Hypergeometric local systems / $\mathbb{Q}$

with Hodge vector $(1,1,1,1)$

MITP Youngst@rs - Physics and Number Theory

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ABOUT:

"hypergeometric local systems over $\mathbb{Q}$ with Hodge vector $(1,1,1,1)$"

(hgm)

Jt with Fernando Rodriguez Villegas

arXiv: 2401.13529

Consider all irreducible hgm local systems on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ that support a rational variation of Hodge structure (VHS) with Hodge vector $(1,1,1,1)$

1) How many? 47 cases.

GOALS :

2) A. Are they associated to 1-parameter families of 3 folds? yes, constructive

3) B. Analyze geometry and arithmetic at $t=1$ (conifold pt).

PLAN.

0) Background / Motivation

0) Background and Motivation

hgm over $\mathbb{C}$

$n \geq 1$ $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ multisets of rational numbers.

the hgm operator $H = H(\alpha, \beta) = \prod_{i=1}^{n} (D + \beta_i - 1) - t \prod_{i=1}^{n} (D + \alpha_i)$

(t a coordinate on $\mathbb{P}^1$, $D = t\frac{d}{dt}$)

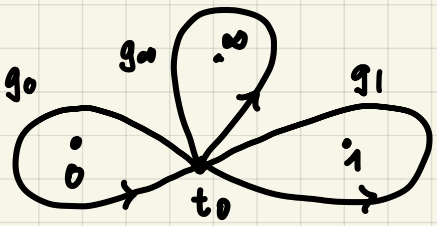
a differential operator with regular singularities at $t = 0, 1, \infty$.

the local system $\mathbb{H}$ of slns to $H \cdot \varphi = 0$ is a complex local system of rank n on $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

hgm local systems on U.

$\rho : \pi_1(U, t_0) \to GL_n(\mathbb{C})$ the monodromy representation.

$h_i := \rho(g_i)$ local monodromy at i



$g_\infty g_1 g_0 = 1$

Properties.

1) characteristic polynomials of h_0^{-1} and h_∞ are :

$q_0 = \prod_{j=1}^{n} (x - e^{2\pi i \beta_j})$ $q_\infty = \prod_{j=1}^{n} (x - e^{2\pi i \alpha_j})$

2) h_1 is a pseudoreflection i.e. $\mathrm{rk}(h_1 - \mathrm{Id}) = 1$

if $\beta_i = 1$ for some i,

the hgm function $F(\alpha, \beta; t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_n)_k} t^k$ solves $H(\alpha, \beta) \cdot \varphi = 0$.

EXAMPLES.

- $\alpha = (1/2, 1/2)$, $\beta = (1, 1)$ $\qquad F(\alpha, \beta; t) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k$

- $\alpha = (1/5, 2/5, 3/5, 4/5)$ $\beta = (1,1,1,1)$ $\qquad F(\alpha, \beta; t) = \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5} \left(\frac{t}{5^5}\right)^k$

Irreducible case.

$\mathbb{H}$ (or ρ) is irreducible $\iff \forall i, j \in \{1, \dots, n\}\ \alpha_i - \beta_j \notin \mathbb{Z}$

$\rho \simeq$ representation $\rho_L : \pi_1(U, t_0) \to GL_n(\mathbb{C})$

(Levelt) $\quad g_0^{-1} \mapsto M_0 \quad$ companion matrices of g_0

$\quad g_\infty \mapsto M_\infty \quad g_\infty$

ρ, up to isomorphism, is determined by α_i, β_j mod $\mathbb{Z}$. (rigid)

M_0, M_∞ lie in $GL_n(E)$, $E \subset \mathbb{C}$ subfield generated by coefficients of g_0, g_∞. "$\mathbb{H}$ (or ρ) defined over E."

Convenient format to encode the isomorphism class of $\mathbb{H}$: $\quad Q(x) = \dfrac{g_\infty(x)}{g_0(x)} \in E(x)$ "family parameter"

$\mathbb{H}$ underlies a 1-parameter family of irreducible motives $\mathcal{H}(t)$. Katz, 90's

$\mathbb{H}$ is 'a piece of' the variation of cohomology of X_t^n, $t \in \mathbb{P}^1$

hgm function as Euler integrals.

if $\beta_i = 1$ for some i,

the hgm function $F(\alpha, \beta; t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_n)_k} t^k$ solves $H(\alpha,\beta) \cdot \varphi = 0$.

EXAMPLES.

- $\alpha = (1/2, 1/2)$, $\beta = (1,1)$

  $Q(x) = \frac{(T+1)^2}{(T-1)^2}$

  $$F(\alpha,\beta;t) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k$$

  $$= \frac{1}{2\pi i} \oint \frac{1}{\sqrt{x(1-x)(x-t)}} dx$$

  a period of the family $E_t : y^2 = x(1-x)(x-t)$ (Legendre)

- $\alpha = (1/5, 2/5, 3/5, 4/5)$ $\beta = (1,1,1,1)$

  $$F(\alpha,\beta;t) = \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5} \left(\frac{t}{5^5}\right)^k$$

  = a period of the mirror quintic family.

  $Q(x) = \frac{T^4+T^3+T^2+T+1}{(T-1)^4}$

restrict to $E = \mathbb{Q}$ (i.e. q_∞, q_0 are $\prod$ of cyclotomic polynomials) as the examples

then:

- (Corti-Golyshev, Fedorov)

  $\mathbb{H}$ supports a rational VHS (pure) whose Hodge weight w and numbers $h^{i,j}$ can be computed combinatorially from α, β Hodge vector $h = (h^{w,0}, \dots, h^{0,w})$

  only depends on interlacing of $e^{2\pi i \alpha_j}$, $e^{2\pi i \beta_j}$ on S^1.

- Frobenious traces, polynomial ... of $\mathcal{H}(t)$ are given by explicit formulas in α, β

  (see Roberts - Rodriguez Villegas 2022).

tractable (+ broad)

Natural questions.

Can we classify all $\mathbb{H}$ with a given Hodge vector?

given $\mathbb{H}$ of weight w, does it arise from a family Y_t of varieties of dim w?

up to isomorphism $\qquad$ $\alpha_i, \beta_j \in [0,1)$.

up to pullback by $\iota : t \mapsto 1/t$ $\qquad$ $\iota^*\mathbb{H} \simeq \mathbb{H}(\beta, \alpha)$ $\qquad$ one only between (α, β) and (β, α).

Zig-zag diagram $\subset \mathbb{R}^2$: (as in Roberts - Rodriguez Villegas 2022).

1) order the parameters α_i, β_J
2) draw a pt $(0,0)$ in $\mathbb{R}^2$ corresponding to the smallest parameter
3) proceed left to right, drawing a pt in $\mathbb{R}^2$ $\forall$ parameter, moving upwards after α_i, downwards after β_J.

vertical coordinate corresponding to the parameter c "level of c" $p(c)$.

$w = \max p(\alpha_J) - \min p(\alpha_J) = \max p(\beta_J) - \min p(\beta_J)$

h formed by # of pts on levels between the maximum and the minimum

EASY to SEE that :

- complete intertwining of α_i, β_J gives Hodge vector (n)
- complete separation of α_i, β_J gives Hodge vector $(1,1,1,\dots,1,1)$.

if $\beta_i = 1$ for some i,

the hgm function $F(\alpha, \beta; t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_n)_k} t^k$ solves $H(\alpha,\beta)\cdot \varphi = 0$.

EXAMPLES.

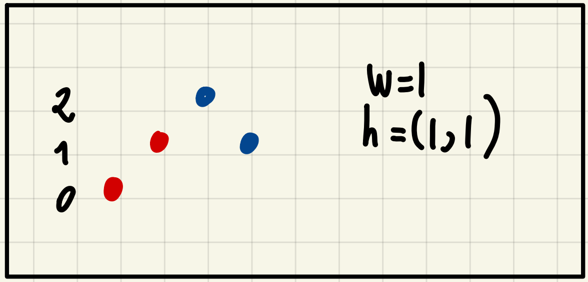
- $\alpha = (1/2, 1/2)$, $\beta = (1,1)$

$$F(\alpha,\beta;t) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k = \frac{1}{2\pi i} \oint \frac{1}{\sqrt{x(1-x)(x-t)}}\, dx$$

a period of the family $E_t : y^2 = x(1-x)(x-t)$

$$Q(x) = \frac{(T+1)^2}{(T-1)^2}$$

$\gamma = (-2,-2,1,1,1,1)$



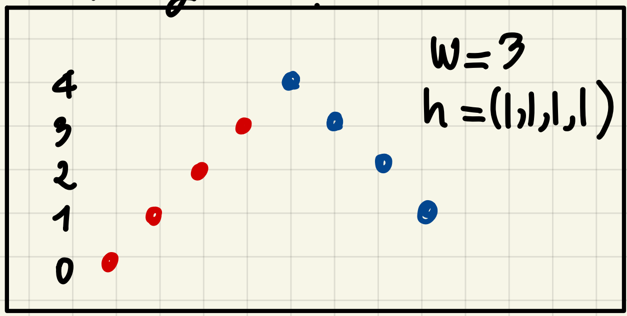
- $\alpha = (1/5, 2/5, 3/5, 4/5)$ $\beta = (1,1,1,1)$

$$F(\alpha,\beta;t) = \sum_{k=0}^{\infty} \frac{(5k)!}{k!^5} \left(\frac{t}{5^5}\right)^k$$

= a period of the mirror quintic family

$$Q(x) = \frac{T^4+T^3+T^2+T+1}{(T-1)^4}$$

$\gamma = (-5,1,1,1,1,1)$



We consider $\mathbb{H}$ defined over $\mathbb{Q}$ with $h=(1,1,1,1)$ (rank 4, $w=3$ $h^{i,j}=1$)

REASONS:

- first $w=$ odd after $h=(1,1)$ (which corresponds to elliptic surfaces)

- the subcase $q_0=(T-1)^4$ ($=h_0$ is a single Jordan block) physics: MUM at $t=0$.
  q_∞ arbitrary deg 4 product of cyclotomic polynomials, but no factor $(T-1)$

14 cases ("MUM" cases)

A. correspond to families of CY 3folds, play an important role in MS.
(Candelas-delaOssa-Green-Parkes, Batyrev-van Straten, Doran-Morgan, ...)

B. conifold pts give rise to modular forms of weight 4
(Kilbourn, McCarthy, Yui, ...)

In total (up to isomorphism and pullback by $t \mapsto 1/t$) 47 cases.
We ask same questions (A & B).

① CLASSIFICATION:

14 MUM cases

n°	α	γ^{red}	D	f	E
1	$(\frac12,\frac12,\frac12,\frac12)$	$(-2^4,1^8)$	1	8.4.a.a	1
2	$(\frac13,\frac12,\frac12,\frac23)$	$(-3,-2^2,1^7)$	12	36.4.a.a	-3
3	$(\frac14,\frac12,\frac12,\frac34)$	$(-4,-2,1^6)$	8	16.4.a.a	-4
4	$(\frac16,\frac12,\frac12,\frac56)$	$(-6,-2,1^5,3)$	1	72.4.a.b	-3
5	$(\frac13,\frac13,\frac23,\frac23)$	$(-3^2,1^6)$	1	27.4.a.a	1
6	$(\frac14,\frac13,\frac23,\frac34)$	$(-4,-3,1^5,2)$	24	9.4.a.a	12
7	$(\frac16,\frac13,\frac23,\frac56)$	$(-6,1^4,2)$	12	108.4.a.a	1
8	$(\frac14,\frac14,\frac34,\frac34)$	$(-4^2,1^4,2^2)$	1	32.4.a.a	1
9	$(\frac16,\frac14,\frac34,\frac56)$	$(-6,-4,1^3,2^2,3)$	8	144.4.a.f	12
10	$(\frac16,\frac16,\frac56,\frac56)$	$(-6^2,1^2,2^2,3^2)$	1	216.4.a.c	1
11	$(\frac15,\frac25,\frac35,\frac45)$	$(-5,1^5)$	5	25.4.a.b	5
12	$(\frac18,\frac38,\frac58,\frac78)$	$(-8,1^4,4)$	8	128.4.a.b	1
13	$(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$	$(-10,1^3,2,5)$	1	200.4.a.f	5
14	$(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12})$	$(-12,-2,1^4,4,6)$	1	864.4.a.a	1

Table 1: The 14 MUM cases. We omit $\beta = (0,0,0,0)$.

Def. $\mathbb{H}$ hgm local system associated to α, β. "the total twist of $\mathbb{H}$" is:

the hgm local system $\tilde{\mathbb{H}}$ associated to $\tilde\alpha, \tilde\beta$ $\quad \tilde\alpha_i = \alpha_i + 1/2$, $\tilde\beta_i = \beta_i + 1/2$

Note. if $\mathbb{H}$ is irreducible, the total twist operation is an involution.

if $\mathbb{H}$ def over $\mathbb{Q}$, so is $\tilde{\mathbb{H}}$, and $\tilde h = h$.

33 cases with no MUM pt.

(= it's its own total twist up to equivalence)

$\Downarrow$

25 pairs $((\alpha,\beta),(\tilde\alpha,\tilde\beta))$

$n°$	α	γ^{red}	D	f	E
15 ($\widetilde{2}$)	$(0,0,\frac16,\frac56),(\frac12,\frac12,\frac12,\frac12)$	$(-6,-1^7,2^5,3)$	-4	12.4.a.a	-3
16 ($\widetilde{3}$)	$(0,0,\frac14,\frac34),(\frac12,\frac12,\frac12,\frac12)$	$(-4,-1^6,2^5)$	-8	8.4.a.a	-4
17 ($\widetilde{4}$)	$(0,0,\frac13,\frac23),(\frac12,\frac12,\frac12,\frac12)$	$(-3,-1^5,2^4)$	-3	24.4.a.a	-3
18 ($\widetilde{5}$)	$(\frac16,\frac16,\frac56,\frac56),(\frac12,\frac12,\frac12,\frac12)$	$(-6^2,-1^6,2^6,3^2)$	1	27.4.a.a	1
19 ($\widetilde{6}$)	$(\frac14,\frac34,\frac16,\frac56),(\frac12,\frac12,\frac12,\frac12)$	$(-6,-4,-1^5,2^6,3)$	8	144.4.a.d	12
20 ($\widetilde{7}$)	$(\frac13,\frac23,\frac16,\frac56),(\frac12,\frac12,\frac12,\frac12)$	$(-6,-1^4,2^5)$	12	108.4.a.a	1
21 ($\widetilde{8}$)	$(\frac14,\frac14,\frac34,\frac34),(\frac12,\frac12,\frac12,\frac12)$	$(-4^2,-1^4,2^6)$	1	32.4.a.a	1
22 ($\widetilde{9}$)	$(\frac13,\frac23,\frac14,\frac34),(\frac12,\frac12,\frac12,\frac12)$	$(-4,-3,-1^3,2^5)$	24	72.4.a.a	12
23 ($\widetilde{10}$)	$(\frac13,\frac13,\frac23,\frac23),(\frac12,\frac12,\frac12,\frac12)$	$(-3^2,-1^2,2^4)$	1	216.4.a.c	1
24 ($\widetilde{11}$)	$(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10}),(\frac12,\frac12,\frac12,\frac12)$	$(-10,-1^5,2^5,5)$	1	25.4.a.a	5
25 ($\widetilde{12}$)	$(\frac18,\frac38,\frac58,\frac78),(\frac12,\frac12,\frac12,\frac12)$	$(-8,-1^4,2^4,4)$	8	128.4.a.b	1
26 ($\widetilde{13}$)	$(\frac15,\frac25,\frac35,\frac45),(\frac12,\frac12,\frac12,\frac12)$	$(-5,-1^3,2^4)$	5	200.4.a.e	5
27 ($\widetilde{14}$)	$(\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}),(\frac12,\frac12,\frac12,\frac12)$	$(-12,-1^4,2^3,4,6)$	1	864.4.a.a	1
28	$(0,0,\frac14,\frac34),(\frac13,\frac12,\frac12,\frac23)$	$(-4,-1^5,2^3,3)$	-24	18.4.a.a	12
29	$(0,0,\frac14,\frac34),(\frac13,\frac13,\frac23,\frac23)$	$(-4,-1^4,2,3^2)$	-8	54.4.a.c	-4
30	$(0,0,\frac16,\frac56),(\frac13,\frac12,\frac12,\frac23)$	$(-6,-1^6,2^3,3^2)$	-3	6.4.a.a	1
31	$(0,0,\frac16,\frac56),(\frac13,\frac13,\frac23,\frac23)$	$(-6,-1^5,2,3^3)$	-4	54.4.a.d	-3
32	$(0,0,\frac16,\frac56),(\frac14,\frac13,\frac23,\frac34)$	$(-6,-1^4,3^2,4)$	-24	12.4.a.a	-4
33	$(0,0,\frac16,\frac56),(\frac14,\frac14,\frac34,\frac34)$	$(-6,-1^3,-2,3,4^2)$	-4	96.4.a.	-3
34	$(0,0,\frac16,\frac56),(\frac15,\frac25,\frac35,\frac45)$	$(-6,-1^4,2,3,5)$	-20	150.4.a.h	-15
35	$(\frac13,\frac13,\frac23,\frac23),(\frac14,\frac14,\frac34,\frac34)$	$(-3^2,-2^2,1^2,4,4)$	1	864.4.a.a	1
36	$(\frac13,\frac13,\frac23,\frac23),(\frac16,\frac14,\frac34,\frac56)$	$(-3^3,-2^2,1^3,4,6)$	8	432.4.a.k	12
37	$(\frac13,\frac13,\frac23,\frac23),(\frac16,\frac16,\frac56,\frac56)$	$(-3^4,-2^2,1^4,6^2)$	1	72.4.a.b	1
38	$(\frac13,\frac13,\frac23,\frac23),(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$	$(-5,-3^2,-2,1^3,10)$	1	5400.4.?.?	5
39 ($\widetilde{28}$)	$(\frac14,\frac12,\frac12,\frac34),(0,0,\frac16,\frac56)$	$(-4,-3,-2^2,1^5,6)$	-8	48.4.a.c	12
40 ($\widetilde{29}$)	$(\frac14,\frac12,\frac12,\frac34),(\frac16,\frac16,\frac56,\frac56)$	$(-4,-3^2,-2^3,1^4,6^2)$	8	432.4.a.j	-4
41 ($\widetilde{31}$)	$(\frac13,\frac12,\frac12,\frac23),(\frac16,\frac16,\frac56,\frac56)$	$(-2^4,-3^3,1^5,6^2)$	12	54.4.a.a	-3
42 ($\widetilde{32}$)	$(\frac13,\frac12,\frac12,\frac23),(\frac16,\frac14,\frac34,\frac56)$	$(-2^4,-3^2,1^4,4,6)$	24	48.4.a.a	-4
43 ($\widetilde{33}$)	$(\frac13,\frac12,\frac12,\frac23),(\frac14,\frac14,\frac34,\frac34)$	$(-3,-2^4,1^3,4^2)$	12	288.4.a.f	-3
44 ($\widetilde{34}$)	$(\frac13,\frac12,\frac12,\frac23),(\frac{1}{10},\frac{3}{10},\frac{7}{10},\frac{9}{10})$	$(-5,-3,-2^3,1^4,10)$	12	450.4.a.o	-15
45 ($\widetilde{35}$)	$(\frac16,\frac16,\frac56,\frac56),(\frac14,\frac14,\frac34,\frac34)$	$(-6^2,-1^2,3^2,4^2)$	1	864.4.a.a	1
46 ($\widetilde{36}$)	$(\frac16,\frac16,\frac56,\frac56),(\frac14,\frac13,\frac23,\frac34)$	$(-6^2,-1^3,2,3^3,4)$	24	216.4.a.a	12
47 ($\widetilde{38}$)	$(\frac16,\frac16,\frac56,\frac56),(\frac15,\frac25,\frac35,\frac45)$	$(-6^2,-1^3,2^2,3^2,5)$	5	5400.4.?.?	5

Table 2: The 33 cases without a MUM point. We write n ($\widetilde{m}$) to indicate that the case n is the total twist of a previous case m.

② GOAL A: families of 3 folds.

gamma vectors and associated pairs

$\mathbb{H}$ defined over $\mathbb{Q}$

$$Q(x) = \prod_{i=1}^{e} \left(x^{|\gamma_i|} - 1 \right)^{-\operatorname{sign}(\gamma_i)} \quad \bigstar$$

for a multiset $\gamma := (\gamma_1, \dots, \gamma_e)$ of non ~~zero~~ integers with zero sum. "a gamma vector"

entries cannot all be grouped in pairs of integers of opposite sign.

e.g.: $(-3,2,1)$ yes, $(-3,-1,2,1,1)$ yes, $(-2,\cancel{2},\cancel{-1},1)$ No.

(look back at what γ is in the examples at page 2)

"A gamma vector γ corresponds to $\mathbb{H}$" if it satisfies $\bigstar$.

If $\gamma = (\gamma_1, \dots, \gamma_e)$ corresponds to $\mathbb{H}$, so does $\gamma^{(n)} = (\gamma_1, \dots, \gamma_e, -n, +n)$,

but $\exists!$ γ^{red} with no pair of integers summing to 0. For the 47 cases, we list them in the table.

If γ corresponds to $\mathbb{H}$, then $\tilde{\gamma}$: replace odd γ_i with $(2\gamma_i, -\gamma_i)$ corresponds to $\tilde{\mathbb{H}}$

We call $\tilde{\gamma}$ the total twist of γ. In general $\tilde{\gamma} \neq \tilde{\gamma}^{red}$

Assume $\gcd(\gamma_1, \dots, \gamma_\ell) = 1$ (for us γ^{red} satisfies this) 10

The pair associated to γ is $(\mathcal{Z}, \pi)$ where.

$d = \ell - 2$

$$\mathcal{Z} = \left(\sum_{i=1}^{\ell} z_i = 0 \right) \subset \mathbb{P}^{\ell-1} \quad \text{homogeneous coord}$$

$$\mathcal{Z} \xrightarrow{\pi = z^{\gamma} / \prod \gamma_i^{\gamma_i}} \mathbb{C}^\times$$

$k = \ell - 3$

It defines the 1 par family $\mathcal{Z}_t = \begin{cases} \sum_i z_i = 0 \\ z^{\gamma} = t \prod \gamma_i^{\gamma_i} \end{cases} \subset \mathbb{P}^{\ell-1}$ $\quad$ $Z_t, t \neq 1$, smooth; Z_1 a unique ODP

Beukers-Cohen-Mellit '15

Corti-Golyshev '10

EXPECTATION.

$$\mathbb{H} \simeq \mathrm{gr}_k^W R^k \pi_{U!} \mathbb{C} \quad * \qquad \text{stalk: } \mathrm{gr}_k^W H_c^k(Z_t, \mathbb{C})$$

Thm (Abdelraouf, G) If the local monodromy at 1 of π is not trivial, then $*$ holds.

different presentations: "toric models" 11

$d = \ell - 2$, $\quad \ell \times \ell$ matrix $A = \begin{pmatrix} 1, \dots & , 1 \\ m_{11} & m_{1\ell} \\ m_{d1} & m_{d\ell} \\ k_1 & \dots k_\ell \end{pmatrix}$ st
- first $\ell - 1$ rows span ker $\gamma : \mathbb{Z}^\ell \to \mathbb{Z}$
- $(k_1, \dots, k_\ell)$ st $\gamma \cdot k = 1$.

$m_j := m_{*j} \in \mathbb{Z}^d$, $\mathbb{T}^d$ torus with coordinates $x_1, \dots, x_d$, u a coordinate on $\mathbb{C}^\times$

via $z_j \mapsto u^{k_j} x^{m_j}$

$$Z = \left(\sum_{j=1}^{\ell} u^{k_j} x^{m_j} \right) \subset \mathbb{T}^d \times \mathbb{C}^\times \quad , \quad \pi = u / \prod x_i^{\gamma_i}$$

FRV : formulae for Hodge #s coincide

however, in general $w < k = \ell - 3$ | can we reduce dimension ?

in our case, $\quad w < k = \ell - 3$ every time $\ell > 6$.

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Example 2. Consider case 28 in table 2. The vector γ^{red} has length $l = 10$, thus the pair (Z, π) is a one-parameter family of 7-folds. We construct a toric model for (Z, π).

We can choose M as: $\gamma = (-4 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad 2 \quad 2 \quad 2 \quad 3)$

(matrix with columns the m_J)

$$M = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

and $\kappa = (0, 0, 0, 0, 0, -1, 0, 0, 0, 0)$. We obtain the hypersurface:

$$\left(1 + \frac{x_5 x_6 x_7}{x_1} + \frac{x_1}{x_2 x_3 x_4 x_8} + x_2 x_5 x_6 x_7 + x_3 + \frac{1}{u} x_4 + x_5 x_8 + x_6 + x_7 + \frac{1}{x_8} = 0\right) \subset \mathbb{T}^8 \times \mathbb{C}^\times$$

with the projection onto $u/\mathcal{M}_0$, where $\mathcal{M}_0 = -27/4$.

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THEOREM (G–RV) for each case, $\exists\,(Y, w)$ family of affine 3folds st.

$$\mathrm{gr}_k\, R^k \pi_! \, \mathbb{Q} \hookrightarrow \mathrm{gr}_3\, R^3 w_! \, \mathbb{Q}$$

stalkwise : $\mathrm{gr}_k\, H^k_c(Z_t, \mathbb{Q}) \hookrightarrow \mathrm{gr}_3\, H^3_c(Y_t, \mathbb{Q})$

How?

start from (Z, π) associated to γ^{red}

apply a range of dimension reduction techniques

- include BCM technique: (Beukers-Cohen-Mellit '15, finite hypergeometric functions, §6)

when $\gamma = \gamma^0 \cup \dots \cup \gamma^s$, $\gcd(\gamma^i_j) = 1$, then another model is $(Y, W) = (Z_0 \times \dots \times Z_s, \pi_0 \cdots \pi_s)$

- using toric models.

explain geometric and Hodge-theoretic relation between (Z, π) & (Y, W)

extend the technique to union of arbitrary gamma vectors

associate a pair (Z, π) to an arbitrary gamma vector

- interpret $\tilde{\mathbb{H}}$ in terms of double covers of the pair (Z, π) associated to a gamma vector γ for $\mathbb{H}$.

③ GOAL B: $t=1$.

GEOMETRY.

Fact. If w is odd, then ρ has determinant 1.

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_2 \quad (\text{pseudoreflection} + \det = 1)$$

$$\mathrm{rk}(h_1 - \mathrm{Id}) = 1$$

$V \subset \mathbb{C}^4$ invariants has dim 3.

monodromy filtration $\Rightarrow$ SES

$$0 \subseteq W_2 \subseteq W_3 = V \qquad 0 \to gr_2 V = W_2 \to V \to gr_3 V = W_3/W_2 \to 0 \quad ☆$$

$\dim W_2 = 1 \quad \dim W_3/W_2 = 2$

We interpret this in terms of:

$d = 2n$

$(\mathcal{Z}, \pi)$ pair associated to γ.

$$\mathcal{Z}_1 = \big(\underbrace{F(\cdot, \textstyle\prod \gamma_i^{\gamma_i}) = 0}_{*}\big) \subset \mathbb{T}^d$$

$$\begin{tikzcd}
\tilde{\mathcal{Z}} \arrow[d, "\text{blow-up } p"'] & \tilde{\mathcal{Z}}_1 \arrow[d, "\phi_1"] \arrow[r, phantom, "\supset"] & E = (2n-2) \text{ dim'l quadric given by} \\
\mathcal{Z} \ni & \mathcal{Z}_1 \ni p &
\end{tikzcd}$$

$$Q = (x^T H x = 0) \subset \mathbb{P}^{2n-1}$$

$\uparrow$

Hessian of $*$

$$\begin{tikzcd} & \tilde{V} \arrow[d] & & \tilde{V}_1 \arrow[d, "\phi_1"] \\ \text{In fact, first take } Z \subset & V & \text{smooth, relative nc} \supset & V_1 \end{tikzcd}$$

middle cohomology $H^{2n-2}(Q)$ generated by the 2 rulings of Q, i.e. families $\mathcal{L}_1$ and $\mathcal{L}_2$ of cod n linear subspaces of $\mathbb{P}^{n-1}$ $\subset Q$.

$\partial, \eta :=$ difference, sum of the 2 rulings.

$i: Q \hookrightarrow \tilde{V}_1$ inclusion. Then: $i_* \partial = 0$ & $H^{2n-2}(\tilde{V}_1)$ gen by η. So you get l.e.s.:

$$\cdots \to H^{2n-2}(\tilde{V}_1) \to H^{2n-2}(Q) \to H^{2n-1}(\tilde{V}_1, Q) \to H^{2n-1}(\tilde{V}_1) \to H^{2n-1}(Q) \to \cdots$$

$$\begin{tikzcd} & & H^{2n-1}(\tilde{V}_1, Q) \arrow[d, phantom, "\cong" rotate=90] & & \\ 0 \arrow[r] & \mathbb{Q}(-(n-1)) \arrow[r] & H^{2n-1}(V_1) \arrow[r] & H^{2n-1}(\tilde{V}_1) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Q}(-(n-1)) \arrow[r] \arrow[u] & H^{2n-1}_c(Z_1) \arrow[r] \arrow[u] & H^{2n-1}_c(\tilde{Z}_1) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

matches ☆ up to a Tate twist by $n-2$.

ARITHMETIC

L-series of $\mathcal{H}(1)$ Euler factor for good primes p (MAGMA computes this)

$$L_p(T) = (1-\delta(p)pT)\cdot(1-a_pT+p^3T^2)$$

quadratic character weight 2. ①

conjecturally the pth coefficient of a modular form f of weight 4. ②

Results:

① δ corresponds to quadratic extension $K/\mathbb{Q}$ determined by $\sqrt{(-1)^{\frac{d}{2}}\prod_1^{\ell}\gamma_i}$ ← we list these in the table "D"

we interpret this in terms of ∂.

Hessian H has $\det(H) = -\prod_1^{\ell}\gamma_i$.

for a $m\times m$ symmetric matrix A,

signed discriminant $\operatorname{disc} A := (-1)^{\frac{m(m-1)}{2}}\det A$

$d = 2n$ even. for $(2n-2)$ dime quadric $(X^T A X = 0) \circ \mathbb{P}^{2n-1} / F$ 17

field of def of 2 rulings Z_1 & Z_2 (and so of $\partial = Z_1 - Z_2$) is :

$$K := F(\sqrt{\operatorname{disc} A}) = F(\sqrt{(-1)^n \det A})$$

In our case

$F = \mathbb{Q}$, and the quadratic ext given by ∂ is $K = \mathbb{Q}(\sqrt{\operatorname{disc} H}) / \mathbb{Q}$,
the field of definition of ∂

(2) identified modular forms f by computing a_p's numerically + LMFDB. (we list these in the table)

(3) finite hypergeometric sum associated to $\mathcal{H}(t)$

$$\tilde{\mathcal{H}}(t) \simeq \varepsilon_t \otimes \mathcal{H}(t)$$

$\downarrow$

character associated to quadratic field E_t determined by $\sqrt{(-1)^{e_0/2} \prod \gamma_i^{\gamma_i} t}$

(we list these in the table "E")

thank you!