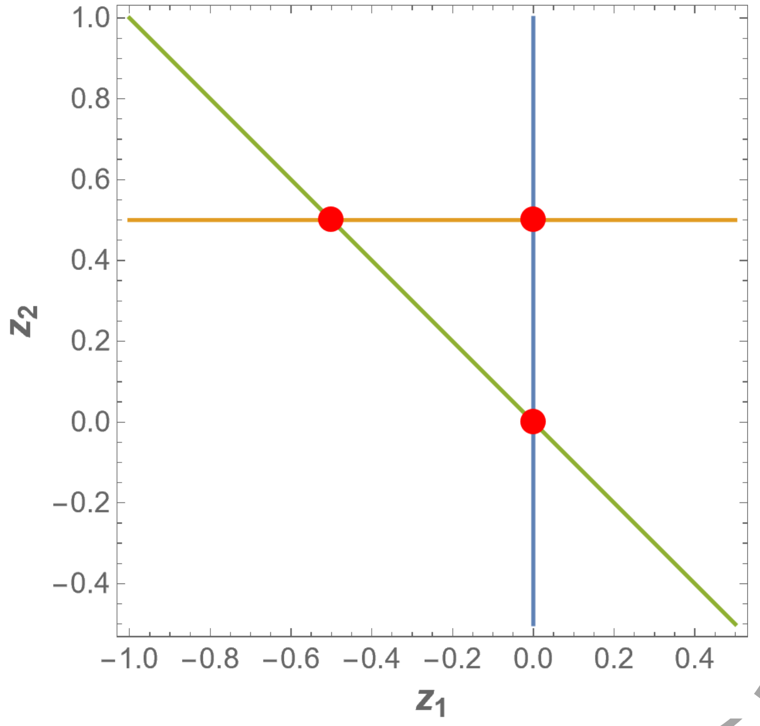
2 consecutive residues at

$(z_1, z_2) = \{(0,0), (0, x), (-x, 0)\}$

PO 2 at

$(z_1, z_2) = \{(0,0), (0, x), (-x, 0)\}$

$|\boldsymbol{\mu}| = 1 + 1 + 1 = 3$



Number of residues of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

$\varepsilon \to 0$

Pole order of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

Denominator power of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

Ordering!

$(a, r, o, |\mu|, \ldots)$

Number of **residues** of $\Psi^0_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}]$

Pole order of $\Psi^0_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}]$

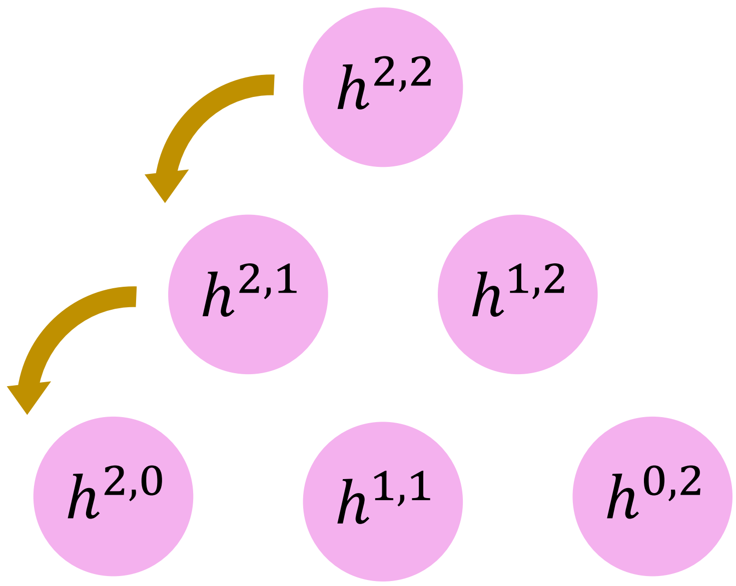
Denominator power of $\Psi^0_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}]$

Localization level of $\Psi^0_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}]$

$\vec{J}$**-basis**

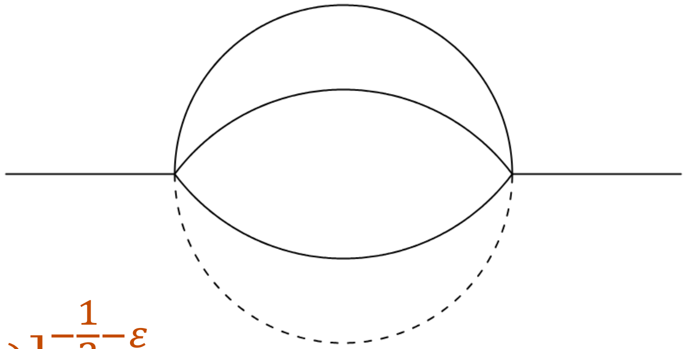
Example

Find basis of subsystems



Step 1 ✓

Example: Electron Self-Energy



$$U^0 = \underbrace{z_0^{4\varepsilon} z_2^{\varepsilon} (z_0 - z_2)^{-2\varepsilon}}_{P_{\text{even}}} \underbrace{z^{-\frac{1}{2}} (z_1 - 4xz_0)^{-\frac{1}{2}-\varepsilon} \left[x^2 z_0^2 + (z_1 - z_2)^2 - 2xz_0(z_1 + z_2)\right]^{-\frac{1}{2}-\varepsilon}}_{P_{\text{odd}}}$$

$$\Phi_1 = \frac{1}{z_2}$$

$r = 2, o = 2, |\mu| = 1$

$$\Phi_2 = \frac{z_1}{z_0(z_0 - z_2)}$$

$r = 2, o = 2, |\mu| = 2$

$$\Phi_3 = \frac{1}{z_0}$$

$r = 2, o = 2, |\mu| = 1$

$$\Phi_4 = \frac{1}{z_0 - z_2}$$

$r = 1, o = 1, |\mu| = 1$

$$\Phi_5 = \frac{z_0}{(z_1 - 4xz_0)(z_0 - z_2)}$$

$r = 1, o = 2, |\mu| = 1$

$$\Phi_6 = \frac{1}{z_1 - 4xz_0}$$

$r = 0, o = 1, |\mu| = 0$

3

1 1

0 1 0

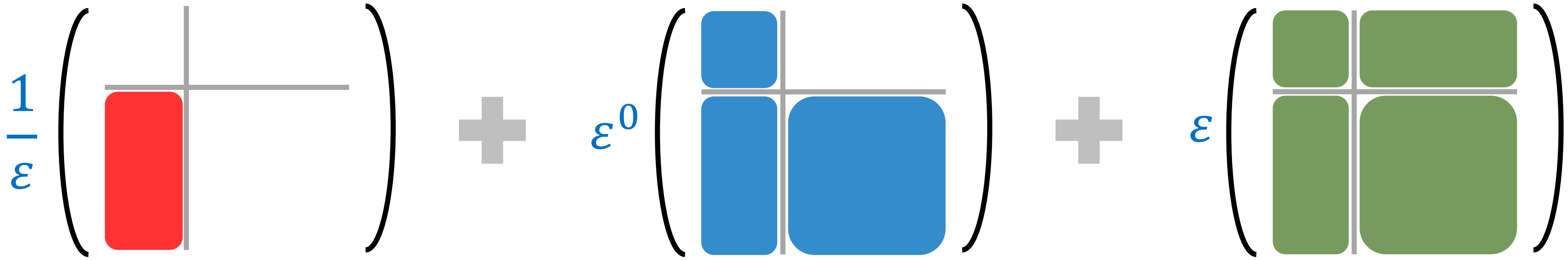
6 Forms $\Rightarrow$ **3 Integrals**

Example: Electron Self-Energy

$$I_{n_1,\dots,n_N} \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

$$\vec{J} \propto \left\{ \mathrm{I}_{1,1,1,1,\vec{0}}, \mathrm{I}_{1,2,1,1,\vec{0}}, \mathrm{I}_{1,1,1,1,-1,\vec{0}} \right\}$$

$$d_x \vec{J} = \underbrace{\left(\begin{array}{c|cc} 0 & 6\varepsilon & 0 \\ \hline \frac{1-3x}{2F(x)}\frac{1}{\varepsilon} + \frac{5-21x}{2F(x)} + \frac{1+9x-18x^2}{xF(x)}\varepsilon & -\frac{1-20x+27x^2}{F(x)} + \frac{1+30x-63x^2}{F(x)}\varepsilon & \frac{3}{2xF(x)}\varepsilon \\ -4 - \frac{8(1+6x)}{3x}\varepsilon & -8\,(1+3x)\varepsilon & -\frac{4}{x}\varepsilon \end{array} \right)}_{} \vec{J}$$



"Rotation"

$$\vec{K} = R^{-1}(\varepsilon, x)\vec{J} = \left[\boxed{R^{(-1)}(\varepsilon, x)}\ \boxed{R^{(0)}(\varepsilon, x)}\right]\vec{J}$$

remove ε^{-1}

$$\begin{pmatrix} \varepsilon^0 g_1 & & \\ \varepsilon^{-1} g_2 & \varepsilon^0 g_4 & \varepsilon^0 g_6 \\ \varepsilon^{-1} g_3 & \varepsilon^0 g_5 & \varepsilon^0 g_7 \end{pmatrix}$$

remove ε^0

$$\begin{pmatrix} 1 & & \\ \varepsilon^0 f_1 & 1 & \\ \varepsilon^0 f_2 & & 1 \end{pmatrix}$$

Functions g_j, f_j depend only on kinematics x

Integration Kernels

$$F(x) = x(x-1)(x-9)$$

Period of an elliptic curve

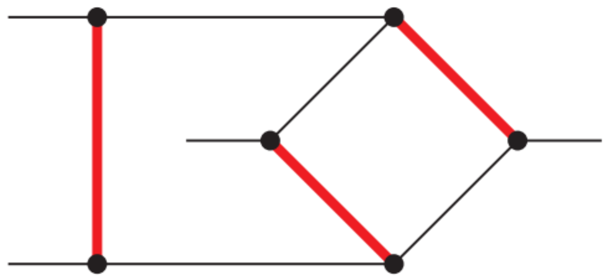
Applied to the example:

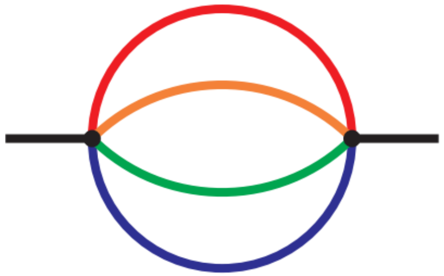
$$d_x\vec{K} = \boldsymbol{\varepsilon} \begin{pmatrix} \frac{1+30x-63x^2}{2F(x)} & \frac{6}{xF(x)\psi^2(x)} & 0 \\ \frac{\cdots}{24F(x)} & \frac{1+30x-63x^2}{2F(x)} & \frac{3\psi(x)}{2x} \\ \frac{8\psi(x)}{3x} & 0 & -\frac{4}{x} \end{pmatrix}\vec{K}$$

Summary

- Feynman Integrals are **building blocks** of scattering amplitudes
- Their analytic computation can be achieved using **DEQ and ε-factorization**
- Proposed approach constructs this transformation **algorithmically**

More examples to follow!

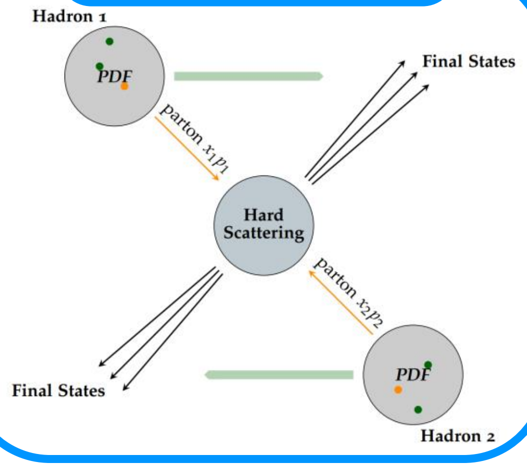




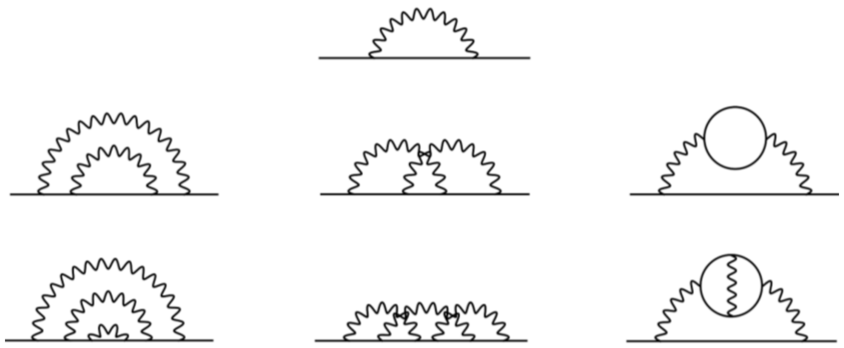
Back-Up Slides

Motivation

Hadron Collision



Loop Corrections



Feynman Integrals

$$I[n_1, \dots, n_N] \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

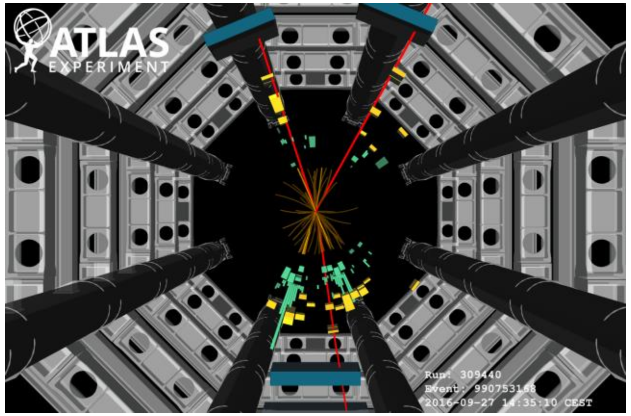
Computation Methods

Analytic: ε-factorization, Mellin-Barnes

Semi-analytic: DiffExp, AmFlow

Numeric: Sector Decomposition, Tropicalisation

Comparison to Experiment



Motivation IBPs

$$I_n(\alpha) = \int_0^\infty e^{-\alpha x^2}\, x^n dx$$

n even

$$\boxed{I_n = (-\partial_\alpha)^{\frac{n}{2}} I_0}$$

n odd

$$\boxed{I_n = (-\partial_\alpha)^{\frac{n-1}{2}} I_1}$$

- "Master Integrals": $\left\{ I_0(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}},\, I_1(\alpha) = \frac{1}{2\alpha} \right\}$

Baikov Representation

$$I_{n_1,\ldots,n_N} \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

Momentum Rep

Integrate in loop momenta k_j

$\Longleftrightarrow$

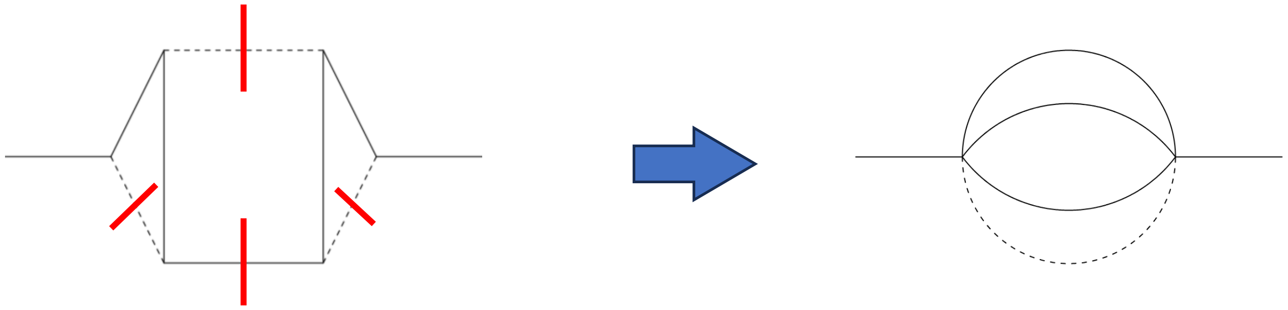
Baikov Rep

Integrate in propagators σ_j

$$I_{n_1,\ldots,n_N} \propto \int_C \frac{B(\sigma)^\beta d^n \sigma}{\prod_j \sigma_j^{a_j}}$$

$\mathcal{B} = \mathsf{Gram}(k_1, \ldots, k_L, p_1, \ldots, p_E)$ $\qquad$ $\beta = (d - E - L - 1)/2$

Self-Energy Sectors



Scaleless Integrals

- To show the above expectation is too generous, consider an example of a *Scaleless Integral*:

$$I \equiv \int_{k_1,k_2} \frac{1}{[k_1 \cdot p_i]^m [k_2 \cdot p_i]^n \cdot \ldots} \xrightarrow[k_j \to \lambda k_j]{k_i \to \lambda k_i} \int_{k_1,k_2} \frac{1}{\left[k_1 \cdot \lambda p_j\right]^m \left[k_2 \cdot \lambda\left(p_i + p_j\right)\right]^n \cdot \ldots}$$

rescale back k_1, k_2

$$= I \qquad\qquad = \underbrace{\lambda^{-m-n+\cdots}}_{\equiv \lambda^{\alpha}} \cdot I$$

- $\alpha \neq 0$ in dimensional regularization, hence:

$$\boxed{(\lambda^{\alpha} - 1) I = 0 \quad \Rightarrow \quad I = 0}$$

scaleless integrals vanish in dimreg $=$ **fully reducible sectors**

More on Differential Equations

- Instead of computing **integrals**, we will solve a system of **differential equations**!

Main idea:

1. Derivative of a Master Integral will give a different integral within the same family
2. This integral can again be reduced to a linear combination of Master Integrals
3. We obtain a relation between a derivative of an MI and a linear combination of MIs

- Integrals depend on scalar values (e.g. u, s, t), it makes sense to take derivatives with respect to them

vector of Master Integrals

$$\overbrace{p_{j,\mu}\frac{\partial \vec{I}}{\partial p_{k,\mu}}}^{\text{to be reduced using IBPs}} = \sum_{\alpha=1}^{N}\left(p_{j,\mu}\cdot\frac{\partial s_\alpha}{\partial p_{k,\mu}}\right)\frac{\partial \vec{I}}{\partial s_\alpha}$$

with $p_j, p_k \in \{p_i, p_j\}$

scalar variable