BPS structures and attractor flow in 4d $\mathcal{N} = 2$ theories

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Introduction

We will first review the concept of BPS states and 4d supersymmetric field theories.

We will then describe a joined work involving the counting of BPS states using attractor flow methods.

This is titled:

Special geometry, quasi-modularity and attractor flow for BPS structures

Authors:

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BPS states

- This talk involves the counting of BPS states in special 4d $\mathcal{N}=2$ field theories.
- BPS stands for Bogomol'nyi-Prasad-Sommerfeld.
- BPS states are topologically protected under deformations of masses and couplings.
- They describe many rich mathematical structures that arise in $\mathcal{N}=2$ theories including Quivers, Lie algebras, Donaldson-Thomas invariants.

BPS states

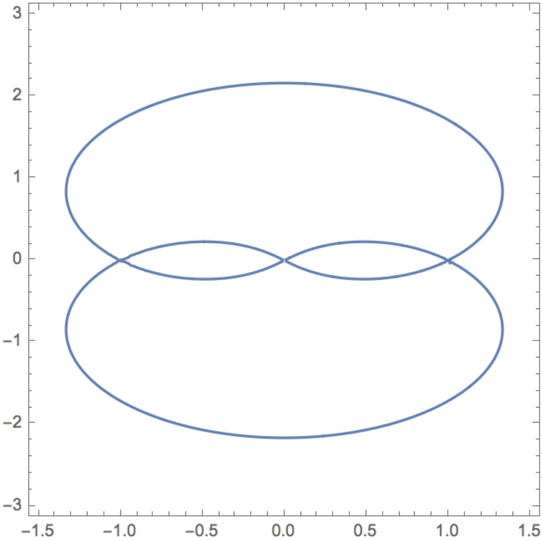
- In supersymmetric field theories they saturate the BPS bound $M \geq |Z|$ such that $M = |Z|$ (central charge).

- In this talk we focus on $\frac{1}{2}$BPS hypermultiplets and vectormultiplets in 4d $\mathcal{N} = 2$ supersymmetric field theories.

- These theories have a moduli space $\mathcal{B}$ parameterised by VEVs of scalars.

- At special loci in the moduli space the number of BPS states can jump discontinuously. These are called walls of marginal stability.

Walls of marginal stability



BPS states

- One can define a charge lattice of electric and magnetic charges Γ and define a basis $\gamma_1, \gamma_2 \in \Gamma$.

- We can also define an antisymmetric pairing between charges $\langle \gamma_i, \gamma_j \rangle$.

$$\langle \gamma_i, \gamma_j \rangle = \sum_a (p_i)^a (q_j)_a - (p_j)^a (q_i)_a \,, \quad i, j \in \{1, 2\} \,.$$

- The BPS degeneracies counting the states are defined as $\Omega(\gamma)$ where γ is a linear combination of the basis charges.

- This data when taken together can be defined as a BPS structure.

BPS structures

- For theories with a a rank r gauge group one can denote any local coordinate on $\mathcal{B}$ by $(u_1, \ldots, u_r)$

  Γ carries a symplectic pairing

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}.$$

- The moduli space (or coulomb branch) $\mathcal{B}$ carries a local system Γ with fiber $\Gamma_u \cong \mathbb{Z}^{2r}$ at each $u \in \mathcal{B}$.

- Γ comes with a holomorphic map and mass function

$$Z \colon \mathcal{B} \to \mathrm{Hom}(\Gamma, \mathbb{C}) \,, \qquad M \colon \mathcal{B} \to \mathrm{Map}(\Gamma, \mathbb{R}) \,,$$

- The BPS invariants are included as $\Omega(\gamma)$.

Note on Donaldson-Thomas invariants

- The BPS invariant of a particle is the same as that of an antiparticle $\Omega(\gamma) = \Omega(-\gamma)$.
- The BPS numbers $\Omega(\gamma) \in \mathbb{Q}$ also have an interpretation as Donaldson-Thomas invariants.
- In particular the explicit DT invariants can be written as (Bridgeland 2018)

$$\mathrm{DT}(\gamma) = \sum_{\gamma = m\alpha} \frac{1}{m^2} \Omega(\alpha) \in \mathbb{Q},$$

4d $\mathcal{N} = 2$ theories

- There is a class of 4d $\mathcal{N} = 2$ theories for which the BPS spectra and wall crossing have been studied extensively.
- These include ADE type Argyres-Douglas theories and Seiberg-Witten SU(2) theory.
- These theories in general do not have a Lagrangian description (although Seiberg-Witten theory has a low energy effective action).
- One can also study these theories by taking the low energy effective supergravity limit and then taking a further decopling limit.

4d $\mathcal{N}=2$ Argyres-Douglas and Seiberg-Witten theories

- The masses and couplings can be parameterised on a moduli space $\mathcal{B}$ with local coordinates $u \in \mathcal{B}$ describing a family of elliptic curves Σ_u called the Seiberg-Witten curves.
- An ADE type Argyres-Douglas theory arises when deforming ADE type singularities on the curve.

A_2 first realisation:

$$\Sigma^{I}_{A_2} := \{y^2 = 4z^3 - 3\Lambda^2 z + u \in \mathbb{C}^2\}, \quad u \in \mathcal{B} = \mathbb{P}^1 \setminus \{\pm\Lambda^3, \infty\}$$

A_2 second realisation:

$$\Sigma^{II}_{A_2} := \{y^2 = (z-\Lambda^2)(z+\Lambda^2)(z-u) \in \mathbb{C}^2\}, \quad u \in \mathcal{B} = \mathbb{P}^1 \setminus \{\pm\Lambda^2, \infty\}$$

Seiberg–Witten $SU(2)$ theory:

$$\Sigma_{SW} := \{y^2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \in \mathbb{C} \times \mathbb{C}^*\}, \quad u \in \mathcal{B} = \mathbb{P}^1 \setminus \{\pm\Lambda^2, \infty\}$$

Embedding in CY

- The elliptic curve can be embedded in the CY: $\Sigma \hookrightarrow X$
- If we write the elliptic curve as:

$$y^2 = f(z)$$

- Then the CY 3-fold takes the form:

$$X := \{(y, z, v_1, v_2) \in \mathbb{C}^4 \mid z^4(v_1^2 + v_2^2) + y^2 = f(z)\}.$$

- This CY also contains special Lagrangian submanifolds L_γ.

String theory description

- These theories are constructed by wrapping branes on the CY 3-fold in type IIA or type IIB string theory.
- This CY 3-fold contains the family of elliptic curves Σ_u that defines the theory.
- For example in type IIB string theory one can wrap a D3 brane on a 2-sphere and a 1-cycle γ in Σ_u.

Central charges

- The central charge of a BPS state with charge γ can be derived by integrating over a meromorphic 1-forms λ_u associated to Σ_u:

$$Z_\gamma(u) = \int_\gamma \lambda_u \ , \qquad \lambda_u = y dz, \qquad u \in \mathcal{B}.$$

String theory description

- This means in the full string theory we have a deformation family X_u of CY 3-folds wrapped by D3 branes

- For these CY 3-folds one can now define

the central charge and mass $$Z_\gamma(u) = \int_{X_u} \gamma \wedge \Omega_u \equiv \int_\gamma \Omega_u\,, \quad M_\gamma(u) = \int_\gamma |\Omega_u|\,,$$

the BPS bound $$\int_\gamma |\Omega_u| \geq |\int_\gamma \Omega_u|\,, \qquad \gamma \in H_3(X_u, \mathbb{Z}),\ u \in \mathcal{B},$$

- Ω_u is a holomorphic (3,0)-form on the CY 3-fold.

- The meromorphic 1-forms λ_u, which are used to compute the period integrals over 1-cycles in the Riemann surface realising the theory, are now reductions of this holomorphic (3,0)-form.

Picard-Fuchs equations

- To determine the BPS spectrum we need to derive the moduli dependent central charges.
- These can be derived by solving a set of differential equations called Picard-Fuchs equations.
- The Picard-Fuchs equations themselves can be found by deriving relations in cohomology using Griffiths pole order reduction methods.
- There are Picard-Fuchs operators $\mathcal{L}_h$ and $\mathcal{L}_m$ that annihilate holomorphic and meromorphic 1-forms ω_0 and λ.

Picard-Fuchs equations and solutions

For the first realisation $\Sigma^I_{A_2}$ of the A_2 theory the Picard Fuchs equations for the periods are

$$(1-u^2)\partial_u^2 g(u) - \frac{5}{36} g(u) = 0.$$

These have solutions

$$g_1(u) = \frac{3}{5\pi^{\frac{3}{2}}}(6\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})F^1_2(-\frac{5}{12},-\frac{1}{12},\frac{1}{2},u^2) - u\Gamma(\frac{1}{12})\Gamma(\frac{17}{12})F^1_2(\frac{1}{12},\frac{5}{12},\frac{3}{2},u^2))$$

$$g_2(u) = \frac{-3i}{5\pi^{\frac{3}{2}}}(6\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})F^1_2(-\frac{5}{12},-\frac{1}{12},\frac{1}{2},u^2) + u\Gamma(\frac{1}{12})\Gamma(\frac{17}{12})F^1_2(\frac{1}{12},\frac{5}{12},\frac{3}{2},u^2))$$

Picard-Fuchs equations and solutions

For the second realisation $\Sigma^{II}_{A_2}$ we have

$$(u^2 - 1)\partial_u^2\pi(u) - 2u\partial_u\pi(u) + \frac{5}{4}\pi(u) = 0.$$

With solutions

$$\pi_1(u) = -\frac{8}{15}\sqrt{2}(-1 + u^2)Q^2_{\frac{1}{2}}(u),$$
$$\pi_2(u) = -\frac{4}{15}\sqrt{2}i(-1 + u^2)\pi P^2_{\frac{1}{2}}(u).$$

Picard-Fuchs equations for Seiberg-Witten theory

Finally for Seiberg-Witten theory Σ_{SW} the Picard Fuchs equations are

$$(1-u^2)\partial_u^2 h(u) - \frac{1}{4}h(u) = 0.$$

and have solutions

$$h_1(u) = -\frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\Gamma(\frac{3}{4})^2 F_2^1(-\frac{1}{4},-\frac{1}{4},\frac{1}{2},u^2) - u\frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}}F_2^1(\frac{1}{4},\frac{1}{4},\frac{3}{2},u^2),$$

$$h_2(u) = -\frac{-i\sqrt{\pi}\Gamma(-\frac{1}{4})}{8\Gamma(\frac{5}{4})}F_2^1(-\frac{1}{4},-\frac{1}{4},\frac{1}{2},u^2) - \frac{i\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}uF_2^1(\frac{1}{4},\frac{1}{4},\frac{3}{2},u^2).$$

Relation to the central charges

One can show that the solutions to these Picard Fuchs equations are in fact the central charges $Z_{\gamma_1}(u)$ and $Z_{\gamma_2}(u)$ of the basis BPS states with charges γ_1 and γ_2.

- One can check that they vanish at $u = \pm 1$ as expected.
- Also the monodromies around the singular points can be matched from those obtained from Picard Lefschetz transformations of the cycles.

It must now be checked which linear combinations of central charges are now also BPS states.

Monodromies

- There are 3 singular points ± 1 and the point at infinity
- There is a monodromy around each of these points denoted M_{+1}, M_{-1} and M_∞
- They are related by $M_\infty = (M_{+1}M_{-1})^{-1}$
- This results in brach cuts in the u-plane.
- These can be obtained by expanding the central charges around the singular points.

Example

A_2 second realisation on $\Sigma_{A_2}^{II}$ expansions around $+1$:

$$Z_{\gamma_1}(u) = \frac{7i\pi v^5}{16384\sqrt{2}} + \frac{799v^5}{983040\sqrt{2}} - \frac{35v^5\log(2)}{16384\sqrt{2}} - \frac{5i\pi v^4}{2048\sqrt{2}} - \frac{89v^4}{24576\sqrt{2}} + \frac{25v^4\log(2)}{2048\sqrt{2}} + \frac{i\pi v^3}{32\sqrt{2}} - \frac{v^3}{192\sqrt{2}} - \frac{5v^3\log(2)}{32\sqrt{2}} + \frac{i\pi v^2}{4\sqrt{2}} + \frac{5v^2}{8\sqrt{2}} - \frac{5v^2\log(2)}{4\sqrt{2}} +$$

$$\left(\frac{7v^5}{16384\sqrt{2}} - \frac{5v^4}{2048\sqrt{2}} + \frac{v^3}{32\sqrt{2}} + \frac{v^2}{4\sqrt{2}}\right)\log(v) + \frac{2\sqrt{2}v}{3} + \frac{16\sqrt{2}}{15} + O\left(v^6\right)$$

$$Z_{\gamma_2}(u) = -\frac{i\pi v^2}{4\sqrt{2}} - \frac{i\pi v^3}{32\sqrt{2}} + \frac{5i\pi v^4}{2048\sqrt{2}} - \frac{7i\pi v^5}{16384\sqrt{2}} + O\left(v^6\right)$$

Monodromies

For the first realisation $\Sigma^{I}_{A_2}$ of the A_2 theory we find the monodromies

$$M_{+1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_{\infty} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

For the second realisation $\Sigma^{II}_{A_2}$ we have

$$M_{+1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad M_{\infty} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$$

Monodromies

For pure SU(2) Seiberg-Witten theory Σ_{SW} we have

$$M_{+1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad M_{\infty} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

Wall-crossing

Wall crossing occurs when a BPS state becomes unstable and decays into its constituents e.g. $\gamma \longrightarrow \gamma_1 + \gamma_2$ as special loci in the moduli space are crossed.

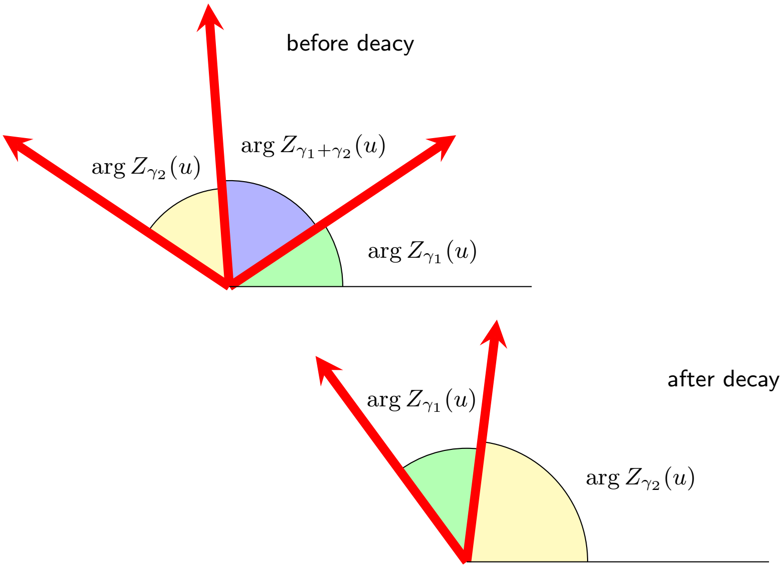
- The loci for this decay are called walls of marginal stability MS_{γ_1,γ_2}.
- The decay into charges γ_1, γ_2 occurrs when the central charges align

$$Z_{\gamma_1}(u) = |Z_{\gamma_1}(u)|e^{i\theta_1}, \quad Z_{\gamma_2}(u) = |Z_{\gamma_1}(u)|e^{i\theta_2}, \qquad \text{Im}[Z_{\gamma_1}(u)\bar{Z}_{\gamma_2}(u)] = 0.$$

- There are also wall crossing formulae relating the BPS degeneracies $\Omega(\gamma)$ on both sides of the wall of marginal stability (Denef, Moore 2011).

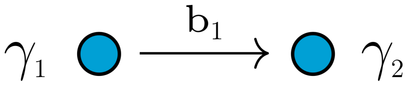
$$\Delta\Omega(\gamma, u) = (-1)^{\langle\gamma_1,\gamma_2\rangle - 1}|\langle\gamma_1, \gamma_2\rangle|\Omega(\gamma_1, u_{ms})\Omega(\gamma_2, u_{ms}).$$

Wall-crossing



Quiver description

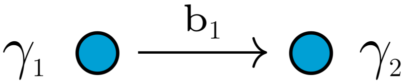
- A quiver is a diagram with charged BPS states at its nodes and the arrows given by the product of the charges.
- In the ADE examples the BPS charges at the nodes can be represented as roots of the Lie algebra associated to the quiver.



- The BPS spectrum can be counted using distinct quiver mutation sequences in the different chambers (Alim, Cecotti, Espahbodi, Rastogi, Vafa 2011).

Example: Argyres-Douglas A_2 model

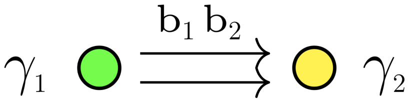
This theory is described by the quiver, where $\langle \gamma_1, \gamma_2 \rangle = 1$:



- There are electric and magnetic monopoles $\gamma_1 = (1, 0)$ and $\gamma_2 = (0, 1)$ existing in every region of the moduli space.
- On the other side of the wall of marginal stability (weak coupling region) they combine to form a dyon $\gamma_3 = (1, 1)$.

Seiberg-Witten theory

This time the quiver takes this form with $\langle \gamma_1, \gamma_2 \rangle = 2$:



- This time we have a monopole $\gamma_1 = (0, 1)$ and a dyon $\gamma_2 = (2, -1)$ on one side of the wall.
- On the other side there is an infinite spectrum of dyons of the form $n\gamma_1 + (n+1)\gamma_2$ and $(n+1)\gamma_1 + n\gamma_2$
- There is also a W-boson $\gamma_1 + \gamma_2$.

Supergravity description of wall-crossing

- One can take the low energy supergravity limit of type IIB string theory.
- From which one obtains an effective action (Denef 2000):

$$S_{eff} = \frac{S}{\Delta t} = -\frac{1}{2}\int_0^\infty d\tau \Big(\dot{U}(\tau)^2 + \frac{1}{2} g_{a\bar{b}} \dot{u}^a \dot{\bar{u}}^{\bar{b}} + e^{2U(\tau)} V(u)\Big) - \Big(e^{2U(\tau)} |Z_\gamma(u)|\Big)_{\tau=\infty},$$

where $V(u) = |Z_\gamma(u)|^2 + 4 g^{a\bar{b}} \partial_a |Z_\gamma(u)| \bar{\partial}_{\bar{b}} |Z_\gamma(u)|$.

- $g_{a\bar{b}}$ is the metric in the moduli space, $U(\tau)$ is a radial function from the spacetime metric at inverse radius τ.

Metric and Kähler potential

- The metric on the moduli space takes the form of:

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}, \quad \text{where} \quad \mathcal{K} = -i \ \ln\Big(\int_X \Omega_0 \wedge \bar{\Omega}_0 \Big).$$

- Here Ω_0 is the normalised holomorphic volume form $\Omega = e^{\frac{\mathcal{K}}{2}} \Omega_0$.

Supergravity equations of motion

Equations of motion can be derived from the low energy type IIB supergravity action (Denef 2000)

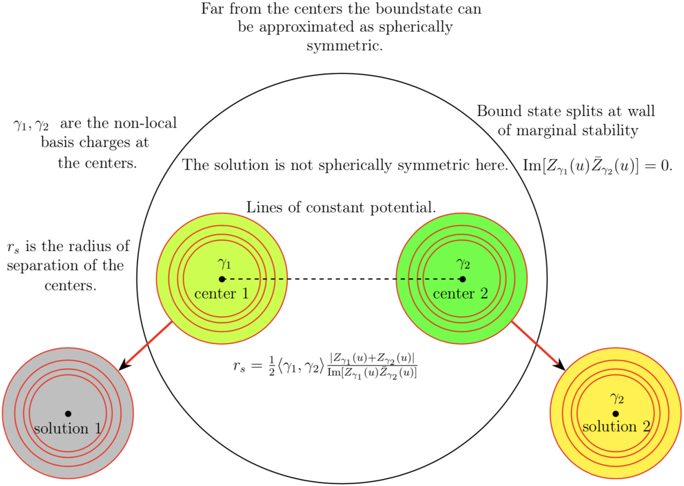
$$\partial_\tau U(\tau) = -e^{U(\tau)}|Z_\gamma(u)|,$$
$$\partial_\tau u^a = -2e^{U(\tau)} g^{a\bar{b}} \bar{\partial}_{\bar{b}} |Z_\gamma(u)|.$$

If the moduli space is 1-dimensional, and the inverse 4d radius $\tau = 1/r$ can be reparameterised, the equations can be simplified to:

$$\partial_\tau u = -\bar{\partial}_{\bar{u}} |Z_\gamma(u)|.$$

This is the gradient or attractor flow of the BPS state with this central charge.

Wall-crossing in supergravity



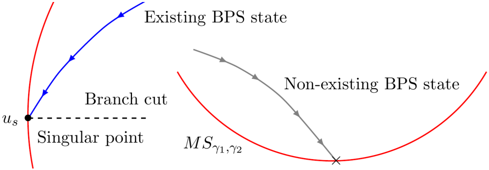
Attractor flow existence conditions

In general the existence of a BPS state with a particular attractor flow line is determined by the endpoint of the flow lines.

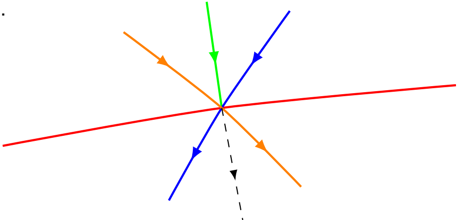
- If $|Z_\gamma(u)| \neq 0$ at the modulus u_* (where $*$ denotes the attractor point) then the flow always exists as $\tau \to \infty$ and the equation describes a BPS black hole.
- If $|Z_\gamma(u)| = 0$ at u_* and u_* is a regular point in the moduli space then the above inequality is violated and no BPS states exist.
- If $|Z_\gamma(u)| = 0$ and u_* is a singular or boundary point in the moduli space then massless BPS states can exist. In this case one can interpret the resulting solution as an empty hole solution. Unlike a black hole this has no horizon but still has a core region.

Attractor flow existence conditions

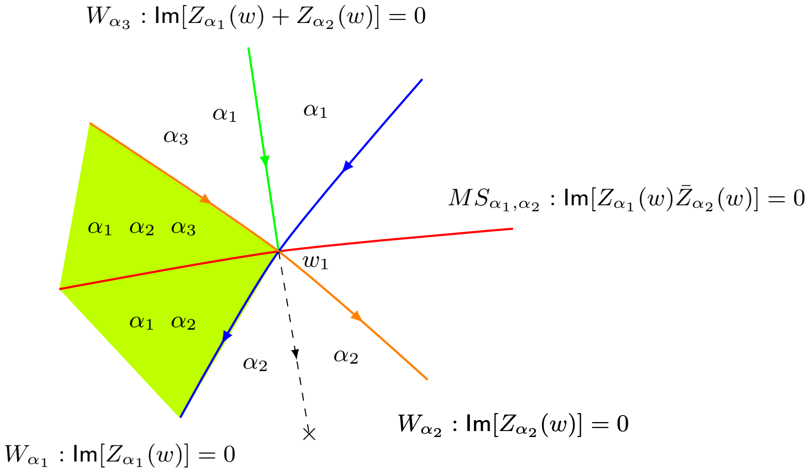
There are various conditions on the existence of BPS states with particular attractor flow lines:



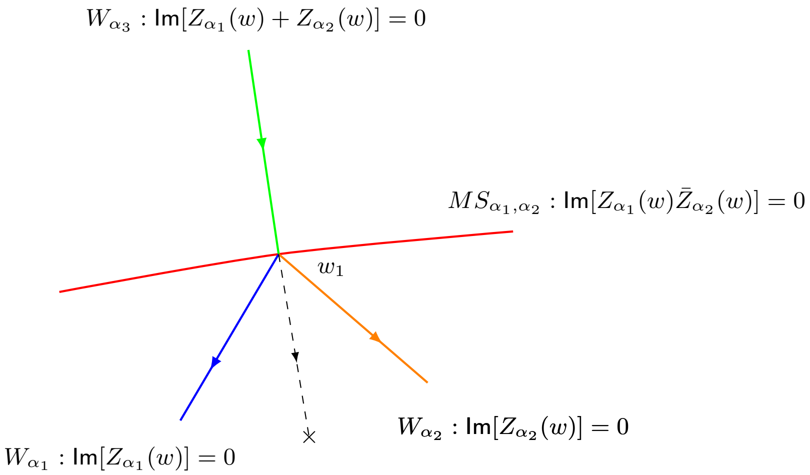
- If the flow line terminates at a regular point in the moduli space where the central charge vanishes the BPS state cannot exist.
- If the flow line terminates at a singular point this is not a problem and the BPS state can exist.



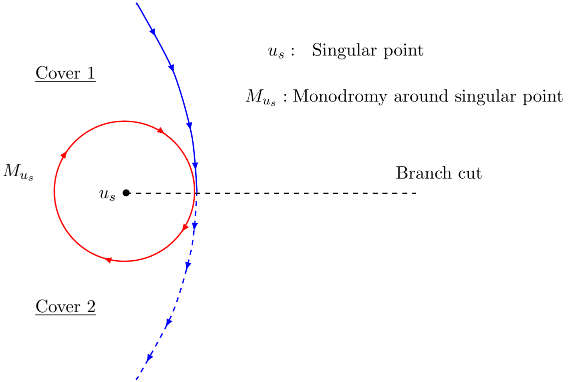
Chambers


W_{α_3} : $\text{Im}[Z_{\alpha_1}(w) + Z_{\alpha_2}(w)] = 0$
α_1
α_1
α_3
MS_{α_1,α_2} : $\text{Im}[Z_{\alpha_1}(w)\bar{Z}_{\alpha_2}(w)] = 0$
α_1 α_2 α_3
w_1
α_1 α_2
α_2
α_2
W_{α_2} : $\text{Im}[Z_{\alpha_2}(w)] = 0$
W_{α_1} : $\text{Im}[Z_{\alpha_1}(w)] = 0$

Split flow lines

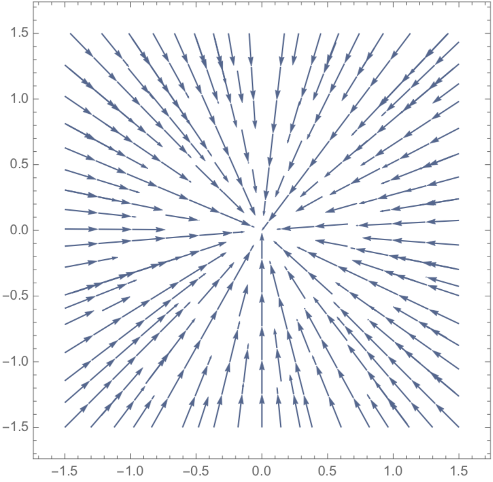


Branch cuts and monodromies



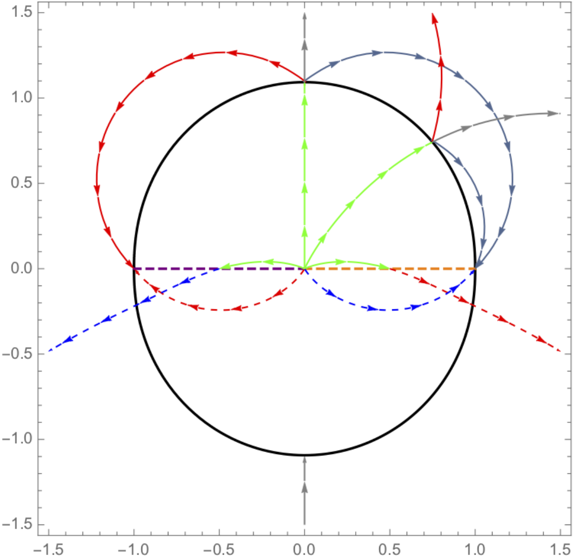
- Single flow line passing through branch cut after acting with monodromy around singular point.
- This in combination with the existence conditions recovers the BPS spectra in all regions of the moduli and their wall crossing phenomena.

Argyres-Douglas A_1 example



Argyres-Douglas A_2 first realisation

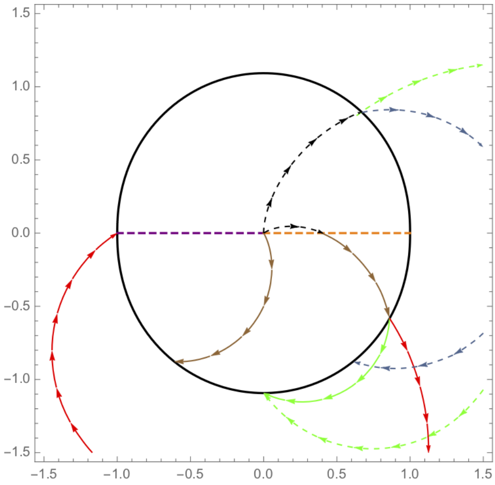
$$y^2 = 4z^3 - 3z + u$$



BPS spectrum of Argyres-Douglas theory

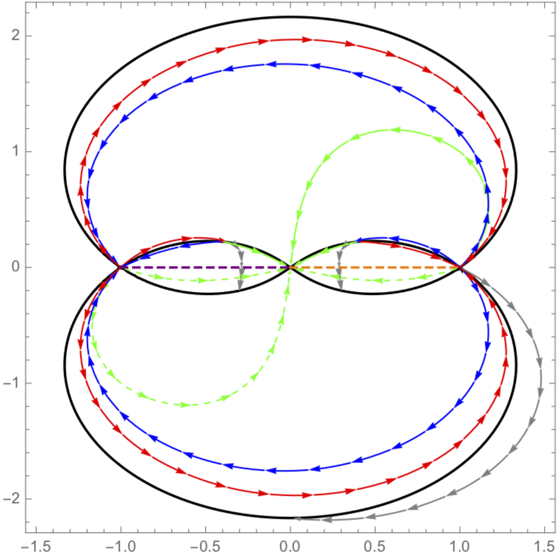
- So the existing BPS states have electric and magnetic charges γ_1, γ_2.
- There is also a dyonic charge which can be written as $\gamma_1 + \gamma_2$ or $\gamma_1 - \gamma_2$ depending on what side of the branch cuts one is on.
- This has now recovered the known BPS spectrum that can also be derived by other methods such as quiver mutation sequences.

Exclusion of higher linear combinations

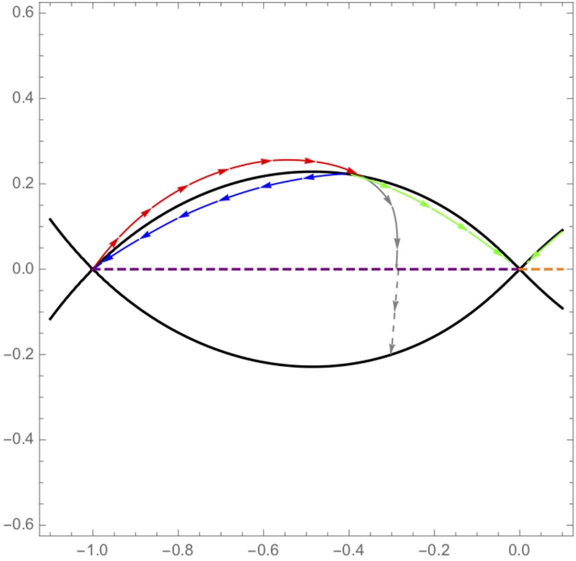


Argyres-Douglas A_2 second realisation

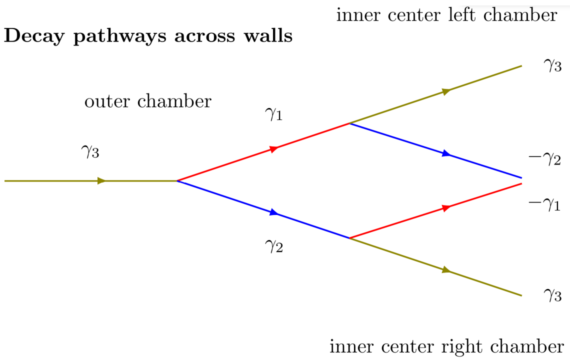
$$y^2 = (z-1)(z+1)(z-u)$$



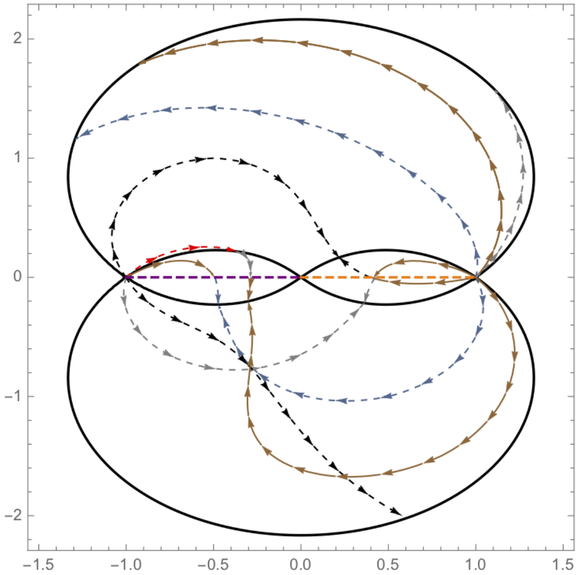
Zoomed in on central chamber



All possible decay pathways of the BPS states

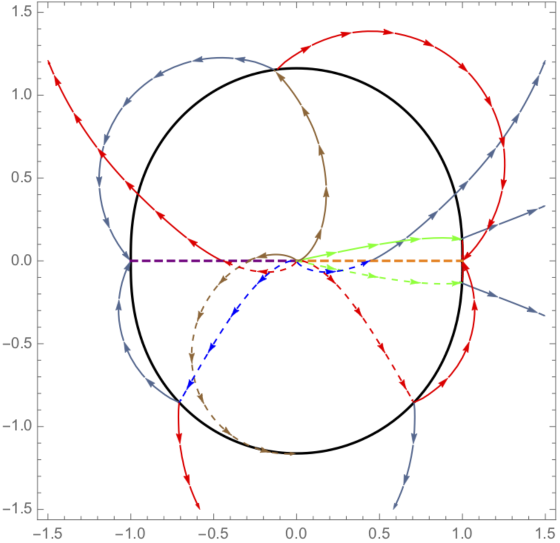


Exclusion of higher linear combinations

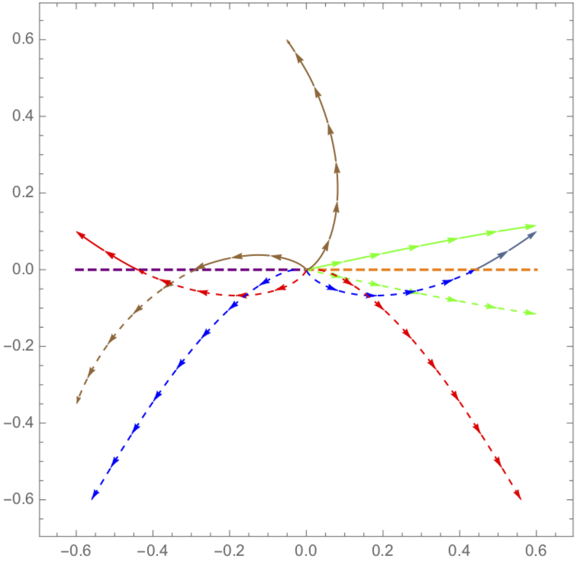


Seiberg-Witten theory

$$y^2 = \frac{1}{z^3} + \frac{2u}{z^2} + \frac{1}{z}$$



Lines flowing through branch cut

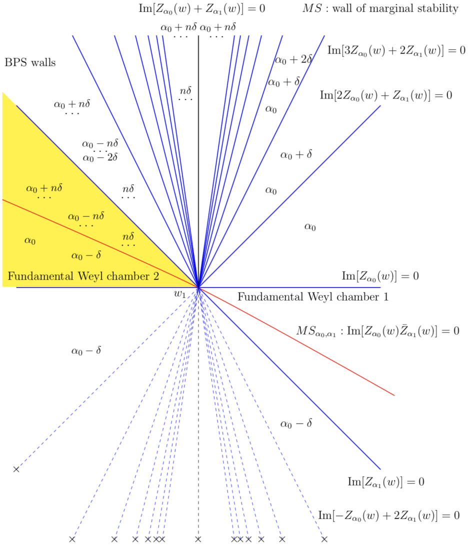


BPS spectrum

So using the attractor flow and monodromies we recover the BPS spectrum:

- This is the monopole $\gamma_1 = (0, 1)$ and a dyon $\gamma_2 = (2, -1)$ on one side of the wall
- and the infinite spectrum of dyons of the form $n\gamma_1 + (n+1)\gamma_2$ and $(n+1)\gamma_1 + n\gamma_2$
- with the W-boson $\gamma_1 + \gamma_2$ on the other side.

Connection to scattering diagrams and Lie algebras



Conclusions and future directions

Main conclusions

- We have found the central charges as linear combinations of periods of the Seiberg-Witten curve.
- We have used the attractor flow method involving gradient descent and existence conditions to determine the BPS spectrum.

Future directions of this research

- This can be generalised to a larger class of theories including ADE type Argyres-Douglas theories and theories with added flavors.
- This is already being used to count BPS black hole microstates.

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