UNIVERSITÄT BONN
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bctp Bethe Center for Theoretical Physics

(Hyperelliptic) Feynman Integrals From Differential Equations

Franziska Porkert

with Claude Duhr, Cathrin Semper & Sven Stawinski

arXiv: 2408.04904

arXiv: 2407.17175

arXiv: 2412.02300

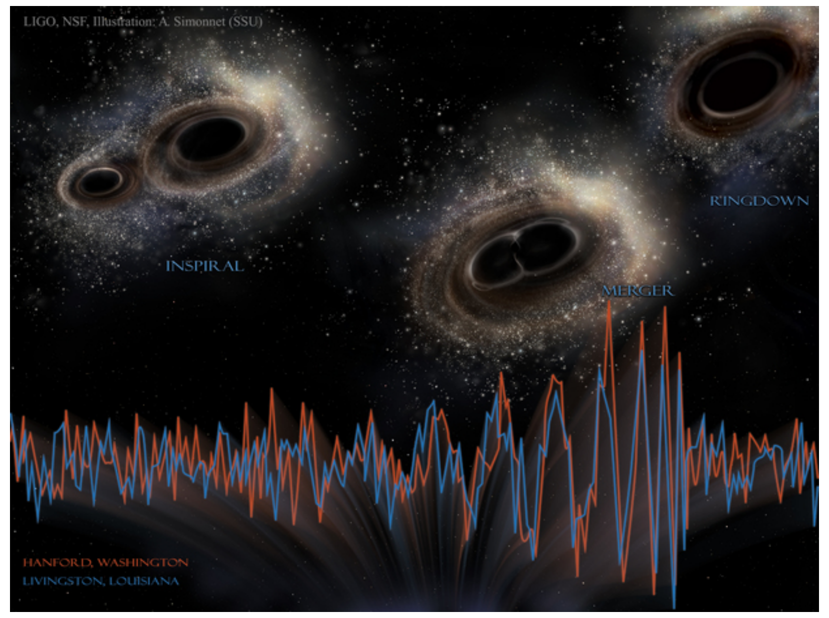
+ work in progress (also with Gaia Fontana, Sara Maggio)

Physics and Number Theory Workshop, 23.01.2025

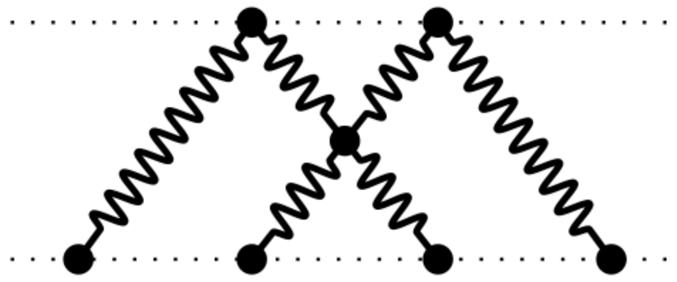
MOTIVATION

Feynman(-like) integrals are the building blocks for scattering amplitudes in …

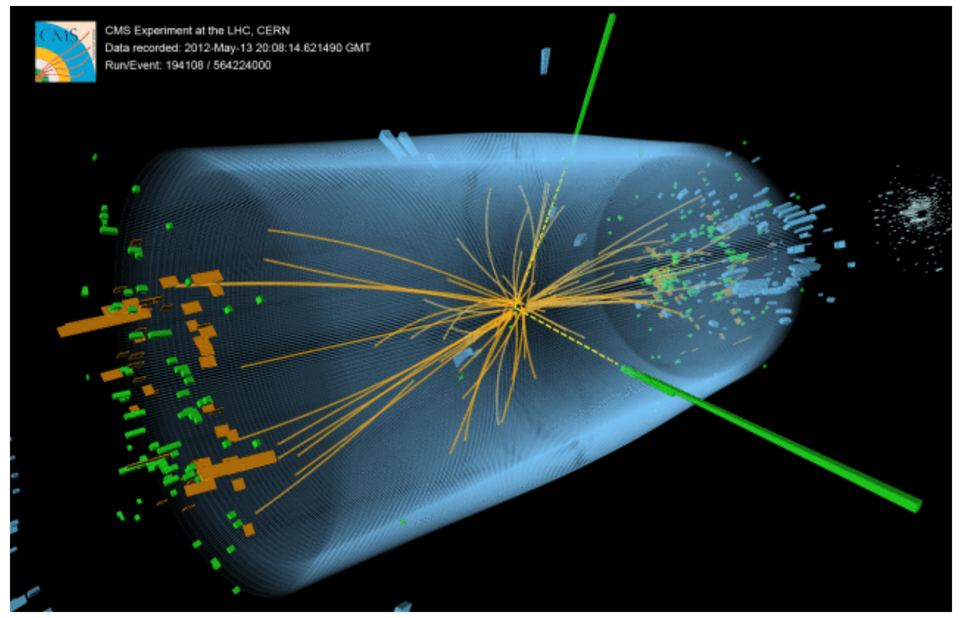
Gravitational wave physics



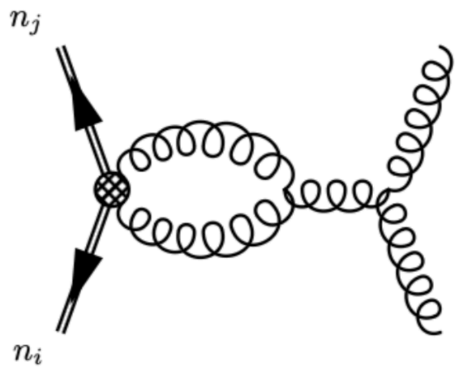
Aurore Simones (Sonoma State University)

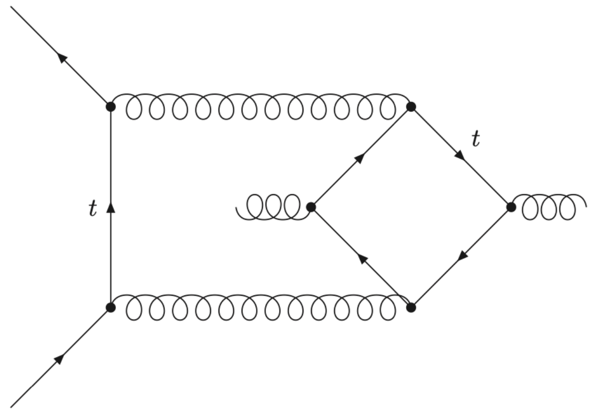


Collider phenomenology

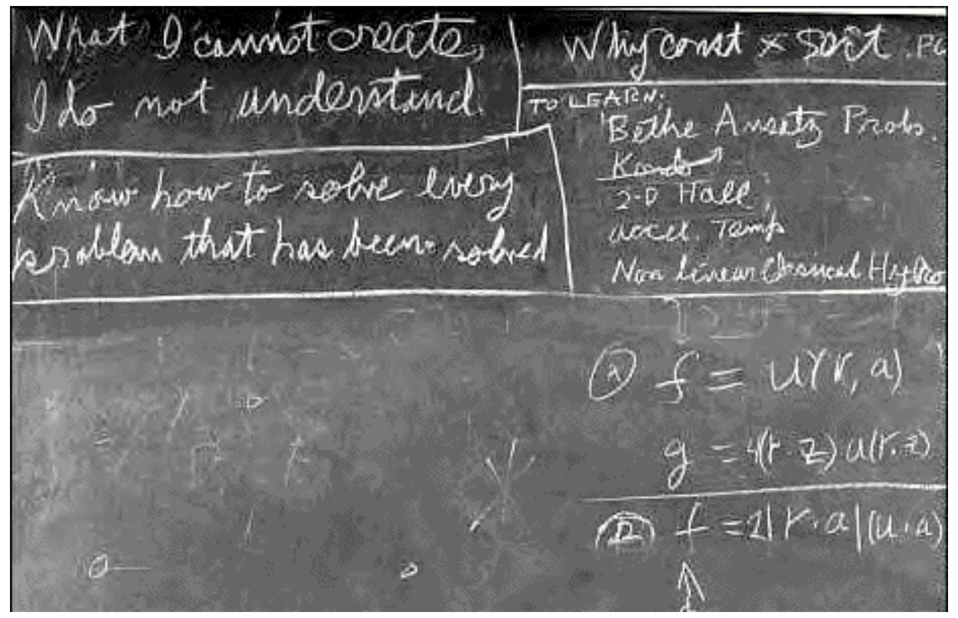


CERN





Integrability



What I cannot create, I do not understand.

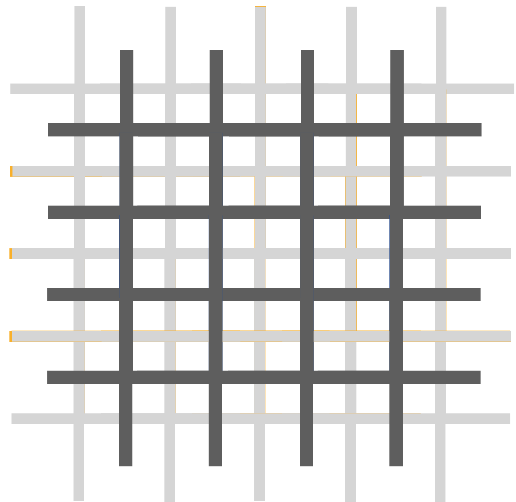
Why const × sort .PC

Know how to solve every problem that has been solved

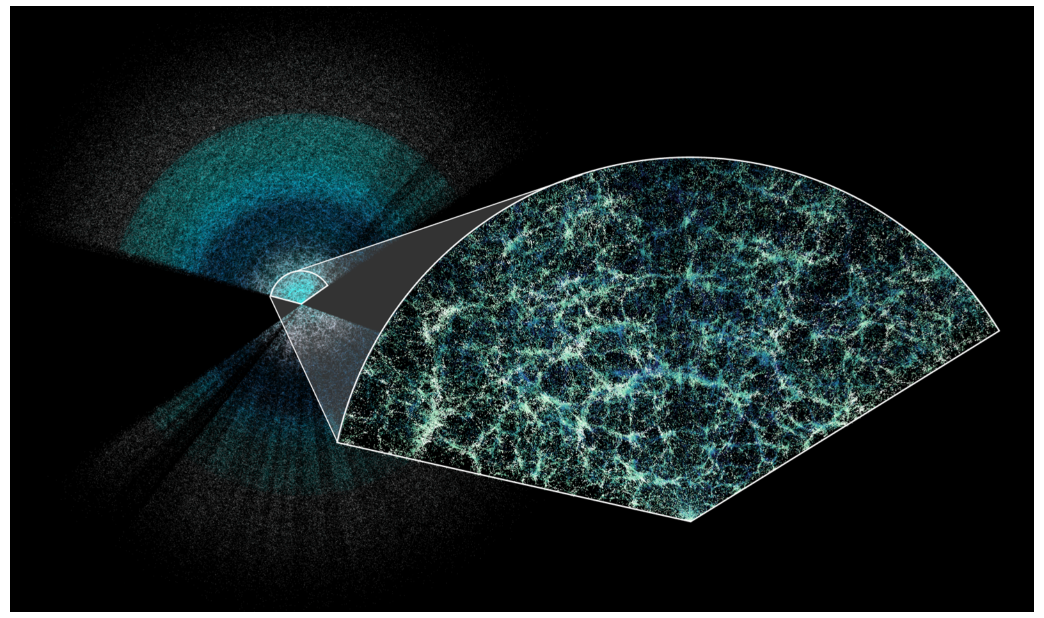
TO LEARN:
Bethe Ansatz Probs.
~~Kondo~~
2-D Hall
accel. Temp
Non Linear Classical Hydro

(A) $f = u(r, a)$
$g = \psi(r \cdot z)\, u(r \cdot z)$
(B) $f = 2|r \cdot a|(u \cdot a)$

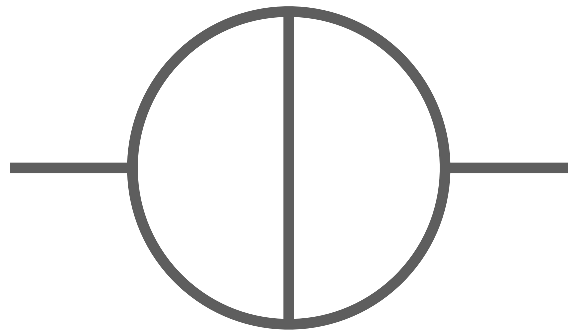
Feynman



Cosmology



Claire Lamman/DESI collaboration

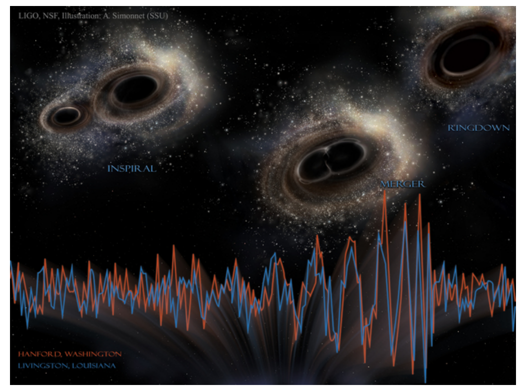


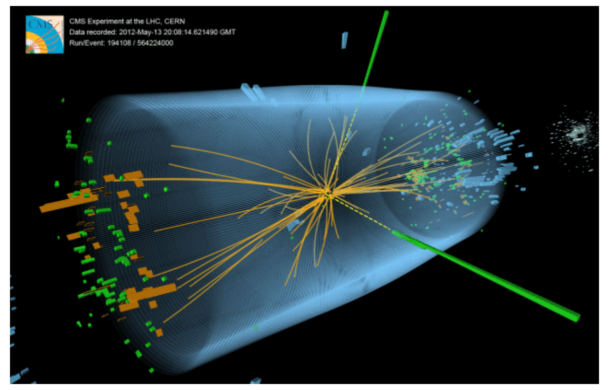
MOTIVATION

Number Theory

The (canonical) differential equation

FEYNMAN INTEGRALS



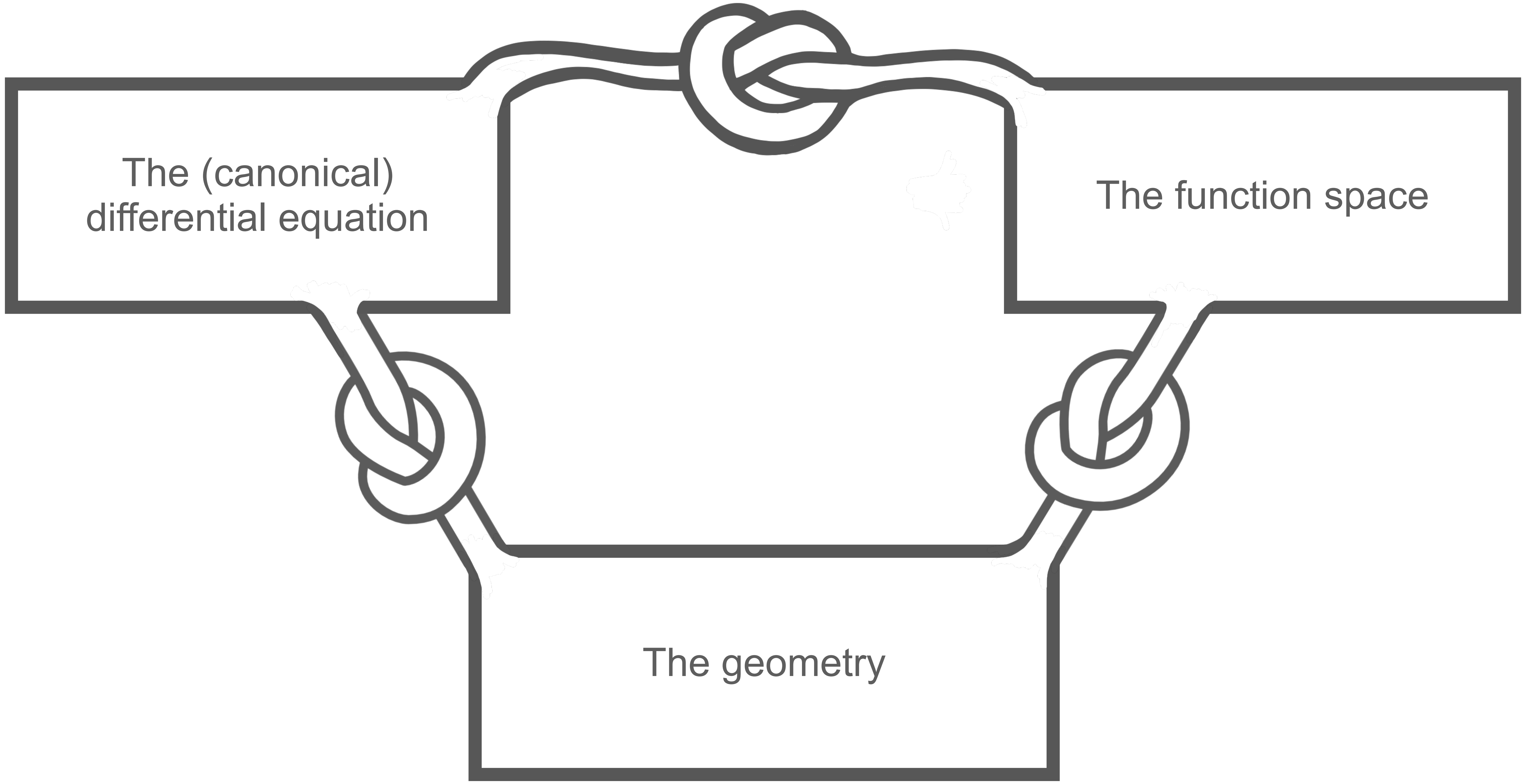


The function space

The geometry

CONTENTS

BASICS ON FEYNMAN INTEGRALS

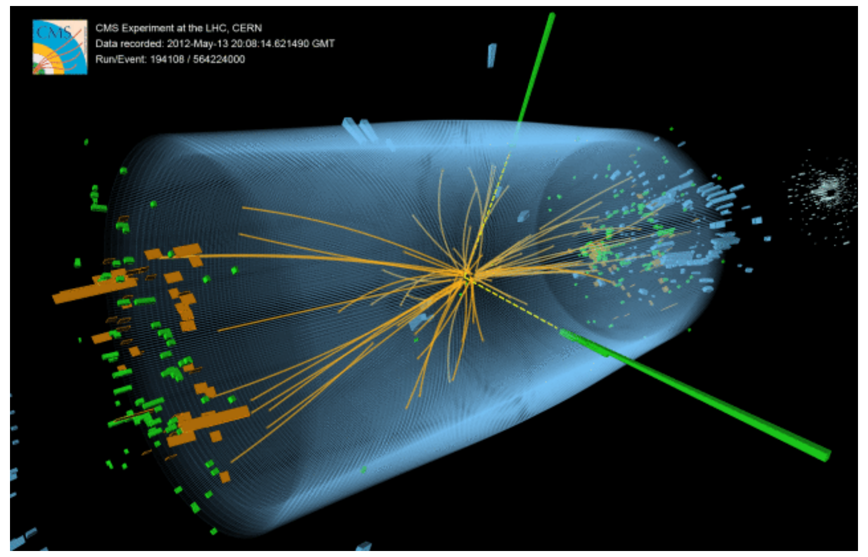


The (canonical)
differential equation

The function space

The geometry

WHAT IS A FEYNMAN INTEGRAL?


CERN

Collider experiment:

Probability for certain outcome = | Scattering Amplitude = $\mathcal{A} |^2$

Perturbation theory: $\mathcal{A} = \mathcal{A}^{(0)} + g\mathcal{A}^{(1)} + g^2\mathcal{A}^{(2)} + \ldots$

coupling constant

$\mathcal{A}^{(L)}$: contributions from all allowed L-loop Feynman diagrams

→ Translate to analytic expressions with Feynman rules

Building blocks: L-loop Feynman integrals

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{D_i^{\nu_i}}$$

Dimensional regularization: $D = d - 2\varepsilon$

Propagators: $D_i = (q_i^2 - m_i^2)$

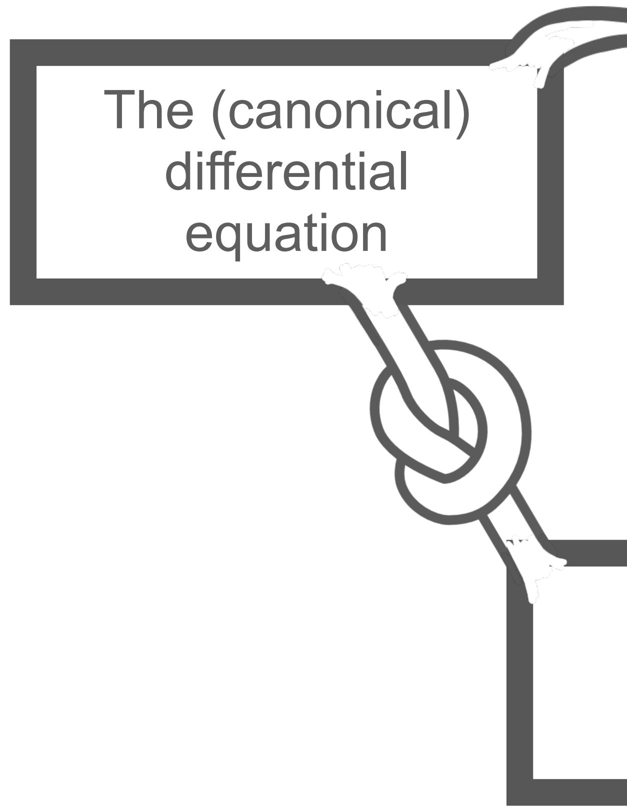
WHAT IS A FEYNMAN INTEGRAL?

Family of Feynman integrals
in
Momentum representation:

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{D_i^{\nu_i}}$$

Vector space with a basis $\boldsymbol{I} = (I_{\nu_1} \ldots I_{\nu_n})$
that fulfills a differential equation
$\mathrm{d}\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon) = \boldsymbol{A}(\boldsymbol{\lambda}, \varepsilon)\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon)$ with $\mathrm{d} = \sum_{\mathrm{i}=1} \mathrm{d}\lambda_{\mathrm{i}} \partial_{\lambda_{\mathrm{i}}}$

FEYNMAN INTEGRALS FROM DIFFERENTIAL EQUATIONS



Family of Feynman integrals
in
Momentum representation:

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}}\right) \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{D_i^{\nu_i}}$$

We want to compute a **Feynman integral family** analytically with ***differential equations***.

- Use IBPs to find a **basis of master integrals** for the integral family
- Set up a **differential equation** w.r.t the external (kinematic) parameters

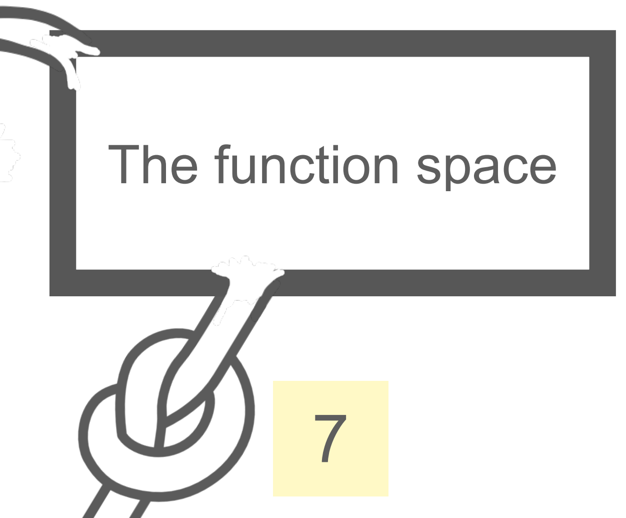
$\mathrm{d}\mathbf{I}(\mathbf{X}) = A(\mathbf{X},\varepsilon)\mathbf{I}(\mathbf{X})$ with $\mathrm{d} = \sum \mathrm{d}X_i \partial_{X_i}$ where X_i are kinematic variables

- Find a **canonical differential equation** & solve in terms of **iterated integrals.**
  [Henn]

$\mathbf{J}(\mathbf{X}) = \mathbf{U}\cdot\mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$

and $\varepsilon B(\boldsymbol{X}) = (\mathrm{d}\boldsymbol{U})\cdot \boldsymbol{U}^{-1} + \boldsymbol{U}\cdot \boldsymbol{A}(\boldsymbol{X},\varepsilon)\cdot\boldsymbol{U}^{-1}$

$$\mathbf{J}(\mathbf{X}) = \mathbb{P}\mathrm{exp}\left(\varepsilon\int_\gamma B\right)\cdot \mathbf{J}\left(\text{some point } \mathbf{X}^0\right) = \left(1+\varepsilon\int_\gamma B + \varepsilon^2\int_\gamma B\int_\gamma B + \dots\right)\cdot\mathbf{J}(\mathbf{X}^0)$$



WHAT IS A FEYNMAN INTEGRAL?

Family of Feynman integrals
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Momentum representation:

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{D_i^{\nu_i}}$$

Baikov representation:

$$I_{\underline{\nu}}(\underline{X}) = \int_\Gamma \boxed{\mathcal{B}(\underline{z})^{\mu}} \cdot \mathrm{d}^n \underline{z} \cdot \prod_i z_i^{-\nu_i}$$

non-integer, contains ε

Baikov polynomial

FEYNMAN INTEGRALS: RELATIVE TWISTED PERIODS

multivalued function in $\underline{z}$

Feynman Integrals: Baikov polynomial

single-valued form in $\underline{z}$, only poles @ branch points of Φ

function with additional poles in $\underline{z}$

Feynman Integrals: Propagators

$$\int \boxed{\Phi} \cdot \varphi \cdot \varpi$$

Period of **twisted cohomology**

Period of **relative twisted cohomology**

$$\begin{tikzcd}
\textbf{cohomology}\text{ group with basis } \{\varphi_i\} \arrow[rr, leftrightarrow, "C_{ij} \sim \int \varphi_i \wedge \check{\varphi}_j"] \arrow[d, leftrightarrow, "P_{ij} = \int_{\gamma_i} \boxed{\Phi} \cdot \varphi_i"'] & & \textbf{dual cohomology}\text{ group with basis } \{\check{\varphi}_i\} \arrow[d, leftrightarrow, "\check{P}_{ij} = \int_{\check{\gamma}_i} \boxed{\Phi}^{-1} \cdot \check{\varphi}_i"] \\
\textbf{homology}\text{ group with basis } \{\gamma_i\} \arrow[rr, leftrightarrow, "H_{ij} \sim \text{weighted topological intersection of } \gamma_i \,\&\, \check{\gamma}_j"'] & & \textbf{dual homology}\text{ group with basis } \{\check{\gamma}_i\}
\end{tikzcd}$$

WHAT IS A (MAXIMAL) CUT?

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\text{int}}} \frac{1}{D_i^{\nu_i}} \qquad \mathrm{Cut}_{j_1,\ldots,j_r}\left[I_{\boldsymbol{\nu}}\right] \sim \mathrm{Res}_{D_{j_1}=0,\ldots,D_{j_r}=0} \left[\int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\text{int}}} \frac{1}{D_i^{\nu_i}} \right]$$

Cut

in propagators $j_1, \ldots, j_r$

Baikov representation: $I_{\underline{\nu}}(\underline{X}) = \int_\Gamma \boxed{\mathcal{B}(\underline{z})^\mu} \cdot \mathrm{d}^n \underline{z} \cdot \xcancel{\prod_i z_i^{-\nu_i}}$

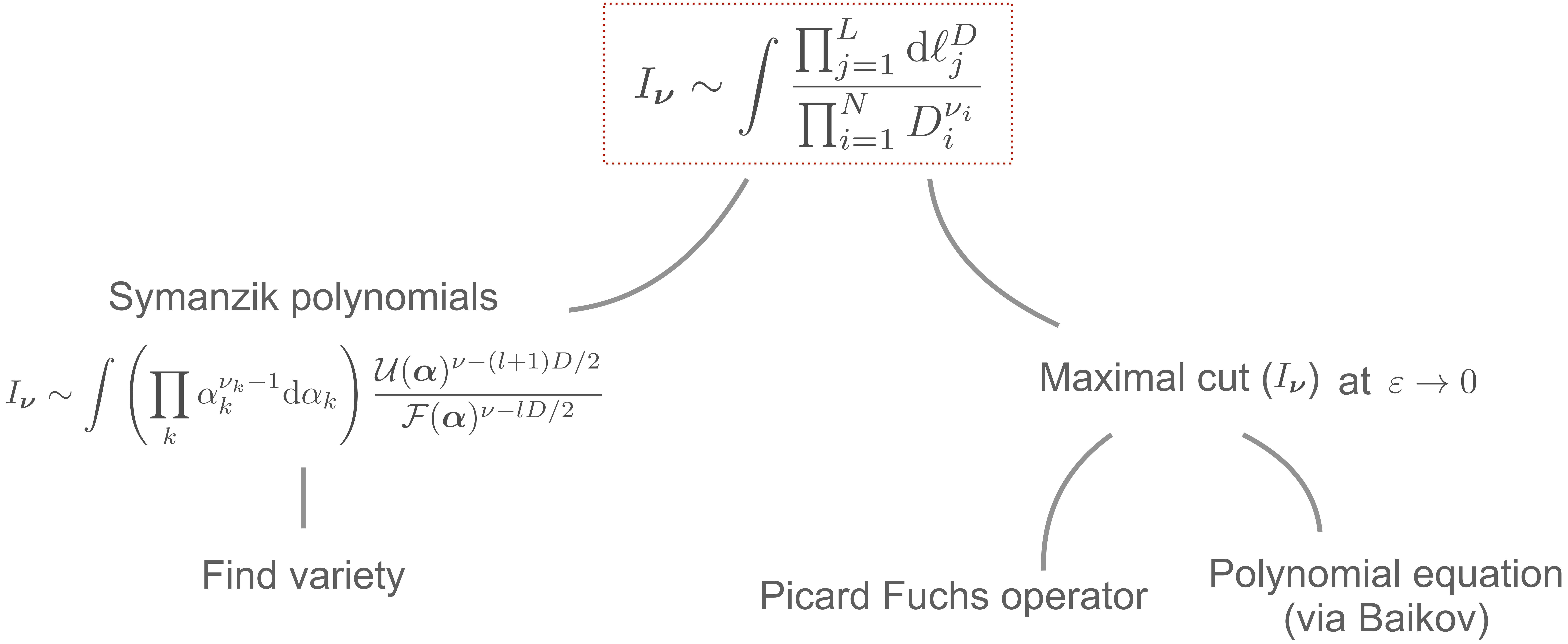
$z_1, \ldots, z_N \mapsto 0$ $\quad$ $\mathrm{d}^n \underline{z} \mapsto \mathrm{d}^{n-N} \underline{z}$

Period matrix = matrix of maximal cuts: $\boxed{\boldsymbol{P}} = \left(\int_{\gamma_i} \Phi \, \varpi_j \right)_{ij}$

Bases of twisted (co-)homology groups defined by $\mathcal{B}(\underline{z})^\mu$

FEYNMAN INTEGRALS AND *THEIR* GEOMETRIES

How do we associate one (or multiple) geometries to a Feynman integral (family)?



$$I_{\boldsymbol{\nu}} \sim \int \frac{\prod_{j=1}^{L} \mathrm{d}\ell_j^D}{\prod_{i=1}^{N} D_i^{\nu_i}}$$

Symanzik polynomials

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_k \alpha_k^{\nu_k-1}\mathrm{d}\alpha_k\right)\frac{\mathcal{U}(\boldsymbol{\alpha})^{\nu-(l+1)D/2}}{\mathcal{F}(\boldsymbol{\alpha})^{\nu-lD/2}}$$

Find variety

Maximal cut $(I_{\boldsymbol{\nu}})$ at $\varepsilon \to 0$

Picard Fuchs operator

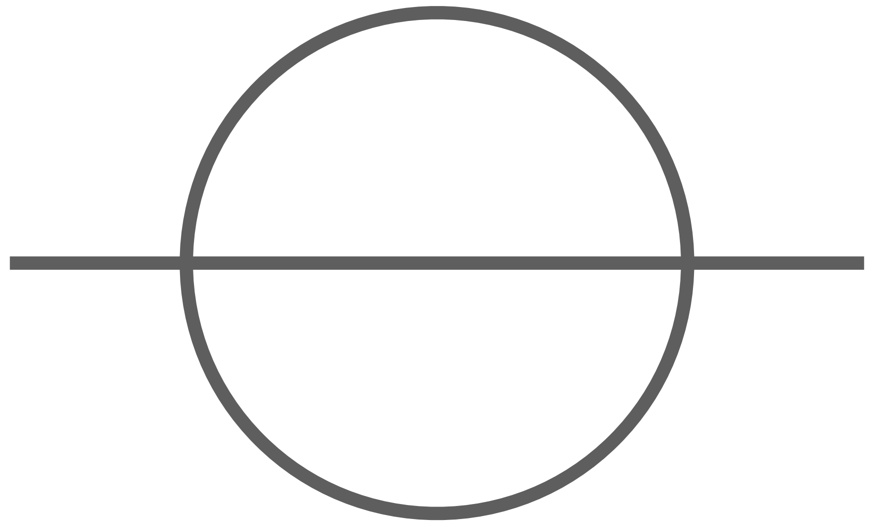
Polynomial equation
(via Baikov)

Not necessarily unique! [Marzucca, McLeod, Page, Pögel, Weinzierl | Jockers, Kotlewski, Kuusela, McLeod, Pögel, Sarve, Wang, Weinzierl]

EXAMPLES FOR HYPERELLIPTIC FEYNMAN INTEGRALS

ELLIPTIC EXAMPLE: SUNRISE

$$D = 2 - 2\varepsilon$$



maximal cut
loop - by - loop

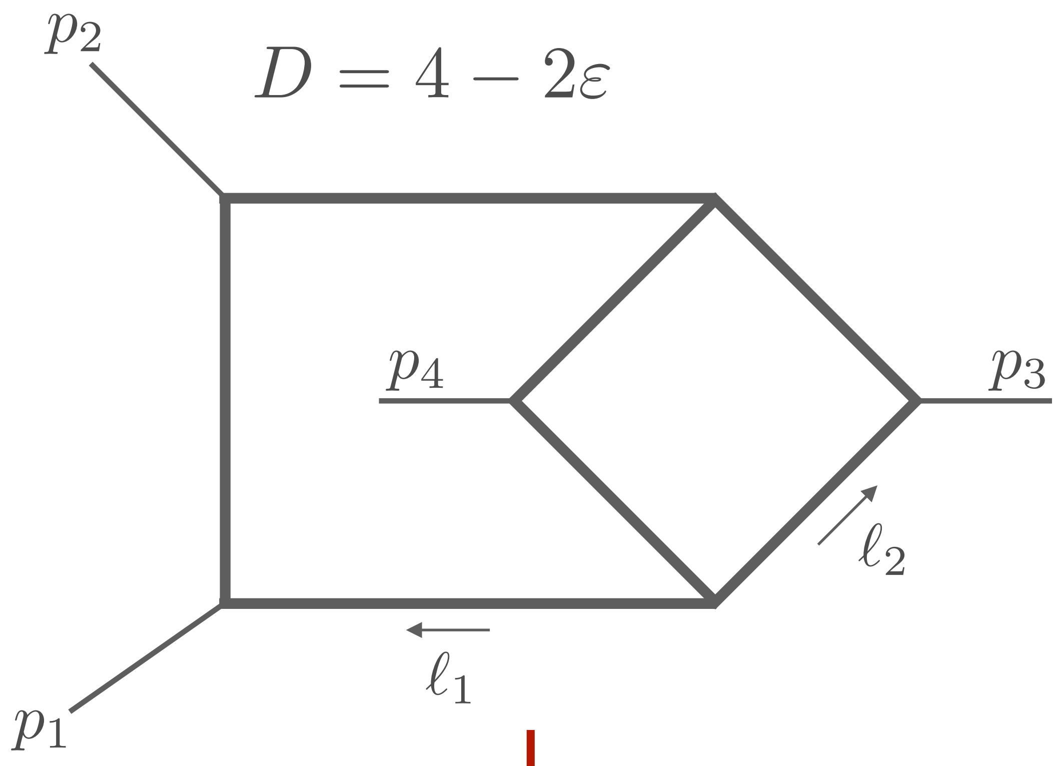
$$\int_\Gamma \mathrm{d}x\, x^\varepsilon \left[\boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}\right]^{-\frac{1}{2}-\varepsilon}$$

even **elliptic** curve of **genus 1**:

$$y^2 = \boxed{(x-\mu_1)(x-\mu_2)(x-\mu_3)(x-\mu_4)}$$

HYPERELLIPTIC EXAMPLE: NON-PLANAR CROSSED BOX

$$D = 4 - 2\varepsilon$$



maximal cut
loop - by - loop

$$\int_\Gamma \mathrm{d}x\, \boxed{[(x-\lambda_1)(x-\lambda_2)]^{-\frac{1}{2}}\,[(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)(x-\lambda_6)]}^{-\frac{1}{2}-\varepsilon}$$

[Huang, Zhang | Georgoudis, Zhang | Marzucca, McLeod, Page, Pögel, Weinzierl]

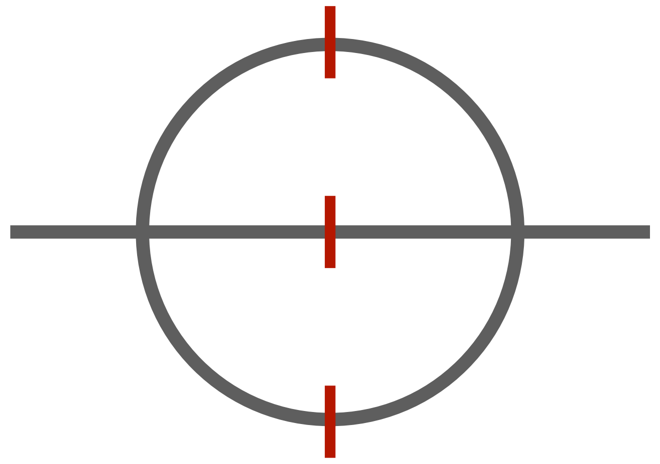
even **hyperelliptic** curve of **genus 2**:

$$y^2 = \boxed{(x-\lambda_1)\,(x-\lambda_2)\,(x-\lambda_3)\,(x-\lambda_4)\,(x-\lambda_5)\,(x-\lambda_6)}$$

WHAT IS A (MAXIMAL) CUT? SECOND PERSPECTIVE

MAXIMAL CUT

The **fundamental solution** for the homogenous differential equation of the top sector.



$$\mathrm{d}\begin{pmatrix} I_{110} \\ I_{101} \\ I_{011} \\ I_{111} \\ I_{211} \\ I_{121} \\ I_{112} \end{pmatrix} = \begin{bmatrix} \bullet & - & - & - & - & - & - \\ - & \bullet & - & - & - & - & - \\ - & - & \bullet & - & - & - & - \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \\ - & - & - & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \begin{pmatrix} I_{110} \\ I_{101} \\ I_{011} \\ I_{111} \\ I_{211} \\ I_{121} \\ I_{112} \end{pmatrix}$$

$$\mathrm{d}\boldsymbol{P} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} \boldsymbol{P}$$

2. MORE ON CANONICAL DIFFERENTIAL EQUATIONS

C - FORM

FINDING CANONICAL DEQs

$\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \longrightarrow$ **How do we find this (systematically)?**

Different methods for finding canonical DEQ of Feynman integrals with elliptic curve or CY geometry.

[Brösel, Duhr, Dulat, Penante, Tancredi | Pögel, Wang, Weinzierl | Görges, Nega, Tancredi, Wagner]

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2. Compute the period matrix at $\varepsilon = 0$ and split it in semi-simple and unipotent parts.
   Rotate the initial basis with the **inverse of the semi-simple part**.
   (*Geometry inspired* step)

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3. Make further simple rotations (exchanges of basis elements + powers of ε)
   to make the remaining **non-canonical part lower-triangular**.
   (*Adjustment* step)

FINDING CANONICAL DEQs

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   (*Geometry inspired* step)
3. Make further simple rotations (exchanges of basis elements + powers of ε)
   to make the remaining **non-canonical part lower-triangular**.
   (*Adjustment* step)
4. Make **ansatz** to remove these remaining **non-canonical** entries and
   solve the resulting differential equations.
   (*New objects* step)

THE C - FORM

Short version: All known (to us) canonical DEQS for Feynman integrals are also in C-form!

Long version:

$$\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \quad \text{with} \quad B(\boldsymbol{X})_{ij} = \sum_{k=1}^{n} \mathrm{d}X_k f_{ijk}$$

$\mathcal{A} = \mathbb{K}$ - algebra of functions that contains all f_{ijk} and:

- Differentially closed ($f \in \mathcal{A} \Rightarrow \partial_{X_i} f \in \mathcal{A} \forall i$)
- Constants = $\mathbb{K}$ ($\partial_{X_i} f = 0 \forall i \Rightarrow f \in \mathbb{K}$)

$\mathbb{A} = \mathbb{K}$ - vector space of closed differential forms generated by the forms appearing in $B(\boldsymbol{X})$

$\mathcal{F}_{\mathbb{C}} = \mathrm{Frac}\,(\mathbb{C} \otimes_{\mathbb{K}} \mathcal{A})$

An ε- factorised differential equation is in C-form, if $\mathbb{A} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$.

[Duhr, Semper, Stawiński, FP]

Example:

$B(\boldsymbol{X})$ in dLog-form with

$$f_{ij} = \sum_r \frac{1}{a_{ijr} - X}$$

- $\mathcal{A}_{\mathrm{dLog}} =$ Rational functions in X with singularities at the a_{ijr}
- $\mathbb{A}_{\mathrm{dLog}} = \left\langle \frac{\mathrm{d}X}{a_{ijr} - X} \middle| \ \text{all } i, j, r \right\rangle$

Elements of $\mathrm{d}\mathcal{F}_{\mathbb{C}}$:
no pole/ pole of order > 1

$$\Rightarrow \mathbb{A}_{\mathrm{dLog}} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$$

THE C - FORM : MORE EXAMPLES

Elliptic ${}_2F_1$

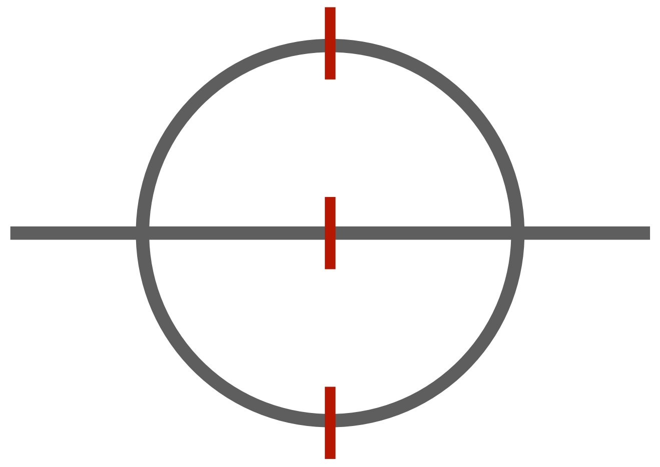
$$I_{\boldsymbol{\nu}}^1 = \int_0^1 \mathrm{d}\mathrm{x}\,\mathrm{x}^{-\frac{1}{2}+\nu_1+\mathrm{a}_1\varepsilon}(\mathrm{x}-1)^{-\frac{1}{2}+\nu_2+\mathrm{a}_2\varepsilon}(\mathrm{x}-\lambda)^{-\frac{1}{2}+\nu_3+\mathrm{a}_3\varepsilon}$$

→ 2 master integrals

→ Elliptic curve $y^2 = x(x-1)(x-\lambda)$

$$\mathcal{A} = \text{Differential closure of } \mathbb{Q}\left[i\pi^{\pm}, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \lambda, \mathrm{K}(\lambda), \mathrm{E}(\lambda), \frac{1}{\mathrm{K}(\lambda)}\right]$$

Sunrise



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→ Specific modular and quasi-modular form

THE C - FORM FOR MAXIMAL CUTS

Reminder: Feynman integral family defines $H^{n-N}_{\mathrm{dR}}(X,\Phi)$ and $H_{n-N}(X,\check{\mathcal{L}}_\Phi)$ with $\Phi = \mathcal{B}(\underline{z})^\mu$

Period matrix = matrix of cuts:

Slogan:

Basis and dual basis are in $\mathcal{E}$**-form** and **C-form** $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}\boldsymbol{C} = 0$.

[Duhr, Semper, Stawiński, FP]

Proof idea: Use $\boldsymbol{C} = \frac{1}{(2\pi i)^n}\boldsymbol{P}\left(\boldsymbol{H}^{-1}\right)^T\check{\boldsymbol{P}}^T$ and linear independence of iterated integrals

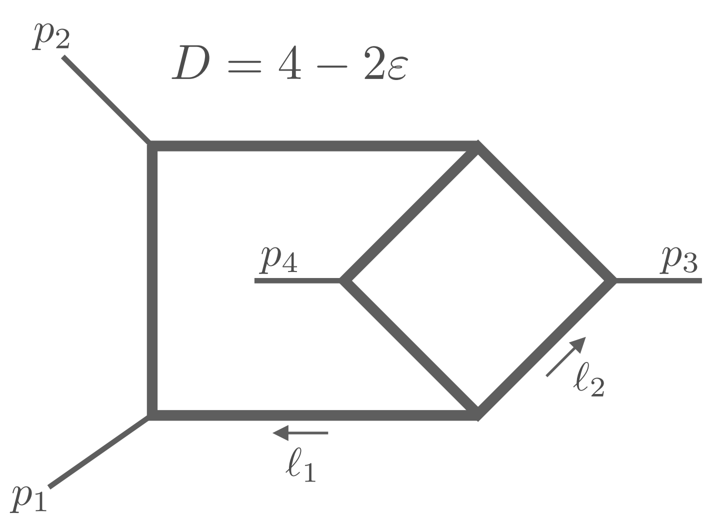
For maximal cuts: Can choose $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$

- Conceptual: Condition for canonical form and classification of appearing functions
- Practical: Helps us find relations for new functions in canonical DEQ

3. EXAMPLE: INTEGRAL FAMILY RELATED TO GENUS 2 CURVE

EXAMPLE: MAXIMAL CUT OF THE NON-PLANAR CROSSED BOX

MC



$$L(\boldsymbol{\lambda},\boldsymbol{a}) = \int_{\lambda_1}^{\lambda_2} \mathrm{d}x\,\left(1-\lambda_1^{-1}x\right)^{-\frac{1}{2}+a_1\varepsilon}\dots\left(1-\lambda_6^{-1}x\right)^{-\frac{1}{2}+a_6\varepsilon}$$

Twist = $\frac{\Phi}{y}$ with $\Phi = \prod_{i=1}^{6}(1-\lambda_i^{-1}x)^{a_i\varepsilon}$ & $y = \prod_{i=1}^{6}\sqrt{(1-\lambda_i^{-1}x)}$

BASIS OF DIFFERENTIALS:

$$\varphi_1^{(0)} = \frac{\mathrm{d}x}{y}\Phi\,,\quad \varphi_2^{(0)} = \frac{x\mathrm{d}x}{y}\Phi\,,\quad \varphi_3^{(0)} = \frac{\Phi_1(x)\mathrm{d}x}{y}\Phi\,,\quad \varphi_4^{(0)} = \frac{\Phi_2(x)\mathrm{d}x}{y}\Phi \quad \text{and} \quad \varphi_5^{(0)} = \frac{x^2\mathrm{d}x}{y}\Phi$$

„first kind“ „second kind“ „third kind“

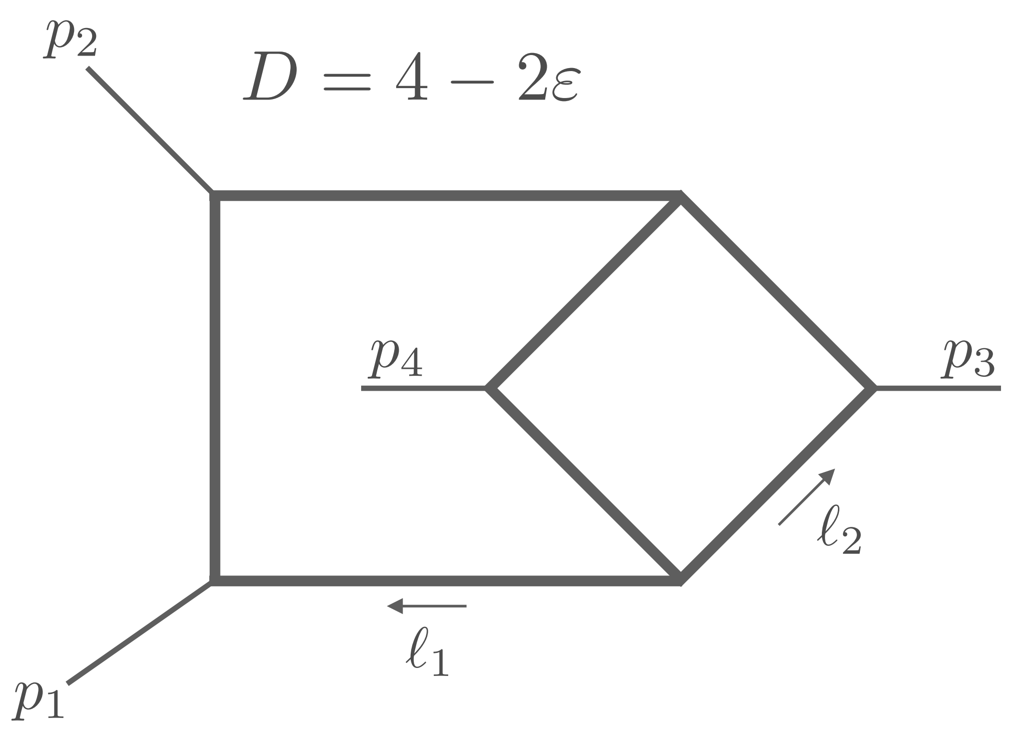
BASIS OF CYCLES:

$$[\lambda_1,\lambda_2]\,,\,[\lambda_3,\lambda_4] \quad [\lambda_2,\lambda_3]+[\lambda_4,\lambda_5]\,,\,[\lambda_4,\lambda_5] \quad [\lambda_1,\lambda_2]+[\lambda_3,\lambda_4]+[\lambda_5,\lambda_6]$$

$a_1\,,\,a_2$ $b_1\,,b_2$

EXAMPLE: MAXIMAL CUT OF THE NON-PLANAR CROSSED BOX

MC



$$\Big)_{\varepsilon \to 0} = \mathcal{P}\Big|_{\epsilon \to 0} = \begin{array}{c|cc|l} \text{a-cycles} & \text{b-cycles} & & \\ \mathcal{A} & \mathcal{B} & 0 & \text{first kind} \\ \hline \tilde{\mathcal{A}} & \tilde{\mathcal{B}} & 0 & \text{second kind} \\ \star & \star & \star & \end{array}$$

„normalized“ period

$$\Omega = \mathcal{A}^{-1} \cdot \mathcal{B}$$

Genus 1:

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

$$\tau = \omega_2 \cdot \omega_1^{-1}$$

STEP 1: DERIVATIVE BASIS

$$\varphi_1^{(0)} = \frac{\mathrm{d}x}{y}\Phi\,,\ \varphi_2^{(0)} = \frac{x\mathrm{d}x}{y}\Phi\ ,\ \varphi_3^{(0)} = \frac{\Phi_1(x)\mathrm{d}x}{y}\Phi\,,\ \varphi_4^{(0)} = \frac{\Phi_2(x)\mathrm{d}x}{y}\Phi \text{ and } \varphi_5^{(0)} = \frac{x^2\mathrm{d}x}{y}\Phi$$

$$\mathrm{d}\boldsymbol{\varphi}^{(0)} = \left[\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix}\varepsilon\right]\boldsymbol{\varphi}^{(0)}$$

$$\downarrow$$

$$\boldsymbol{\varphi}^{(1)} = U_6^{(1)}\boldsymbol{\varphi}^{(0)}$$

$$\varphi_1^{(1)} = \varphi_1^{(0)},\quad \varphi_2^{(1)} = \varphi_2^{(0)} \qquad \varphi_3^{(1)} = \frac{\partial}{\partial\lambda_1}\varphi_1^{(0)}\ ,\ \varphi_4^{(1)} = \frac{\partial}{\partial\lambda_2}\varphi_2^{(0)} \quad \text{and} \quad \varphi_5^{(1)} = \varphi_5^{(0)}$$

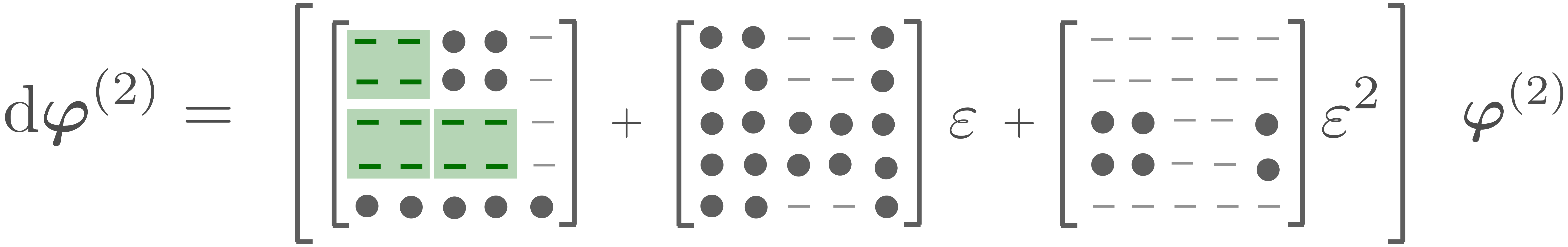
$$\mathrm{d}\boldsymbol{\varphi}^{(1)} = \left[\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & - \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & - & - & \bullet \\ \bullet & \bullet & - & - & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ - & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & - & - & \bullet \end{bmatrix}\varepsilon + \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & \bullet \\ \bullet & \bullet & - & - & \bullet \\ - & - & - & - & - \end{bmatrix}\varepsilon^2\right]\boldsymbol{\varphi}^{(1)}$$

STEP 2: SEMI-SIMPLE ROTATION

$$\lim_{\varepsilon \to 0} \mathcal{P}_{(1)} = \lim_{\varepsilon \to 0} U_6^{(1)} \mathcal{P}_{(0)} = \lim_{\varepsilon \to 0} U_6^{(1)} \begin{pmatrix} \mathcal{A} & \mathcal{B} & 0 \\ \tilde{\mathcal{A}} & \tilde{\mathcal{B}} & 0 \\ \star & \star & \star \end{pmatrix} = \underbrace{\lim_{\varepsilon \to 0} U_6^{(1)} \begin{pmatrix} \mathcal{A} & 0 & 0 \\ \tilde{\mathcal{A}} & 2\pi i \cdot \mathbf{1} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \begin{pmatrix} \mathbf{1} & \Omega & 0 \\ \mathbf{0} & \mathbf{1} & 0 \\ \star & \star & \star \end{pmatrix}$$

↓

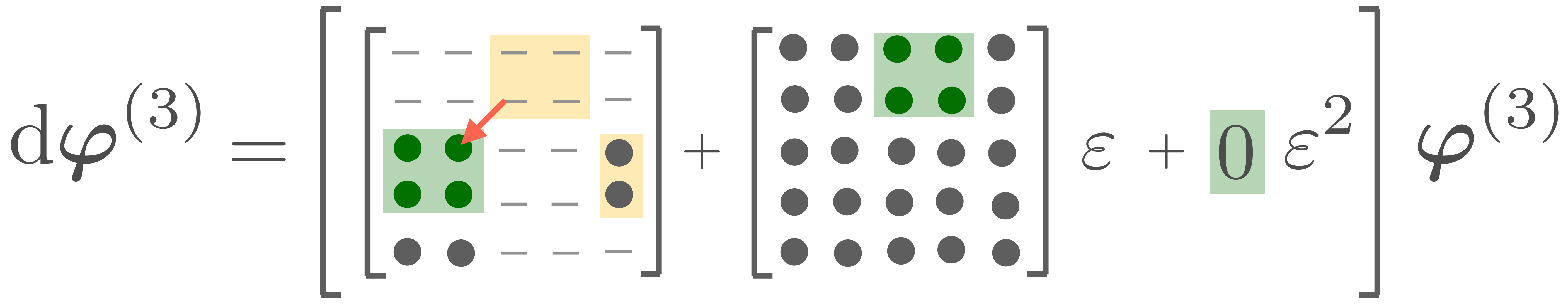
$$\varphi^{(2)} = \boldsymbol{S}^{-1} \varphi^{(1)}$$



STEP 3: ADJUSTMENTS

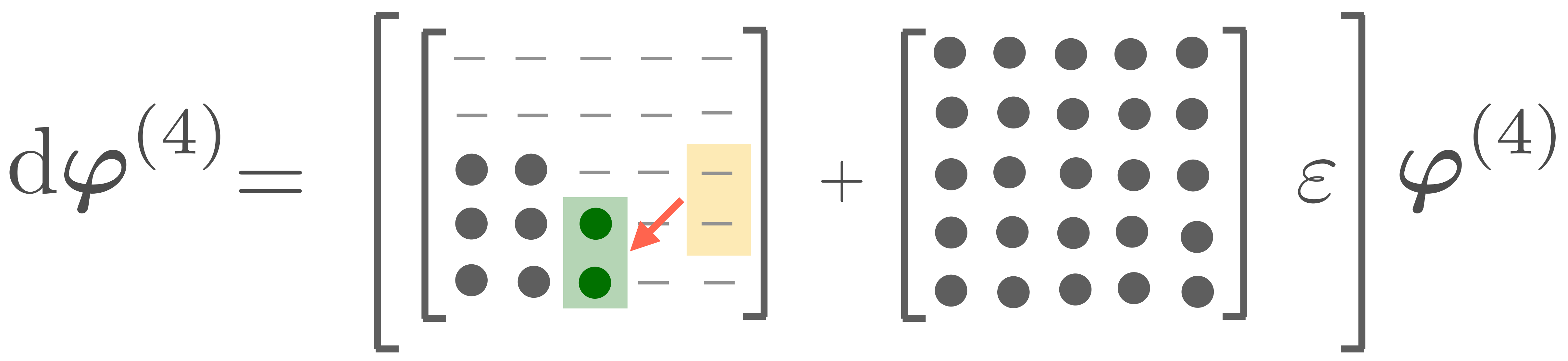
Remove ε^2 - terms:

$$\varphi^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \varphi^{(2)} \longrightarrow$$

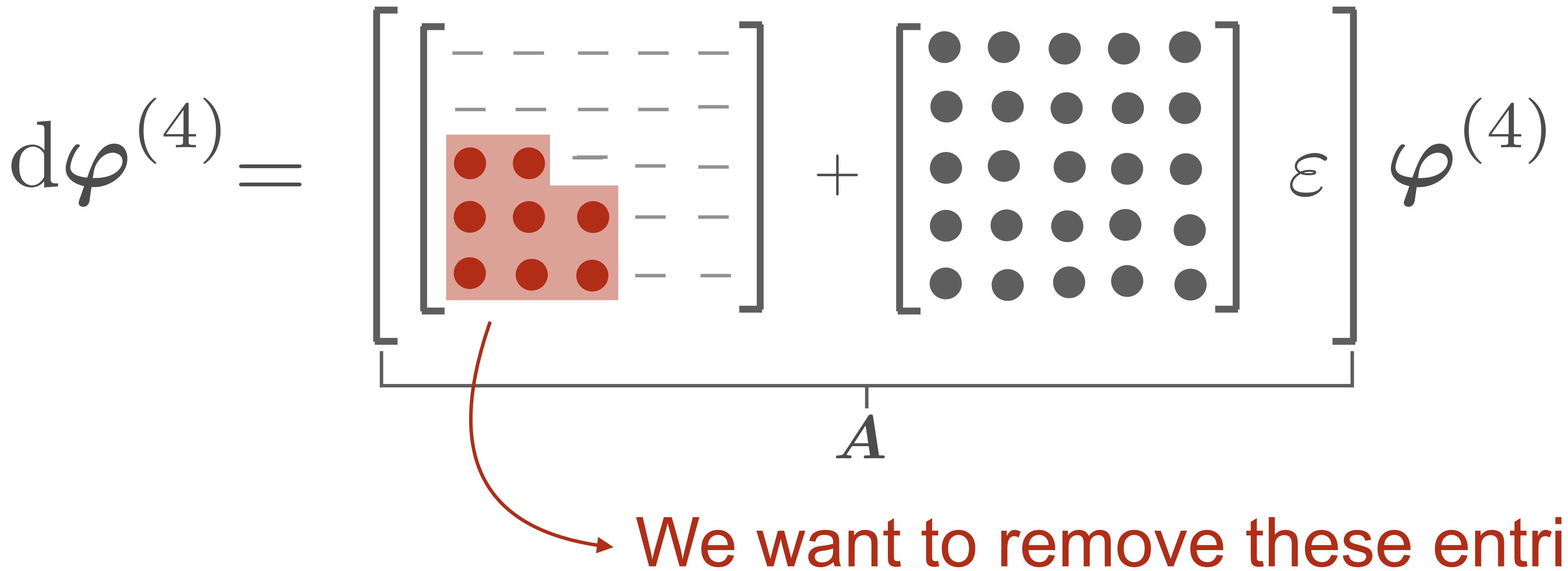


Lower triangular ε^0 - terms:

$$\varphi^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \varphi^{(3)} \longrightarrow$$



STEP 4: ANSATZ



$$\mathrm{d}\boldsymbol{\varphi}^{(4)} = \underbrace{\left[\begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & - \\ \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & \bullet & - & - \end{bmatrix} + \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{bmatrix} \varepsilon \right]}_{\boldsymbol{A}} \boldsymbol{\varphi}^{(4)}$$

We want to remove these entries!

Find final transformation:

1. Make an ansatz:

$$\mathrm{d}\varphi^{(5)} = U_6^{(5)} \varphi^{(4)} \quad \text{with} \quad U_6^{(5)} = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \star & \star & - & - & - \\ \star & \star & \star & - & - \\ \star & \star & \star & - & - \end{bmatrix}$$

unknowns

2. Transform the differential equation:

$$\mathrm{d}\boldsymbol{\varphi}^{(5)} = \left[\left(\mathrm{d}U_6^{(5)}\right) \left(U_6^{(5)}\right)^{-1} + U_6^{(5)} \boldsymbol{A} \left(U_6^{(5)}\right)^{-1} \right] \boldsymbol{\varphi}^{(5)}$$

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

STEP 4: ANSATZ

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

- Non-trivial to solve !

- Undetermined (at most 8) number of *new* functions !
  (not expressible just in periods and branch points)

STEP 4: ANSATZ

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

↓

We can simplify this, using the **intersection matrix**!

Slogan:

Basis and dual basis are in **$\mathcal{E}$-form** and **C-form** $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}\boldsymbol{C} = 0$.

[Duhr, Semper, Stawiński, FP]

Use this condition constructively:

1. Choose basis $\varphi^{(5)}$ & dual basis $\check{\varphi}^{(5)}$, so that $\check{\boldsymbol{P}}(\varepsilon) = \boldsymbol{P}(-\varepsilon)$.
2. Compute intersection matrix $\boldsymbol{C}$ [Contains the 8 unknowns ★ of $U_6^{(5)}$]
3. Require all entries of $\boldsymbol{C}$ to be constant in parameters λ_i and solve for (some) ★ .

STEP 4: ANSATZ

3. Require that the ε^0- entries vanish

8 coupled differential equations of 8 **unknowns** ★

Slogan:

Basis and dual basis are in **$\mathcal{E}$-form** and **C-form** $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}C = 0$.

[Duhr, Semper, Stawiński, FP]

Use this condition constructively:

$$U_6^{(5)} = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & - \\ \star & \bullet & \star & - & - \\ \bullet & \bullet & \star & - & - \end{bmatrix} \longrightarrow$$

All but three entries of the final transformation
(expressed in periods, branch points & the three remaining new functions)

Find from (now) simpler differential equations

$$C = \begin{bmatrix} - & - & - & - & \bullet \\ - & - & - & \bullet & - \\ - & - & \bullet & - & - \\ - & \bullet & - & - & - \\ \bullet & - & - & - & - \end{bmatrix} \longrightarrow$$

A constant skew-diagonal intersection

The requirement, that the **intersection matrix** is constant, can be used ***constructively***!

RESULTS

$$\boldsymbol{\varphi}^{(5)} = U_6^{(5)}\ U_6^{(4)}\ U_6^{(3)}\ U_6^{(2)}\ U_6^{(1)}\ \boldsymbol{\varphi} \rightarrow$$

Initial basis
(inspired by geometry)

Ansatz for final rotation
(Use intersection matrix)

Derivative basis

Reordering

Rotation with the inverse of the semi-simple part of the period matrix

$\mathcal{E}$ - factors

$$\mathrm{d}\varphi^{(5)} = \varepsilon \boldsymbol{B}(\underline{\lambda})\ \varphi^{(5)}$$ in $\mathcal{E}$-form and C-form

SUMMARY: THREE TAKEAWAYS

Compute Feynman integrals in terms of specific iterated integrals (from DEQ)

Differential equation for maximal cut in ε- form and C-form $\Longrightarrow$ constant intersection matrix! Can be used constructively!

The algorithm by [Görges,Nega,Tancredi,Wagner] also works for hyperelliptic maximal cuts!

OUTLOOK

- Better understanding of the appearing (partially Siegel modular) forms
- Numerical evaluation of hyperelliptic Feynman integrals
- Better understanding of the role of the C-form (more generally)

BACKUP

WHY? PROOF!

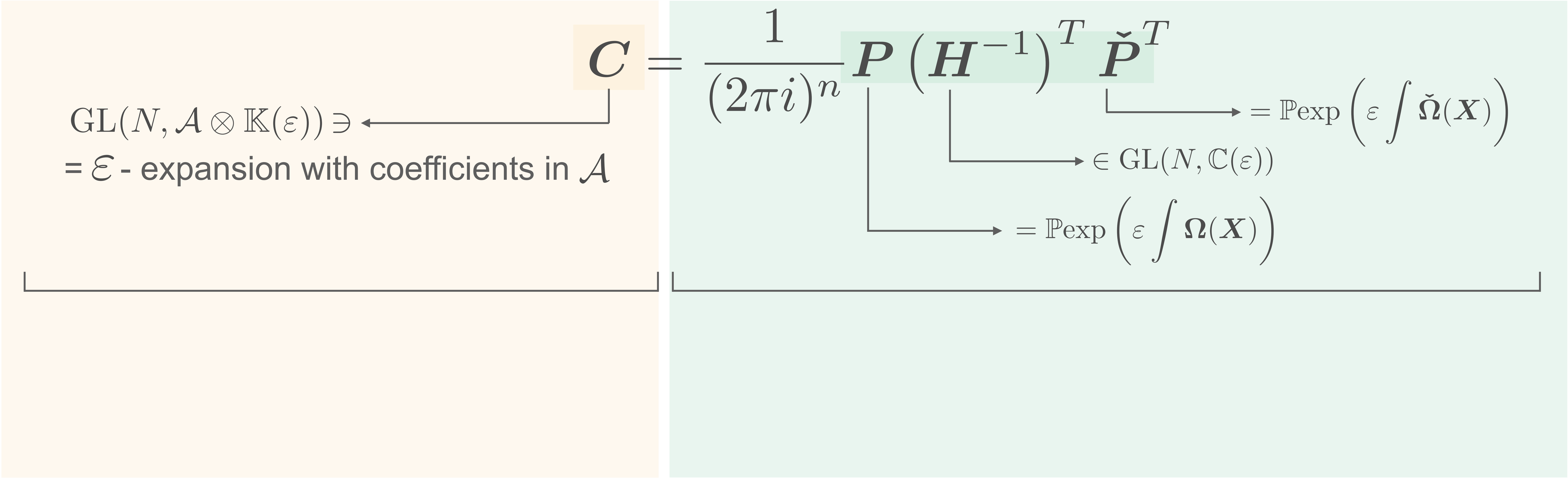
Assumption: Period matrices $\boldsymbol{P}$ and $\check{\boldsymbol{P}}$ with differential equations in ε-form and C-form (same algebra)

$$\boldsymbol{C} \in \mathbb{C}(\varepsilon) \longrightarrow \mathrm{d}\boldsymbol{C} = 0$$

WHY? PROOF!

Assumption: Period matrices $\boldsymbol{P}$ and $\check{\boldsymbol{P}}$ with differential equations in $\mathcal{E}$-form and C-form (same algebra)

Twisted Riemann bilinear relations:



$$\boldsymbol{C} = \frac{1}{(2\pi i)^n} \boldsymbol{P} \left(\boldsymbol{H}^{-1}\right)^T \check{\boldsymbol{P}}^T$$

$\mathrm{GL}(N, \mathcal{A} \otimes \mathbb{K}(\varepsilon)) \ni$

= $\mathcal{E}$ - expansion with coefficients in $\mathcal{A}$

$= \mathbb{P}\mathrm{exp}\left(\varepsilon \int \check{\boldsymbol{\Omega}}(\boldsymbol{X})\right)$

$\in \mathrm{GL}(N, \mathbb{C}(\varepsilon))$

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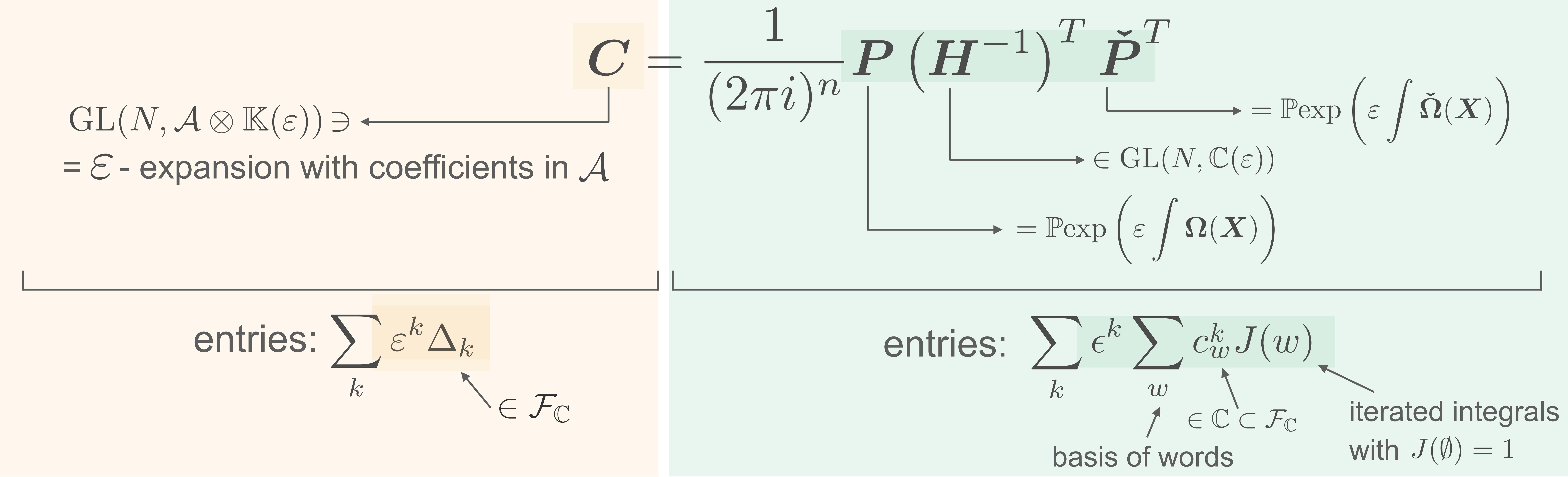

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$= \mathbb{P}\mathrm{exp}\left(\varepsilon \int \boldsymbol{\Omega}(\boldsymbol{X})\right)$

entries: $\sum_k \varepsilon^k \Delta_k$

$\in \mathcal{F}_{\mathbb{C}}$

entries: $\sum_k \epsilon^k \sum_w c_w^k J(w)$

$\in \mathbb{C} \subset \mathcal{F}_{\mathbb{C}}$

basis of words

iterated integrals with $J(\emptyset) = 1$

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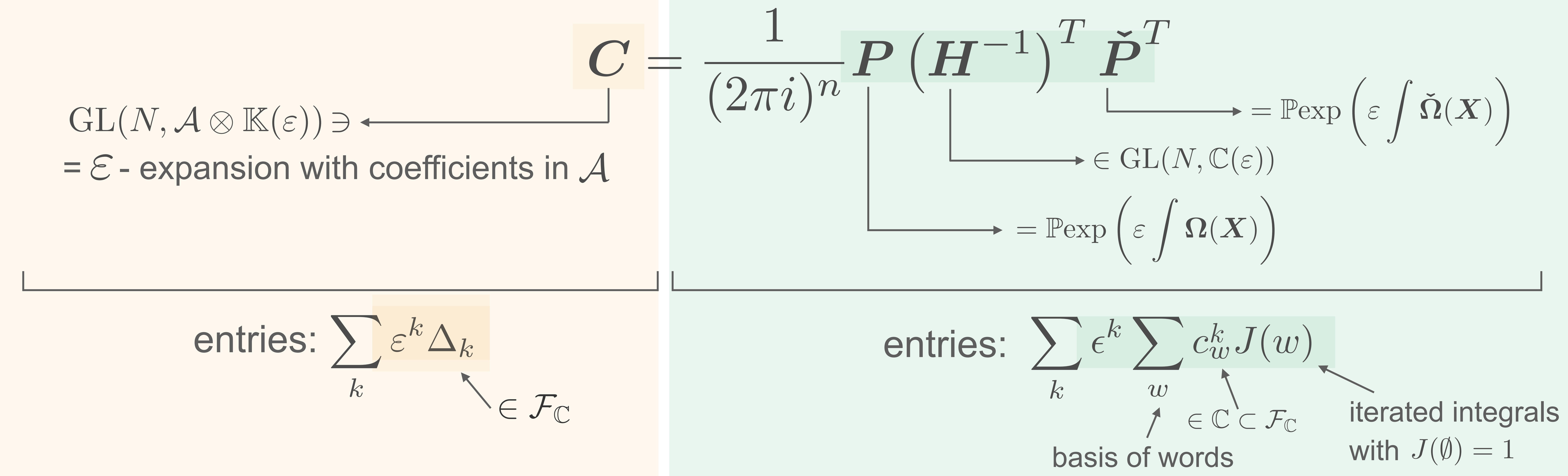

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$\in \mathbb{C} \subset \mathcal{F}_\mathbb{C}$

basis of words

iterated integrals with $J(\emptyset) = 1$

$$\Delta_k J(\emptyset) = \sum_w c_w^k J(w) \Rightarrow 0 = (c_\emptyset^k - \Delta_k) J(\emptyset) + \sum_{w \neq \emptyset} c_w^k J(w)$$

$$\boldsymbol{C} \in \mathbb{C}(\varepsilon) \longrightarrow \mathrm{d}\boldsymbol{C} = 0$$

WHY? PROOF!

Assumption: Period matrices $\boldsymbol{P}$ and $\check{\boldsymbol{P}}$ with differential equations in $\mathcal{E}$-form and C-form (same algebra)

Twisted Riemann bilinear relations:

$$\boldsymbol{C} = \frac{1}{(2\pi i)^n}\boldsymbol{P}\left(\boldsymbol{H}^{-1}\right)^T\check{\boldsymbol{P}}^T$$

$\mathrm{GL}(N,\mathcal{A}\otimes\mathbb{K}(\varepsilon)) \ni$

= $\mathcal{E}$ - expansion with coefficients in $\mathcal{A}$

$= \mathbb{P}\mathrm{exp}\left(\varepsilon\int\check{\boldsymbol{\Omega}}(\boldsymbol{X})\right)$

$\in \mathrm{GL}(N,\mathbb{C}(\varepsilon))$

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entries: $\sum_k \varepsilon^k\Delta_k$

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$\in\mathbb{C}\subset\mathcal{F}_{\mathbb{C}}$

basis of words

iterated integrals with $J(\emptyset)=1$

$$\Delta_k J(\emptyset) = \sum_w c^k_w J(w) \Rightarrow 0 = (c^k_\emptyset - \Delta_k)J(\emptyset) + \sum_{w\neq\emptyset} c^k_w J(w)$$

C-form $\Leftrightarrow$ $J(w)$ are linearly independent over $\mathcal{F}_{\mathbb{C}}$

[Deneufchatel, Duchamp, Hoang Ngoc Minh, Solomon]

$\Rightarrow c^k_w = 0$ for $w\neq\emptyset$ and $c_\emptyset\in\mathbb{C}$

$$\boxed{\boldsymbol{C}\in\mathbb{C}(\varepsilon) \longrightarrow \mathrm{d}\boldsymbol{C}=0}$$