

# THE ARITHMETIC OF RESURGENT TOPOLOGICAL STRINGS

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*21 January 2025*

*Based on arXiv:2212.10606, 2404.10695, and 2404.11550*

# INTRODUCTION

# Topological strings beyond perturbation theory

Perturbative computations in quantum theories rely on *approximation schemes* in a small parameter. The resulting formal power series have *zero radius of convergence* and do not determine the original functions uniquely due to the presence of *non-analytic terms*.

[Dyson, 1952 - Bender, Wu, 1971 - Gross, Periwal, 1988]

The free energy of the *A-model topological string theory* on a Calabi–Yau (CY) threefold  $X$  is defined perturbatively by a worldsheet genus expansion

$$F^{\text{WS}}(\vec{t}, g_s) = \sum_{g \geq 0} F_g(\vec{t}) g_s^{2g-2}.$$

The fixed-genus free energies  $F_g(\vec{t})$  converge in a common region near the large-radius limit  $\Re(t_i) \gg 1$  in the moduli space of Kähler structures of  $X$ . Yet, for fixed  $\vec{t}$  in that region,  
[Shenker, 1990]

$$F_g(\vec{t}) \sim (2g)! \text{ for } g \rightarrow \infty,$$

signaling the presence of exponentially small corrections in  $g_s$ .

The divergence of perturbation theory can sometimes be tamed by applying *resurgence*. Observables can then be written as unambiguous Borel–Laplace resummed *trans-series*.

# Enumerative invariants from resurgence

The resurgence of asymptotic series unveils a universal structure involving a collection of *exponentially small corrections* paired with complex numbers called *Stokes constants*.  
[Écalle, 1981]

The Stokes constants capture information about the large-order behavior of the perturbative series and its *non-perturbative sectors*.

In some remarkable cases, the Stokes constants are (conjecturally) interpreted in terms of *enumerative invariants* based on BPS counting.

- Seiberg–Witten curve of 4d  $\mathcal{N} = 2$  super Yang–Mills theory  
[Grassi, Gu, Mariño, 2019]
- Complex Chern–Simons theory on complements of hyperbolic knots  
[Garoufalidis, Gu, Mariño, 2020]
- Refined/standard/Nekrasov–Shatashvili topological string theory on (toric) CY threefolds  
[Alim, Saha, Teschner, Tulli, 2021 - Gu, Mariño, 2021 - 2022 - Rella, 2022 - Gu, Kashani-Poor, Klemm, Mariño, 2023 - Alexandrov, Mariño, Pioline, 2023 - Fantini, Rella, 2024 - Douaud, Kashani-Poor, 2024]

*Growing evidence indicates that the theory of resurgence can be applied to obtain a systematic understanding of the hidden non-perturbative sectors of topological string theory.*

# THE RESURGENCE TOOLBOX

# Basic notions in resurgence — I

Let  $\varphi(z)$  be a *simple resurgent Gevrey-1* asymptotic series of the form

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]], \quad a_n \sim A^{-n} n! \quad n \gg 1, \quad A \in \mathbb{R}.$$

Its *Borel–Laplace resummation* along the line  $\rho_\theta = e^{i\theta} \mathbb{R}_+$  is the two-step process

$$\varphi(z) \longrightarrow \underbrace{\hat{\varphi}(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \zeta^k}_{\text{locally analytic at } \zeta = 0 \text{ with singularities at } \zeta = \zeta_\omega, \omega \in \Omega} \longrightarrow \underbrace{s_\theta(\varphi)(z) = \int_{\rho_\theta} e^{-\zeta} \hat{\varphi}(\zeta z) d\zeta}_{\text{locally analytic in the complex } z\text{-plane with discontinuities at } \arg(z) = \arg(\zeta_\omega), \omega \in \Omega}$$

If the Borel transform  $\hat{\varphi}(\zeta)$  has a logarithmic branch point at  $\zeta = \zeta_\omega$ , then

$$\hat{\varphi}(\zeta) = -\frac{S_\omega}{2\pi i} \log(\zeta - \zeta_\omega) \hat{\varphi}_\omega(\zeta - \zeta_\omega) + \dots,$$

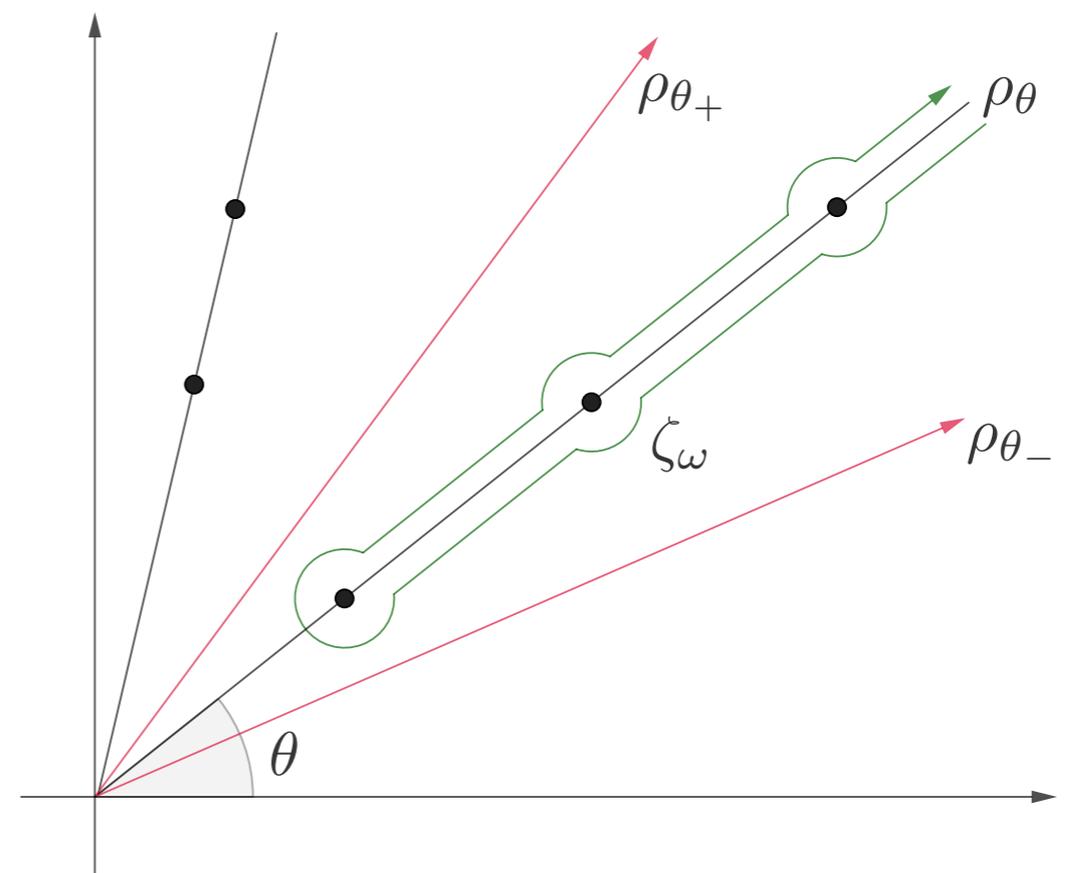
where  $S_\omega \in \mathbb{C}$  is the *Stokes constant* and  $\hat{\varphi}_\omega(\zeta - \zeta_\omega)$  is locally analytic at  $\zeta = \zeta_\omega$ .

When  $\theta = \arg(\zeta_\omega)$  for some  $\omega \in \Omega$ , the line  $\rho_\theta$  is called a *Stokes ray*.

# Basic notions in resurgence — II

The *discontinuity* and the *median resummation* across the Stokes ray  $\rho_\theta$  are

$$\begin{aligned} \text{disc}_\theta \varphi(z) &= s_{\theta_+}(\varphi)(z) - s_{\theta_-}(\varphi)(z) \\ &= \sum_{\omega} S_{\omega} e^{-\zeta_{\omega}/z} s_{\theta_-}(\varphi_{\omega})(z), \\ \mathcal{S}_{\theta}^{\text{med}} \varphi(z) &= \frac{s_{\theta_+}(\varphi)(z) + s_{\theta_-}(\varphi)(z)}{2}. \end{aligned}$$



Schematically,  $\varphi \rightarrow \{\varphi_{\omega}, S_{\omega}\} \rightarrow \{\varphi_{\omega'}, S_{\omega\omega'}\}$ .

Each series is promoted to a *basic trans-series* as  $\Phi_{\omega}(z) = e^{-\zeta_{\omega}/z} \varphi_{\omega}(z)$ .

The *Stokes automorphism*  $\mathfrak{S}_{\theta}$  across the Stokes ray  $\rho_{\theta}$  is defined by  $s_{\theta_+} = s_{\theta_-} \circ \mathfrak{S}_{\theta}$ .

The *minimal resurgent structure* and the matrix of Stokes constants of  $\varphi(z)$  are

$$\mathfrak{B}_{\varphi} = \{\Phi_{\omega}(z)\}_{\omega \in \bar{\Omega}}, \quad \mathcal{S}_{\varphi} = \{S_{\omega\omega'}\}_{\omega, \omega' \in \bar{\Omega}},$$

where  $\bar{\Omega} \subseteq \Omega$  denotes the smallest subset closed under  $\mathfrak{S}$ .

# CALABI–YAU GEOMETRIES AND THE TOPOLOGICAL STRING

# Quantum operators from mirror curves

Let  $X$  be a toric CY threefold. Local mirror symmetry pairs  $X$  with an algebraic curve  $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$  of genus  $g_\Sigma$  describing the *B-model topological string theory* on the mirror  $\tilde{X}$ .  
[Katz, Klemm, Vafa, 1996 - Chiang, Klemm, Yau, Zaslow, 1999]

The Weyl quantization of the mirror curve  $\Sigma$  leads to *quantum-mechanical operators*

$$\rho_j, \quad j = 1, \dots, g_\Sigma,$$

acting on  $L^2(\mathbb{R})$  and conjectured to be positive-definite and of trace class under some assumptions on the mass parameters  $\vec{\xi}$ .

[Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015 - Kashaev, Mariño, 2015]

Their *generalized Fredholm determinant*  $\Xi(\vec{\kappa}, \vec{\xi}, \hbar)$  is an entire function of the true complex deformation parameters  $\kappa_j$ . Its local expansion at  $\vec{\kappa} = 0$  is

$$\Xi(\vec{\kappa}, \vec{\xi}, \hbar) = \sum_{N_1 \geq 0} \cdots \sum_{N_{g_\Sigma} \geq 0} \underbrace{Z(\vec{N}, \vec{\xi}, \hbar)}_{\text{analytic function of } \hbar \in \mathbb{R}_{>0}} \kappa_1^{N_1} \cdots \kappa_{g_\Sigma}^{N_{g_\Sigma}},$$

where the *fermionic spectral traces*  $Z(\vec{N}, \vec{\xi}, \hbar)$  can be analytically continued to  $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ .

# The Topological String/Spectral Theory correspondence

The conjectural *Topological String/Spectral Theory (TS/ST)* correspondence states  
[Hatsuda, Moriyama, Okuyama, 2012 - Grassi, Hatsuda, Mariño, 2014 - Codesido, Grassi, Mariño, 2015]

$$Z(\vec{N}, \vec{\xi}, \hbar) = \frac{1}{(2\pi i)^{g_\Sigma}} \int_{-i\infty}^{i\infty} d\mu_1 \cdots \int_{-i\infty}^{i\infty} d\mu_{g_\Sigma} e^{J(\vec{\mu}, \vec{\xi}, \hbar) - \vec{N} \cdot \vec{\mu}}, \quad \kappa_j = e^{\mu_j},$$

where the *total grand potential* of the A-model topological string on X

$$J(\vec{\mu}, \vec{\xi}, \hbar) = J^{\text{WS}}(\vec{\mu}, \vec{\xi}, \hbar) + J^{\text{WKB}}(\vec{\mu}, \vec{\xi}, \hbar)$$

encodes the standard and Nekrasov-Shatashvili topological string free energies. These can be regarded as non-perturbative corrections of one another in the appropriate regimes.

[Hatsuda, Mariño, Moriyama, Okuyama, 2013]

The string coupling constant  $g_s$  is related to the quantum deformation parameter  $\hbar$  by

$$g_s = 4\pi^2/\hbar \quad (\text{strong-weak coupling duality}).$$

*By the TS/ST correspondence, the fermionic spectral traces  $Z(\vec{N}, \vec{\xi}, \hbar)$  give access to the non-perturbative effects associated with the factorial divergence of the topological string perturbation series in the spirit of large- $N$  gauge/string dualities.*

# RESURGENCE OF THE SPECTRAL THEORY

# Resurgent structures at strong and weak coupling — I

For fixed  $\vec{N} \in \mathbb{N}^{g_\Sigma}$ , we consider the *dual asymptotic expansions*

$$\log Z(\vec{N}, \vec{\xi}, \hbar) \sim \phi_{\vec{N}}(\hbar) \quad \text{for } \hbar \rightarrow 0,$$

$$\log Z(\vec{N}, \vec{\xi}, \hbar) \sim \psi_{\vec{N}}(\hbar^{-1}) \quad \text{for } \hbar \rightarrow \infty,$$

which are expected to be Gevrey-1 and simple resurgent. We conjecture their *minimal resurgent structures*.

[Gu, Mariño, 2021 - Rella, 2022]

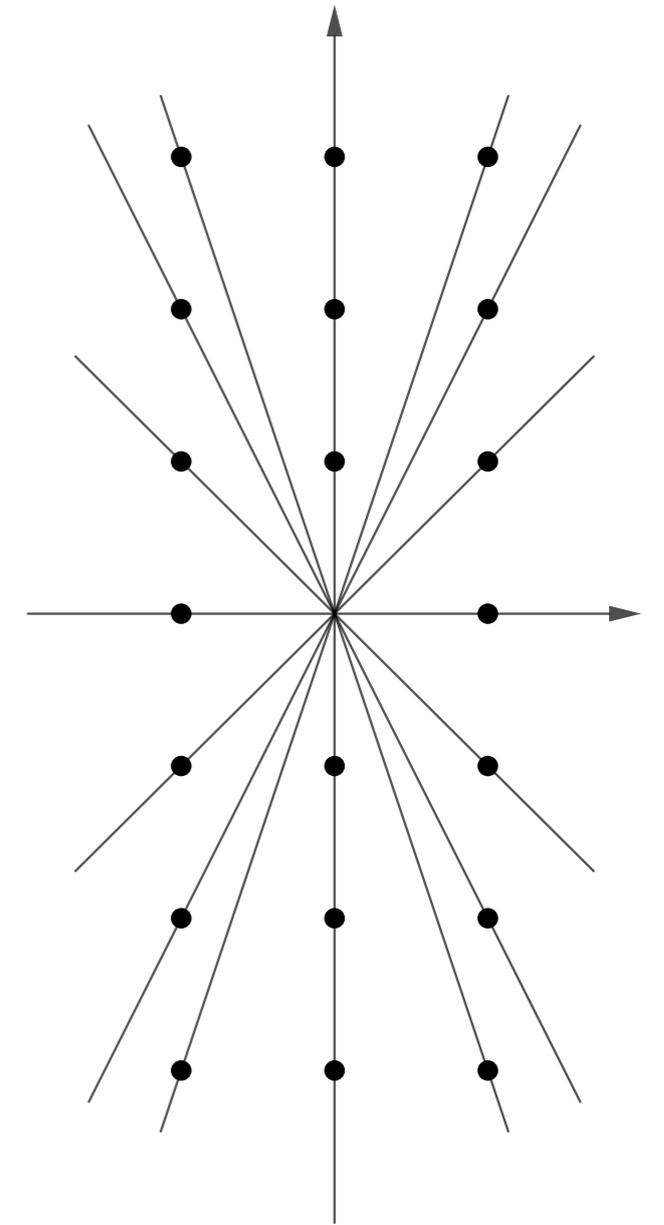
In the semiclassical limit  $\hbar \rightarrow 0$ ,

$$\mathfrak{B}_{\phi_{\vec{N}}} = \{ \Phi_{\sigma, n; \vec{N}}(\hbar) = e^{-n \frac{\mathcal{A}_0}{\hbar}} \phi_{\sigma; \vec{N}}(\hbar) \} \quad (\textit{peacock pattern}),$$

where  $n \in \mathbb{Z}$ ,  $\sigma \in \{0, \dots, l_0\}$ ,  $l_0 \in \mathbb{N}$ , and  $\mathcal{A}_0 \in \mathbb{C}$ . The Gevrey-1 asymptotic series  $\phi_{\sigma; \vec{N}}(\hbar)$  resurge from  $\phi_{\vec{N}}(\hbar) = \phi_{0; \vec{N}}(\hbar)$ .

After fixing a canonical normalization of  $\phi_{\sigma; \vec{N}}(\hbar)$ , the matrix of Stokes constants satisfies

$$\mathcal{S}_{\phi_{\vec{N}}} = \{ S_{\sigma, \sigma', n; \vec{N}} \in \mathbb{Q} \} \quad (\textit{enumerative invariants}).$$



# Resurgent structures at strong and weak coupling — II

Analogously, in the weakly interacting regime  $g_s \propto \hbar^{-1} \rightarrow 0$ ,

$$\mathfrak{B}_{\psi_{\vec{N}}} = \{ \Psi_{\sigma, n; \vec{N}}(g_s) = e^{-n\mathcal{A}_\infty/g_s} \psi_{\sigma; \vec{N}}(g_s) \},$$

where  $n \in \mathbb{Z}$ ,  $\sigma \in \{0, \dots, l_\infty\}$ ,  $l_\infty \in \mathbb{N}$ , and  $\mathcal{A}_\infty \in \mathbb{C}$ . The Gevrey-1 asymptotic series  $\psi_{\sigma; \vec{N}}(g_s)$  resurge from  $\psi_{\vec{N}}(g_s) = \psi_{0; \vec{N}}(g_s)$ .

After fixing a canonical normalization of  $\psi_{\sigma; \vec{N}}(g_s)$ , the matrix of Stokes constants satisfies

$$\mathcal{S}_{\psi_{\vec{N}}} = \{ R_{\sigma, \sigma', n; \vec{N}} \in \mathbb{Q} \}.$$

In both limits, the Stokes constants can be naturally organized into *q-series*

$$S_{\sigma, \sigma'; \vec{N}}(q) = \sum_{n \in \mathbb{Z}} S_{\sigma, \sigma', n; \vec{N}} q^n, \quad R_{\sigma, \sigma'; \vec{N}}(q) = \sum_{n \in \mathbb{Z}} R_{\sigma, \sigma', n; \vec{N}} q^n.$$

*peacock pattern* in the Borel plane  $\longrightarrow$  *infinitely many* Stokes constants in  $\mathbb{Q}$   $\longrightarrow$  *enumerative invariants* of the geometry

**Peacock patterns** are observed in theories controlled by quantum curves.

[Grassi, Gu, Mariño, 2019 - Garoufalidis, Gu, Mariño, 2020 - 2022 - Gu, Mariño, 2021 - Rella, 2022]

# LOCAL $\mathbb{P}^2$ — A CASE STUDY

[Rella, 2022 - Fantini, Rella, 2024]

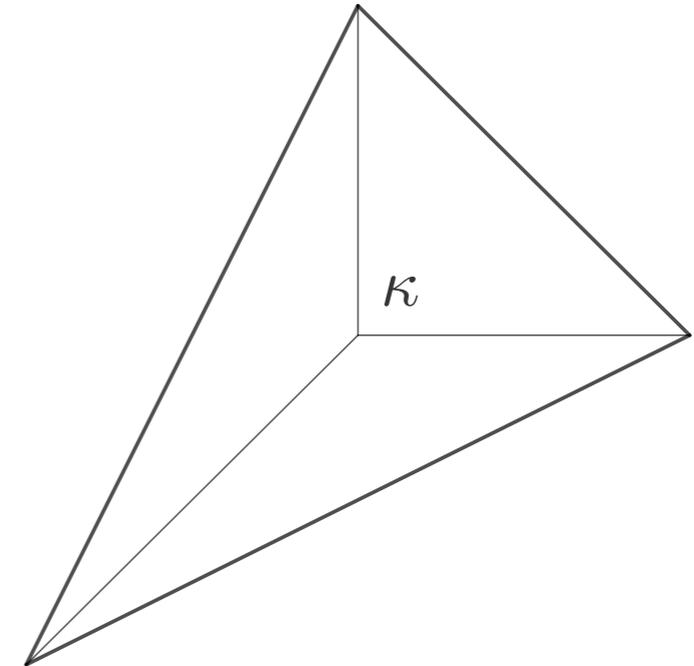
# An exactly solvable example

Local  $\mathbb{P}^2$  is the total space of the canonical bundle over  $\mathbb{P}^2$ , which is a toric del Pezzo CY threefold with one complex modulus  $\kappa$  and no mass parameters.

The spectral trace  $Z_{\mathbb{P}^2}(1, \hbar) = \text{Tr}(\rho_{\mathbb{P}^2})$  factorizes into *holomorphic/anti-holomorphic blocks* as

$$\text{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3}b} e^{-\frac{\pi i}{36}b^2 + \frac{\pi i}{12}b^{-2} + \frac{\pi i}{4}} \frac{(q^{2/3}; q)_{\infty}^2}{(q^{1/3}; q)_{\infty}} \frac{(w; \tilde{q})_{\infty}}{(w^{-1}; \tilde{q})_{\infty}^2},$$

where  $2\pi b^2 = 3\hbar$ ,  $q = e^{2\pi i b^2}$ ,  $\tilde{q} = e^{-2\pi i b^{-2}}$ , and  $w = e^{2\pi i/3}$ .  
 [Kashaev, Mariño, 2015 - Mariño, Zakany, 2015 - Gu, Mariño, 2021]



The *all-orders perturbative expansions* of  $\log \text{Tr}(\rho_{\mathbb{P}^2})$  at weak ( $\hbar \rightarrow 0$ ) and strong ( $\tau = -\mathcal{A}_{\infty}/\hbar \rightarrow 0$ ) coupling give the Gevrey-1 asymptotic series

$$\begin{aligned} \phi(\hbar) &= \sum_{n=1}^{\infty} a_{2n} \hbar^{2n} \in \mathbb{Q}[[\hbar]], & \psi(\tau) &= \sum_{n=1}^{\infty} b_{2n} \tau^{2n-1} \in \mathbb{Q}[\pi, \sqrt{3}][[\tau]], \\ a_{2n} &\sim (-1)^n (2n)! \mathcal{A}_0^{-2n}, & b_{2n} &\sim (-1)^n (2n)! \mathcal{A}_{\infty}^{-2n}, \end{aligned}$$

where  $\mathcal{A}_0 = \frac{4\pi^2}{3}$  and  $\mathcal{A}_{\infty} = \frac{2\pi}{3}$ .

# Arithmetic properties of the Stokes constants — I

We obtain the *exact resurgent structures* at **weak** and **strong** coupling.

The Borel transforms  $\hat{\phi}(\zeta)$ ,  $\hat{\psi}(\zeta)$  are simple resurgent functions with logarithmic branch points at  $\zeta_n = n\mathcal{A}_0 i$  and  $\eta_n = n\mathcal{A}_\infty i$ ,  $n \in \mathbb{Z}_{\neq 0}$ .

The secondary resurgent series are trivial, that is,  $\hat{\phi}_n(\zeta) = 1$  and  $\hat{\psi}_n(\zeta) = 1$ .

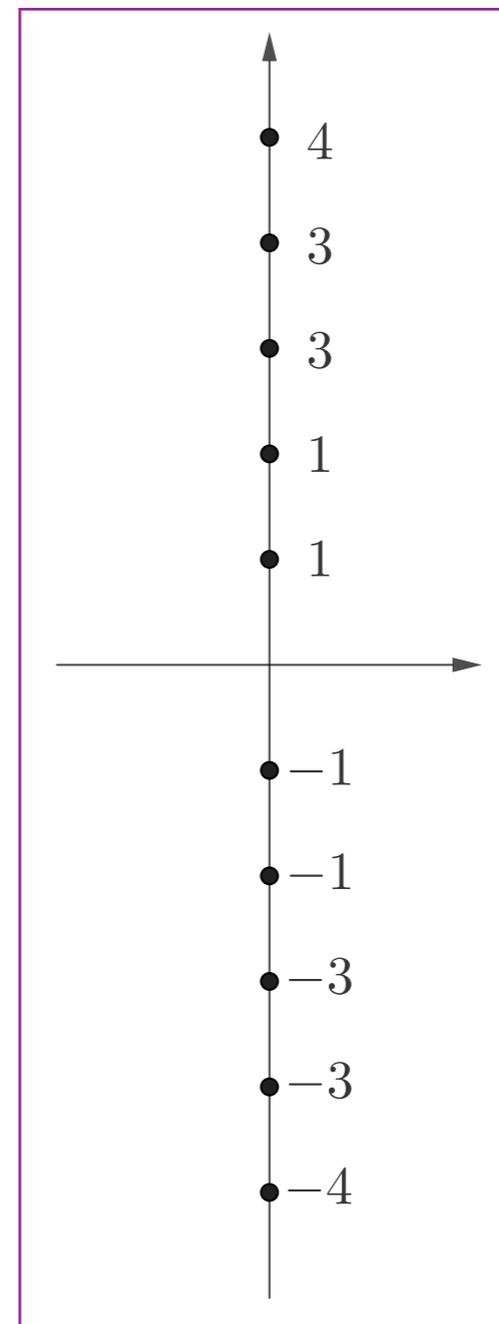
The normalized Stokes constants  $S_n/S_1$ ,  $R_n/R_1$ ,  $n \in \mathbb{Z}_{\neq 0}$ , are the *divisor sum functions*

$$\frac{S_n}{S_1} = \sum_{d|n} \frac{1}{d} \chi_{3,2}(d) \in \mathbb{Q}_{>0}, \quad S_1 = 3\sqrt{3}i,$$

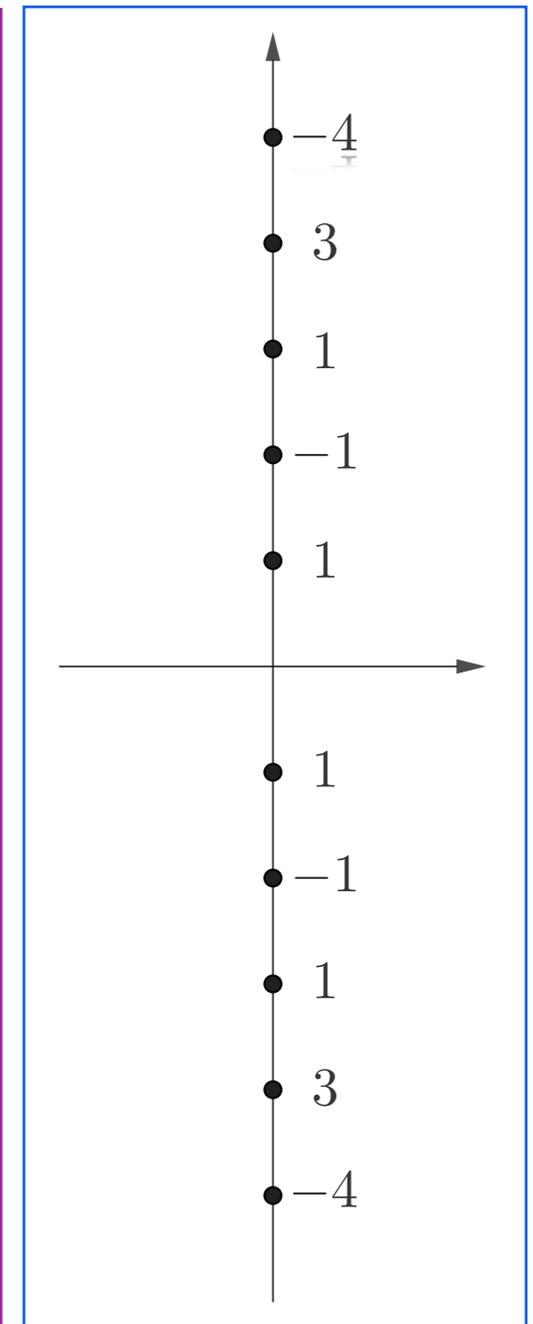
$$\frac{R_n}{R_1} = \sum_{d|n} \frac{d}{n} \chi_{3,2}(d) \in \mathbb{Q}_{\neq 0}, \quad R_1 = 3,$$

where  $\chi_{3,2}(n) = [n]_3$  is the non-principal Dirichlet character modulo 3.

$$n \frac{S_n}{S_1} \in \mathbb{Z}_{\neq 0}$$



$$n \frac{R_n}{R_1} \in \mathbb{Z}_{\neq 0}$$



## Arithmetic properties of the Stokes constants — II

*Theorem:* The Dirichlet series of the Stokes constants  $S_n, R_n, n \in \mathbb{Z}_{>0}$ , can be analytically continued to *weak and strong coupling L-functions*  $L_0(s), L_\infty(s), s \in \mathbb{C}$ , and factorize as

$$L_0(s) = \sum_{m=1}^{\infty} \frac{S_m}{m^s} = S_1 L(s+1, \chi_{3,2}) \zeta(s), \quad L_\infty(s) = \sum_{m=1}^{\infty} \frac{R_m}{m^s} = R_1 L(s, \chi_{3,2}) \zeta(s+1).$$

*The two L-functions are related by a symmetric unitary shift in the arguments of the factors.*

*Theorem:* The generating series of the Stokes constants  $S_n, R_n, n \in \mathbb{Z}_{>0}$ , are given by the  $\tilde{q}, q$ -series in the *anti-holomorphic/holomorphic blocks* of the spectral trace

$$\text{disc}_{\frac{\pi}{2}} \phi(\hbar) = \sum_{n=1}^{\infty} S_n \tilde{q}^n = 3 \log \frac{(w^{-1}\tilde{q}; \tilde{q})_\infty}{(w\tilde{q}; \tilde{q})_\infty}, \quad \text{disc}_{\frac{\pi}{2}} \psi(\tau) = \sum_{n=1}^{\infty} R_n q^{n/3} = 3 \log \frac{(q^{2/3}; q)_\infty}{(q^{1/3}; q)_\infty}.$$

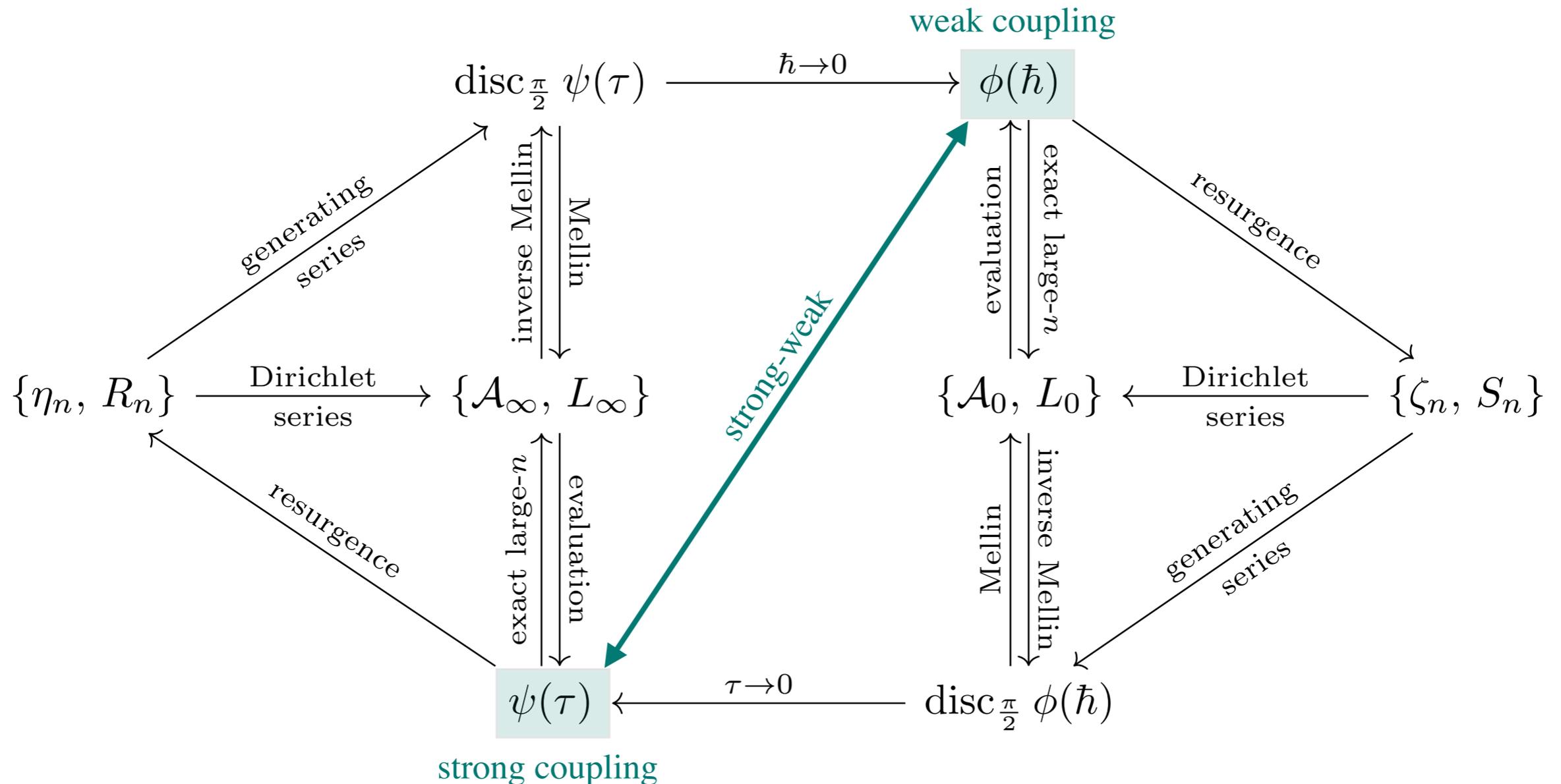
*Each of the two blocks determines the perturbative content in one regime and the non-perturbative content in the other.*

The relations connecting the exact resurgent structures of the dual perturbative expansions of  $\log \text{Tr}(\rho_{\mathbb{P}^2})$  embed into a global *commutative diagram*.

# A full-fledged analytic number-theoretic symmetry — I

The two-way exchange of perturbative/non-perturbative information between the dual regimes in  $\hbar$  takes a mathematically precise form (*strong-weak resurgent symmetry*).

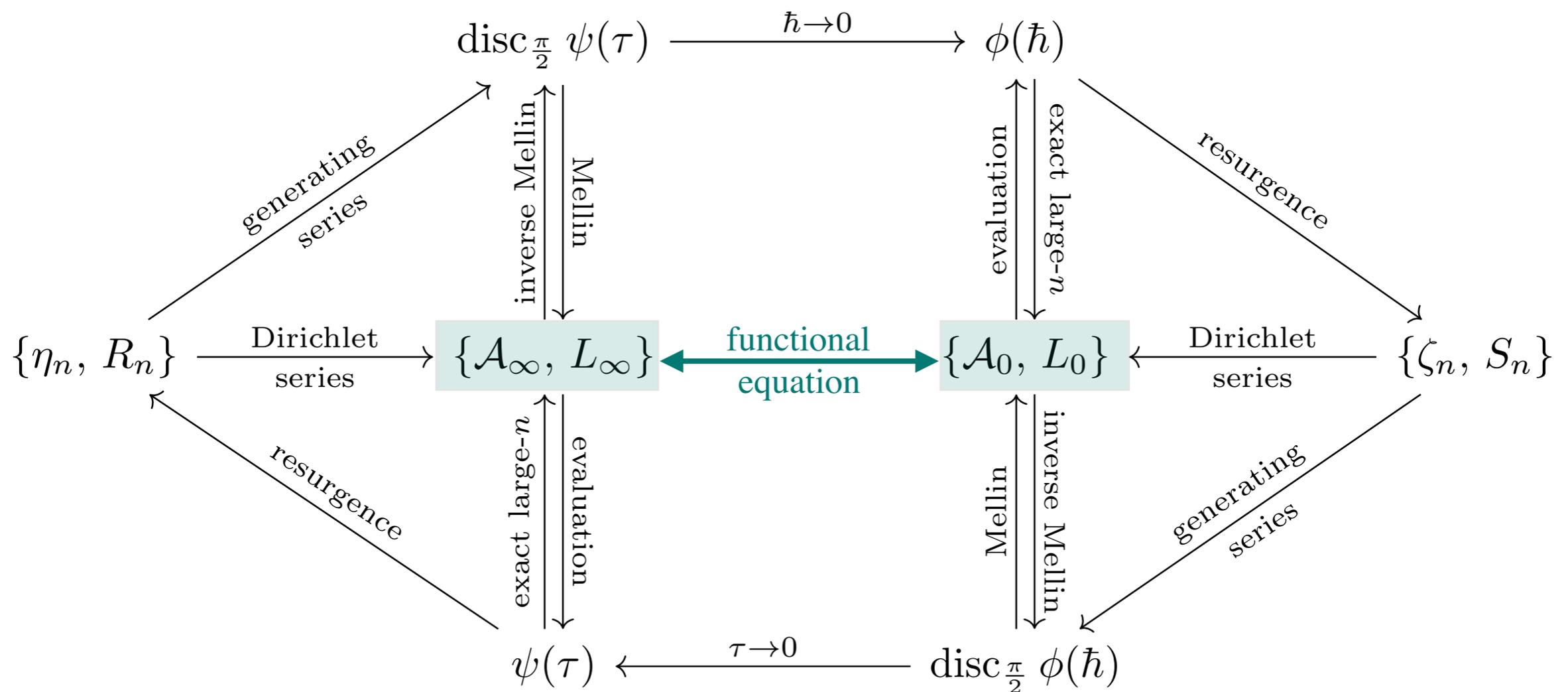
This is a realization of underlying physical mechanisms that can be intuitively traced back to the *S-type duality* between the worldsheet and WKB contributions to the total grand potential of the topological string on local  $\mathbb{P}^2$ .



# A full-fledged analytic number-theoretic symmetry — II

The completions of the weak and strong coupling  $L$ -functions  $\Lambda_0(s), \Lambda_\infty(s), s \in \mathbb{C}$ , do not individually satisfy a standard functional equation—e.g.,  $\Lambda_\zeta(s) = \Lambda_\zeta(1-s)$ .

*Theorem:*  $\Lambda_0(s), \Lambda_\infty(s), s \in \mathbb{C}$ , are analytically continued to the whole complex  $s$ -plane through each other. They satisfy the **combined functional equation**  $\Lambda_0(s) = \Lambda_\infty(-s)$ .



# A THEORY OF MODULAR RESURGENCE

[Fantini, Rella, 2024]

# New perspectives on resurgence via $L$ -functions

We identify a new class of asymptotic series whose resurgent structures reproduce the paramount features of  $\log \text{Tr}(\rho_{\mathbb{P}^2})$ .

Definition: A Gevrey-1 asymptotic series  $\varphi(y) \in \mathbb{C}[[y]]$  has a *modular resurgent structure* if

1. The Borel transform  $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$  is singular at  $\zeta_m = m\mathcal{A}$ ,  $m \in \mathbb{Z}_{\neq 0}$ , for some  $\mathcal{A} \in \mathbb{C}$ , and the resurgent series at  $\zeta_m$  is the Stokes constant  $A_m \in \mathbb{C}$ ;
2. The Stokes constants  $A_m$ ,  $m \in \mathbb{Z}_{\neq 0}$ , are the coefficients of two  $L$ -functions

$$L_+(s) = \sum_{m>0} \frac{A_m}{m^s}, \quad L_-(s) = - \sum_{m>0} \frac{A_{-m}}{m^s}.$$

A *modular resurgent series* is equivalently characterized by the generating function

$$f(y) = \begin{cases} \sum_{m>0} A_m e^{2\pi i m y}, & \Im(y) > 0, \\ -\sum_{m<0} A_m e^{2\pi i m y}, & \Im(y) < 0, \end{cases} \quad y \in \mathbb{C} \setminus \mathbb{R}.$$

There is a canonical correspondence between the  $L$ -functions  $L_{\pm}$  and the  $q$ -series  $f$ .

# The modular resurgence paradigm

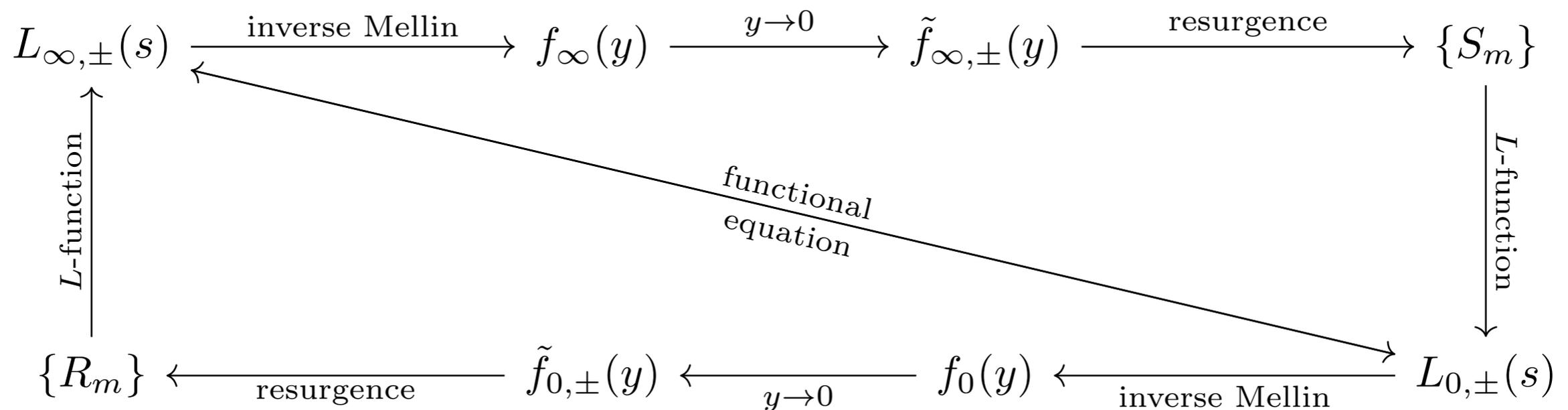
Let us expand the generating function  $f(y)$  for  $y \rightarrow 0$  in  $\mathbb{H}_\pm$ , that is,

$$f|_{\mathbb{H}_\pm}(y) = \pm \sum_{\pm m > 0} A_m e^{2\pi i m y} \xrightarrow{y \rightarrow 0} \tilde{f}_\pm(y) = \sum_{n > 0} \left( \frac{(2\pi i)^n}{n!} L_\pm(-n) \right) y^n.$$

The perturbative expansion  $\tilde{f}_\pm(y)$  is dictated by the analytic continuation of  $L_\pm(s)$ .

A rich analytic number-theoretic fabric underlies the properties of modular resurgent series and motivates us to present a *new paradigm of resurgence*.

A network of exact relations connects *pairs of modular resurgent series* and forms a commutative diagram that generalizes the strong-weak resurgent symmetry of  $\log \text{Tr}(\rho_{\mathbb{P}^2})$ .



# $q$ -series, $L$ -functions, and quantum modular forms

Recall that a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a *holomorphic quantum modular form* of weight  $\omega$  for a subgroup  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ , where  $\omega$  is integer or half-integer, if the cocycle

$$h_\gamma[f](y) := (cy + d)^{-\omega} f\left(\frac{ay + b}{cy + d}\right) - f(y), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

is holomorphic in  $\mathbb{C}_\gamma := \{y \in \mathbb{C} \mid cy + d \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}\}$ .

[Zagier, 2020]

*Conjecture:* Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  be a  $q$ -series where  $q = e^{2\pi iy}$ . If its asymptotic expansion  $\varphi(y)$  as  $y \rightarrow 0$  with  $\Im(y) > 0$  has a modular resurgent structure, then  $f(y)$  is a holomorphic quantum modular form for a subgroup  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$  and

$$\mathcal{S}_\theta^{\mathrm{med}} \varphi(y) = f(y), \quad y \in \mathbb{H} \cap \{\Re(e^{-i\theta y}) > 0\}.$$

*A class of examples is built from Maass cusp forms with spectral parameter 1/2 and more general cusp forms. Further evidence from combinatorics and the quantum invariants of knots and 3-manifolds.*

The conjecture is proven for the generating functions of the weak and strong coupling Stokes constants of  $\log \mathrm{Tr}(\rho_{\mathbb{P}^2})$  with respect to the congruence subgroup  $\Gamma_1(3) \subset \mathrm{SL}_2(\mathbb{Z})$ .

# CONCLUSIONS

# Final remarks

The resurgence of the spectral traces of a toric CY threefold unveils a universal structure of non-perturbative sectors (*peacock patterns*) and Stokes constants (*enumerative invariants*).

The resurgence of the spectral trace of local  $\mathbb{P}^2$  in the weak and strong  $\hbar$ -regimes descends from a unique global number-theoretic construction (*strong-weak resurgent symmetry*).

Our results suggest a new paradigm linking the resurgent properties of  $q$ -series, the analytic properties of  $L$ -functions, and quantum modular forms (*modular resurgent structures*).

## Work in progress

- Explore the geometric/physical interpretation of the non-perturbative sectors and Stokes constants in our conjectural proposal for the resurgence of  $\log Z(\vec{N}, \vec{\xi}, \hbar)$ .
- Translate our results for  $\log Z_{\mathbb{P}^2}(1, \hbar)$  into statements on the topological string and connect with recent works studying the resurgence of the free energies via the HAEs.
- Extend our results for  $\log Z_{\mathbb{P}^2}(1, \hbar)$  to other toric CY threefolds and possibly higher-order fermionic spectral traces.
- Broaden the supporting basis for the paradigm of modular resurgence and fully characterize the admissible resurgent  $L$ -functions and their functional equation.

THANK YOU!