Zeta Functions via Periods of Calabi-Yau Hypersurfaces in Non-Fano Toric Varieties

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Main Question:

How can one obtain $\zeta_p(X_\varphi, T)$ from $\vartheta_1(\mathbb{P}_\Delta, X)$?

"Zeta from theta"

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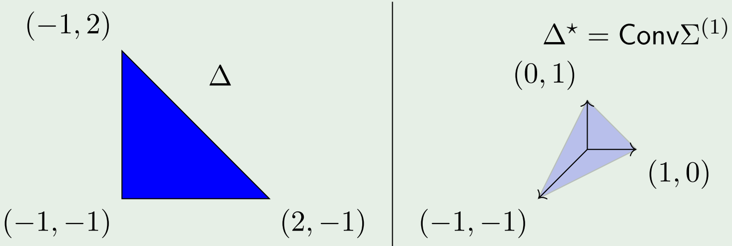
Mirror symmetry

Toric Varieties

Toric varieties are algebraic varieties that contain an dense algebraic torus [Dan78, CLS11].

- Two constructions:
  - Newton polytope $\Delta \rightsquigarrow \mathbb{P}_\Delta$ (vertices $\iff$ monomials)
  - Fan or spanning polytope $\Sigma \rightsquigarrow \mathbb{P}_\Sigma$ (vectices $\iff$ divisors)
- When $\Sigma^{(1)}$ are normal to facets of Δ, we have $\mathbb{P}_\Delta \cong \mathbb{P}_\Sigma$.

Example 1.1 ($\mathbb{P}^2 \cong \mathbb{P}_\Delta \cong \mathbb{P}_\Sigma$)



Calabi-Yau hypersurfaces in toric varieties

- In string compactifications, spacetime is of the form $\mathbb{R}^{3,1} \times X$ where X is a compact Calabi-Yau manifold [CHSW85].
- By the adjunction formula, the anticanonical divisor $-K_{\mathbb{P}_\Delta} = \sum_{\rho=1}^{r} D_\rho$ has trivial canonical class and hence $-K_{\mathbb{P}_\Delta} = X$ is a Calabi-Yau manifold.

Example 1.2 (Anticanonical hypersurfaces in projective space)

$$-K_{\mathbb{P}^n} = (n+1)H \implies \begin{cases} -K_{\mathbb{P}^2} \text{ is the cubic elliptic curve} \\ -K_{\mathbb{P}^3} \text{ is the quartic K3} \\ -K_{\mathbb{P}^4} \text{ is the quintic CY3} \\ \quad \vdots \end{cases}$$

where H is the hyperplane class.

Gauged linear sigma models

Witten introduced gauged linear sigma models (GLSMs) – $U(1)^s$ gauge theories whose charge matrix is given by the *Mori vectors* of a toric variety [Wit93].

Definition 1.3 (Mori vectors of a toric variety)

Let $\Sigma^{(1)} = \{v_1, \ldots, v_r\}$ be the 1 dimensional cones of a toric variety. Let $\overline{v}_i = (v_i, 1)$ and $v_0 = \vec{0}$. Then the *Mori vectors* $\ell_i^{(a)}$ are defined by

$$\sum_{i=0}^{r} \ell_i^{(a)} \overline{v}_i = 0$$

Example 1.4 ($\mathbb{P}^2$)

$$\Sigma_{\mathbb{P}^2}^{(1)} = \{(1,0),(0,1),(-1,-1)\} \implies \ell^{(1)} = (-3;1,1,1)$$

Fano, Semi-Fano, Calabi-Yau, non-Fano

$c_1(\mathbb{P}_\Delta)$ is the Poincarè dual to $-K_{\mathbb{P}_\Delta}$. Write

$$-K_{\mathbb{P}_\Delta} = \sum_{a=1}^{h^{1,1}(\mathbb{P}_\Delta)} (-\ell_0^{(a)}) D_a$$

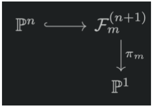
in a basis of ample divisors.

Definition 1.5 (Fano, Semi-Fano, Calabi-Yau, non-Fano)

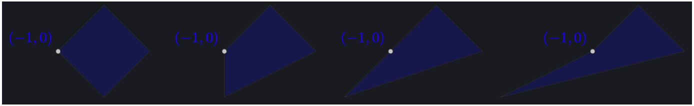
- If $(-\ell_0^{(a)}) > 0$ then $\mathbb{P}_\Delta$ is *Fano*.
- If $(-\ell_0^{(a)}) \geq 0$ then $\mathbb{P}_\Delta$ is *semi-Fano*.
- If $\ell_0^{(a)} = 0$ then $\mathbb{P}_\Delta$ is *Calabi-Yau*.
- If $\ell_0^{(a)} \in \mathbb{Z}$ then $\mathbb{P}_\Delta$ is “of general type” or *non-Fano*.

Hirzebruch scrolls

- The Hirzebruch scrolls $\mathcal{F}_m^{(n+1)} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(m) \oplus \mathcal{O}_{\mathbb{P}^1})$are $\mathbb{P}^n$ fibrations over $\mathbb{P}^1$.



- $-K_{\mathcal{F}_m^{(n+1)}} = (n+1)S + (2-m)F$
  - Non-Fano for $m > 2$



Mori vectors for $\mathcal{F}_m^{(2)}$:

$$\ell^{(1)} = (\quad -2; \quad 1,\ 1,\ 0,\ 0)$$
$$\ell^{(2)} = (\ -(2-m);\ -m,\ 0,\ 1,\ 1)$$

Kreuzer-Skarke database of reflexive polytopes

The Kreuzer-Skarke database lists all reflexive polytopes in 3 and 4 dimensions [KS].

Definition 1.6

A polytope Δ is reflexive if

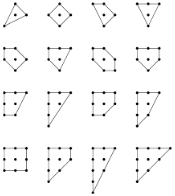
- All facets are supported by an affine hyperplane of the form $\{m \in \mathbb{R}^n \mid \langle m, n \rangle = -1\}$
- Δ has the orgin as its unique interior point.

Theorem 1.7

Δ is reflexive if and only if $\mathbb{P}_\Delta$ is Fano.

Numbers of reflexive polytopes

- 16 in two dimensions



- 4,319 in three dimensions
- 473,800,776 in four dimensions
- ?? in five+ dimensions

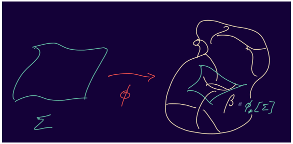
Sigma models

GLSMs have phases, among which are NLSMs and LG models.

Definition 1.8 (Non-linear Sigma Model)

Given an embedding $\phi : \mathsf{WS} \to X$ into a Calabi-Yau,

$$S_{\mathsf{NLSM}} = i\int\Bigg(\frac{1}{2}(g_{j\bar{k}} + iB_{j\bar{k}})\partial_\tau\phi^j\partial_{\bar{\tau}}\overline{\phi}^{\bar{k}} + \frac{1}{2}(g_{j\bar{k}} - iB_{j\bar{k}})\partial_{\bar{\tau}}\phi^j\partial_\tau\overline{\phi}^{\bar{k}}$$
$$+ \frac{i}{2}g_{\bar{j}j}\overline{\psi}^{\bar{j}}_-\nabla_\tau\psi^j_- + \frac{i}{2}g_{\bar{j}j}\overline{\psi}^{\bar{j}}_+\nabla_{\bar{\tau}}\psi^j_+$$
$$+ \frac{1}{4}R_{\bar{j}jk\bar{k}}\psi^j_+\overline{\psi}^{\bar{j}}_+\psi^k_-\overline{\psi}^{\bar{k}}_-\Bigg)d\tau \wedge d\bar{\tau}.$$



LG models

Landau-Ginzburg models can be thought of as sigma models that are deformed by a potential W which is a function of superfields.

Definition 1.9 (Landau-Ginzburg Model)

Given $\phi : \mathsf{WS} \to X$ and a superpotential $W : (\mathbb{C}^*)^n \to \mathbb{C}$,

$$S_{\mathsf{LG}} = \int \Bigg(\partial_\tau \phi^j \partial_{\bar\tau} \overline{\phi}^{\bar j} + \partial_{\bar\tau} \phi^j \partial_\tau \overline{\phi}^{\bar j} + i\overline{\psi}_-^{\bar j} \nabla_\tau \psi_-^j + i\overline{\psi}_+^{\bar j} \nabla_{\bar\tau} \psi_+^j - \frac{1}{4} \partial_j W \partial_{\bar j} \overline{W} - \frac{1}{2} \partial_j \partial_k W \psi_+^j \psi_-^k - \frac{1}{2} \partial_{\bar j} \partial_{\bar k} \overline{W}\, \overline{\psi}_-^{\bar j} \overline{\psi}_+^{\bar k} \Bigg) d\tau \wedge d\bar\tau$$

BRST cohomology

A sigma model has supersymmetry generated by

$$Q_+ = \psi^j p_j = i\overline{\partial}^\dagger \qquad Q_- = \overline{\psi}^j p_j = -i\partial$$
$$\overline{Q}_+ = \overline{\psi}^{\bar{j}} p_{\bar{j}} = -i\overline{\partial} \qquad \overline{Q}_- = \psi^{\bar{j}} p_{\bar{j}} = i\partial^\dagger$$

Given a nilpotent supercharge $Q^2 = 0$, construct the physical operators in a "topologically twisted" theory

$$H^*_Q(\Sigma) = {}^{\text{ker } Q}\big/_{\text{im } Q}$$

A-model and B-model

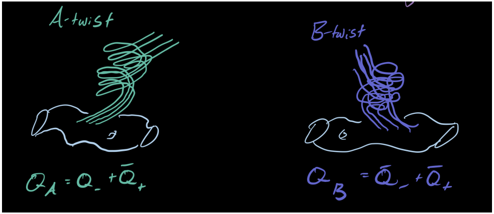
The A-model and the B-model are TQFTs that can be obtained from topologically twisting a sigma model.

- A-model

$$Q_A = -i(\partial + \overline{\partial}) \cong d \implies H^*_{Q_A} \cong H^*_{\mathrm{dR}}(X)$$

- B-model

$$Q_B \cong \overline{\partial} \implies H^*_{Q_B} \cong \bigoplus_{p,q=0}^{n} H^{0,p}(X, \wedge^q TX) \cong H^*_{\overline{\partial}}(X)$$



Invariants from correlation functions

The correlation functions in the A-model and the B-model are topological invariants of the Calabi-Yau.

- A-model: **Gromov-Witten invariants**

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \int_{\mathcal{M}_\Sigma(X,\beta)} e^{-(\omega - iB)\cdot\beta} \mathrm{ev}_1^* \omega_{D_1} \wedge \cdots \wedge \mathrm{ev}_s^* \omega_{D_s}$$

$$= \int_X \omega_1 \wedge \cdots \wedge \omega_s + \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ \beta \neq 0}} n_{\beta, D_1 \ldots D_s} e^{-(\omega - iB)\cdot\beta}.$$

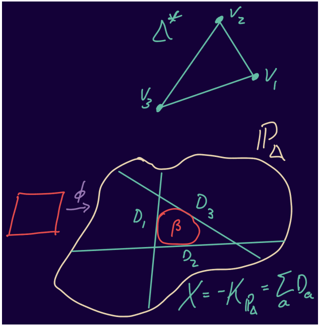
- B-model: **Period integrals**

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega)$$

Gromov-Witten invariants

- GW invariants β are counts of stable maps whose image land on a fixed curve class $\beta \in H_2(X, \mathbb{Z})$.
- Defined as integrals over moduli spaces of maps

$$D_{0,1}(\mathbb{P}_\Delta, \beta) = \int_{[\mathcal{M}_{0,1}((\mathbb{P}_\Delta, X), \beta)]^{virt}} \psi^{\beta \cdot X - 2} \mathrm{ev}^\star[\mathrm{pt}].$$



Mirror symmetry in physics

- Mirror symmetry was first discovered by Greene and Plesser [GP90] for quintic threefold.
- Used to count rational curves by Candelas, de la Ossa, Green, Parkes [CDGP91].
- Mirror symmetry is T-duality [SYZ96] in the large complex structure limit.

Definition 1.10

Two Calabi-Yau manifolds X and $\check{X}$ are a mirror pair (in physics) if

$$\text{A-model}(X) \cong \text{B-model}(\check{X})$$
$$\text{A-model}(\check{X}) \cong \text{B-model}(X)$$

which can be checked by comparing correlation functions.

Mirror symmetry for Calabi-Yau manifolds

- It is a necessary but not sufficent condition that the Hodge diamonds of X and $\check{X}$ are reflections of each other.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad \overset{\sim}{\longmapsto} \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \check{h}^{2,1} & & 0 & \\
1 & & \check{h}^{1,1} & & \check{h}^{1,1} & & 1 \\
 & 0 & & \check{h}^{2,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

- Batyrev: mirror symmetry as polar duality $\Delta \leftrightarrow \Delta^\star$ [Bat].
- Can compute invariants of X in terms of invariants of $\check{X}$.
  - Ratios of periods are generating functions for GW invariants.
  - Deformation method for computing zeta functions [Dwo62].

Mirror symmetry for Fano varieties

- Hori and Vafa [HV] found mirror LG model to sigma models with Fano toric variety target.

Definition 1.11

Let $\mathbb{P}_\Sigma$ be a toric variety. Then the mirror Hori-Vafa potential is given by

$$W_\Sigma = \sum_{\rho \in \Sigma^{(1)}} \varphi^{\beta_\rho} x^{m_\rho}$$

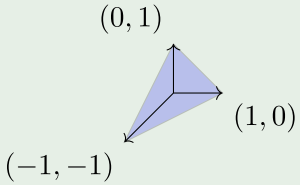
where $m_\rho = (m_1, \ldots, m_n)$ is the primitive ray generator, $x^{m_\rho} = x_1^{m_1} \ldots x_n^{m_n}$, and $\varphi = (\varphi_1, \ldots, \varphi_s)$ are the complex structure moduli of the mirror ($s = h^{1,1}(\mathbb{P}_\Sigma)$).

W_Σ and curve classes

The powers of the complex structure moduli in W_Σ are determined by effective curve classes β_ρ:

- If $(\rho_1, \ldots, \rho_k)$ is a collection of rays with $\sum_{i=1}^k m_{\rho_i} = 0$, then $\sum_{i=1}^k \varphi^\rho x^{m_\rho} = \varphi^\beta := \varphi_1^{d_1} \cdots \varphi_r^{d_r}$, where $\beta = d_1\beta_1 + \ldots + d_r\beta_r$ is the effective curve class whose intersections with toric divisors are given by $(\rho_1, \ldots, \rho_k)$.

Example 1.12 ($W_{\Sigma_{\mathbb{P}^2}}$)



$$W_{\Sigma_{\mathbb{P}^2}} = x + y + \frac{\varphi}{xy}$$

Classical and quantum periods

Definition 1.13

The *classical period* of a Laurent polynomial $W \in \mathbb{C}[\varphi][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is

$$\pi_W(\varphi) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_0} \frac{d\log x_1 \wedge \cdots \wedge d\log x_n}{1 - W} = \sum_{k>0} \mathrm{const}(W^k),$$

where const is with respect to the variables $x_1, \ldots, x_n$.

Definition 1.14

The regularized quantum period of X is

$$G_X(\varphi) = \sum_{\beta} (\beta \cdot (-K_X))! D_{0,1}(X, \beta) \varphi^{\beta}.$$

Periods and Picard-Fuchs equations

The periods of a Calabi-Yau manifold satisfy a system of partial differential equations $\mathcal{L}_a\varpi = 0$ called the Picard-Fuchs operators

$$\mathcal{L}_a \in \mathbb{Q}\left[\varphi_b, \frac{\partial}{\partial\varphi_b}\right]$$

Example 1.15

The local Picard-Fuchs operators [CKYZ99] associated to the canonical bundle $K_{\mathbb{P}_\Sigma}$ are given by the Mori vectors:

$$\mathcal{L}_a = \prod_{\ell_i^{(a)}>0} \prod_{j=0}^{\ell_i^{(a)}-1} \left(\sum_{b=1}^{s} \ell_i^{(b)}\theta_b - j\right) - \varphi_a \prod_{\ell_i^{(a)}<0} \prod_{j=0}^{-\ell_i^{(a)}-1} \left(\sum_{b=1}^{s} \ell_i^{(b)}\theta_b - j\right)$$

where $\theta_a = \varphi_a\partial_{\varphi_a}$ (no summation).

Picard-Fuchs equations for $K_{\mathcal{F}_3^{(2)}}$

Example 1.16 ($\mathcal{L}_a$ for $\mathcal{F}_3^{(2)}$)

The Picard-Fuchs operators for $K_{\mathcal{F}_3^{(2)}}$ are

$$\mathcal{L}_1 = \theta_1(\theta_1 - 3\theta_2) - \varphi_1(-2\theta_1 + \theta_2)(-2\theta_1 + \theta_2 - 1)$$
$$\mathcal{L}_2 = (-2\theta_1 + \theta_2)\theta_2 - \varphi_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2)$$

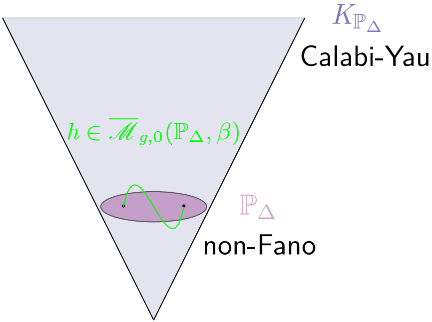
They have the solutions (obtained by recursion)

$$F_1 = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \geq 3n_2}} (-1)^{n_2} \frac{\Gamma(2n_1 - n_2)}{\Gamma(n_1)\Gamma(n_1 - 3n_2 + 1)\Gamma^2(n_2 + 1)} \varphi_1^{n_1} \varphi_2^{n_2}$$

$$F_2 = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \leq 3n_2}} (-1)^{n_1 + n_2} \frac{\Gamma(3n_2 - n_1)}{\Gamma(n_1 + 1)\Gamma(n_2 - 2n_1 + 1)\Gamma^2(n_2 + 1)} \varphi_1^{n_1} \varphi_2^{n_2}$$

Local GW invariants for non-Fano surfaces

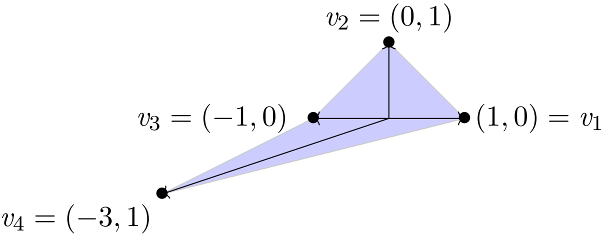
- A curve on a non-Fano surface can be perturbed away from the zero locus away from the canonical bundle.

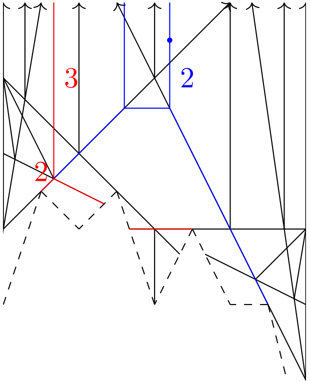


- To obtain the correct GW invariants, we need to include corrections to W_Σ account for such curves.

Scattering diagrams

Obtain $\vartheta_1(\mathbb{P}_\Sigma, X) = W_\Sigma + W'$ via Tim Gräfnitz's `scattering.sage` [Gra].





W_Σ and $\vartheta_1(\mathbb{P}_\Sigma, X)$

Example 1.17 (Corrected superpotentials for Hirzebruch surfaces)

$$\vartheta_1(\mathcal{F}_0^{(2)}) = t \cdot \left(x + y + \frac{\varphi_1}{x} + \frac{\varphi_2}{y} \right)$$

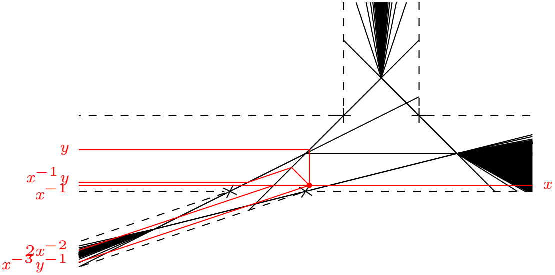
$$\vartheta_1(\mathcal{F}_1^{(2)}) = t \cdot \left(x + y + \frac{\varphi_1}{x} + \frac{\varphi_1\varphi_2}{xy} \right)$$

$$\vartheta_1(\mathcal{F}_2^{(2)}) = t \cdot \left(x + y + \frac{\varphi_1}{x} + \frac{\varphi_1\varphi_2}{x} \left(1 + \frac{\varphi_1}{xy} \right) \right)$$

$$\vartheta_1(\mathcal{F}_3^{(2)}) = t \cdot \left(x + y + \frac{\varphi_1}{x} + \frac{\varphi_1\varphi_2 y}{x} \left(1 + \frac{\varphi_1}{xy} \right)^2 \right)$$

$$\vartheta_1(\mathcal{F}_4^{(2)}) = t \cdot \left(x + y + \frac{\varphi_1}{x} + \varphi_1\varphi_2 y \left(1 + \frac{\varphi_1}{xy} \right) + \frac{\varphi_1\varphi_2 y^2}{x} \left(1 + \frac{\varphi_1}{xy} \right)^3 \right)$$

GW invariants for $\mathcal{F}_3^{(2)}$



n_1, n_2	0	1	2	3	4	5	6	7	8
0	1	1	0	0	0	0	0	0	0
1	0	0	-2	-4	-6	-8	-10	-12	-14
2	0	0	0	0	5	35	135	385	910
3	0	0	0	0	0	0	-32	-400	-2592

Same as for $\mathcal{F}_1^{(2)}$ with $n_1 \mapsto n_1 + n_2$! ($\mathcal{F}_m^{(n)} \cong \mathcal{F}_{m \bmod n}^{(n)}$)

Mirror symmetry for non-Fano varieties

Definition 1.18

A function $W \in \mathbb{C}[\varphi_1, \ldots, \varphi_r][x_1^{-1}, \ldots, x_n^{-1}][\![x_1, \ldots, x_n]\!]$ is called *mirror dual* to (X, D) if the classical period

$$\pi_W(\varphi) = \sum_{k>0} \text{const}(W^k) \in \mathbb{C}[\varphi_1, \ldots, \varphi_r]$$

is equal to the regularized quantum period $G_X(\varphi)$ of X.

Theorem 1.19 ([BGL24])

For every point P inside a chamber in the scattering diagram, $t^{-1}\vartheta_1(X, D)_P$ is a mirror potential for (X, D), in the sense of Definition 1.18,

$$\pi_{t^{-1}\vartheta_1(X,D)_P}(\varphi) = G(\varphi).$$

Mirror symmetry for non-Fano varieties

Theorem 1.20 ([BGL24])

Let (X, D) *be a smooth log Calabi-Yau pair with mirror dual potential* W*. Then, under the change of variables* $Q_i = \varphi_i(t/y)^{d_i}$*, with* $d_i = \beta_i \cdot D$*, we have*

$$\vartheta_1(t, \varphi, y)_\infty = yM_W(Q),$$

where $\vartheta_1(t, \varphi, y)_\infty := \vartheta_1(X, D)_\infty$ *is the theta function at infinity and* M_W *is the open mirror map defined by* W*.*

Number Theory

Calabi-Yau manifolds over finite fields

- Let X_φ be a one parameter family of algebraic varieties defined by a polynomial $P_\varphi \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$ $(x_i \in \mathbb{P}^n)$.
- We clear denominators to get a defnining equation over $\mathbb{Z}$.
- Let $\mathbb{F}_q$ be the finite field with $q = p^k$ (p prime) elements and

$$X_\varphi(\mathbb{F}_q) = |\{x \in \mathbb{F}_q^{n+1} \mid P_\varphi(x) = 0\}|.$$

Zeta functions

Definition 2.1

The *local zeta function* or the *Hasse-Weil zeta function* of the manifold X_φ at the prime p is defined as

$$\zeta_p(X_\varphi, T) = \exp\left(\sum_{k=1}^{\infty} \frac{X_\varphi(\mathbb{F}_{p^k})}{k} T^k\right)$$

Example 2.2

For $X = \{\text{pt}\}$, we have

$$\prod_p \zeta_p(\{\text{pt}\}, p^{-s}) = \zeta(s),$$

the Riemann zeta function.

Weil conjectures

The Weil conjectures – proved by Dwork, Grothendieck, and Deligne – are concerned with the form of local zeta functions.

Theorem 2.3 (Rationality of ζ_p)

$$\zeta_p(X_\varphi, T) = \frac{R_p^{(1)} R_p^{(3)} \dots R_p^{(2d-1)}}{R_p^{(0)} R_p^{(2)} R_p^{(4)} \dots R_p^{(2d)}}$$

where $R_p^{(i)} \in \mathbb{Z}[T]$ *with* $\deg R_p^{(i)} = \dim H^i(X_\varphi)$*.*

Example 2.4

Projective space only has even dimensional cohomology, so

$$\zeta_p(\mathbb{P}^n, T) = \frac{1}{(1-T)(1-pT)\dots(1-p^n T)}$$

Weil conjectures

For a Calabi-Yau 3-fold X_φ, the Weil conjectures imply the local zeta function is of the form

$$\zeta_p(X_\varphi, T) = \frac{R_p(X_\varphi, T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$

In other words, the problem of computing $\zeta_p(X_\varphi, T)$ is reducted to computing the polynomial $R_p(X_\varphi, T)$, called the *Frobenius polynomial*.

$\Longrightarrow$ This can be done with p-*adic cohomology* $H^k(X_\varphi, \mathbb{Q}_p)$

Frobenius map

The Frobenius map acts on coordinates as

$$\begin{aligned}\mathrm{Frob}_p :& \mathbb{K}^n \longrightarrow \mathbb{K}^n \\ & x \longmapsto (x_1^p, x_2^p, \ldots, x_n^p)\end{aligned}$$

This defines a map $\mathrm{Frob}_p : X_\varphi/\mathbb{F}_p \to X_\varphi/\mathbb{F}_p$ since

$$P_\varphi(x^p) = P_\varphi(x)^p = 0 \mod p$$

Therefore

$$|\{\text{Fixed points of } \mathrm{Frob}_p\}| = X_\varphi(\mathbb{F}_p)$$

Lefschetz fixed-point theorem

The pullback of the Frobenius map gives an automorphism of p-adic cohomology

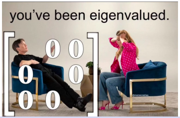
$$\mathsf{Fr}_p = (\mathsf{Frob}_p)_* : H^k(X_\varphi, \mathbb{Q}_p) \longrightarrow H^k(X_\varphi, \mathbb{Q}_p)$$

and applying the Lefschetz fixed-point theorem to this map gives a formula for the point counts

$$X_\varphi(\mathbb{F}_{p^k}) = \sum_{\ell=0}^{2n} (-1)^\ell \mathsf{Tr}\left(\mathsf{Fr}_{p^k} \,\middle|\, H^\ell(X_\varphi, \mathbb{Q}_p) \right)$$

The characteristic polynomial of the inverse Frobenius map acting on the middle cohomology $H^n(X_\varphi, \mathbb{Q}_p)$ is exactly the polynomial $R_p(X_\varphi, T)$:

$$R_p(X_\varphi, T) = \det\left(I - T\mathsf{Fr}_p^{-1} \,\middle|\, H^n(X_\varphi, \mathbb{Q}_p) \right)$$



Computing Frobenius trace with periods

One can compute (following [CdlOK24])

$$R_p(X_\varphi, T) = \det\left(I - T\,\mathsf{Fr}_p^{-1}\middle| H^3(X_\varphi, \mathbb{Q}_p)\right) = \det\left(I - TU_p(\varphi)\right)$$

where

$$U_p(\varphi) = \widetilde{E}(\varphi^p)^{-1}\varphi^{-p\epsilon}\,U_p(0)\varphi^{\epsilon}\widetilde{E}(\varphi).$$

and

$$E(\varphi)_a{}^b = \begin{pmatrix} \theta_a\varpi^b & \theta^a\varpi^b \\ \theta_a\varpi_b & \theta^a\varpi_b \end{pmatrix}$$

is a matrix of periods $\Pi = (\varpi^0, \varpi^a, \varpi_a, \varpi_0)$ in the derivative basis.

Attactor mechanism

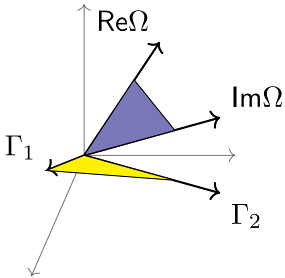
- Four dimensional $\mathcal{N} = 2$ black holes can be constructed by compactifying IIB supergravity on a Calabi-Yau threefold X_φ with complex structure parameter φ.
- The charges of the black hole are determined by a 3-cycle

$$\gamma = q_a A^a - p^a B_a \in H_3(X_\varphi, \mathbb{Z}),$$

  which is wrapped by D3-branes.
- The value of φ at the horizon of the black hole is an *attactor point* $\varphi = \varphi_*$ [Moo07].

Attactor mechanism

If $X = X_{\varphi_*}$ is an attactor variety, the middle cohomology splits as a Hodge structure. This implies Fr_p^{-1} becomes block diagonal and hence $R_p(X, T)$ factorizes.



Factorization of $R_p(X_\varphi, T)$ independent of $p \implies$ Rank 2 attractor point [CdlOEvS20].

Computing zeta functions

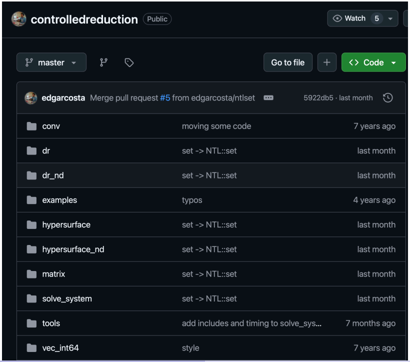
Computational methods

There exists various software to compute periods and zeta functions

- `ore_algebra`: periods of PF operators
- `CY3Zeta`: zeta functions of CY3's
- `controlledreduction`: modified Griffiths-Dwork reduction

controlledreduction

- The package `controlledreduction` by Edgar Costa computes $\zeta_p(X_\varphi, T)$ for non-degenerate fibers X_φ using a special version of Griffiths-Dwork reduction.
- Used this to generate training data for transformer (see my String Data 2024 talk).



Dwork pencil

Example 3.1

The Dwork pencil is the family of K3 surfaces with defined by

$$P_\varphi = x_1^4 + x_2^4 + x_3^4 + x_4^4 + \varphi x_1 x_2 x_3 x_4$$

Recall that for K3s $\dim H^2(X_\varphi) = 22$.

Used `controlledreduction` to compute $R_p(X_\varphi, T)$ for all values of φ at fixed p. For example:

$$R_{11}(X_1, T) = (1 + 11\,T)^6(1 - 11\,T)^{13}(1 + 18\,T + 121\,T^2)$$

```
sage: from pycontrolledreduction import controlledreduction
sage: R.<x,y,z,w> = ZZ[]
sage: controlledreduction(x^4 + y^4 + z^4 + w^4 + x*y*z*w, 11, False).factor()
(-1) * (11*T + 1)^6 * (11*T - 1)^13 * (121*T^2 + 18*T + 1)
```

Note that the Frobenius polynomial has degree 21 here since the factor $(1 - pT)$ from the polarization was omitted.

Training Dwork pencil

Example 3.2 ($p = 7$)

$$R_7(X_0, T) = (1 + 7T)^{10}(1 - 7T)^{11}$$
$$R_7(X_1, T) = (7T + 1)^6(1 - 7T)^{13}(49T^2 + 10T + 1)$$
$$R_7(X_2, T) = (1 - 7T)^9(7T + 1)^{10}(49T^2 - 6T + 1)$$
$$R_7(X_5, T) = (1 - 7T)^9(7T + 1)^{10}(49T^2 - 6T + 1)$$

```
INFO - 12/10/24 12:28:41 - 0:00:24 -      200 -  275.64 equations/s - 30397.72 words/s - ARITHMETIC:  1.4317 - LR: 1.0000e-04
INFO - 12/10/24 12:29:03 - 0:00:46 -      400 -  298.90 equations/s - 32965.76 words/s - ARITHMETIC:  0.0089 - LR: 1.0000e-04
INFO - 12/10/24 12:29:24 - 0:01:07 -      600 -  302.98 equations/s - 33401.93 words/s - ARITHMETIC:  0.0038 - LR: 1.0000e-04
INFO - 12/10/24 12:29:45 - 0:01:29 -      800 -  295.74 equations/s - 32602.79 words/s - ARITHMETIC:  0.0023 - LR: 1.0000e-04
INFO - 12/10/24 12:30:07 - 0:01:50 -     1000 -  294.34 equations/s - 32453.98 words/s - ARITHMETIC:  0.0016 - LR: 1.0000e-04
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INFO - 12/10/24 12:32:59 - 0:04:42 -     2600 -  303.79 equations/s - 33477.18 words/s - ARITHMETIC:  0.0003 - LR: 1.0000e-04
INFO - 12/10/24 12:33:20 - 0:05:03 -     2800 -  305.66 equations/s - 33702.30 words/s - ARITHMETIC:  0.0003 - LR: 1.0000e-04
```
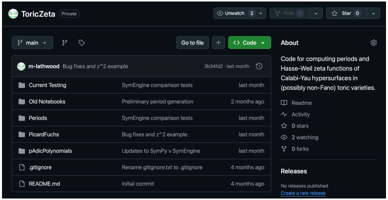

ToricZeta

- In collaboration with Pyry Kuusela at Mainz and Michael Stepniczka at Cornell, we are developing a Python package to compute zeta functions for families of Calabi-Yau hypersurfaces X_φ in toric varieties $\mathbb{P}_\Delta$.
- Motivation is to automate the Mathematica code `CY3Zeta` from arXiv:2405.08067 and port to Python, then include in `CYTools`.
- Our computation involves the period vector Π and linear algebra over $\mathbb{Q}_p$.



New Picard-Fuchs operators

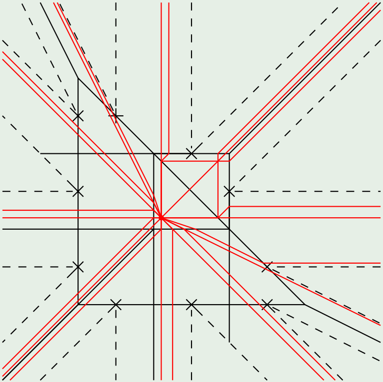
- Recent work producted new PF operators for non-Fano toric varieties [BGL24]. Compute their periods with recursion, then feed to `ToricZeta` to compute new zeta functions.
- Code up Griffiths-Dwork reduction in `Macaulay2` to obtain a map $\Delta \mapsto \{\mathcal{L}_a\}$

Example 3.3 (New $\{\mathcal{L}_a\}$ for $\mathcal{F}_3^{(2)}$)

The new Picard-Fuchs system $\{\mathcal{L}_a^{\vartheta_1}\}$ for $\mathcal{F}_3^{(2)}$ is related to the old system by $\theta_1 \mapsto \theta_1 - \theta_2$.

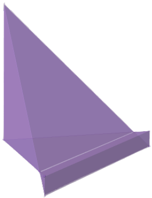
New Picard-Fuchs operators

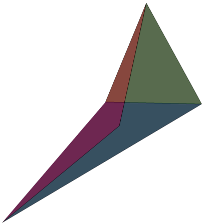
Example 3.4 (New $\{\mathcal{L}_a\}$ for $\mathrm{Bl}^7\mathbb{P}^2 = dP_2$)



Hirzebruch scrolls

Apply ideas of last slide to $\mathcal{F}_3^{(4)}$, a non-Fano toric 4-fold with Calabi-Yau 3-fold as anticanonical divisor





Modularity and GW invariants

Ratios of weight-four special L-values are equal to an infinite series whose summands are formed out of genus-0 Gromov–Witten invariants [CdlOM25].

$\Longrightarrow$ Apply to invariants from ϑ_1!

Thank you!

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