Higher Bessel Product Formulas: Explicit Examples of Multiplication Kernels

Ilia Gaiur

University of Geneva
joint with V. Rubtsov and D. van Straten

23 January 2025

Timeline

Clausen duplication (1828)

$$\left({}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix} \mid x\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a & a+b & 2b \\ & a+b+\frac{1}{2} & 2b \end{matrix} \mid x\right]$$

Schläfli formula (1876)

$${}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{nx}{2}\right] {}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{ny}{2}\right] = \int_{S^{n-1}} {}_0F_1\left[\begin{matrix} - \\ \frac{n}{2} \end{matrix} \mid \frac{n}{2}(x+y+2\sqrt{xy} < e_1, \mu >)\right] d\mu$$

Sonine-Gegenbauer formula (1880; 1884)

$$J_0(x)J_0(y) = \frac{1}{2\pi}\int_{x-y}^{x+y} \frac{J_0(z)zdz}{\sqrt{x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2}}, \; x < y \in \mathbb{R}$$

Kontsevich (2007)

$$(\partial_t \circ f(t) \circ \partial_t + t)\,\phi = \lambda\phi, \quad f(t) = t^3 + At^2 + t$$

$$\phi_\lambda(x)\phi_\lambda(y) = \int \frac{\phi_\lambda(z)}{\sqrt{P(x,y,z)}}dz, \quad P(x,y,z) = \mathrm{discr}_t\left[f(t) - (t-x)(t-y)(t-z)\right]$$

Exhibit the motivic nature of multiplication kernels for the very explicit example of the kernels for these N-Bessel functions

i.e. write these kernels as periods of some algebraic families

Object of study

$N-$Bessel or $\mathbb{P}^{N-1}-$Quantum DO:

$$\boxed{\left(x\frac{d}{dx}\right)^N \Psi - \lambda x \Psi = 0}$$

Analytic solution

$$\Phi_N(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!^N} =$$

$$= \frac{1}{(2\pi i)^{N-1}} \oint \exp\left(\sum_{i=1}^{N-1} Y_i + \frac{x}{Y_1 Y_2 \dots Y_{N-1}}\right) \frac{dY_1}{Y_1}\frac{dY_2}{Y_2}\dots\frac{dY_{N-1}}{Y_{N-1}}$$

Multiplication Kernels:

$$\boxed{\Phi_N(\lambda, x)\Phi_N(\lambda, y) = \oint K_N(x, y|z)\Phi_N(\lambda, z)\frac{dz}{z}}$$

And more general

$$\Phi_N(x_1)\Phi_N(x_2)\dots\Phi_N(x_m) = \oint K_N^{(m)}(x_1, x_2, \dots x_m|z)\Phi_N(z)\frac{dz}{z}$$

Duplication formulas

Consider a "diagonal" $x = y$ in order to get one dimensional families

Convolution of power series and explicit numbers

Duplication kernels

Integral transformation to Sym^2 of DE

$$\Phi_N(x)^2 = \frac{1}{2\pi i}\oint K_N(x/z)\Phi_N(z)\frac{dz}{z}$$

For $N = 2$ – Clausen duplication

$$\Phi_2(x)^2 = \frac{1}{2\pi i}\oint_T \frac{1}{\sqrt{1-4x/z}}\Phi_2(z)\frac{dz}{z}.$$

which leads to

$$(\theta_x^3 - 4x\,\theta_x - 2x)\Phi_2(x)^2 = 0, \quad \theta_x = \frac{xd}{dx}$$

Duplication kernels via multiplicative convolution

Taking the "diagonal" $y = x$ we obtain the Clausen duplication formula

$$\Phi_N(x)^2 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!^N}\right)^2 = \sum_{n=0}^{\infty} \frac{c_n}{n!^N} x^n, \quad c_n = \sum_{k=0}^{n} \binom{n}{k}^N$$

Hadamard product: $f(x) = \sum_{j=0}^{\infty} a_j x^j$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$

$$(f * g)(x) = \frac{1}{2\pi i} \oint f(x/t) g(t) \frac{dt}{t} = \sum_{j=0}^{\infty} (a_j \cdot b_j) x^j.$$

Then

$$\Phi_N(x)^2 = \Phi_N(x) \star \sum_{n=0}^{\infty} c_n x^n = \Phi_N(x) \star K(x) = \frac{1}{2\pi i} \oint K(x/t) \Phi_N(t) dt/t$$

$K(x/z) = K(x, x|z)$ is a generating function of c_n

Binomial coefficient sums

$$K_N(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k}^N \right] t^n, \quad t = x/z$$

$N = 2$ - Clausen duplication for Bessel

$$K_2(x) = \frac{1}{\sqrt{1-4t}}$$

$N = 3$ - Apéry like sequence of type A (D. Zagier). Heun function

$$t(t+1)(8t-1)K^{''} + (24t^2 + 14t - 1)K^{'} + (8t+2)K = 0$$

Generic $N > 1$: period from Landau-Ginzburg model

$$V_N = \prod_{i=1}^{N-1}(1+Y_i) + \prod_{i=1}^{N-1}(1+Y_i^{-1})$$

Corresponding family and its period

$$V_N = 1/t, \quad K_N(t) = \frac{1}{(2\pi i)^{N-1}} \oint_{T^{N-1}} \frac{1}{1-tV_N} \frac{dY_1}{Y_1} \frac{dY_2}{Y_2} \dots \frac{dY_{N-1}}{Y_{N-1}}$$

$$
\begin{aligned}
N=2 \quad & (4t-1)\,\theta_t + 2t,\\
N=3 \quad & (t+1)\,(8t-1)\theta_t^2 + t\,(16t+7)\,\theta_t + 2t(4t+1),\\
N=4 \quad & (16t-1)(4t+1)\theta_t^3 + 6t(32t+3)\theta_t^2 + 2t(94t+5)\theta_t + 2t(30t+1),\\
N=5 \quad & (32t-1)(4t-7)^2(t^2-11t-1)\theta_t^4 + 2t(4t-7)(256t^3-2084t^2+4942t+143)\theta_t^3 +\\
& t(3072t^4-23024t^3+72568t^2-102261t-1638)\theta_t^2 +\\
& t(2048t^4-12896t^3+30072t^2-66094t-637)\theta_t +\\
& 2t(256t^4-1472t^3+1904t^2-7868t-49),\\
N=6 \quad & (t-1)(27t+1)(64t-1)(75t^3+1420t^2+561t+9)\theta_t^6 +\\
& + (842400t^6+18022725t^5-1363487t^4-4622791t^3-127551t^2-1977t-9)\theta_t^5 +\\
& + 5t(452880t^5+10962507t^4+2491544t^3-1779376t^2-46584t-168)\theta_t^4 +\\
& + 5t(644760t^5+17135271t^4+6994741t^3-1716533t^2-56024t-96)\theta_t^3 +\\
& + 2t(1282500t^5+36696915t^4+19164721t^3-2088858t^2-100236t-72)\theta_t^2 +\\
& + 2t(540900t^5+16436910t^4+9826066t^3-428487t^2-38811t-9)\theta_t +\\
& + 12t^2(15750t^4+503175t^3+327205t^2-1845t-1044).
\end{aligned}
$$

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ord$\mathcal{D}_{\mathcal{N}}$	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40

Number of unipotent blocks increase on each fifth iteration

$$M_0^{(7)} \sim \begin{pmatrix} 1 & 1/2! & 1/3! & 1/4! & 1/5! & 1/6! & 0 & 0 \\ 0 & 1 & 1/2! & 1/3! & 1/4! & 1/5! & 0 & 0 \\ 0 & 0 & 1 & 1/2! & 1/3! & 1/4! & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2! & 1/3! & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/2! & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2! \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Mirror maps

$$N=3 \quad q-3q^2+3q^3+5q^4-18q^5+15q^6+24q^7-75q^8+57q^9+86q^{10}+O(q^{11})$$

$$N=4 \quad q-4q^2-6q^3+56q^4-45q^5-360q^6+894q^7+960q^8-6951q^9+4660q^{10}+O\left(q^{11}\right)$$

$$N=5 \quad q-5q^2-40q^3+115q^4-645q^5-12846q^6-177350q^7- \\ -2574585q^8-44198680q^9-736554815q^{10}+O\left(q^{11}\right)$$

$$N=6 \quad q-6q^2-135q^3-380q^4-24960q^5-696366q^6- \\ -26153302q^7-901888104q^8-35369115894q^9-1381135576280q^{10}+O\left(q^{11}\right)$$

$$N=7 \quad q-7q^2-371q^3-4543q^4-378637q^5-20096783q^6-1568975093q^7 \\ -112310305031q^8-9251250532328q^9-758736375700793q^{10}+O\left(q^{11}\right)$$

N-Bessel kernels as periods

$K_N(x, y|z)$ are periods of $\mathbb{P}^2$-families of algebraic varieties

$K_N^{(m)}(x_1, x_2 \ldots x_m|z)$ are periods of $\mathbb{P}^m$-families

Theorem

Let

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

Define

$$K_N(x,y,z) = \frac{1}{(2\pi i)^{N-1}}\oint \frac{1}{W_N(x,y,z|\mathbf{Y})}\prod_{j=1}^{N-1}\frac{dY_j}{Y_j}$$

Theorem (I. Gaiur, V.R., D. van Straten) $K_N(x,y,z)$ is a kernel for N-Bessel function

$$W_N(x,y,z|Y) = \prod_{j=1}^{N-1}(1+Y_j)\left(x + \frac{y}{Y_1Y_2\dots Y_{N-1}}\right) - z = 0$$

Proof: Kontsevich-Odesskii formal power series for kernel + combinatorics

$$K_N(x,y,z) = \sum_{j,k}\binom{j+k}{k}^N \frac{x^j y^k}{z^{j+k}}.$$

Multi-Kernels

Multi-product

$$\Phi_N(x_1)\Phi_N(x_2)\dots\Phi_N(x_{m-1})\Phi_N(x_m) = \frac{1}{2\pi i}\oint K_N(x_1,x_2,\dots x_m\,|z)\,\Phi_N(z)\frac{dz}{z}.$$

$K_N(x_1,x_2,\dots x_m\,|z)$ is a convolution of $K_N(x,y|z)$ with itself $m-2$ times

Theorem (I. Gaiur, V.R., D. van Straten) Kernel $K_N(x_1,x_2,\dots x_m\,|z)$ is a period of

$$W_N^{(m)}(x_1,\dots x_m,z) = \prod_{j=1}^{N-1}\left(1+\sum_{l=1}^{m-1}Y_j^{(l)}\right)\cdot\left(x_1+\sum_{l=1}^{m-1}\prod_{j=1}^{N-1}\frac{x_{l+1}}{Y_j^{(l)}}\right)-z=0,$$

i.e.

$$K_N(x_1,x_2,\dots x_m\,|z) = \frac{1}{(2\pi i)^{N-1}}\oint\frac{1}{W_N^{(m)}(x_1,x_2,\dots x_m,z)}\prod_{j,l}\frac{d\,Y_j^{(l)}}{Y_j^{(l)}}$$

Proof: Similar to the multiplication one

In Watson (13.46, formula (9))

$$\int_0^\infty \prod_{i=1}^4 J_0(a_n\lambda)\lambda d\lambda = \frac{1}{\pi^2}\begin{cases} \frac{1}{\Delta} K\left(\frac{\sqrt{a_1 a_2 a_3 a_4}}{\Delta}\right), & \left|\frac{\sqrt{a_1 a_2 a_3 a_4}}{\Delta}\right| < 1 \\ \frac{1}{\sqrt{a_1 a_2 a_3 a_4}} K\left(\frac{\Delta}{\sqrt{a_1 a_2 a_3 a_4}}\right), \end{cases}$$

$K(k)$ is complete elliptic integral of the first kind

$$16\Delta^2 = \prod_{n=1}^4 (a_1 + a_2 + a_3 + a_4 - 2a_n).$$

LHS is a "pairing"

$$\langle J_0(a_4\lambda),\ J_0(a_1\lambda)J_0(a_2\lambda)J_0(a_3\lambda)\rangle$$

RHS is a product kernel for 3 Bessel functions. Corresponding elliptic curve is

$$(1+X+Y)\left(a_1^2 + \frac{a_2^2}{X} + \frac{a_3^2}{Y}\right) - a_4^2 = 0$$

```
sage: X,Y,a1,a2,a3,a4=QQ["X,Y,a1,a2,a3,a4"].gens()
sage: W = (1+X+Y)*(X*Y*a1^2+Y*a2^2+X*a3^2) - X*Y*a4^2;
sage: from sage.schemes.toric.weierstrass import WeierstrassForm
sage: from sage.schemes.toric.weierstrass import Discriminant
sage: factor(Discriminant(W, [X,Y]))
(-1/16) * (-a1 - a2 + a3 - a4) * (-a1 - a2 + a3 + a4) * (-a1 + a2 + a3 -
a4) * (-a1 + a2 + a3 + a4) * (a1 - a2 + a3 - a4) * (a1 - a2 + a3 + a4) *
(a1 + a2 + a3 - a4) * (a1 + a2 + a3 + a4) * a4^4 * a3^4 * a2^4 * a1^4
```

$N = 3$ Elliptic family

Family

$$xXY(1+X)(1+Y) + y(1+X)(1+Y) - zXY = 0,$$

Deformation of Beauville family of type IV

$$(\tilde{X}+\tilde{Y})(\tilde{Y}+\tilde{Z})(x\,\tilde{Z}+y\,\tilde{X}) + z\tilde{X}\tilde{Y}\tilde{Z} = 0$$

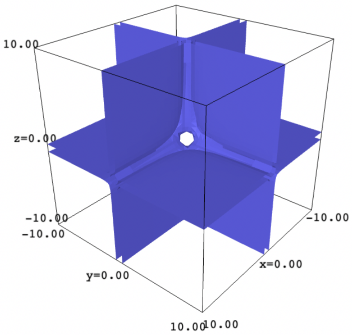
Kernel

$$K_3(x,y,z) = -\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;-\frac{27xyz}{(x+y-z)^3}\right)}{x+y-z}.$$

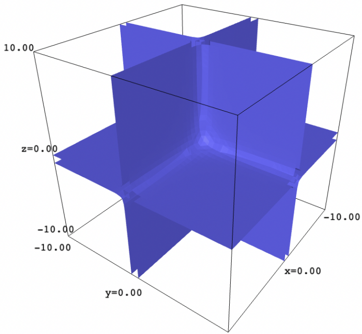
Gauss hypergeometry as expected

$N = 4$ K3 families

Family
$$xXYZ(1+X)(1+Y)(1+Z) + y(1+X)(1+Y)(1+Z) - zXYZ = 0.$$



$[-2:-1:1]$



$[-2:1:1]$

Singularities: Landau Discriminants, Buchstaber-Rees

Appeared families have a very specific geometry of singularity loci

For standard multiplication formulas all singularities are of discriminantal type

Product formulas singularities

The singularities of the family

$$W_N(x,y,z) = x\prod_{j=1}^{N-1}(1+Y_j) + y\prod_{j=1}^{N-1}(1+Y_j^{-1}) - z = 0$$

is a projective curve in $\mathbb{P}^2$ which a union of triangle

$$xyz = 0$$

and irreducible rational curve

$$\Delta_N(x,y,z) = x^N + y^N + z^N + \ldots,$$

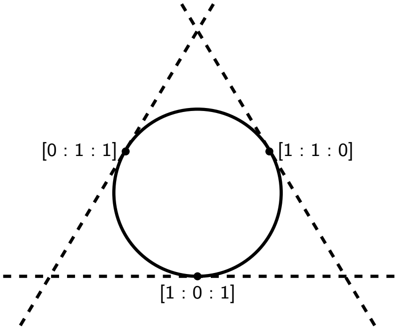
which is given by the property

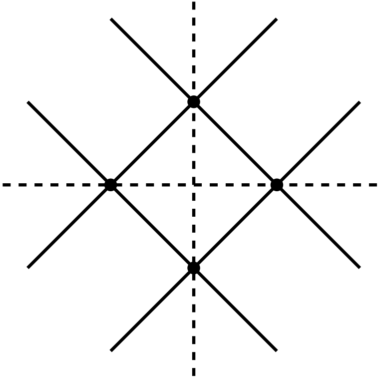
$$\Delta_N(u^N, v^N, w^N) = \prod_{\omega,\eta}(u + \omega v + \eta w),$$

where in the product ω and η run over the N-th roots of unity.

Remark: $\Delta_N(x,y,z)$ us a homogeneous polynomial with integer coefficients

$N = 2$ singularity loci and unfolding





Theorem The polynomials $\Delta_N(x, y, z)$ may be expressed via T-discriminants of the $2N-2$ degree polynomials given by

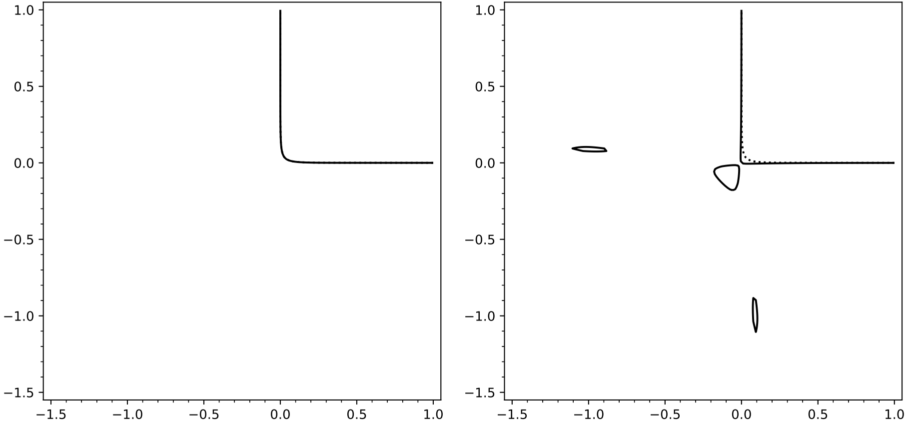
$$P_{x,y,z}(T) = xT^{N-1}(1+T)^{N-1} + y(1+T)^{N-1} - T^{N-1}z.$$

More precisely, the equality holds

$$(-1)^N(N-1)^{2(N-1)}(xyz)^{N-2}\Delta_N(x, y, z) = \mathrm{disc}_T(P_{x,y,z}(T))$$

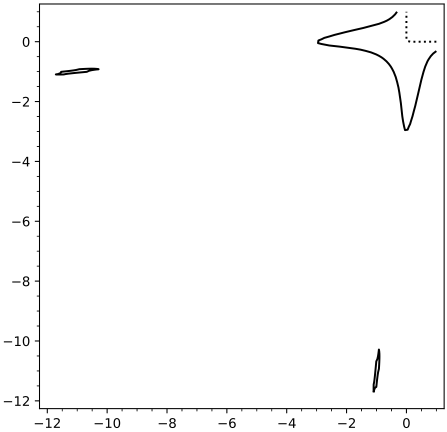
Theorem The projective curve $\Delta_N = 0$ is a rational curve that has $(N-1)(N-2)/2$ double points. All singularities belong to $\mathbb{P}^2(\mathbb{R})$. Moreover, coordinates of the singularities belong to $\mathbb{Q}(\cos(2\pi/N))$.

$N = 5\ \Delta = \epsilon$

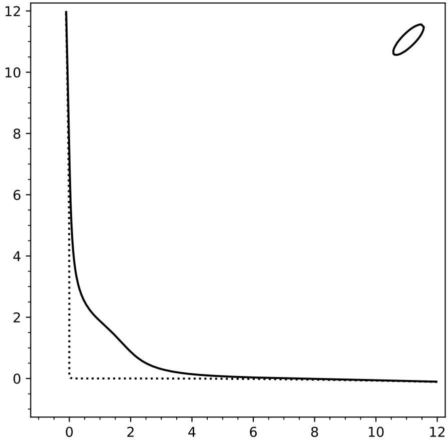


$\epsilon = 0$ $\qquad$ $\epsilon = 1$

$N = 5;\ \Delta = \epsilon$



$\epsilon = 1000$



$\epsilon = -1000$

The singular locus of

$$W_N^{(m)}(x_1, \dots x_m, z) = \prod_{j=1}^{N-1} \left(1 + \sum_{l=1}^{m-1} Y_j^{(l)}\right) \cdot \left(x_1 + \sum_{l=1}^{m-1} \prod_{j=1}^{N-1} \frac{x_{l+1}}{Y_j^{(l)}}\right) - x_0 = 0$$

is given by

$$x_0 x_1 x_2 x_3 \dots x_m \Delta(x_0, x_1, \dots x_m) = 0,$$

where Δ is symmetric polynomial given by

$$\Delta(u_0^N, u_1^N \dots u_m^N) = \prod_{\omega_i}^{\omega_i^N = 1} \left(u_0 + \sum_{i=1}^{m} \omega_i u_i\right), \tag{1}$$

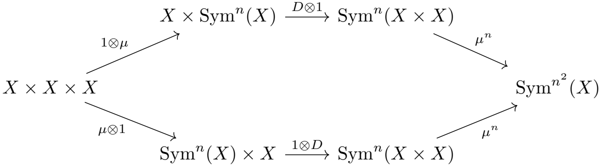
where ω_i runs independently over the N-th roots of unity.

n-valued group is a map

$$\mu : X \times X \to \mathrm{Sym}^n(X)$$

$$\mu(x, y) = x \star y = [z_1, z_2, \ldots, z_n], \quad z_k = (x * y)_k$$

Associative, has unity and inverse element



Coset construction: affine mfd with action of the discrete group => multivalued group

Buchstaber-Rees: Consider $\mathbb{C}$ with action of $\mathfrak{S}_N$

Multiplication by primitive N-root of unity, $\chi^N = 1$

$$x \star y = \left[\left(x^{1/N} + \chi^r y^{1/N} \right)^N, \quad 1 \leq r \leq N \right]$$

Corresponding divisor

$$\Delta_N(x, y, z) = (z - z_1)(z - z_2) \dots (z - z_N), \quad \mu(x, y) = [z_1, z_2, \dots z_N]$$

coincides with singularities of $W_N(x, y, z)$

$$N = 2 \quad x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$$

$$N = 3 \quad (x - z + y)^3 + 27xyz$$

$$N = 4 \quad x^4 - 4x^3y - 4x^3z + 6x^2y^2 - 124x^2yz + 6x^2z^2 - 4y^3x - 124xy^2z - 124xyz^2 - 4z^3x + y^4 - 4y^3z + 6y^2z^2 - 4yz^3 + z^4$$

Links and perspectives

Hecke operators and Kontsevich polynomial

2-Bessel kernel singularities is a degeneration of the Kontsevich polynomial

Kernel may be considered as an analog of the Hecke operator in the analytic context

Higher Bessel Kernels should correpond to the degeneration of the higher rank local systems

Non-Abelian Abel's theorem

Non-Abelian Abel's theorem by V. Golyshev, V.R., A. Mellit and D. van Straten:

Kernels are lifts of the Abel's law on a base curve

Example: Kontsevich polynomial - elliptic curve Abelians law

N-Bessel kernels lift Buchstaber-Rees law

Lift from the corresponding spectral curve

$$\delta = x\frac{d}{dx} - A = x\frac{d}{dx} - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ x & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\boxed{\Gamma(x,\lambda) = (-1)^N \det(A/x - \lambda) = \lambda^N - 1/x = 0}$$

P. Vanhove formula

$$\int_0^\infty I_0(z)K_0(z)^{l+1}\, z\, dz = \frac{1}{2^l}\int_{X_i\geq 0} \frac{1}{1-W(X)}\prod_{k=1}^{l}\frac{dX_k}{X_k}, \quad W = \left(1+\sum_{i=1}^{l+1}X_i\right)\left(1+\sum_{i=1}^{l+1}X_i^{-1}\right)$$

Same potential as for 2-Bessel multi-product restricted to diagonal, BUT different integration path

Studied as one parametric families

$$(X_0+X_1+\ldots+X_m)\left(\frac{x_0}{X_0}+\frac{x_1}{X_1}+\ldots+\frac{x_m}{X_m}\right)=1/t.$$

Appeared in banana graph Feynman integrals (Bloch, Kerr and Vanhove; Klemm, Dühr).
Calabi-Yau case $m=5$ (Hulek and Verrill; Candelas c.s.)
Bessel moments by Broadhurst and Roberts, extended by Fresán, Sabbah and Yu

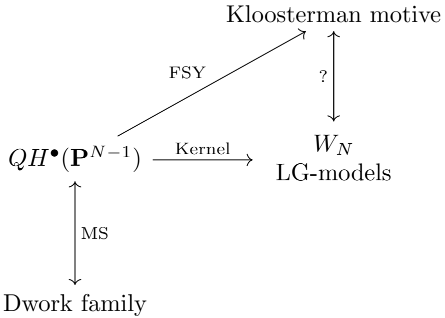
W_N give a universal differential forms which connects oscillatory integrals in LHS and periods in RHS

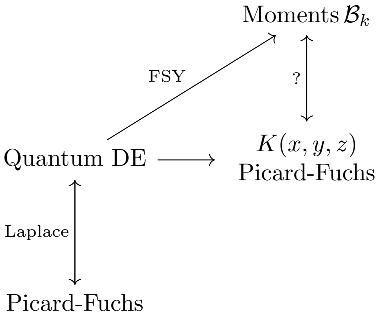
Works not only for diagonal

Reminds us Givental's MS which connects

$$\text{Oscillatory Integrals} \quad \leftrightarrow \quad \text{Periods}$$

Should work also for $N>2$, but need to find a corresponding pairing


Kloosterman motive
FSY
?
$QH^\bullet(\mathbf{P}^{N-1})$
Kernel
W_N
LG-models
MS
Dwork family


Moments $\mathcal{B}_k$
FSY
?
Quantum DE
$K(x, y, z)$
Picard-Fuchs
Laplace
Picard-Fuchs

Thank you for your time