Modularity of Calabi-Yau Fourfolds and Applications to Physics

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Outline

1. Motivation: Flux Compactifications in String Theory
2. Modularity of Calabi-Yau Manifolds
3. Modular Calabi-Yau Fourfolds and M-theory Fluxes
4. Example of a modular Calabi-Yau Fourfold
5. Conclusions

Motivation: Why should we study String Theory?

String theory is a candidate for a Grand unification theory as it

- provides a UV complete quantum theory
- contains naturally gravity and gauge interactions
- may lead to standard model physics in the low energy limit

String theory provides a large landscape of low energy EFTs

- framework to study consistent low energy limits of quantum gravity

Distinct string theories are connected via duality transformations

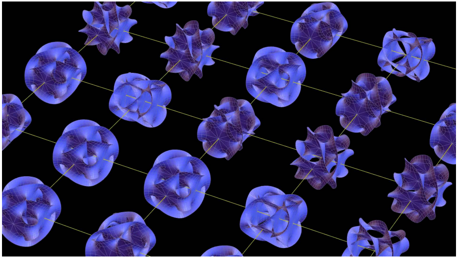
- suggests the existence of an unique underlying theory

Motivation: String Compactifications

Anomaly-free string theory only for 10 spacetime dimensions

Obtain 4d effective field theory by decomposing

$$M^{10} = \mathbb{R}^{1,3} \times X$$



Effective physics on $\mathbb{R}^{1,3}$ is determined by the geometry of X

- Spectrum of the EFT $\Leftrightarrow$ Eigenmodes of Δ on X
- Massless states of the EFT $\Leftrightarrow$ Harmonic modes on X $\Leftrightarrow$ $H^k(X, \mathbb{C})$
- (Quantum-corrected) couplings $\Leftrightarrow$ Moduli space geometry

Motivation: Calabi-Yau Manifolds

Internal spaces of particular interest: Calabi-Yau manifolds

- provide a vacuum solution in the absence of any field content
- restore supersymmetry after compactification

Definition: X n-dimensional Calabi-Yau manifold (CY n-fold) if

- X Kähler manifold of complex dimension n
- X is Ricci-flat, i.e. $R(X) = 0$

Moduli Space: Parameter space consisting of

- $h^{n-1,1}$ complex structure moduli $z_i \Rightarrow$ Holomorphic n-form
$$\Omega = \Omega(z_i) \in H^{n,0}(X, \mathbb{C})$$
- $h^{1,1}$ Kähler moduli $t^i \Rightarrow$ Kähler form
$$J = J(t^i) \in H^{1,1}(X, \mathbb{C})$$

Motivation: Moduli Stabilization and Fluxes

Low energy EFT of string compactification: supergravity

Moduli (z_i, t^i) $\Leftrightarrow$ massless scalar fields

- Phenomenologically: no massless scalar fields observed
- Fundamental: Moduli tune shape and size of internal space

For realistic (or semi-realistic) string models: Moduli Stabilization

Possible Mechanism: Flux Compactifications [Gukov, Vafa, Witten, 2005]

$$W = \int_X \Omega(z_i) \wedge G$$

for $G \in H^n(X, \mathbb{Z})$ topological n-form flux

- For type IIB string theory compactified on CY 3-fold: $G = F - \tau H$ for $F, H \in H^3(X, \mathbb{Z})$ and axio-dilaton τ

Motivation: Supersymmetric Flux Vacua

Supersymmetric vacuum constraints:

$$\partial_{z_i} W = 0 \quad , \quad \partial_\tau W = 0 \quad , \quad W = 0$$

implying

$$\int_X \Omega(z_i) \wedge F = 0 \quad , \quad \int_X \Omega(z_i) \wedge H = 0$$

and

$$\int_X \partial_{z_i} \Omega(z_i) \wedge (F - \tau H) = 0$$

Recall $\Omega(z_i) \in H^{3,0}(X, \mathbb{C})$. Thus,

$$F, H \in \left(H^{2,1}(X, \mathbb{C}) \oplus H^{1,2}(X, \mathbb{C}) \right) \cap H^3(X, \mathbb{Z})$$

Question: Under which conditions does X admit non-trivial fluxes F, H?

$\Rightarrow$ Address this question via arithmetic geometry

[Kachru, Nally, Yang, 2020], [Candelas, De la Ossa, Kuusela, McGovern, 2023]

Modularity: Varieties over Finite Fields

X variety over $\mathbb{C}$:

$$X = \{f_i(x) = 0\} \subset \mathbb{A}^n \text{ or } \mathbb{P}^{n-1}$$

for $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$

Treat now X as variety over the finite field $\mathbb{F}_{p^r}$ with p^r elements

$$X/\mathbb{F}_{p^r} := \{\bar{f}_i(x) = 0\}$$

for $\bar{f}_i \in \mathbb{F}_{p^r}[x_1, \ldots, x_n]$ the canonical projection

(Finite) Number of points:

$$N_{p^r}(X) := |X/\mathbb{F}_{p^r}|$$

Modularity: Elliptic Curves

Elliptic Curve $\mathcal{E}$: defined by cubic equation $y^2 = x^3 + ax + b$

$$\mathcal{E} = \{y^2 z = x^3 + axz^2 + bz^3\} \subset \mathbb{P}^2 \quad 4a^3 + 27b^2 \neq 0$$

Consider the coefficients $a_p = p + 1 - N_p(\mathcal{E})$, then for $q = e^{2\pi i \tau}$

$$f(\tau) = \sum_{p \text{ prime}} a_p q^p \text{ is a weight-two modular form}$$

Note: The same a_p determine

$$R_1^p(\mathcal{E}, T) = \det(\mathbb{1} - \mathrm{Fr}_p^{-1}) = 1 - a_p T + pT^2$$

Frobenius endomorphism

$$\mathrm{Fr}_p : H^1(X, \mathbb{Q}_p) \to H^1(X, \mathbb{Q}_p)$$

$H^1(X, \mathbb{Q}_p)$ suitable p-adic cohomology group

Modularity: Serre's Modularity Conjecture

For a general algebraic variety X:

Serre's Modularity Conjecture: [Serre, 1975]

One-to-one correspondence between two-dimensional representations $\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{p^r})$ and modular forms that are Hecke eigenforms.

$$\mathrm{Gal}(L/K) = \{\phi \in \mathrm{Aut}(L) \mid \phi(K) = K\}$$

$\mathrm{Fr}_p\colon H^k(X, \mathbb{Q}_p) \to H^k(X, \mathbb{Q}_p)$: b^k-dimensional reps. of $\mathrm{fr}_p \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

- For elliptic curves: $b^1 = H^1(\mathcal{E}, \mathbb{Q}) = 2$. Thus, modularity of $\mathcal{E}$ is a special case of this conjecture.
- If $H^k(X, \mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p$ such that $\mathrm{Fr}_p(\Lambda_p) \subseteq \Lambda_p$:
  $\mathrm{Fr}_p|_{\Lambda_p}\colon \Lambda_p \to \Lambda_p$ defines a $\dim(\Lambda_p)$-dimensional (sub-)rep. of fr_p

Modularity: The Zeta-Function

Define local zeta function

$$\zeta_p(X,T) := \exp\left(\sum_{r=1}^{\infty} N_{p^r}(X)\frac{T^r}{r}\right)$$

as generating function for the $N_{p^r}(X)$

Weil Conjectures: [Weil, 1949]

$$\zeta_p(X,T) = \frac{R_1^p(X,T)\cdots R_{2n-1}^p(X,T)}{R_0^p(X,T)\cdots R_{2n}^p(X,T)}$$

$R_k^p(X,T)$ are polynomials of degree $b^k := \dim(H^k(X,\mathbb{Q})$.

More precisely: $R_k^p(X,T) = \det(\mathbb{1} - T\mathrm{Fr}_p^{-1})$ characteristic polynomial

If $H^k(X,\mathbb{Q}_p) = \Lambda \oplus \Sigma$ defines a sub-representation, then

$$R_k^p(X,T) = R_\Lambda^p(X,T)\cdot R_\Sigma^p(X,T)$$

Modularity: Modular Calabi-Yau manifolds

Definition: A CY n-fold X is *modular* if

$$H^n(X,\mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p \quad , \quad \mathrm{Fr}_p(\Lambda_p) \subseteq \Lambda_p \quad , \quad \dim(\Lambda) = 2$$

for almost all primes p.

- If X is modular, $R^p_\Lambda(X,T) = 1 - a_p p^\alpha T + p^\beta T^2$ for some $\alpha, \beta \in \mathbb{N}_0$.

  The a_p determine the corresponding Hecke eigenform via

$$f(\tau) = \sum_{p \text{ prime}} a_p q^p \qquad q = e^{2\pi i \tau}$$

- If $H^n(X,\mathbb{Q}) = \Lambda \oplus \Sigma$, then $H^n(X,\mathbb{Q}_p) = \Lambda_p \oplus \Sigma_p$ for almost all primes p.

  If moreover, $\Lambda \oplus \Sigma$ defines a Hodge substructure of $H^n(X,\mathbb{Q})$, then $\mathrm{Fr}_p(\Lambda_p) \subseteq \Lambda_p$. I.e.

$$\Lambda \otimes \mathbb{C} = \bigoplus \Lambda^{p,q} \quad , \quad \Lambda^{q,p} = \overline{\Lambda^{p,q}} \quad , \quad \Lambda^{p,q} \subseteq H^{p,q}(X,\mathbb{C})$$

Modularity: Relation to Flux Vacua

Recall for type IIB string fluxes on a CY threefold X:

$$F, H \in H^3(X, \mathbb{Z}) \cap \left(H^{2,1}(X, \mathbb{C}) \oplus H^{1,2}(X, \mathbb{C})\right)$$

for a supersymmetric flux vacuum. Hence

$$\Lambda := \langle F, H \rangle_{\mathbb{Q}} \subseteq H^3(X, \mathbb{Q})$$

defines a two-dimensional sub-representation

Necessary condition for supersymmetric flux vacua:

X has to be a modular Calabi-Yau threefold!

Note: For modular CY 3-folds with $h^{2,1} = 1$:

- $\int_X \partial_z \Omega(z) \wedge (F - \tau H) = 0$ fixes the axio-dilaton τ
- The modular form f_Λ determines τ

[Kachru, Nally, Yang, 2020], [Candelas, De la Ossa, Kuusela, McGovern, 2023]

Modular Fourfolds: Why Fourfolds?

So far: 10-dimensional string theory compactified on a CY 3-fold.

More general setup: M-theory on 11 spacetime dimensions

- Compactification on CY 4-fold X gives 3-dimensional EFT
- Similarly, moduli are massless scalar fields $\Rightarrow$ stabilize via fluxes
- Flux Superpotential:

$$W = \int_X \Omega(z_i) \wedge G$$

  for $G \in H^4(X, \mathbb{Z})$ topological four-form flux
- Supersymmetric flux vacua only if

$$G \in H^4(X, \mathbb{Z}) \cap \left(H^{4,0}(X, \mathbb{C}) \cap H^{2,2}(X, \mathbb{C}) \cap H^{0,4}(X, \mathbb{C}) \right)$$

Similar result for F-theory (12 spacetime dimensions) compactifications

Modular Fourfolds: Possibilities for Modularity

Two different choices of two-dimensional sub-reps. $\Lambda \subseteq H^4(X, \mathbb{Q})$:

"Attractor points":

$$\Lambda \subseteq H^{4,0}(X,\mathbb{C}) \cap H^{2,2}(X,\mathbb{C}) \cap H^{0,4}(X,\mathbb{C})$$

Mimic the behaviour of rank-two attractor points of Calabi-Yau threefolds

"Attractive K3-points":

$$\Lambda \subseteq H^{3,1}(X,\mathbb{C}) \cap H^{1,3}(X,\mathbb{C})$$

Geometric origin from an attractive K3 surface

Sufficient condition for M-theory flux vacua:

$$X \text{ attractor point} \Rightarrow \text{any } G \in \Lambda \text{ is a suitable M-theory flux}$$

In contrast to the threefold case:

- Modularity is not a necessary condition

Modular Fourfolds: The Hodge Structure of Fourfolds

Threefolds vs. Fourfolds: The Hodge diamond of a Calabi-Yau threefold

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

- horizontal and vertical Hodge structure are separated

$$\zeta_p(X,T) = \frac{R_3^p(X,T)}{(1-T)(1-pT)^{h^{1,1}}(1-p^2T)^{h^{1,1}}(1-p^3T)}$$

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$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

- $H^{2,2}(X,\mathbb{C}) = H_h^{2,2}(X,\mathbb{C}) \oplus H_v^{2,2}(X,\mathbb{C}) \oplus H_\perp^{2,2}(X,\mathbb{C})$

$$\zeta_p(X,T) = \frac{R_3^p(X,T)R_5^p(X,T)}{(1-T)(1-pT)^{h^{1,1}}R_4^p(X,T)(1-p^3T)^{h^{1,1}}(1-p^4T)}$$

Modular Fourfolds: The Hodge Structure of Fourfolds

Threefolds vs. Fourfolds: The Hodge diamond of a Calabi-Yau fourfold

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

- $H^{2,2}(X,\mathbb{C}) = H_h^{2,2}(X,\mathbb{C}) \oplus H_v^{2,2}(X,\mathbb{C}) \oplus H_\perp^{2,2}(X,\mathbb{C})$

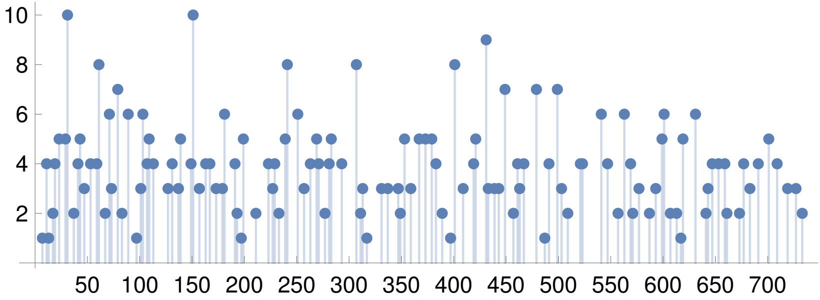
$$\zeta_p(X,T) = \frac{R_3^p(X,T)R_5^p(X,T)}{(1-T)(1-pT)^{h^{1,1}} R_v^p(X,T) R_\perp^p(X,T)(1-p^3T)^{h^{1,1}}(1-p^4T)}$$

Modular Fourfolds: A Modular Example

A one-parameter family of Hulek-Verrill fourfolds

$$\mathrm{HV}_z^4 = \left\{ \left(\frac{z}{x_1} + \cdots + \frac{z}{x_6} \right) (x_1 + \cdots + x_6) = 1 \right\} \subset \mathbb{T}^6$$

Histogram of points $z_p \in \mathbb{F}_p$ s.t. $R_v^p(\mathrm{HV}_{z_p}^4, T)$ factorizes quadratically



Here: $7 \leq p \leq 733$

Modular Fourfolds: A Modular Example

Reconstruction of the modular point $z \in \bar{\mathbb{Q}}$:

$$z_p \equiv z \mod p$$

s.t. $R_v^p(\mathrm{HV}^4_{z_p}, T)$ factorizes for (almost) all primes p

Collection of points $z_p \in \mathbb{F}_p$ with quadratic factorization

prime p	$z_p \in \mathbb{F}_p$			
$p = 11$	1	6	8	10
$p = 13$	1			
$p = 17$	1	15		

prime p	$z_p \in \mathbb{F}_p$			
$p = 19$	1	2	7	17
$p = 23$	1	4	5	12
...				

(Rational) solution $z \in \mathbb{Q}$ s.t. HV^4_z is modular (and smooth): $z = 1$

- Tested for all primes $11 \leq p \leq 733$
- No additional modular point found

Modular Fourfolds: A Modular Example

Consistency checks:

- Coefficients a_p of quadratic factor

$$R^p_\Lambda(\mathrm{HV}^4_1, T) = 1 - a_p p T + p^2 T^2$$

  give q-expansion of a unique Hecke eigenform
- Identified generators of the two-dimensional sublattice

$$\Lambda = \left[H^{3,1}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{1,3}(\mathrm{HV}^4_1, \mathbb{C})\right] \cap H^4(\mathrm{HV}^4_1, \mathbb{Z})$$

How about fluxes?

- Attractive K3-point! $\Rightarrow \Lambda$ does **not** provide suitable fluxes
- As $h^{3,1} = 1$ we have:

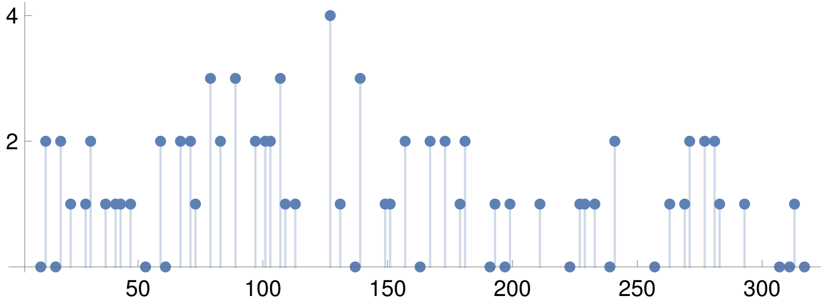
$$\Sigma = \left[H^{4,0}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{2,2}_h(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{0,4}(\mathrm{HV}^4_1, \mathbb{C})\right] \cap H^4(\mathrm{HV}^4_1, \mathbb{Z})$$

- In particular: $G := C \cdot \mathrm{Re}(\Omega(z = 1)) \in \Sigma \; , \quad C \in \mathbb{R}$

Modular Fourfolds: A Non-modular Example

The mirror family of the complete intersection $\mathbb{P}^7[2,2,4]$:

- Number of quadratic factorizations for each prime $7 \leq p \leq 317$:



- Many primes p with no point $z_p \in \mathbb{F}_p$ of quadratic factorization
- The existence of a modular point $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ is highly unlikely

Conclusions

Modularity is a useful tool to search for M-theory flux vacua

- "Attractor-point" $\Rightarrow$ Sublattice Λ realizes fluxes
- Modularity is not a necessary criterion!
- Modular form (L-function) $\Rightarrow$ cosmological constant

Restrictions and Assumptions

- Analysis restricted to horizontal part $H^4_h(X, \mathbb{C})$ of cohomology
- Need to assume that Fr_p factorizes correspondingly
- Search restricted to algebraic moduli space $z \in \bar{\mathbb{Q}}$

Based on numerical experiments:

- Construction of Frobenius action is self-consistent
- Modular structure in accordance with Deligne's conjecture and geometric interpretation

A posteori justification for the assumptions