Borel singularities and Stokes constants of the topological string free energy on one-parameter Calabi-Yau threefolds

Simon Douaud
Based on arXiv:2412.16140 with **Amir-Kian Kashani-Poor**.

ENS | PSL

LPENS
LABORATOIRE DE PHYSIQUE DE L'ÉCOLE NORMALE SUPÉRIEURE

The big picture

Resurgence properties of the **topological string** amplitudes on **compact one parameter Calabi-Yau manifolds** are strongly related to **BPS states** counting. .

Structure of the talk

- One parameter Calabi-Yau manifolds, periods and BPS states
- Topological string and boundary conditions
- Resurgence
- Numerical results

One parameter Calabi-Yau 3-fold and Mirror symmetry

X is a Calabi-Yau 3-fold : $\exists!$ non vanishing $(3,0)$ closed form Ω.
X can be "deformed" in 2 ways :

- Deformation of the Kähler metric : $h^{1,1}(X)$ moduli parameters.
- Deformation of the complex structure : $h^{2,1}(X)$ moduli parameters.

Mirror symmetry : there exist a Calabi-Yau manifold X^* such that

- $h^{1,1}(X^*) = h^{2,1}(X)$
- $h^{2,1}(X^*) = h^{1,1}(X)$

The two deformations are exchanged.
One parameter Calabi-Yau : $h^{2,1}(X^*) = h^{1,1}(X) = 1$.

Period geometry on the mirror

Let z^i be the complex deformation parameters of X^*.
A central role is played by the periods of $\Omega(z)$,

$$X^I(z) = \int_{A_I} \Omega(z), \quad P_I(z) = \int_{B^I} \Omega(z)\,, \tag{1}$$

defined with regard to a symplectic basis of $H_3(X^*, \mathbb{Z})$,

$$A_I \cap A_J = B^I \cap B^J = 0\,, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J\,. \tag{2}$$

In the following, we will organize the periods in a period vector

$$\Pi = (P_I, X^I)\,. \tag{3}$$

$$\mathrm{e}^{-K} = i \int \Omega \wedge \bar{\Omega}\,, \quad C_{ijk} = \int \Omega \wedge \partial_i \partial_j \partial_k \Omega\,, \quad i, j, k = 1, \ldots, h^{1,1}(X)\,, \tag{4}$$

Period geometry on the mirror

We specify the study to the so called one parameter hypergeometric Calabi-Yau manifold where the period on X^* satisfy a Pichard-Fuchs differential equation.

$$[\,\theta^4 - \mu^{-1} z \prod_{k=1}^{4} (\theta + a_k)] f(z) = 0, \quad \theta = z \frac{d}{dz}. \tag{5}$$

$$\mathcal{P} \left\{ \begin{array}{ccc} 0 & \mu & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_4 \end{array} \right\} \quad . \tag{6}$$

$z = 0:$ Large Radius point (X goes to infinite volume)
$z = \mu:$ Conifold point, a cycle of X^* goes to zero size.

Period geometry on the mirror

$z = \infty$ classification depending on the exponents

- $(a_1, a_2, a_3, a_4) = (a, b, c, d)$: R-points, X_5, X_6, X_8, X_{10}, $X_{4,3}$ and $X_{6,4}$.

- $(a_1, a_2, a_3, a_4) = (a, b, b, c)$: C-points, $X_{4,2}$, $X_{6,2}$ and $X_{3,2,2}$.

- $(a_1, a_2, a_3, a_4) = (a, a, b, b)$: K-points , $X_{3,3}$, $X_{4,4}$, and $X_{6,6}$.

- $(a_1, a_2, a_3, a_4) = (a, a, a, a)$: M-points, $X_{2,2,2,2}$.

Period geometry on the mirror

Non trivial monodromy M_{pt} around singular points ($pt = 0, \mu, \infty$).

At $z = 0$ ne can find a special basis of solution for which the monodromies are integral, i.e $M_{pt} \in Sp(4, \mathbb{Z})$. This is the large radius basis $\mathbf{\Pi} = (P_I, X^I)$. A cycle γ in $H_3(X^*, \mathbb{Z})$ is represented by its charge vectors $\mathbf{q}$.

$$\int_\gamma \Omega = \mathbf{q} \cdot \mathbf{\Pi} = (q_0 P_0 + q_1 P_1 + q_2 X^0 + q_3 X^1) \tag{7}$$

where $\mathbf{q} = (q_0, q_1, q_2, q_3)$ are $D_6 D_4 D_0 D_2$ charges.
Orbits of a charge $\mathbf{q}$, $\{M_\infty^k \mathbf{q}\}$ are important.

For some models, very simple monodromy for example : mondromy group of X_{10} around $z = \infty$ is $\mathbb{Z}_{10}$ i.e $M_\infty^{10} = 0$.

Special periods at $z = \infty$

We associate a BPS state to a cycle in H_3.
Expanding in the Large radius basis $\mathbf{\Pi} = (P_I, X^I)$. We have the mass formula

$$M(\mathbf{q}) = e^{K/2}\mathbf{q}\cdot\mathbf{\Pi} \tag{8}$$

For certain model can build a 2 dimensional lattice of massless periods $\mathbb{Z}\mathbf{q}_1 + \mathbb{Z}\mathbf{q}_2$ (relates to number theory).

- for R-points : $X_6, X_{4,3}$ and $X_{6,4}$ i.e for $X_{4,3}$ $\mathbf{q}_1 = (1,0,0,-2)$, $\mathbf{q}_2 = (1,1,-3,1)$.
- All K-point i.e for $X_{3,3}$ $\mathbf{q}_1 = (1,0,0,-3)$, $\mathbf{q}_2 = (1,1,-3,1)$.
  On this subspace M_∞ acts as $\mathbb{Z}_n$ (n=3 for $X_{3,3}$).
- All C-points i.e for $X_{6,2}$ $\mathbf{q}_1 = (2,1,-4,0)$, $\mathbf{q}_2 = (1,0,0,-1)$.

BPS states counting

Around $z = 0$, we can compute the BPS invariant associated to the charge $\mathbf{q}$, $\Omega(\mathbf{q}, z)$.
[2023, *Alexandrov*, *Pioline*, *Klemm*, *Feyzbakhshf*, *Schimannek*]

Known for $\mathbf{q}$ of the form $(\pm 1, q_2, q_3, q_4)$(rank 1) and of the form $(0, q_2, q_3, q_4)$.

For $\mathbf{q} = (0, 0, m, d)$, $\Omega(\mathbf{q}, z = 0) = n_0^d$, genus 0 GV invariant.

For $\mathbf{q} = (1, 0, 0, 0)$, $\Omega(\mathbf{q}, z = 0) = 1$.

Topological A and B model

Non-linear sigma model on X (Calabi-Yau metric $g_{i\bar{j}}$ and coordinates ϕ^i)

$$S(\Sigma_g) = \int_{\Sigma_g} g_{i\bar{j}} \bar{\partial}\phi^i \partial\bar{\phi}^{\bar{j}} + \text{fermionic.} \tag{9}$$

Make the theory topological (independent of world-sheet metric) by twisting : lead to two models, A and B.

- A-model : space of marginal deformation of the theory $\cong$ space of Kähler deformation of X.
- B-model : space of marginal deformation of the theory $\cong$ space of complex structure deformation of X.

From mirror symmetry : A model on X = B model on X^*.

Holomorphic anomaly equation

Let z be a deformation parameter of the theory. One deform the action by adding marginal operator to the action

$$S[z,\bar{z}] = S + z^i O_i + \bar{z}^{\bar{i}} \bar{O}_{\bar{i}}. \tag{10}$$

Topological amplitudes

$$F_g(z,\bar{z}) = \int_{\mathcal{M}_g} \int D[\phi,\psi] O_g^{BRST} \mathrm{e}^{\mathrm{i}S[z,\bar{z}]}. \tag{11}$$

Satisfies an holomorphic anomaly equations

$$\partial_{\bar{k}} F_g = \frac{1}{2} C_{\bar{k}}^{ij} [D_i D_j F_{g-1} + \sum_{r=1}^{g-1} D_i F_r D_j F_{g-r}], \quad D_i F_g = (\partial_i + (2g-2)\partial_i K) F_g. \tag{12}$$

F_g can be computed up to an holomorphic ambiguity f_g.
Associated with Boundary conditions : can solve for F_g to high genus (up to 64 for X_5). [2006, *Huang*, *Klemm*, *Quackenbush*]

Frame, Boundary conditions

An holomorphic limit of $F_g(z,\bar{z}) \to F_g(X^I_*)$ is specified by a choice of A-period (X^I_*).
At $z = 0, \mu$: universal behaviour of F_g in certain frames.

- $z = 0$: Gopakumar-Vafa formula, in the large radius frame specified by the A periods (X^I).

$$F_g(X^0, X^1) = \sum_{d>0,m} n_0^d \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{dX^1 + mX^0} \right)^{2g-2} + \ldots \quad (13)$$

  (d, m) are D_2, D_0 charges.

- $z = \mu$: the period P_0 goes to 0 at this point, in the frame (P_0, X^1) we have the singular "gap" behaviour

$$F_g(P_0, X^1) = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} + \text{regular}. \quad (14)$$

Resurgence in a nutshell

From Boundary conditions $F_g \sim (2g-2)!$. $F = \sum_g F_g g_s^{2g-2}$ is asymptotic.
Borel transform

$$B[F] = \sum_g \frac{F_g}{(2g-2)!} g_s^{2g-2} \sim -\frac{S_\mathcal{A}}{2\pi}\left(\frac{F_{-1}^\mathcal{A}}{g_s - \mathcal{A}} + \mathrm{Log}(g_s - \mathcal{A}) B[F_\mathcal{A}(g_s - \mathcal{A})]\right) \tag{15}$$

$$F_\mathcal{A}(g_s) = \sum_{n \geq 0} F_n^\mathcal{A} g_s^n. \tag{16}$$

$\mathcal{A}$ = Borel singularity.
$S_\mathcal{A}$ =Stokes Constant.
$(\mathcal{A}, S_\mathcal{A})$= Resurgence data.

$F_n^\mathcal{A}$ can be computed in term of F_g using an operator formalism.
[2023, Gu, Kashani-Poor, Klemm, Marino]

Resurgence in a nut-shell

Borel resummation provides a family of well defined functions whose expansion equal to F

$$s_\theta(F) = \int_0^{e^{i\theta}\infty} B[F](x) e^{-x/g_s}. \tag{17}$$

Discontinuity when $\theta = \text{Arg}(\mathcal{A}_0)$

$$\begin{aligned} \text{Disc}_{\text{Arg}(\mathcal{A}_0)}(F) &= s_{\text{Arg}(\mathcal{A}_0)^+}(F) - s_{\text{Arg}(\mathcal{A}_0)^-}(F) \\ &= iS_{\mathcal{A}_0} s_{\text{Arg}(\mathcal{A}_0)^-}\left(\sum_{n\geq -1, \ell>0} F_n^{\ell\mathcal{A}_0} g_s^n e^{-\ell\mathcal{A}_0/g_s}\right). \end{aligned}$$

Resurgence data of F from broundary conditions

Asymptotics of F_g from resurgence data

$$F_g \sim (2g-2)! \sum_{\mathcal{A}} \frac{S_{\mathcal{A}}}{\mathcal{A}^{2g-1}} \left[F_{-1}^{\mathcal{A}} + \frac{F_0^{\mathcal{A}} \mathcal{A}}{2g-2} + \ldots \right]. \tag{18}$$

From boundary conditions one can can reads off some of the resurgence data at $z=0$ and $z=\mu$, $(\ell \in \mathbb{N})$

- at $z=0$, $\mathcal{A} = \aleph \ell (d X^1 + m X^0)$, $S_{\ell,d,m} = n_0^d = \Omega((0,0,m,d), z=0)$
- at $z=\mu$, $\ell \mathcal{A} = \aleph P_0$, $S_\ell = 1 = \Omega(P_0, z=0)$.

Resurgence data at $z=0,\mu$ is of the form

$$(\aleph \ell \mathbf{q} \cdot \mathbf{\Pi}, \Omega(\mathbf{q}, z=0)). \tag{19}$$

Hypothesis : For any z the resurgence data is of the form

$$(\aleph \ell \mathbf{q} \cdot \mathbf{\Pi}, \Omega(\mathbf{q}, z)). \tag{20}$$

And if $S_{\mathbf{q}} \neq \Omega(\mathbf{q}, z=0)$ it is because of wall-crossing.

Questions

- What about $z = \infty$?

- What about the resurgence of

$$F_g^{red} = F_g - \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} \tag{21}$$

at $z = \mu$?

$\Rightarrow$ no analytical control for most models : need to use numerics.

Numerical tools to extract Resurgence Data

Padé approximant : $B[F] \sim \frac{P_g(g_s)}{Q_g(g_s)}$, P_g, Q_g polynomial.
Roots of Q_g accumulate around line starting at $\mathcal{A}$.

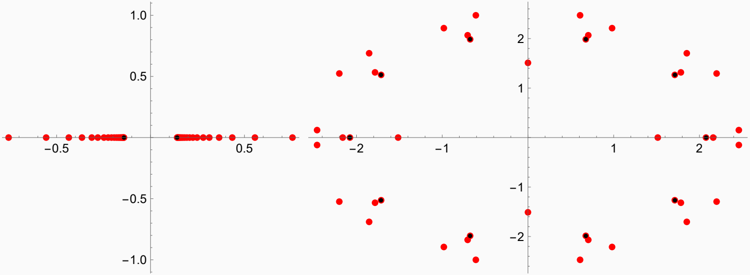

Figure 1: Example of Borel plane for X_5 around $z = \mu$ (left) and around $z = \infty$ (right).

Numerical tools to extract resurgence data

From asymptotic

$$S_{\mathcal{A}} \sim \frac{2\pi}{2}\left(\frac{(2g-2)!F_{-1}^{\mathcal{A}}}{\mathcal{A}^{2g-1}} + \frac{(2g-3)!F_{0}^{\mathcal{A}}}{\mathcal{A}^{2g-2}} + \dots\right)^{-1} F_g\,, \qquad (22)$$

when $\mathcal{A}$ is leading (i.e smallest singularity).

$$S_{\mathcal{A}} \sim -\mathrm{i}e^{\mathcal{A}/g_s}\frac{\mathrm{Disc}_{\arg(\mathcal{A})}(F)}{\sum_{n\geq -1} F_n^{\mathcal{A}} g_s^n}. \qquad (23)$$

Disc_θ is computed numerically from the residue of the Padé approximant. Removing contribution of singularity

$$F_g - (2g-2)!\frac{S_{\mathcal{A}}}{\mathcal{A}^{2g-1}}\left[F_{-1}^{\mathcal{A}} + \frac{F_0^{\mathcal{A}}\mathcal{A}}{2g-2} + \dots\right]. \qquad (24)$$

Numerical results : X_6

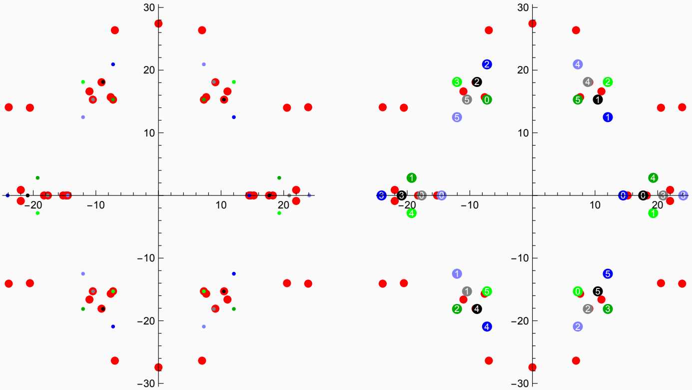

Figure 2: Borel plane of X_6 at $z = 10^4\mu$ in the frame $\left(\mathbf{q}_1, (0,1,-1,0)\right)$. The (light) blue points are orbits of $(-)\aleph P_0$ under M_∞, the (light) green points of $(-)\aleph X^0$, the (gray) black points of $(-)\aleph X^1$.

Numerical results : X_6

- The masless period are not Borel singularities i.e $S_{\mathcal{A}} = 0$ but at $z = 0$, $\Omega(\mathbf{q}, z = 0) = 0$ so not stable at $z = 0$ and not stable at $z = \infty$ as well.
- Monodromy invariance : if $\mathbf{q} \cdot \mathbf{\Pi}$ is a singularity then $M_{\infty}^{k}(\mathbf{q}) \cdot \mathbf{\Pi}$ is.
- Stokes constant of orbits of X^0, P_0 and X^1 are equal to $\Omega(X^0/P_0/X^1, z = 0)$.

Numerical results : X_{10}

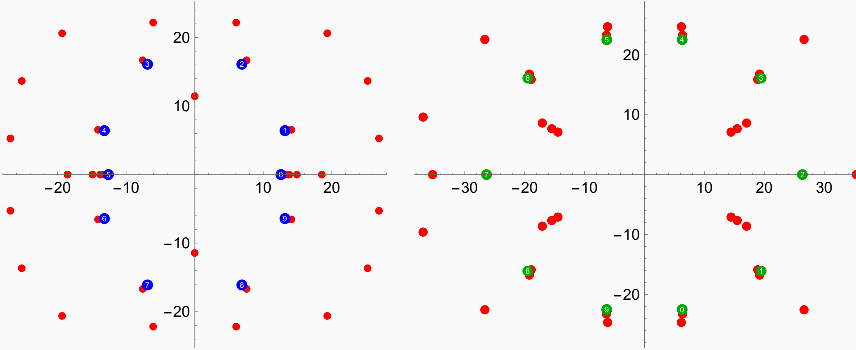

Figure 3: Borel plane of X_{10} in the frame (P_0, X^1) at $z = 10^4\mu$. On the right, the leading singularities are subtracted. Blue dots indicate orbit elements of $\aleph P_0$ under M_∞, green dots orbit elements of $\aleph X^0$.

Stokes constant of orbits of X^0, P_0 and X^1 are equal to $\Omega(P_0/X^0, z = 0)$.

Numerical results : $X_{4,3}$

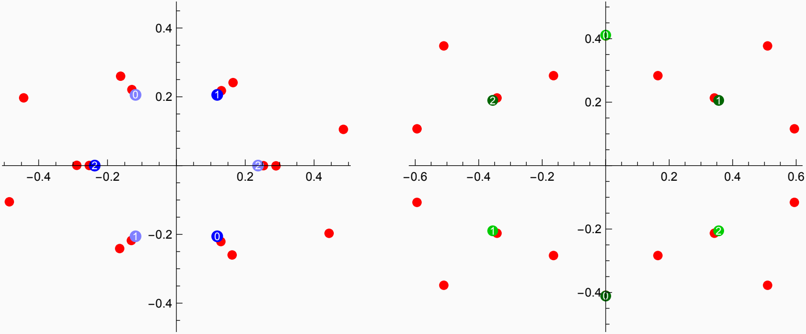

Figure 4: Borel plane of $X_{4,3}$ at $z = 10^8\mu$ in the frame $\big(\mathbf{q}_1, (0,1,-2,0)\big)$. On the right, the leading singularities are subtracted. The (light) blue dots indicate the orbit elements of $\aleph\mathbf{q}_1 \cdot \mathbf{\Pi}$ $(-\aleph\mathbf{q}_1 \cdot \mathbf{\Pi})$ under the monodromy M_∞, the (light) green dots the orbit elements of $\aleph(\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{\Pi}$ $(\aleph(\mathbf{q}_2 - \mathbf{q}_1) \cdot \mathbf{\Pi})$.

$$S_{M_\infty^k \mathbf{q}_1} = 27 = \Omega(\mathbf{q}_1, z = 0).$$

Numerical results : $X_{4,4}$

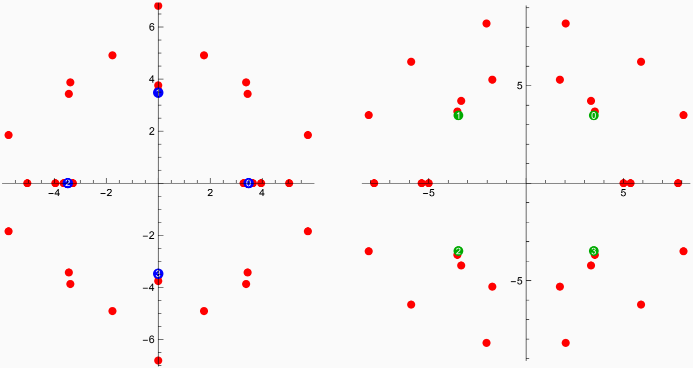

Figure 5: Borel plane of $X_{4,4}$ at $z = 10^6\mu$ in the frame $(\mathbf{q}_1 \cdot \mathbf{\Pi}, P_0)$. On the right, the leading singularities are subtracted. The blue dots indicate the orbit elements of $\aleph\mathbf{q}_1 \cdot \mathbf{\Pi}$ $(-\mathbf{q}_1 = \mathbf{q}_1 \cdot M_\infty^2)$ under the monodromy M_∞, the green dots the orbit elements of $\aleph(\mathbf{q}_1 - \mathbf{q}_2) \cdot \mathbf{\Pi}$.

$S_{M_\infty^k \mathbf{q}_1} = 1408 = \Omega(\mathbf{q}_1, z = 0)$. $S_{M_\infty^k (\mathbf{q}_1+\mathbf{q}_2)} = 9984 \neq \Omega(\mathbf{q}_1 + \mathbf{q}_2, z = 0)$.
Wall crossing ?

C-point : analytical behaviour

For those models : lattice of vanishing periods $\mathbb{Z}\mathbf{q}_1 + \mathbb{Z}\mathbf{q}_2$.
In a A-frame containing $\mathbf{q}_1 \cdot \Pi_{LR}$. we have the following behaviour of F_g for X_{42} and X_{62}

$$F_g = S\frac{(-1)^{g-1}B_{2g}}{2g(2g-2)}\left(\frac{(2\pi i)^{1/2}}{\mathbf{q}_1 \cdot \Pi_{LR}}\right)^{2g-2} + \text{regular}\,, \tag{25}$$

with $S = 2$ for $X_{4,2}$ and $S = 1$ for $X_{6,2}$. $\Rightarrow \mathcal{A} = \aleph\ell\mathbf{q}_1 \cdot \Pi_{LR}, S_\ell = S$.
No more singular terms $\Rightarrow S_{\mathbf{q}_2} = 0$, coherent with the fact that for the two models $\Omega(\mathbf{q}_2, z = 0) = 0$, no wall crossing. Also verified numerically.

Wall-crossing around conifold point

$$F_g^{red} = F_g - \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)} \left(\frac{(2\pi i)^{1/2}}{P_0} \right)^{2g-2} \tag{26}$$

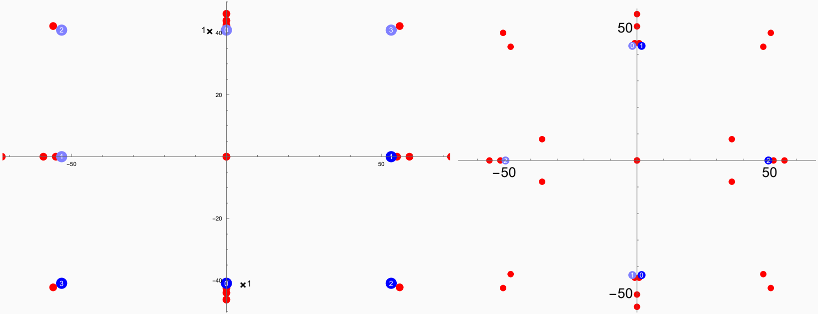

Figure 6: Borel plane of X_5 in the frame (P_0, X^1) upon removing the leading gap singularity, at $z = 1.3\mu$ on the right, $z = 0.7\mu$ on the left . (Light) blue dots : $(-)\aleph\{X^0,\ P_0 - X^0,\ X^1,\ X^1 + X^0,\ X^1 - X^0\}$. At $z = 0.7\mu$ there is no red points associated to $\pm(P_0 - X^0)$ (number 1). At $z = 1.3\mu$ we have enough precision to identify $\aleph(X^1 + X^0)$ and $\aleph(X^1 - X^0)$.

Wall-crossing around conifold point

Decay of $X^0 - P_0 = \gamma_1 + \gamma_2$, $\gamma_1 = X^0$, $\gamma_2 = -P_0$.
$\Omega(\gamma_1 + \gamma_2, \mu^-) = 0$.

$$\Omega(\gamma_1+\gamma_2,\mu^+)-\Omega(\gamma_1+\gamma_2,\mu^-) = (-1)^{\langle\gamma_1,\gamma_2\rangle-1}|\langle\gamma_1,\gamma_2\rangle|\Omega(\gamma_1,\mu)\Omega(\gamma_2,\mu) \tag{27}$$

$$\Rightarrow \Omega(X^0 - P_0, \mu^+) = \Omega(X^0, \mu), \tag{28}$$

verified numerically for almost all model.

Conclusion and questions

- Numerical evidence for our hypothesis :
  For any z the resurgence data is of the form

$$(\aleph\ell\mathbf{q}\cdot\Pi_{LR},\Omega(\mathbf{q},z)). \tag{29}$$

  And if $\Omega(\mathbf{q},z)=S_{\mathbf{q}}\neq\Omega(\mathbf{q},z=0)$ it is because of wall-crossing.
- Can we get those results analytically ?
- Are all BPS states present in the resurgence data ? Or only a "topological " sub-sector ?
- Importance of choice of frame