B-branes in $\mathcal{N} = (2, 2)$ Hybrid Models

Robert Pryor

The University of Melbourne

YOUNGST@RS - Physics and Number Theory

21/01/2025

Based on work in collaboration with Johanna Knapp [SciPostPhys.17.6.165].

Introduction

- The goal of this work is to understand the category of
  B-branes in $\mathcal{N} = (2, 2)$ hybrid models.

---

[1]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[2]Bertolini, Melnikov, and Plesser, 2014; Bertolini and Romo, 2018.

Introduction

- The goal of this work is to understand the category of B-branes in $\mathcal{N} = (2, 2)$ hybrid models.
- This category is understood in some special cases[1], but a more general framework is missing.

[1]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[2]Bertolini, Melnikov, and Plesser, 2014; Bertolini and Romo, 2018.

Introduction

- The goal of this work is to understand the category of B-branes in $\mathcal{N} = (2, 2)$ hybrid models.
- This category is understood in some special cases[1], but a more general framework is missing.
- These hybrid models appear as limiting points in the moduli space of some interesting Calabi-Yaus, and can also be considered intrinsically.

---

[1]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[2]Bertolini, Melnikov, and Plesser, 2014; Bertolini and Romo, 2018.

Introduction

- The goal of this work is to understand the category of B-branes in $\mathcal{N} = (2, 2)$ hybrid models.
- This category is understood in some special cases[1], but a more general framework is missing.
- These hybrid models appear as limiting points in the moduli space of some interesting Calabi-Yaus, and can also be considered intrinsically.
- We build on prior work in the closed string setting[2].

---

[1]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[2]Bertolini, Melnikov, and Plesser, 2014; Bertolini and Romo, 2018.

1. Background

2. Hybrid B-branes

3. Examples

4. Lifting to the GLSM

Non-linear sigma models

- The string worldsheet action is that of an 2d $\mathcal{N} = (2,2)$ non-linear sigma model (NLSM), on a Calabi-Yau manifold X

$$S_{\text{NLSM}} = \int d^2x \bigg(-g_{\alpha\bar{\beta}}\partial^\mu\phi^\alpha\partial_\mu\bar{\phi}^{\bar{\beta}} + 2ig_{\alpha\bar{\beta}}\bar{\psi}_-^{\bar{\beta}}\overleftrightarrow{D_+}\psi_-^\alpha$$
$$+ 2ig_{\alpha\bar{\beta}}\bar{\psi}_+^{\bar{\beta}}\overleftrightarrow{D_-}\psi_+^\alpha + R_{\alpha\bar{\beta}\gamma\bar{\delta}}\psi_+^\alpha\psi_-^\gamma\bar{\psi}_-^{\bar{\beta}}\bar{\psi}_+^{\bar{\delta}} \bigg).$$

Non-linear sigma models

- The string worldsheet action is that of an 2d $\mathcal{N} = (2,2)$ non-linear sigma model (NLSM), on a Calabi-Yau manifold X

$$S_{\text{NLSM}} = \int d^2x \bigg(- g_{\alpha\bar{\beta}}\partial^\mu\phi^\alpha\partial_\mu\bar{\phi}^{\bar{\beta}} + 2ig_{\alpha\bar{\beta}}\bar{\psi}_-^{\bar{\beta}}\overleftrightarrow{D_+}\psi_-^\alpha$$
$$+ 2ig_{\alpha\bar{\beta}}\bar{\psi}_+^{\bar{\beta}}\overleftrightarrow{D_-}\psi_+^\alpha + R_{\alpha\bar{\beta}\gamma\bar{\delta}}\psi_+^\alpha\psi_-^\gamma\bar{\psi}_-^{\bar{\beta}}\bar{\psi}_+^{\bar{\delta}} \bigg).$$

- This action is invariant under the following $\mathcal{N} = (2,2)$ SUSY

$$\delta\phi^\alpha = \epsilon_+\psi_-^\alpha - \epsilon_-\psi_+^\alpha,$$
$$\delta\psi_\pm^\alpha = \pm 2i\bar{\epsilon}_\mp\partial_\pm\phi^\alpha + \epsilon_\pm\Gamma_{\beta\gamma}^\alpha\psi_+^\beta\psi_-^\gamma.$$

Non-linear sigma models

- The string worldsheet action is that of an 2d $\mathcal{N} = (2,2)$ non-linear sigma model (NLSM), on a Calabi-Yau manifold X

$$S_{\text{NLSM}} = \int d^2x \bigg(-g_{\alpha\bar{\beta}}\partial^\mu\phi^\alpha\partial_\mu\bar{\phi}^{\bar{\beta}} + 2ig_{\alpha\bar{\beta}}\bar{\psi}_-^{\bar{\beta}}\overleftrightarrow{D_+}\psi_-^\alpha$$
$$+ 2ig_{\alpha\bar{\beta}}\bar{\psi}_+^{\bar{\beta}}\overleftrightarrow{D_-}\psi_+^\alpha + R_{\alpha\bar{\beta}\gamma\bar{\delta}}\psi_+^\alpha\psi_-^\gamma\bar{\psi}_-^{\bar{\beta}}\bar{\psi}_+^{\bar{\delta}} \bigg).$$

- This action is invariant under the following $\mathcal{N} = (2,2)$ SUSY

$$\delta\phi^\alpha = \epsilon_+\psi_-^\alpha - \epsilon_-\psi_+^\alpha,$$
$$\delta\psi_\pm^\alpha = \pm 2i\bar{\epsilon}_\mp\partial_\pm\phi^\alpha + \epsilon_\pm\Gamma_{\beta\gamma}^\alpha\psi_+^\beta\psi_-^\gamma.$$

- For X a Calabi-Yau, the NLSM has a non-anomalous $U(1)_A \times U(1)_V$ R-symmetry.

Landau-Ginzburg orbifold models

- A 2d $\mathcal{N} = (2,2)$ LG orbifold model with target space $\mathbb{C}^n/\Gamma$ and superpotential W is given by

$$S_{\mathsf{LG}} = \int d^2x \Bigg(-\partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar\alpha} + 2i\bar{\psi}_-^{\bar\alpha} \overleftrightarrow{\partial_+} \psi_-^\alpha + 2i\bar{\psi}_+^{\bar\alpha} \overleftrightarrow{\partial_-} \psi_+^\alpha$$
$$- \frac{1}{4}\partial_\alpha W \partial_{\bar\alpha} \overline{W} - \frac{1}{2}\partial_\alpha \partial_\beta W \psi_+^\alpha \psi_-^\beta - \frac{1}{2}\partial_{\bar\alpha} \partial_{\bar\beta} \overline{W} \bar{\psi}_-^{\bar\alpha} \bar{\psi}_-^{\bar\beta} \Bigg).$$

Landau-Ginzburg orbifold models

- A $2d$ $\mathcal{N} = (2,2)$ LG orbifold model with target space $\mathbb{C}^n/\Gamma$ and superpotential W is given by

$$
S_{\text{LG}} = \int d^2x \bigg(-\partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar{\alpha}} + 2i\bar{\psi}_-^{\bar{\alpha}} \overleftrightarrow{\partial_+} \psi_-^\alpha + 2i\bar{\psi}_+^{\bar{\alpha}} \overleftrightarrow{\partial_-} \psi_+^\alpha
$$
$$
- \frac{1}{4}\partial_\alpha W \partial_{\bar{\alpha}} \overline{W} - \frac{1}{2}\partial_\alpha \partial_\beta W \psi_+^\alpha \psi_-^\beta - \frac{1}{2}\partial_{\bar{\alpha}} \partial_{\bar{\beta}} \overline{W} \psi_-^{\bar{\alpha}} \bar{\psi}_-^{\bar{\beta}} \bigg).
$$

- This model is invariant under the following $\mathcal{N} = (2,2)$ SUSY

$$
\delta\phi^\alpha = \epsilon_+ \psi_-^\alpha - \epsilon_- \psi_+^\alpha,
$$
$$
\delta\psi_\pm^\alpha = \pm 2i\bar{\epsilon}_\mp \partial_\pm \phi^\alpha - \frac{1}{2}\epsilon_\pm \delta^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \overline{W}.
$$

Landau-Ginzburg orbifold models

- A 2d $\mathcal{N} = (2,2)$ LG orbifold model with target space $\mathbb{C}^n/\Gamma$ and superpotential W is given by

$$S_{\text{LG}} = \int d^2x \bigg(-\partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar{\alpha}} + 2i\bar{\psi}_-^{\bar{\alpha}} \overleftrightarrow{\partial_+} \psi_-^\alpha + 2i\bar{\psi}_+^{\bar{\alpha}} \overleftrightarrow{\partial_-} \psi_+^\alpha$$
$$- \frac{1}{4}\partial_\alpha W \partial_{\bar{\alpha}} \overline{W} - \frac{1}{2}\partial_\alpha \partial_\beta W \psi_+^\alpha \psi_-^\beta - \frac{1}{2}\partial_{\bar{\alpha}}\partial_{\bar{\beta}} \overline{W} \psi_-^{\bar{\alpha}} \bar{\psi}_-^{\bar{\beta}} \bigg).$$
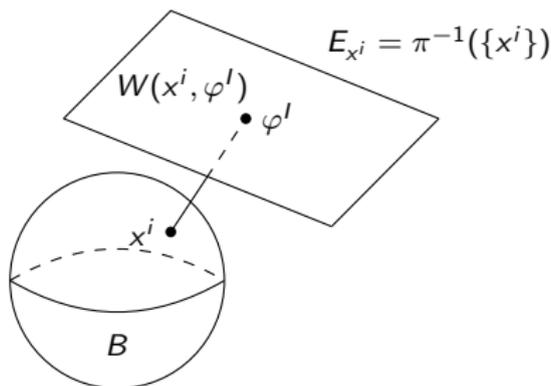
- This model is invariant under the following $\mathcal{N} = (2,2)$ SUSY

$$\delta\phi^\alpha = \epsilon_+ \psi_-^\alpha - \epsilon_- \psi_+^\alpha,$$
$$\delta\psi_\pm^\alpha = \pm 2i\bar{\epsilon}_\mp \partial_\pm \phi^\alpha - \frac{1}{2}\epsilon_\pm \delta^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \overline{W}.$$

- LG models have a non-anomalous $U(1)_A \times U(1)_V$ R-symmetry if W is quasi-homogeneous of degree 2 under $U(1)_V$.

Hybrid models

- A hybrid model "looks like", an LG orbifold fibred over a geometric base. This allows a non-zero superpotential in the fibre, and NLSM-like physics in the base.

$$E_{x^i} = \pi^{-1}(\{x^i\})$$

$W(x^i, \varphi^I)$ • φ^I

x^i

B

Hybrid models

- A hybrid model "looks like", an LG orbifold fibred over a geometric base. This allows a non-zero superpotential in the fibre, and NLSM-like physics in the base.

- A certain class of hybrid models are believed to flow to SCFTs at low energies.

$$E_{x^i} = \pi^{-1}(\{x^i\})$$
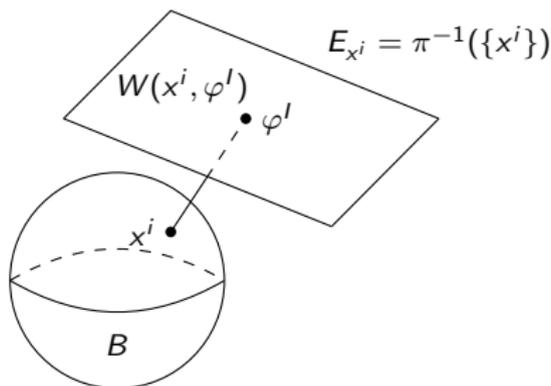
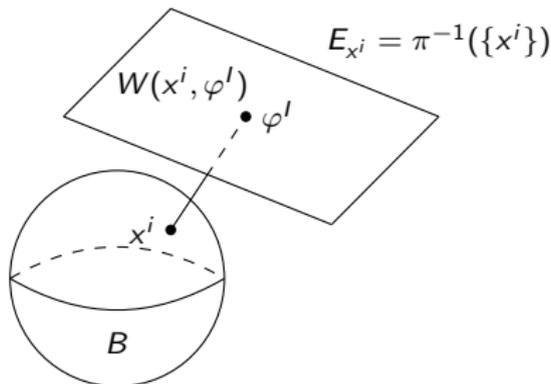$W(x^i, \varphi^I)$

φ^I

x^i

B

Hybrid models

- A hybrid model "looks like", an LG orbifold fibred over a geometric base. This allows a non-zero superpotential in the fibre, and NLSM-like physics in the base.

- A certain class of hybrid models are believed to flow to SCFTs at low energies.

- NLSMs and LG models are special cases of hybrid models.

$$E_{x^i} = \pi^{-1}(\{x^i\})$$

$W(x^i, \varphi^I)$ $\bullet \, \varphi^I$

$x^i \bullet$

B

Definition of a hybrid model

- At the level of actions, a 2d $\mathcal{N} = (2,2)$ hybrid model with target space Y just looks like a generic NLSM with superpotential

$$S_{\text{hyb}} = \int d^2x \left(-g_{\alpha\bar{\beta}} \partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar{\beta}} + \cdots - \frac{1}{4} g^{\alpha\bar{\beta}} \partial_\alpha W \partial_{\bar{\beta}} \overline{W} + \cdots \right).$$

Definition of a hybrid model

- At the level of actions, a 2d $\mathcal{N} = (2,2)$ hybrid model with target space Y just looks like a generic NLSM with superpotential

$$S_{\text{hyb}} = \int d^2x \left(-g_{\alpha\bar{\beta}} \partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar{\beta}} + \cdots - \frac{1}{4} g^{\alpha\bar{\beta}} \partial_\alpha W \partial_{\bar{\beta}} \overline{W} + \cdots \right).$$

- The key differences only become apparent when considering the geometry of Y.

Definition of a hybrid model

- At the level of actions, a 2d $\mathcal{N} = (2,2)$ hybrid model with target space Y just looks like a generic NLSM with superpotential

$$S_{\text{hyb}} = \int d^2x \left(-g_{\alpha\bar{\beta}}\partial^\mu \phi^\alpha \partial_\mu \bar{\phi}^{\bar{\beta}} + \cdots - \frac{1}{4}g^{\alpha\bar{\beta}}\partial_\alpha W \partial_{\bar{\beta}}\overline{W} + \cdots \right).$$

- The key differences only become apparent when considering the geometry of Y.

Hybrid Geometry

Let B be a compact Kähler manifold of dimension d, and let $Y : X \to B$ be a rank r holomorphic vector bundle. If there exists a holomorphic function $W : Y \to \mathbb{C}$ such that $dW^{-1}(0) = B$, we call Y a hybrid geometry.

The Good Hybrid Condition

- A hybrid which flows to an SCFT is said to be a "good hybrid". This is believed to be the case when we can consistently assign both non-anomalous axial and vector $U(1)$ R-symmetries[3].

---

[3]Bertolini, Melnikov, and Plesser, 2014.

The Good Hybrid Condition

- A hybrid which flows to an SCFT is said to be a "good hybrid". This is believed to be the case when we can consistently assign both non-anomalous axial and vector $U(1)$ R-symmetries[3].

- For this to be possible, we require the existence of a vertical holomorphic Killing vector V on Y.

$$\mathcal{L}_V W = W \text{ , and } \mathcal{L}_V \pi^*(\omega) = 0 \text{ , for all } \omega \in \Omega^*(B).$$

---

[3]Bertolini, Melnikov, and Plesser, 2014.

The Good Hybrid Condition

- A hybrid which flows to an SCFT is said to be a "good hybrid". This is believed to be the case when we can consistently assign both non-anomalous axial and vector $U(1)$ R-symmetries[3].

- For this to be possible, we require the existence of a vertical holomorphic Killing vector V on Y.

$$\mathcal{L}_V W = W \text{ , and } \mathcal{L}_V \pi^*(\omega) = 0 \text{ , for all } \omega \in \Omega^*(B).$$

- S_{hyb} is then invariant under δ_A and $\delta_V + 2\delta_{KV}$, where δ_A and δ_V are the NLSM R-symmetries.

---

[3]Bertolini, Melnikov, and Plesser, 2014.

An example of hybrid geometry

- A nice explicit example of hybrid geometry[4] is the total space
  $Y = \text{tot}(\mathcal{O}(-2) \to \mathbb{P}^1)$.

---

[4]Bertolini, Melnikov, and Plesser, 2014.

An example of hybrid geometry

- A nice explicit example of hybrid geometry[4] is the total space
  $Y = \text{tot}(\mathcal{O}(-2) \to \mathbb{P}^1)$.
- We choose an open cover $\{U_u, U_v\}$ of Y with base
  coordinates $u = v^{-1}$ and fibre coordinates $\varphi_u = v^2 \varphi_v$.

---

[4]Bertolini, Melnikov, and Plesser, 2014.

An example of hybrid geometry

- A nice explicit example of hybrid geometry[4] is the total space $Y = \mathrm{tot}(\mathcal{O}(-2) \to \mathbb{P}^1)$.
- We choose an open cover $\{U_u, U_v\}$ of Y with base coordinates $u = v^{-1}$ and fibre coordinates $\varphi_u = v^2 \varphi_v$.
- The superpotential can be given locally by

$$W_u = (\alpha + u^8)\varphi_u^4 \text{ , and } W_v = (\alpha v^8 + 1)\varphi_v^4 \text{ , } \alpha \in \mathbb{C}^\times,$$

and indeed, $dW^{-1}(0) = \mathbb{P}^1$.

---

[4]Bertolini, Melnikov, and Plesser, 2014.

An example of hybrid geometry

- A nice explicit example of hybrid geometry[4] is the total space $Y = \text{tot}(\mathcal{O}(-2) \to \mathbb{P}^1)$.

- We choose an open cover $\{U_u, U_v\}$ of Y with base coordinates $u = v^{-1}$ and fibre coordinates $\varphi_u = v^2 \varphi_v$.

- The superpotential can be given locally by

$$W_u = (\alpha + u^8)\varphi_u^4 \text{ , and } W_v = (\alpha v^8 + 1)\varphi_v^4 \text{ , } \alpha \in \mathbb{C}^\times,$$

and indeed, $dW^{-1}(0) = \mathbb{P}^1$.

- A vertical, holomorphic Killing vector is $V = \frac{1}{4}\varphi_u \partial_{\varphi_u}$.

---

[4]Bertolini, Melnikov, and Plesser, 2014.

An example of hybrid geometry

- A nice explicit example of hybrid geometry[4] is the total space $Y = \text{tot}(\mathcal{O}(-2) \to \mathbb{P}^1)$.

- We choose an open cover $\{U_u, U_v\}$ of Y with base coordinates $u = v^{-1}$ and fibre coordinates $\varphi_u = v^2 \varphi_v$.

- The superpotential can be given locally by

$$W_u = (\alpha + u^8)\varphi_u^4 \ , \text{ and } W_v = (\alpha v^8 + 1)\varphi_v^4 \ , \ \alpha \in \mathbb{C}^\times,$$

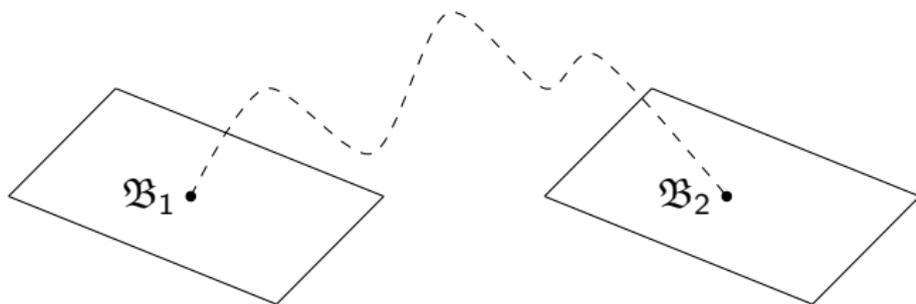and indeed, $dW^{-1}(0) = \mathbb{P}^1$.

- A vertical, holomorphic Killing vector is $V = \frac{1}{4}\varphi_u \partial_{\varphi_u}$.

- Indeed, $\mathcal{L}_V W = W$.

---

[4]Bertolini, Melnikov, and Plesser, 2014.

D-branes and B-branes

- The set of branes in a given string theory forms a category.
  The objects are given by the branes, and the morphisms are
  given by the open strings which join them.

D-branes and B-branes

- The set of branes in a given string theory forms a category. The objects are given by the branes, and the morphisms are given by the open strings which join them.

- Worldsheet boundaries can preserve at most one of the $\mathcal{N} = 2$ subalgebras of $\mathcal{N} = (2, 2)$. We choose to preserve the $\mathcal{N} = 2_B$ subalgebra, and specifically consider D-branes in B-twisted theories. These are the B-branes.

Elementary B-branes in NLSMs

- The data of an elementary B-brane[5] in an NLSM is
  $\mathfrak{B} = (E, A, Q)$, where E is a $\mathbb{Z}_2$-graded vector bundle over X,
  A is a connection with a $(1,1)$-curvature form and Q is an
  odd, nilpotent, holomorphic endomorphism of E.

---

[5]Aspinwall, 2005.

Elementary B-branes in NLSMs

- The data of an elementary B-brane[5] in an NLSM is
  $\mathfrak{B} = (E, A, Q)$, where E is a $\mathbb{Z}_2$-graded vector bundle over X,
  A is a connection with a $(1,1)$-curvature form and Q is an
  odd, nilpotent, holomorphic endomorphism of E.

- This structure arises by constructing general $\mathcal{N} = 2_B$ SUSY
  preserving boundary interactions.

---

[5]Aspinwall, 2005.

Elementary B-branes in NLSMs

- The data of an elementary B-brane[5] in an NLSM is
  $\mathfrak{B} = (E, A, Q)$, where E is a $\mathbb{Z}_2$-graded vector bundle over X,
  A is a connection with a $(1,1)$-curvature form and Q is an
  odd, nilpotent, holomorphic endomorphism of E.

- This structure arises by constructing general $\mathcal{N} = 2_B$ SUSY
  preserving boundary interactions.

- The morphisms are given by open string states, which are
  cohomology classes of the boundary supercharge acting on the
  space of NLSM zero-modes

$$H^p_{Q_{\partial\Sigma}} \left(\mathcal{H}_{zero}(\mathfrak{B}) \right) = H^p_{Q_{\partial\Sigma}} \left(\bigoplus_{i=0}^{n} \Omega^{0,i}(X, End(E)) \right) = Ext^p(\mathcal{E}, \mathcal{E}).$$

---

[5]Aspinwall, 2005.

General B-branes in NLSMs

- More generally, a B-brane in an NLSM is a complex of holomorphic vector bundles, with differential Q

$$\mathcal{C} = \cdots \xrightarrow{Q} \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^0 \xrightarrow{Q} \mathcal{E}^1 \xrightarrow{Q} \cdots .$$

General B-branes in NLSMs

- More generally, a B-brane in an NLSM is a complex of holomorphic vector bundles, with differential Q

$$\mathcal{C} = \cdots \xrightarrow{Q} \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^{0} \xrightarrow{Q} \mathcal{E}^{1} \xrightarrow{Q} \cdots.$$

- By the locally free resolution property, this is equivalent to a coherent sheaf on X.

General B-branes in NLSMs

- More generally, a B-brane in an NLSM is a complex of holomorphic vector bundles, with differential Q

$$\mathcal{C} = \cdots \xrightarrow{Q} \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^{0} \xrightarrow{Q} \mathcal{E}^{1} \xrightarrow{Q} \cdots .$$

- By the locally free resolution property, this is equivalent to a coherent sheaf on X.

- The cohomology classes defining the morphisms are given by cochain maps up to quasi-isomorphism

$$H^{p}_{iQ}(\mathcal{H}_{zero}(\mathfrak{B})) = Hom_{\mathbf{D}}(\mathcal{C}, \mathcal{C}[p]).$$

General B-branes in NLSMs

- More generally, a B-brane in an NLSM is a complex of holomorphic vector bundles, with differential Q

$$\mathcal{C} = \cdots \xrightarrow{Q} \mathcal{E}^{-1} \xrightarrow{Q} \mathcal{E}^0 \xrightarrow{Q} \mathcal{E}^1 \xrightarrow{Q} \cdots .$$

- By the locally free resolution property, this is equivalent to a coherent sheaf on X.

- The cohomology classes defining the morphisms are given by cochain maps up to quasi-isomorphism

$$H_{iQ}^p(\mathcal{H}_{zero}(\mathfrak{B})) = Hom_{\mathbf{D}}(\mathcal{C}, \mathcal{C}[p]).$$

- This is the definition of the (bounded) derived category of coherant sheaves on X, $D^b(X)$.

B-branes in LG models

- The data of a B-brane in an LG model[6] is (V, Q), where V is a $\mathbb{Z}_2$-graded vector space and Q is an odd, holomorphic endomorphism of V such that $Q^2 = W\mathbb{1}_V$.

---

[6]Brunner et al., 2006; Kapustin and Li, 2003.
[7]Warner, 1995.

B-branes in LG models

- The data of a B-brane in an LG model[6] is (V, Q), where V is a $\mathbb{Z}_2$-graded vector space and Q is an odd, holomorphic endomorphism of V such that $Q^2 = W\mathbb{1}_V$.

- The construction of a boundary action realising such a Q restores $\mathcal{N} = 2_B$ supersymmetry on $\partial\Sigma$. This is the solution to the so-called Warner problem[7].

---

[6]Brunner et al., 2006; Kapustin and Li, 2003.
[7]Warner, 1995.

B-branes in LG models

- The data of a B-brane in an LG model[6] is (V, Q), where V is a $\mathbb{Z}_2$-graded vector space and Q is an odd, holomorphic endomorphism of V such that $Q^2 = W \mathbb{1}_V$.

- The construction of a boundary action realising such a Q restores $\mathcal{N} = 2_B$ supersymmetry on $\partial\Sigma$. This is the solution to the so-called Warner problem[7].

Matrix factorisations

Writing $Q = \text{antidiag}(f(x), g(x))$, we have

$$Q^2 = W \mathbb{1}_V \iff f(x)g(x) = g(x)f(x) = W(x)\mathbb{1}_V.$$

Such a pair (f, g) is called a matrix factorisation of W.

---

[6]Brunner et al., 2006; Kapustin and Li, 2003.
[7]Warner, 1995.

Example - LG B-branes of Clifford type

- An important class of matrix factorisations can be realised in terms of Clifford algebras. They have the following form

$$Q = \sum_{i=1}^{n} \left(f_i(x)\eta_i + g_i(x)\overline{\eta}_i \right), \; \{\eta_i, \overline{\eta}_j\} = \delta_{ij}.$$

Example - LG B-branes of Clifford type

- An important class of matrix factorisations can be realised in terms of Clifford algebras. They have the following form

$$Q = \sum_{i=1}^{n} \left(f_i(x)\eta_i + g_i(x)\overline{\eta}_i \right), \; \{\eta_i, \overline{\eta}_j\} = \delta_{ij}.$$

- The corresponding Chan-Paton space V is then a highest weight module for the Clifford algebra generated by the action of the $\overline{\eta}_i$s on a vacuum vector $|0\rangle$.

Example - LG B-branes of Clifford type

- An important class of matrix factorisations can be realised in terms of Clifford algebras. They have the following form

$$Q = \sum_{i=1}^{n} \left(f_i(x)\eta_i + g_i(x)\overline{\eta}_i \right), \ \{\eta_i, \overline{\eta}_j\} = \delta_{ij}.$$

- The corresponding Chan-Paton space V is then a highest weight module for the Clifford algebra generated by the action of the $\overline{\eta}_i$s on a vacuum vector $|0\rangle$.

- For example, we always have the so-called canonical matrix factorisation

$$Q = \sum_{i=1}^{n} \left(x_i\eta_i + \frac{1}{d_i}\frac{\partial W}{\partial x_i}\overline{\eta}_i \right).$$

Background
○○○○○○○○○○○

Hybrid B-branes
●○○○○○○○○○

Examples
○○○○○○

Lifting to the GLSM
○○○○○

Hybrid B-branes

- Far less is known about the category of B-branes in hybrid models.

---

[8]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.

[9]Addington, 2009; Kuznetsov, 2005, 2013.

Hybrid B-branes

- Far less is known about the category of B-branes in hybrid models.
- Some results[8] are known for hybrids which can be realised as non-commutative resolutions of branched double covers of $\mathbb{P}^3$. In particular, this includes the hybrid phase of $\mathbb{P}^7[2, 2, 2, 2]$.

---

[8]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[9]Addington, 2009; Kuznetsov, 2005, 2013.

Hybrid B-branes

- Far less is known about the category of B-branes in hybrid models.
- Some results[8] are known for hybrids which can be realised as non-commutative resolutions of branched double covers of $\mathbb{P}^3$. In particular, this includes the hybrid phase of $\mathbb{P}^7[2, 2, 2, 2]$.
- In this case, the hybrid B-branes are known to be described by sheaves of even parts of Clifford algebras[9].

---

[8]Katz and Schimannek, 2023; Katz et al., 2023; Lee, Lian, and Romo, 2023.
[9]Addington, 2009; Kuznetsov, 2005, 2013.

The hybrid Warner problem and B-branes

- We consider an $\mathcal{N} = (2, 2)$ hybrid model (Y, W) defined on a worldsheet of the form

$$\Sigma = \mathbb{R} \times [0, \pi].$$

Background
0000000000

Hybrid B-branes
0●00000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- We consider an $\mathcal{N} = (2,2)$ hybrid model (Y, W) defined on a worldsheet of the form

$$\Sigma = \mathbb{R} \times [0, \pi].$$

- We see that supersymmetry is broken on $\partial\Sigma$. In particular, under an $\mathcal{N} = 2_B$ transformation

$$\delta S_{\mathsf{hyb}} = \frac{1}{2} \int dx^0 \left(i\bar{\epsilon}(\psi_+^\alpha + \psi_-^\alpha)\partial_\alpha W + \mathsf{c.c} \right)\Big|_0^\pi \neq 0,$$

this is the hybrid Warner problem.

Background
0000000000

Hybrid B-branes
0●00000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- We consider an $\mathcal{N} = (2,2)$ hybrid model (Y, W) defined on a worldsheet of the form

$$\Sigma = \mathbb{R} \times [0, \pi].$$

- We see that supersymmetry is broken on $\partial\Sigma$. In particular, under an $\mathcal{N} = 2_B$ transformation

$$\delta S_{\text{hyb}} = \frac{1}{2} \int dx^0 \left(i\bar{\epsilon}(\psi_+^\alpha + \psi_-^\alpha)\partial_\alpha W + \text{c.c} \right)\Big|_0^\pi \neq 0,$$

this is the hybrid Warner problem.

- Hybrid B-branes will correspond to boundary interactions which restore $\mathcal{N} = 2_B$ on $\partial\Sigma$.

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Background
00000000000

Hybrid B-branes
000●0000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

Background
○○○○○○○○○○○

Hybrid B-branes
○○●○○○○○○○

Examples
○○○○○○

Lifting to the GLSM
○○○○○

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

- E is a $\mathbb{Z}_2$-graded vector bundle over Y.

Background
0000000000

Hybrid B-branes
0000000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

- E is a $\mathbb{Z}_2$-graded vector bundle over Y.
- A is a connection with a $(1, 1)$-curvature form.

Background
0000000000

Hybrid B-branes
000000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

- E is a $\mathbb{Z}_2$-graded vector bundle over Y.
- A is a connection with a $(1, 1)$-curvature form.
- Q is an odd, holomorphic endomorphism of E with $Q^2 = W \mathbb{1}_E$.

Background
0000000000

Hybrid B-branes
0●00000000

Examples
000000

Lifting to the GLSM
00000

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

- E is a $\mathbb{Z}_2$-graded vector bundle over Y.
- A is a connection with a $(1, 1)$-curvature form.
- Q is an odd, holomorphic endomorphism of E with $Q^2 = W\mathbb{1}_E$.

- We refer to this structure as a global matrix factorisation, Q over a holomorphic vector bundle $\mathcal{E} = (E, A)$.

The hybrid Warner problem and B-branes

- A general $\mathcal{N} = 2_B$-restoring boundary interaction can be produced using a triple of data $\mathfrak{B} = (E, A, Q)$.

Hybrid B-brane Data

- E is a $\mathbb{Z}_2$-graded vector bundle over Y.
- A is a connection with a $(1, 1)$-curvature form.
- Q is an odd, holomorphic endomorphism of E with $Q^2 = W \mathbb{1}_E$.

- We refer to this structure as a global matrix factorisation, Q over a holomorphic vector bundle $\mathcal{E} = (E, A)$.

- This data corresponds to the following boundary interaction

$$\mathcal{A}_t = \dot{\phi}^\alpha A_\alpha - \frac{i}{4} F_{\alpha\bar{\beta}} \eta^\alpha \bar{\eta}^{\bar{\beta}} - \frac{1}{2} \psi^\alpha D_\alpha Q + \frac{1}{2} Q Q^\dagger + \text{c.c.}$$

Boundary R-symmetry and orbifolding

- We are interested in orbifolded, R-symmetric hybrid models.

Boundary R-symmetry and orbifolding

- We are interested in orbifolded, R-symmetric hybrid models.

Orbifolding hybrid B-branes

For an orbifold group Γ, and a representation $\rho : \Gamma \to GL(E)$, a hybrid brane in an orbifolded model must satisfy

$$\rho(g^{-1})Q(g \cdot x)\rho(g) = Q(x) , \; g \in \Gamma.$$

Boundary R-symmetry and orbifolding

- We are interested in orbifolded, R-symmetric hybrid models.

Orbifolding hybrid B-branes

For an orbifold group Γ, and a representation $\rho : \Gamma \to GL(E)$, a hybrid brane in an orbifolded model must satisfy

$$\rho(g^{-1})Q(g \cdot x)\rho(g) = Q(x) , \ g \in \Gamma.$$

R-symmetry for hybrid B-branes

To ensure boundary R-symmetry, E must admit a $U(1)$-action R_λ such that

$$R_\lambda Q(x^i, \lambda^{q_I}\varphi^I)R_\lambda^{-1} = \lambda Q(x^i, \varphi^I),$$
$$R_\lambda A_\alpha(x^i, \lambda^{q_I}\varphi^I)R_\lambda^{-1} = \lambda^{q_\alpha} A_\alpha(x^i, \varphi^I).$$

Building hybrid B-branes I - Chan-Paton bundles

- To construct explicit examples of hybrid B-branes, we must construct triples $\mathfrak{B} = (E, A, Q)$.

Background
○○○○○○○○○○○

Hybrid B-branes
○○○○●○○○○○

Examples
○○○○○○

Lifting to the GLSM
○○○○○

Building hybrid B-branes I - Chan-Paton bundles

- To construct explicit examples of hybrid B-branes, we must construct triples $\mathfrak{B} = (E, A, Q)$.
- The first thing we need is a source of vector bundles over hybrid geometries. We focus on the case of $B = \mathbb{P}^n$.

Background
0000000000

Hybrid B-branes
0000●00000

Examples
000000

Lifting to the GLSM
00000

Building hybrid B-branes I - Chan-Paton bundles

- To construct explicit examples of hybrid B-branes, we must construct triples $\mathfrak{B} = (E, A, Q)$.

- The first thing we need is a source of vector bundles over hybrid geometries. We focus on the case of $B = \mathbb{P}^n$.

- Vector bundles $\tilde{E} \to B$ can be pulled back along $\pi : Y \to B$ to bundles over Y.

$$
\begin{array}{ccc}
\pi^* \tilde{E} & \longrightarrow & \tilde{E} \\
\downarrow & & \downarrow{\pi_{E'}} \\
Y & \xrightarrow{\ \pi\ } & B
\end{array}
$$

Background
○○○○○○○○○○○

Hybrid B-branes
○○○○●○○○○○

Examples
○○○○○○

Lifting to the GLSM
○○○○○

Building hybrid B-branes I - Chan-Paton bundles

- To construct explicit examples of hybrid B-branes, we must construct triples $\mathfrak{B} = (E, A, Q)$.

- The first thing we need is a source of vector bundles over hybrid geometries. We focus on the case of $B = \mathbb{P}^n$.

- Vector bundles $\tilde{E} \to B$ can be pulled back along $\pi : Y \to B$ to bundles over Y.

$$
\begin{array}{ccc}
\pi^* \tilde{E} & \longrightarrow & \tilde{E} \\
\downarrow & & \downarrow{\scriptstyle \pi_{E'}} \\
Y & \xrightarrow{\ \pi\ } & B
\end{array}
$$

- We then take Chan-Paton bundles of the form

$$E \cong \pi^* \mathcal{O}(k_1) \oplus \pi^* \mathcal{O}(k_2) \oplus \ldots \oplus \pi^* \mathcal{O}(k_{2\ell}), \quad k_i \in \mathbb{Z}.$$

Background
0000000000

Hybrid B-branes
0000000000

Examples
000000

Lifting to the GLSM
00000

Building hybrid B-branes II - Connections

- We start with the Chern connection $\nabla^{(k)} = d + A^{(k)}$ on $\mathcal{O}(k) \to \mathbb{P}^n$. It has a $(1,1)$ curvature form, and is given locally by

$$A_i^{(k)} = \frac{-k(\overline{u}_1 du_1 + \cdots + \overline{u}_n du_n)}{1 + |u_1|^2 + \cdots + |u_n|^2}.$$

Building hybrid B-branes II - Connections

- We start with the Chern connection $\nabla^{(k)} = d + A^{(k)}$ on $\mathcal{O}(k) \to \mathbb{P}^n$. It has a $(1,1)$ curvature form, and is given locally by

$$A_i^{(k)} = \frac{-k(\overline{u}_1 du_1 + \cdots + \overline{u}_n du_n)}{1 + |u_1|^2 + \cdots + |u_n|^2}.$$

- It can then be pulled back to obtain a corresponding connection on $\pi^* \mathcal{O}(k) \to Y$

$$(\pi^* \nabla^{(k)})_X(\pi^* s) = \pi^*(\nabla_{d\pi[X]}^{(k)} s) \, , \, s \in \Gamma(\mathcal{O}(k)).$$

Building hybrid B-branes II - Connections

- We start with the Chern connection $\nabla^{(k)} = d + A^{(k)}$ on $\mathcal{O}(k) \to \mathbb{P}^n$. It has a $(1,1)$ curvature form, and is given locally by

$$A_i^{(k)} = \frac{-k(\overline{u}_1 du_1 + \cdots + \overline{u}_n du_n)}{1 + |u_1|^2 + \cdots + |u_n|^2}.$$

- It can then be pulled back to obtain a corresponding connection on $\pi^* \mathcal{O}(k) \to Y$

$$(\pi^* \nabla^{(k)})_X(\pi^* s) = \pi^*(\nabla_{d\pi[X]}^{(k)} s) \, , \, s \in \Gamma(\mathcal{O}(k)).$$

- For a general sum of line bundles, we then have

$$\pi^* A^{(E)} = \begin{pmatrix} \pi^* A^{(k_1)} & 0 & \cdots & 0 \\ 0 & \pi^* A^{(k_2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \pi^* A^{(k_{2\ell})} \end{pmatrix}.$$

Background
0000000000

Hybrid B-branes
000000●000

Examples
000000

Lifting to the GLSM
00000

Building hybrid B-branes III - Global matrix factorisations

- We now need to construct the global matrix factorisations, Q.
  We do so by patching together local data.

Building hybrid B-branes III - Global matrix factorisations

- We now need to construct the global matrix factorisations, Q. We do so by patching together local data.

Local matrix factorisations

A local matrix factorisation Q_i over a chart $U_i \cong \mathbb{C}^{d+r}$ of Y is an odd, holomorphic endomorphism which squares to the local superpotential, W_i

$$Q_i^2 = W_i \mathbb{1}.$$

Background
00000000000

Hybrid B-branes
0000000●000

Examples
00000

Lifting to the GLSM
00000

Building hybrid B-branes III - Global matrix factorisations

- We now need to construct the global matrix factorisations, Q. We do so by patching together local data.

Local matrix factorisations

A local matrix factorisation Q_i over a chart $U_i \cong \mathbb{C}^{d+r}$ of Y is an odd, holomorphic endomorphism which squares to the local superpotential, W_i

$$Q_i^2 = W_i \mathbb{1}.$$

- Since Q is an odd holomorphic section of the bundle $End(E)$, we obtain a gluing condition for the local matrix factorisations

$$Q_j = \tau_{ij} Q_i \tau_{ij}^{-1},$$

where the τ_{ij} are the transition functions of E.

Building hybrid B-branes III - Global matrix factorisations

- Many explicit examples of hybrid B-branes can be obtained by globalising local matrix factorisations of Clifford type.

Building hybrid B-branes III - Global matrix factorisations

- Many explicit examples of hybrid B-branes can be obtained by globalising local matrix factorisations of Clifford type.
- We start with a series of local Clifford type matrix factorisations on an open cover $\{U_i\}$ of Y

$$Q_i = \sum_{p=1}^{n} \left(f_p^{(i)}(u,x)\eta_p + g_p^{(i)}(u,x)\bar{\eta}_p \right) \text{ , where } \{\eta_p, \bar{\eta}_q\} = \delta_{pq}.$$

Background
0000000000

Hybrid B-branes
0000000000

Examples
000000

Lifting to the GLSM
00000

Building hybrid B-branes III - Global matrix factorisations

- Many explicit examples of hybrid B-branes can be obtained by globalising local matrix factorisations of Clifford type.

- We start with a series of local Clifford type matrix factorisations on an open cover $\{U_i\}$ of Y

$$Q_i = \sum_{p=1}^{n} \left(f_p^{(i)}(u, x)\eta_p + g_p^{(i)}(u, x)\bar{\eta}_p \right) \text{ , where } \{\eta_p, \bar{\eta}_q\} = \delta_{pq}.$$

- To obtain global matrix factorisations, we impose compatibility with the gluing condition and global holomorphicity.

Building hybrid B-branes III - Global matrix factorisations

- We make the ansatz that the Chan-Paton space has the form

$$E = \pi^* \mathcal{O}(k_1) \oplus \cdots \oplus \pi^* \mathcal{O}(k_{2l}).$$

Building hybrid B-branes III - Global matrix factorisations

- We make the ansatz that the Chan-Paton space has the form

$$E = \pi^* \mathcal{O}(k_1) \oplus \cdots \oplus \pi^* \mathcal{O}(k_{2l}).$$

- This tells us that the transition functions are of the form

$$\tau_{ij} = \text{diag}\left(u_i^{k_1}, \cdots, u_i^{k_{2l}} \right).$$

Building hybrid B-branes III - Global matrix factorisations

- We make the ansatz that the Chan-Paton space has the form

$$E = \pi^* \mathcal{O}(k_1) \oplus \cdots \oplus \pi^* \mathcal{O}(k_{2l}).$$

- This tells us that the transition functions are of the form

$$\tau_{ij} = \text{diag}\left(u_i^{k_1}, \cdots, u_i^{k_{2l}}\right).$$

- We construct a set of local matrix factorisations $\{Q_i\}$ on an open cover $\{U_i\}$ of Y.

Building hybrid B-branes III - Global matrix factorisations

- We make the ansatz that the Chan-Paton space has the form

$$E = \pi^* \mathcal{O}(k_1) \oplus \cdots \oplus \pi^* \mathcal{O}(k_{2l}).$$

- This tells us that the transition functions are of the form

$$\tau_{ij} = \text{diag}\left(u_i^{k_1}, \cdots, u_i^{k_{2l}}\right).$$

- We construct a set of local matrix factorisations $\{Q_i\}$ on an open cover $\{U_i\}$ of Y.

- The allowed values of the weights k_i will then be constrained by the compatibility of the set $\{Q_i\}$ with the gluing condition and holomorphicity.

Building hybrid B-branes III - Global matrix factorisations

- The integers k_i can then be regarded as gauge charges for states in the local Clifford module

$$E_i = \text{span}_{\mathbb{C}}\{|0\rangle, \overline{\eta}_i |0\rangle, \cdots, \overline{\eta}_1 \cdots \overline{\eta}_n |0\rangle\}.$$

Building hybrid B-branes III - Global matrix factorisations

- The integers k_i can then be regarded as gauge charges for states in the local Clifford module

$$E_i = \mathrm{span}_{\mathbb{C}}\{|0\rangle, \overline{\eta}_i |0\rangle, \cdots, \overline{\eta}_1 \cdots \overline{\eta}_n |0\rangle\}.$$

- Under rather mild assumptions, the charges corresponding to the Clifford generators are fixed by holomorphicity and the gluing condition.

Building hybrid B-branes III - Global matrix factorisations

- The integers k_i can then be regarded as gauge charges for states in the local Clifford module

$$E_i = \text{span}_{\mathbb{C}}\{|0\rangle, \overline{\eta}_i |0\rangle, \cdots, \overline{\eta}_1 \cdots \overline{\eta}_n |0\rangle\}.$$

- Under rather mild assumptions, the charges corresponding to the Clifford generators are fixed by holomorphicity and the gluing condition.

- The remaining charges are determined by the Clifford algebra structure. This fixes E up to a choice of vacuum charges.

Example 1 - The octic hybrid

- Our first example is the hybrid phase of the octic hypersurface in the toric resolution of $\mathbb{P}^4_{11222}$.

Example 1 - The octic hybrid

- Our first example is the hybrid phase of the octic hypersurface in the toric resolution of $\mathbb{P}^4_{11222}$.

- This model is built upon the following hybrid geometry

$$Y = \text{tot}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-2) \to \mathbb{P}^1).$$

Example 1 - The octic hybrid

- Our first example is the hybrid phase of the octic hypersurface in the toric resolution of $\mathbb{P}^4_{11222}$.

- This model is built upon the following hybrid geometry

$$Y = \text{tot}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-2) \to \mathbb{P}^1).$$

- The local superpotentials on the cover $\{U_u, U_v\}$ can be taken to be

$$W_u = x_{3,u}^4 + x_{4,u}^4 + x_{5,u}^4 + (1 + u^8)x_{6,u}^4,$$
$$W_v = x_{3,v}^4 + x_{4,v}^4 + x_{5,v}^4 + (1 + v^8)x_{6,v}^4.$$

Example 1 - The octic hybrid

- Our first example is the hybrid phase of the octic hypersurface in the toric resolution of $\mathbb{P}^4_{11222}$.

- This model is built upon the following hybrid geometry

$$Y = \text{tot}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-2) \to \mathbb{P}^1).$$

- The local superpotentials on the cover $\{U_u, U_v\}$ can be taken to be

$$W_u = x_{3,u}^4 + x_{4,u}^4 + x_{5,u}^4 + (1 + u^8)x_{6,u}^4,$$
$$W_v = x_{3,v}^4 + x_{4,v}^4 + x_{5,v}^4 + (1 + v^8)x_{6,v}^4.$$

- A basis of hybrid B-branes can be obtained by globalising local matrix factorisations of Clifford type.

Example 1 - The octic hybrid

- The canonical local matrix factorisation of the octic hybrid is

$$Q_u = x_{6,u}\eta_6 + (u^8 + 1)x_{6,u}^3\bar{\eta}_6 + \sum_{a=3}^{5} \left(x_{a,u}\eta_a + x_{a,u}^3\bar{\eta}_a \right).$$

Example 1 - The octic hybrid

- The canonical local matrix factorisation of the octic hybrid is

$$Q_u = x_{6,u}\eta_6 + (u^8 + 1)x_{6,u}^3\bar{\eta}_6 + \sum_{a=3}^{5} \left(x_{a,u}\eta_a + x_{a,u}^3\bar{\eta}_a \right).$$

- We construct a local Chan-Paton space as a highest weight module for the corresponding Clifford algebra

$$E_u = \text{Span}\{|0\rangle, \bar{\eta}_i |0\rangle, \cdots, \bar{\eta}_3 \cdots \bar{\eta}_6 |0\rangle \}.$$

Example 1 - The octic hybrid

- The canonical local matrix factorisation of the octic hybrid is

$$Q_u = x_{6,u}\eta_6 + (u^8 + 1)x_{6,u}^3\bar{\eta}_6 + \sum_{a=3}^{5}\left(x_{a,u}\eta_a + x_{a,u}^3\bar{\eta}_a\right).$$

- We construct a local Chan-Paton space as a highest weight module for the corresponding Clifford algebra

$$E_u = \text{Span}\{|0\rangle, \bar{\eta}_i |0\rangle, \cdots, \bar{\eta}_3 \cdots \bar{\eta}_6 |0\rangle \}.$$

- This module can be depicted as a twisted complex

$$\begin{array}{ccccccc}
 & \bar{\eta}_6 \oplus_{a=3}^{5} \bar{\eta}_3\hat{\bar{\eta}}_a\bar{\eta}_5 |0\rangle & & & & \bar{\eta}_6 |0\rangle & \\
\bar{\eta}_3 \cdots \bar{\eta}_6 |0\rangle & \underset{\longleftarrow}{\overset{\longrightarrow}{}} \oplus \underset{\longleftarrow}{\overset{\longrightarrow}{}} & \cdots & \underset{\longleftarrow}{\overset{\longrightarrow}{}} \oplus \underset{\longleftarrow}{\overset{\longrightarrow}{}} |0\rangle . \\
 & \bar{\eta}_3\bar{\eta}_4\bar{\eta}_5 |0\rangle & & & & \oplus_{a=3}^{5}\bar{\eta}_a |0\rangle & \\
\end{array}$$

Example 1 - The octic hybrid

- Holomorphicity and compatibility with the gluing conditions uniquely fix the gauge charges of the Clifford generators

$$a_6 = 2 \ , \ a_3 = a_4 = a_5 = 0.$$

Example 1 - The octic hybrid

- Holomorphicity and compatibility with the gluing conditions uniquely fix the gauge charges of the Clifford generators

$$a_6 = 2 \ , \ a_3 = a_4 = a_5 = 0.$$

- The Clifford algebra structure then fixes the gauge charges of the remaining states

$$\overline{\eta}_{i_1} \cdots \overline{\eta}_{i_p} |0\rangle \leftrightarrow \left(a_{i_1} + \cdots + a_{i_p} \right) + a_0.$$

Example 1 - The octic hybrid

- Holomorphicity and compatibility with the gluing conditions uniquely fix the gauge charges of the Clifford generators

$$a_6 = 2 \ , \ a_3 = a_4 = a_5 = 0.$$

- The Clifford algebra structure then fixes the gauge charges of the remaining states

$$\overline{\eta}_{i_1} \cdots \overline{\eta}_{i_p} \, |0\rangle \leftrightarrow \left(a_{i_1} + \cdots + a_{i_p}\right) + a_0.$$

- Similar considerations also yield the R and orbifold charges of the Clifford states.

Example 1 - The octic hybrid

- The only remaining degrees of freedom are the charges of the vacuum state $|0\rangle$. We denote them by a_0, r_0, and α_0 for the vacuum gauge, R and orbifold charges respectively.

Example 1 - The octic hybrid

- The only remaining degrees of freedom are the charges of the vacuum state $|0\rangle$. We denote them by a_0, r_0, and α_0 for the vacuum gauge, R and orbifold charges respectively.

- We thus obtain a family of hybrid B-branes $\mathfrak{B}^{(a_0, r_0, \alpha_0)} = \left(E^{(a_0)}, A^{(a_0)}, Q, \rho^{(\alpha_0)}, R^{(r_0)}\right)$ represented by the following twisted complex

$$\mathcal{O}(a_0+2)^{\oplus 3}_{r_0-\frac{3}{2}, \alpha_0-3} \qquad \mathcal{O}(a_0+2)_{r_0-\frac{1}{2}, \alpha_0-1}$$

$$\mathcal{O}(a_0+2)_{r_0-2, \alpha_0-4} \xleftarrow{\qquad} \oplus \xleftarrow{\qquad} \dots \xleftarrow{\qquad} \oplus \xleftarrow{\qquad} \mathcal{O}(a_0)_{r_0, \alpha_0} \cdot$$

$$\mathcal{O}(a_0)_{r_0-\frac{3}{2}, \alpha_0-3} \qquad \mathcal{O}(a_0)^{\oplus 3}_{r_0-\frac{1}{2}, \alpha_0-1}$$

Example 2 - The bicubic hybrid

- Our second example is the hybrid phase of $\mathbb{P}^5[3,3]$. This model is built upon the following hybrid geometry

$$
Y = \mathrm{tot}\left(\mathcal{O}\left(-\frac{1}{3}\right)^{\oplus 6} \to \mathbb{P}^1 \right).
$$

Example 2 - The bicubic hybrid

- Our second example is the hybrid phase of $\mathbb{P}^5[3,3]$. This model is built upon the following hybrid geometry

$$Y = \text{tot}\left(\mathcal{O}\left(-\frac{1}{3}\right)^{\oplus 6} \to \mathbb{P}^1\right).$$

- The associated local superpotentials on $\{U_u, U_v\}$ are given by

$$W_u = G^1_{3,u} + u G^2_{3,u} \text{ , and } W_v = v G^1_{3,v} + G^2_{3,v},$$

where the $G^i_{3,u}$ and $G^i_{3,v}$ are generic cubic polynomials in the corresponding fibre fields.

Example 2 - The bicubic hybrid

- Our second example is the hybrid phase of $\mathbb{P}^5[3,3]$. This model is built upon the following hybrid geometry

$$Y = \text{tot}\left(\mathcal{O}\left(-\frac{1}{3}\right)^{\oplus 6} \to \mathbb{P}^1\right).$$

- The associated local superpotentials on $\{U_u, U_v\}$ are given by

$$W_u = G_{3,u}^1 + u G_{3,u}^2 \text{ , and } W_v = v G_{3,v}^1 + G_{3,v}^2,$$

where the $G_{3,u}^i$ and $G_{3,v}^i$ are generic cubic polynomials in the corresponding fibre fields.

Remark

The potential condition is satisfied exactly when the corresponding Calabi-Yau is smooth.

Example 2 - The bicubic hybrid

- An interesting hybrid B-brane can be constructed by intersecting the canonical matrix factorisation with a linear divisor on the base

$$Q_u = \sum_{i=1}^{6} \left(x_{i,u} \eta_i + \frac{1}{3} \frac{\partial W_u}{\partial x_{i,u}} \overline{\eta}_i \right) + (\alpha + \beta u) \overline{\eta}_7 \ , \ \alpha, \beta \in \mathbb{C}.$$

Example 2 - The bicubic hybrid

- An interesting hybrid B-brane can be constructed by intersecting the canonical matrix factorisation with a linear divisor on the base

$$Q_u = \sum_{i=1}^{6} \left(x_{i,u}\eta_i + \frac{1}{3}\frac{\partial W_u}{\partial x_{i,u}}\overline{\eta}_i \right) + (\alpha + \beta u)\overline{\eta}_7 \ , \ \alpha, \beta \in \mathbb{C}.$$

- Holomorphicity and the gluing condition no longer uniquely fix the gauge charges of the Clifford generators

$$a_1 = \cdots = a_6 = \frac{1}{3} \ , \ a_7 \in \frac{1}{3}\mathbb{Z}_{\geq 3}.$$

Example 2 - The bicubic hybrid

- An interesting hybrid B-brane can be constructed by intersecting the canonical matrix factorisation with a linear divisor on the base

$$Q_u = \sum_{i=1}^{6} \left(x_{i,u}\eta_i + \frac{1}{3}\frac{\partial W_u}{\partial x_{i,u}}\overline{\eta}_i \right) + (\alpha + \beta u)\overline{\eta}_7 \ , \ \alpha, \beta \in \mathbb{C}.$$

- Holomorphicity and the gluing condition no longer uniquely fix the gauge charges of the Clifford generators

$$a_1 = \cdots = a_6 = \frac{1}{3} \ , \ a_7 \in \frac{1}{3}\mathbb{Z}_{\geq 3}.$$

- Using these charges to transform to the other chart, we see that this ambiguity can be interpreted as the ability to stack $n = 3(a_7 - 1) \in \mathbb{Z}_{\geq 0}$ D0-branes over the north pole of $B = \mathbb{P}^1$

$$Q_v = \sum_{i=1}^{6} \left(x_{i,v}\eta_i + \frac{1}{3}\frac{\partial W_v}{\partial x_{i,v}}\overline{\eta}_i \right) + v^n(\alpha v + \beta)\overline{\eta}_7.$$

Lifting to the GLSM

- Many hybrid models arise as phases of gauged linear sigma models.

---

[10]Herbst, Hori, and Page, 2008.

Lifting to the GLSM

- Many hybrid models arise as phases of gauged linear sigma models.

- A hybrid B-brane $\mathfrak{B} = (E, A, Q, \rho, R)$ defined on such a hybrid can be lifted to a GLSM brane $\mathcal{B} = (M, Q, \rho_M, r_*)$.

---

[10]Herbst, Hori, and Page, 2008.

Lifting to the GLSM

- Many hybrid models arise as phases of gauged linear sigma models.
- A hybrid B-brane $\mathfrak{B} = (E, A, Q, \rho, R)$ defined on such a hybrid can be lifted to a GLSM brane $\mathcal{B} = (\mathsf{M}, \mathsf{Q}, \rho_\mathsf{M}, \mathsf{r}_*)$.
- We denote this lift via a projection from the category of GLSM B-branes to the category of hybrid B-branes

$$\pi^{hyb}(\mathcal{B}) = \mathfrak{B}.$$

<hr>

[10]Herbst, Hori, and Page, 2008.

Lifting to the GLSM

- Many hybrid models arise as phases of gauged linear sigma models.

- A hybrid B-brane $\mathfrak{B} = (E, A, Q, \rho, R)$ defined on such a hybrid can be lifted to a GLSM brane $\mathcal{B} = (\mathsf{M}, \mathsf{Q}, \rho_{\mathsf{M}}, r_*)$.

- We denote this lift via a projection from the category of GLSM B-branes to the category of hybrid B-branes

$$\pi^{hyb}(\mathcal{B}) = \mathfrak{B}.$$

- These lifts can be used to identify hybrid B-branes as analytic continuations of familiar geometric B-branes[10].

---

[10]Herbst, Hori, and Page, 2008.

Robert Pryor (The University of Melbourne)   B-branes in $\mathcal{N} = (2, 2)$ Hybrid Models

Lifting - The octic

- A possible lift of our octic global matrix factorisation is

$$Q = \sum_{i=3}^{6} \left(x_i \eta_i + \frac{1}{4} \frac{\partial W}{\partial x_i} \bar{\eta}_i \right).$$

Lifting - The octic

- A possible lift of our octic global matrix factorisation is

$$Q = \sum_{i=3}^{6} \left(x_i \eta_i + \frac{1}{4} \frac{\partial W}{\partial x_i} \bar{\eta}_i \right).$$

- A corresponding complex of Wilson line branes is given by

$$
\begin{array}{ccccc}
 & \mathcal{W}(1,0)^{\oplus 3}_{\frac{1}{2}} & & \mathcal{W}(3,0)_{\frac{3}{2}} & \\
\mathcal{W}(0,0)_0 \underset{\longleftarrow}{\overset{\longrightarrow}{}} \oplus \underset{\longleftarrow}{\overset{\longrightarrow}{}} \cdots \underset{\longleftarrow}{\overset{\longrightarrow}{}} \oplus \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathcal{W}(4,-2)_2 & & \\
 & \mathcal{W}(1,-2)_{\frac{1}{2}} & & \mathcal{W}(3,-2)^{\oplus 3}_{\frac{3}{2}} &
\end{array} \cdot
$$

Lifting - The octic

- A possible lift of our octic global matrix factorisation is

$$
Q = \sum_{i=3}^{6} \left(x_i \eta_i + \frac{1}{4} \frac{\partial W}{\partial x_i} \overline{\eta}_i \right).
$$

- A corresponding complex of Wilson line branes is given by

$$
\mathcal{W}(0,0)_0 \; \underset{\longleftarrow}{\overset{\longrightarrow}{}} \; \begin{matrix} \mathcal{W}(1,0)_{\frac{1}{2}}^{\oplus 3} \\ \oplus \\ \mathcal{W}(1,-2)_{\frac{1}{2}} \end{matrix} \; \underset{\longleftarrow}{\overset{\longrightarrow}{}} \cdots \underset{\longleftarrow}{\overset{\longrightarrow}{}} \; \begin{matrix} \mathcal{W}(3,0)_{\frac{3}{2}} \\ \oplus \\ \mathcal{W}(3,-2)_{\frac{3}{2}}^{\oplus 3} \end{matrix} \; \underset{\longleftarrow}{\overset{\longrightarrow}{}} \; \mathcal{W}(4,-2)_2 \; .
$$

- The first $U(1)$ factor corresponds to the hybrid orbifold charge, and the second to the hybrid gauge charge.

Lifting - The octic

- A possible lift of our octic global matrix factorisation is

$$
Q = \sum_{i=3}^{6} \left(x_i \eta_i + \frac{1}{4} \frac{\partial W}{\partial x_i} \bar{\eta}_i \right).
$$

- A corresponding complex of Wilson line branes is given by

$$
\mathcal{W}(0,0)_0 \xrightleftharpoons \begin{array}{c} \mathcal{W}(1,0)_{\frac{1}{2}}^{\oplus 3} \\ \oplus \\ \mathcal{W}(1,-2)_{\frac{1}{2}} \end{array} \xrightleftharpoons \cdots \xrightleftharpoons \begin{array}{c} \mathcal{W}(3,0)_{\frac{3}{2}} \\ \oplus \\ \mathcal{W}(3,-2)_{\frac{3}{2}}^{\oplus 3} \end{array} \xrightleftharpoons \mathcal{W}(4,-2)_2 \cdot
$$

- The first $U(1)$ factor corresponds to the hybrid orbifold charge, and the second to the hybrid gauge charge.

- We thus see that this brane corresponds to the analytic continuation of the structure sheaf of the large volume Calabi-Yau to the hybrid phase.

Lifting - The bicubic

- A GLSM-lift of our bicubic global matrix factorisation is given by

$$Q = \sum_{i=1}^{6} \left(x_i \eta_i + \frac{1}{3} \frac{\partial W}{\partial x_i} \overline{\eta}_i \right) + p_1^n (\alpha p_1 + \beta p_2) \overline{\eta}_7.$$

Lifting - The bicubic

- A GLSM-lift of our bicubic global matrix factorisation is given by

$$
Q = \sum_{i=1}^{6} \left(x_i \eta_i + \frac{1}{3} \frac{\partial W}{\partial x_i} \overline{\eta}_i \right) + p_1^n (\alpha p_1 + \beta p_2) \overline{\eta}_7.
$$

- A corresponding complex of Wilson line branes is given by

$$
\mathcal{W}(n)_0 \underset{\longleftarrow}{\overset{\longrightarrow}{}} \oplus \underset{\longleftarrow}{\overset{\mathcal{W}(1-n)_{\frac{1}{3}}^{\oplus 6}}{\overset{\longrightarrow}{}}} \cdots \underset{\longleftarrow}{\overset{\mathcal{W}(2-n)_{\frac{2}{3}}^{\oplus 15}}{\overset{\longrightarrow}{}}} \oplus \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathcal{W}(9)_1 \cdot
$$

$$
\mathcal{W}(3)_{-1} \qquad\qquad \mathcal{W}(4)_{-\frac{2}{3}}^{\oplus 6}
$$

Lifting - The bicubic

- A GLSM-lift of our bicubic global matrix factorisation is given by

$$Q = \sum_{i=1}^{6} \left(x_i \eta_i + \frac{1}{3} \frac{\partial W}{\partial x_i} \overline{\eta}_i \right) + p_1^n (\alpha p_1 + \beta p_2) \overline{\eta}_7.$$

- A corresponding complex of Wilson line branes is given by

$$
\begin{array}{ccc}
& \mathcal{W}(1-n)^{\oplus 6}_{\frac{1}{3}} & \mathcal{W}(2-n)^{\oplus 15}_{\frac{2}{3}} \\
\mathcal{W}(n)_0 \xrightarrow{\;\;\;\longleftarrow\;\;\;} \oplus \xrightarrow{\;\;\;\longleftarrow\;\;\;} \cdots \xrightarrow{\;\;\;\longleftarrow\;\;\;} \oplus \xrightarrow{\;\;\;\longleftarrow\;\;\;} \mathcal{W}(9)_1 \\
& \mathcal{W}(3)_{-1} & \mathcal{W}(4)^{\oplus 6}_{-\frac{2}{3}}
\end{array}
$$

- This confirms that our hybrid B-brane corresponds to a $D0$ brane at $u = -\frac{\alpha}{\beta}$ along with n D0-branes stacked at the north pole of the base $\mathbb{P}^1$.

Summary of results

- We solved the Warner problem for hybrid models, thus obtaining a general formulation of the data defining a hybrid B-brane.

Summary of results

- We solved the Warner problem for hybrid models, thus obtaining a general formulation of the data defining a hybrid B-brane.

- We developed a method for constructing B-branes in general hybrid models by globalising local matrix factorisations.

Summary of results

- We solved the Warner problem for hybrid models, thus obtaining a general formulation of the data defining a hybrid B-brane.
- We developed a method for constructing B-branes in general hybrid models by globalising local matrix factorisations.
- Globalising Clifford-type local matrix factorisations allowed us to obtain bases of hybrid B-branes in several examples.

Summary of results

- We solved the Warner problem for hybrid models, thus obtaining a general formulation of the data defining a hybrid B-brane.

- We developed a method for constructing B-branes in general hybrid models by globalising local matrix factorisations.

- Globalising Clifford-type local matrix factorisations allowed us to obtain bases of hybrid B-branes in several examples.

- By lifting these branes to the GLSM and utilising D-brane transport, we were able to identify them as analytic continuations of particular large volume branes.

Some further directions

- More examples - non projective bases, more general notions of hybrid.

Some further directions

- More examples - non projective bases, more general notions of hybrid.
- Computing the boundary cohomology to find the morphisms in the hybrid B-brane category.

Some further directions

- More examples - non projective bases, more general notions of hybrid.
- Computing the boundary cohomology to find the morphisms in the hybrid B-brane category.
- Applying supersymmetric localisation to study bulk-boundary correlators.

Background
00000000000

Hybrid B-branes
0000000000

Examples
000000

Lifting to the GLSM
00000

Some further directions

- More examples - non projective bases, more general notions of hybrid.
- Computing the boundary cohomology to find the morphisms in the hybrid B-brane category.
- Applying supersymmetric localisation to study bulk-boundary correlators.
- Studying the categorical structure of hybrid B-branes - A_∞ structures.

Some further directions

- More examples - non projective bases, more general notions of hybrid.
- Computing the boundary cohomology to find the morphisms in the hybrid B-brane category.
- Applying supersymmetric localisation to study bulk-boundary correlators.
- Studying the categorical structure of hybrid B-branes - A_∞ structures.
- Making a connection with the sheaf of algebras language used in the known special case.