Optimal Quantum Control for Precision Frequency Measurement

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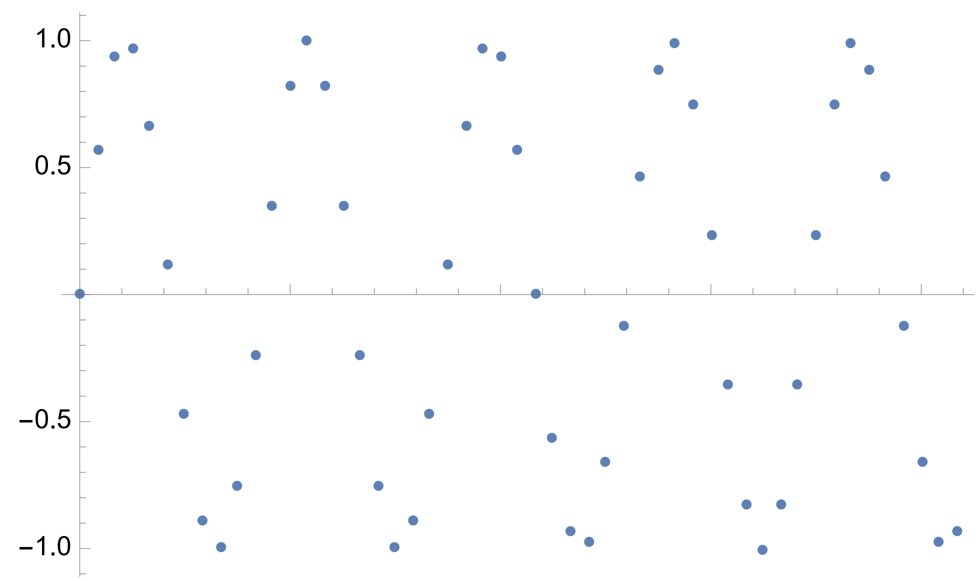
Optimal control work (not my own!):
Pang and Jordan, Nature Communications 8:14695 (2017)
Naghiloo, Jordan, Murch, PRL 119, 180801 (2017)
Schmitt et al., Science 356 832-837 (2017)

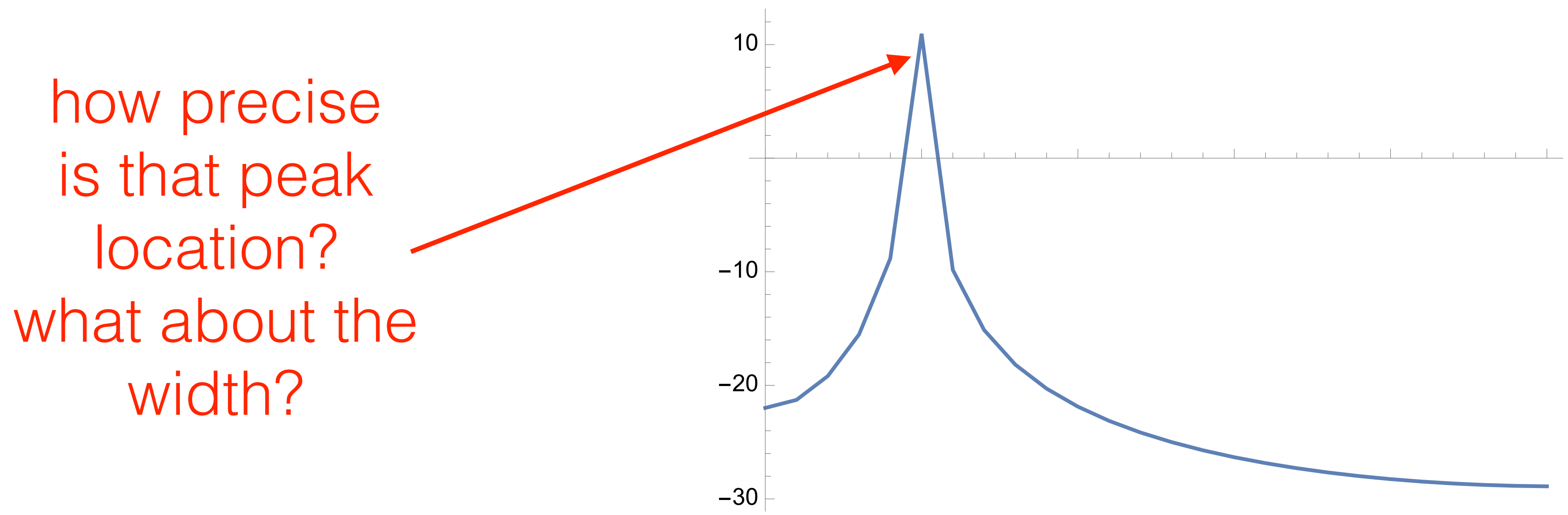
Application to axion detection:
Foster, Gao, Halperin, YK, Mande, Nguyen, Schütte-Engel, Scott, arXiv:2310.07791

Frequency estimation

You have some time-series data. You think it's described by an oscillation at a single frequency ω. How do you estimate ω?



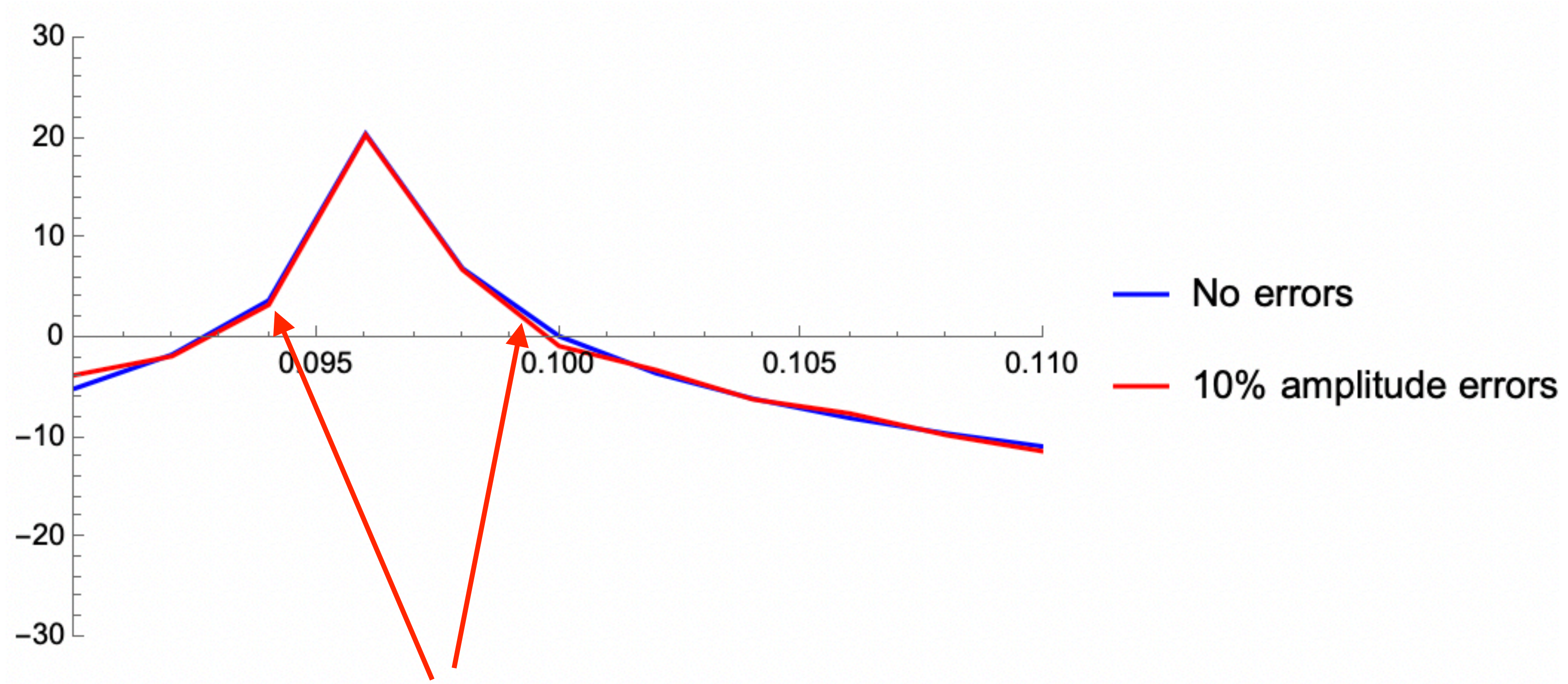
The obvious solution: discrete Fourier transform



Problems with the classical approach

Even with perfect data, Nyquist's theorem says $\Delta\omega \sim 1/T$

But what if your data is polluted by amplitude noise?



width of peak shifts: small, but it's there!

Quantum lets us do better

With optimal quantum control, we can gain a factor of T:

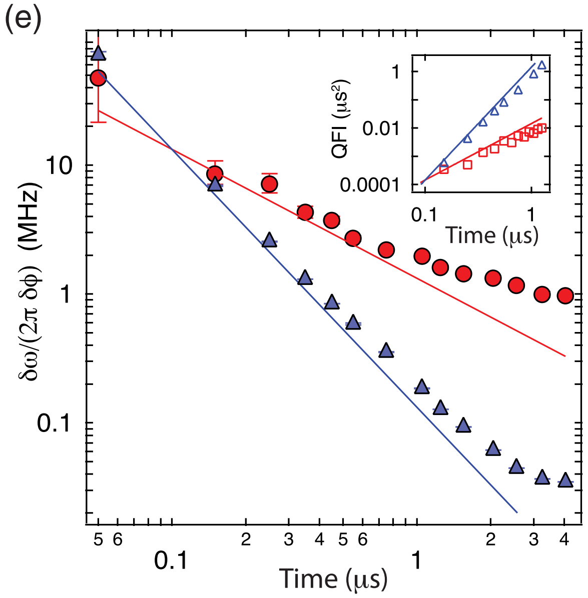
$$\Delta\omega \sim 1/T \rightarrow 1/T^2$$

classical, or SQL   optimal control

Even better for a frequency drift: $\Delta\dot{\omega} \sim 1/T^3$

The trick is to let ω modulate the level splitting of a qubit, and use a time-dependent control Hamiltonian to poke the system at judiciously-chosen intervals. Amplitude noise never enters!

(e)



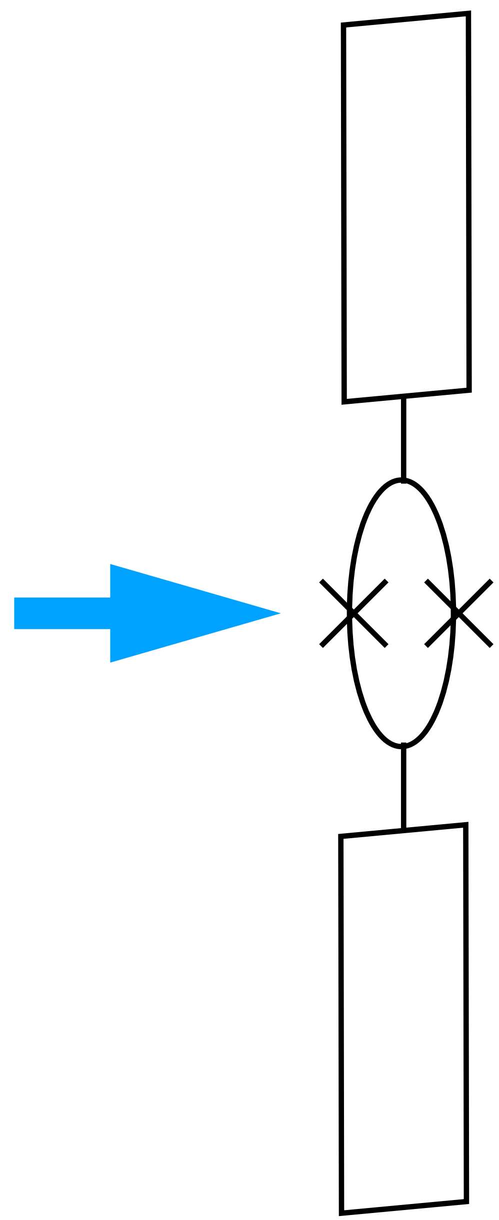
This actually works!! Let's see how.

no control

control

Setup: B-field and superconducting qubit

$$B_z = B_0 \sin(\omega t)$$



Flux modulates qubit level spacing:

$$H_\omega(t) = \frac{A\hbar}{2}\sin(\omega t)\sigma_z$$

Goal is quantum parameter estimation:
use measurements of the qubit to estimate Hamiltonian parameter ω

Metrology and Fisher Information

Suppose we are trying to measure g, which is a parameter controlling the probability distribution of an observable X: $X \sim p_g(X)$

Cramér-Rao bound tells us the best we can do:

$$\langle \delta^2 \hat{g} \rangle \geq \frac{1}{v I_g}$$

mean squared deviation

unbiased estimator

amount of data

Fisher information:

$$I_g = \int p_g(X) [\partial_g \ln p_g(X)]^2 \, dX$$

Quantum metrology

In a quantum measurement, $p_g(X)$ comes from measurements on quantum states parameterized by g, i.e. $|\psi_g\rangle$

The relevant quantity is the quantum Fisher information:

$$I_g^{(Q)} = 4\left[\langle\partial_g\psi_g|\partial_g\psi_g\rangle - \left|\langle\psi_g|\partial_g\psi_g\rangle\right|^2\right]$$

Measures distinguishably of two states under small variations of g

SQL: $I_g^{(Q)} \propto N$

Heisenberg: $I_g^{(Q)} \propto N^2$

What about scaling with measurement time?

Time-dependent metrology

Suppose our parameter multiplies a time-independent Hamiltonian,

$$H_g = gH_0$$

If we let the system evolve for time T, we have $|\psi_g\rangle = e^{-igH_0T}|\psi_0\rangle$

Easy to show that $I_g^{(Q)} = 4T^2\left[\langle H_0^2\rangle_g - \langle H_0\rangle_g^2\right]$

Lessons: we don't want to measure in an eigenstate of H_0 where the variance vanishes, but no matter what, $\sigma_g \propto 1/T$

But we can let the system evolve with a time-dependent Hamiltonian!

$$I_g^{(Q)} = 4\,\mathrm{Var}[h_g(T)]_{|\psi_0\rangle}$$

$$h_g(T) = iU_g^\dagger(0\to T)\partial_g U_g(0\to T)$$

unitary evolution (time-ordered exponential) under $H_g(t)$

Optimal control for frequency

Let's go back to our original problem:

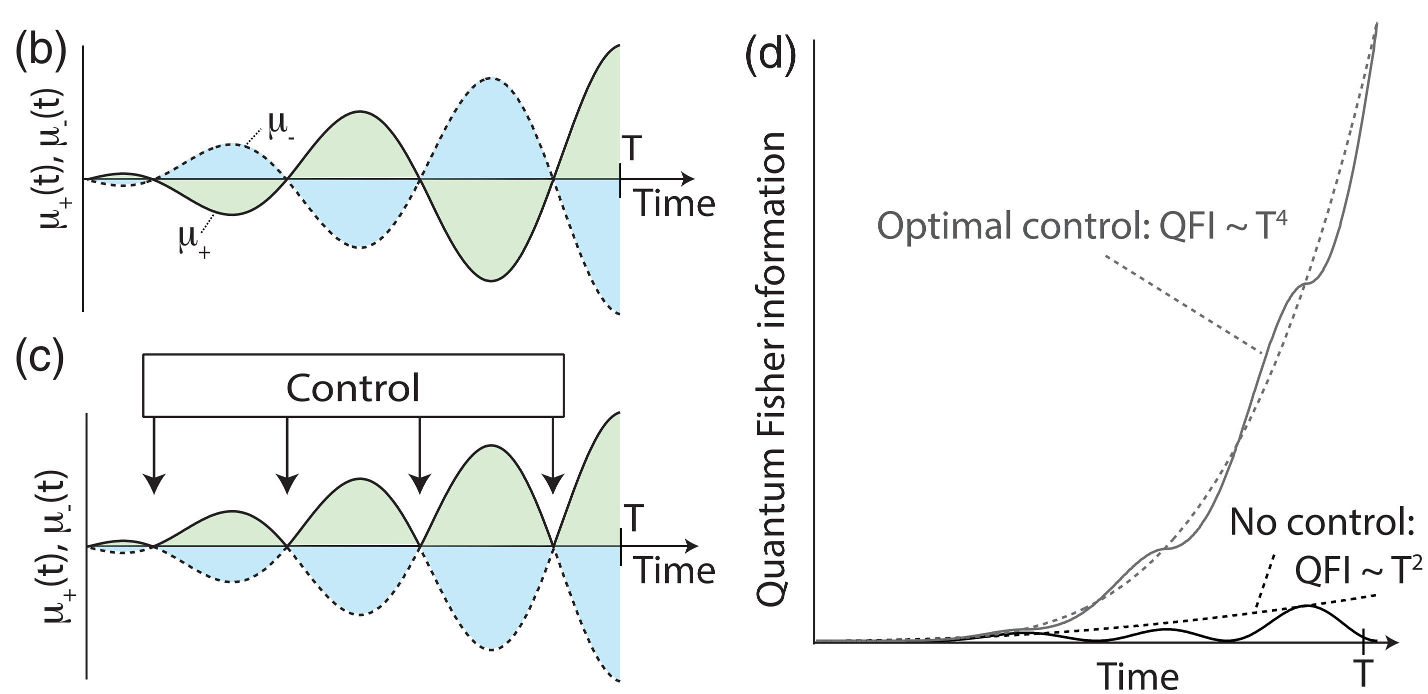
$$H_\omega(t) = \frac{A\hbar}{2}\sin(\omega t)\sigma_z \quad \Longrightarrow \quad h_\omega(t) = \frac{A\hbar}{2} t\cos(\omega t)\sigma_z$$

H commutes with itself at different times, no time-ordering necessary

Eigenvalues: $\mu_\pm = \pm\frac{A\hbar}{2} t\cos(\omega t) \quad \Longrightarrow \quad I_\omega^{(Q)} = \left(\int_0^T [\mu_+(t) - \mu_-(t)]\, dt\right)^2$

The control step is best explained in pictures:

(Time-dependent) control maximally separates eigenvalues to maximize QFI



Optimal frequency estimation

Prepare qubit in a superposition of H_ω eigenstates, which maximizes QFI:

$$|\psi_\omega\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

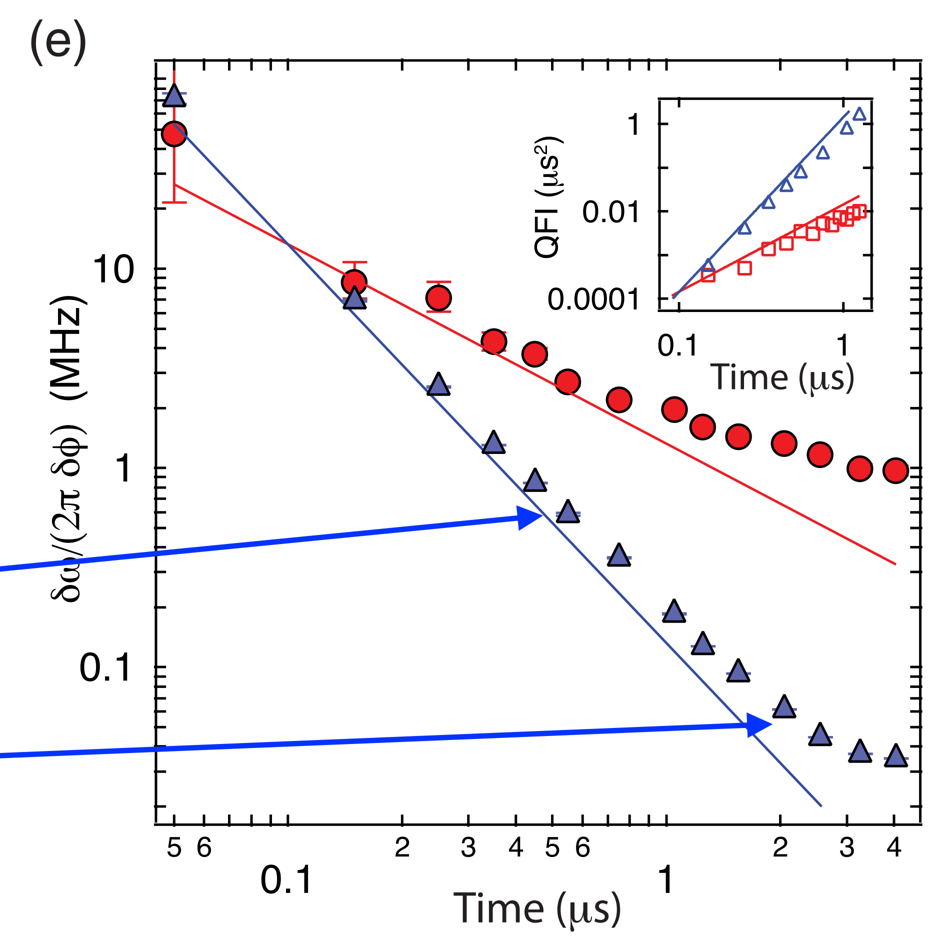
Evolution under H_ω and control will make the two eigenstates acquire a relative phase ϕ. Measure ϕ (ideally, non-destructively):

$$\frac{\delta\omega}{\omega} = \frac{\pi}{\omega A T^2}\delta\phi = \frac{\pi}{\omega A T^2 \sqrt{4N_q}}$$

This is a true quantum advantage! (Up to the qubit coherence time)

$\omega T \gtrsim \pi$

$T > T_2^*$

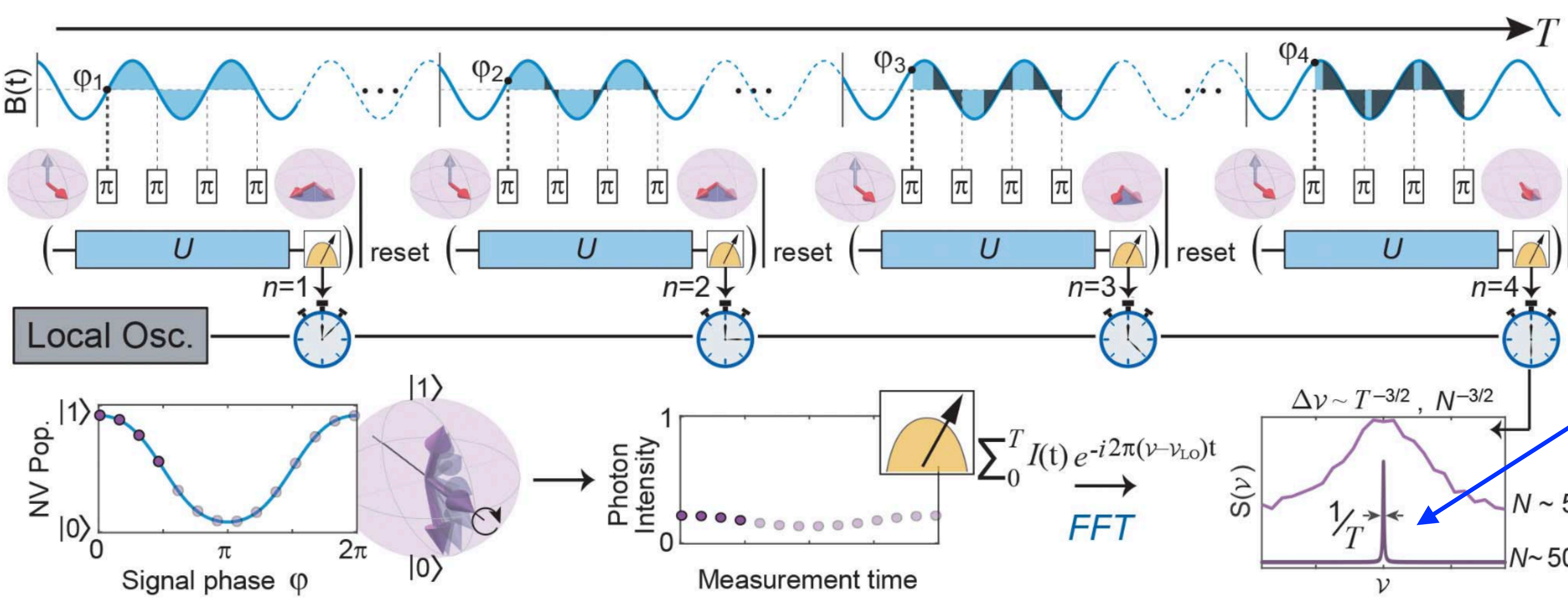


What about longer times?

If the signal is very coherent, just time-stamp the measurements at intervals of Δt with the best clock you can find!

$$I_\omega^{(Q)} = \frac{4A^2}{\omega^2}\sum_{j=1}^{T/\Delta t}(\Delta t)^2\left[\sin(\omega j\Delta t) - \sin(\omega(j-1)\Delta t)\right] \sim \frac{A^2}{\omega^2}T^3$$

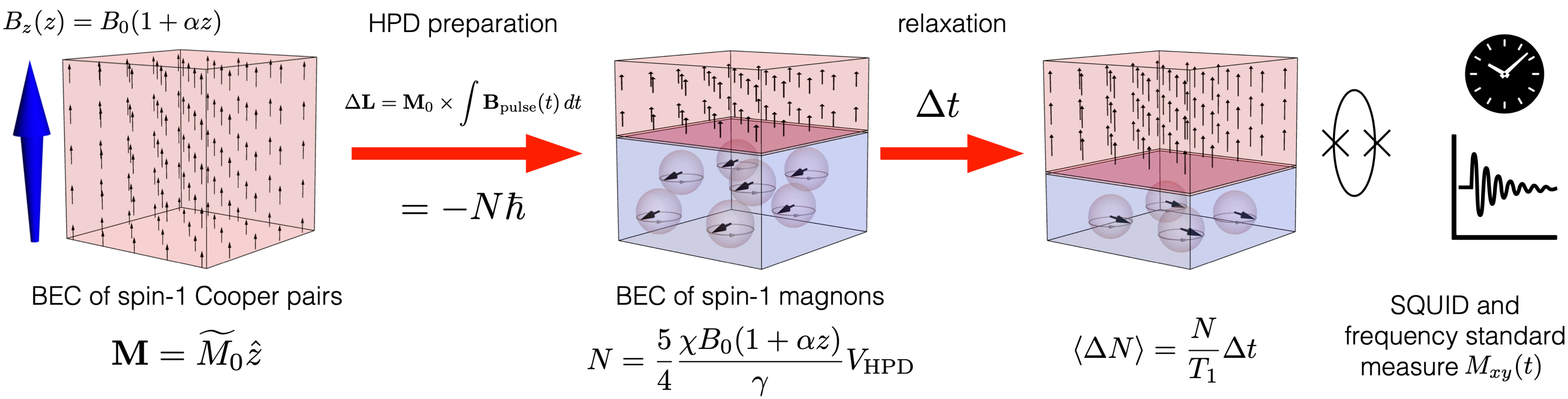
This lets you beat the qubit coherence time limit:



$$\frac{\delta\omega}{\omega} \sim \frac{1}{\omega A T^{3/2}\sqrt{T_2^*}}$$

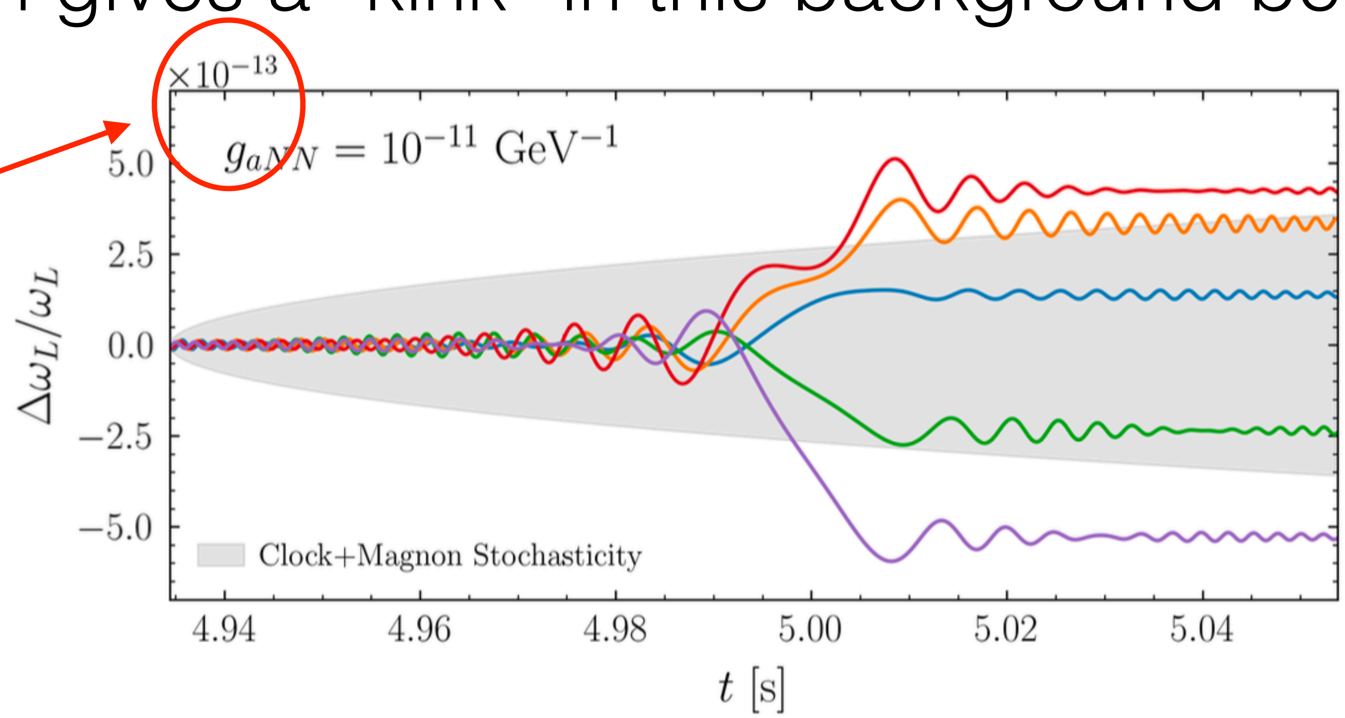
Application: axions + HPD

HPD review in one slide:



In absence of an axion, slowly drifting precession frequency.
Axion gives a “kink” in this background behavior

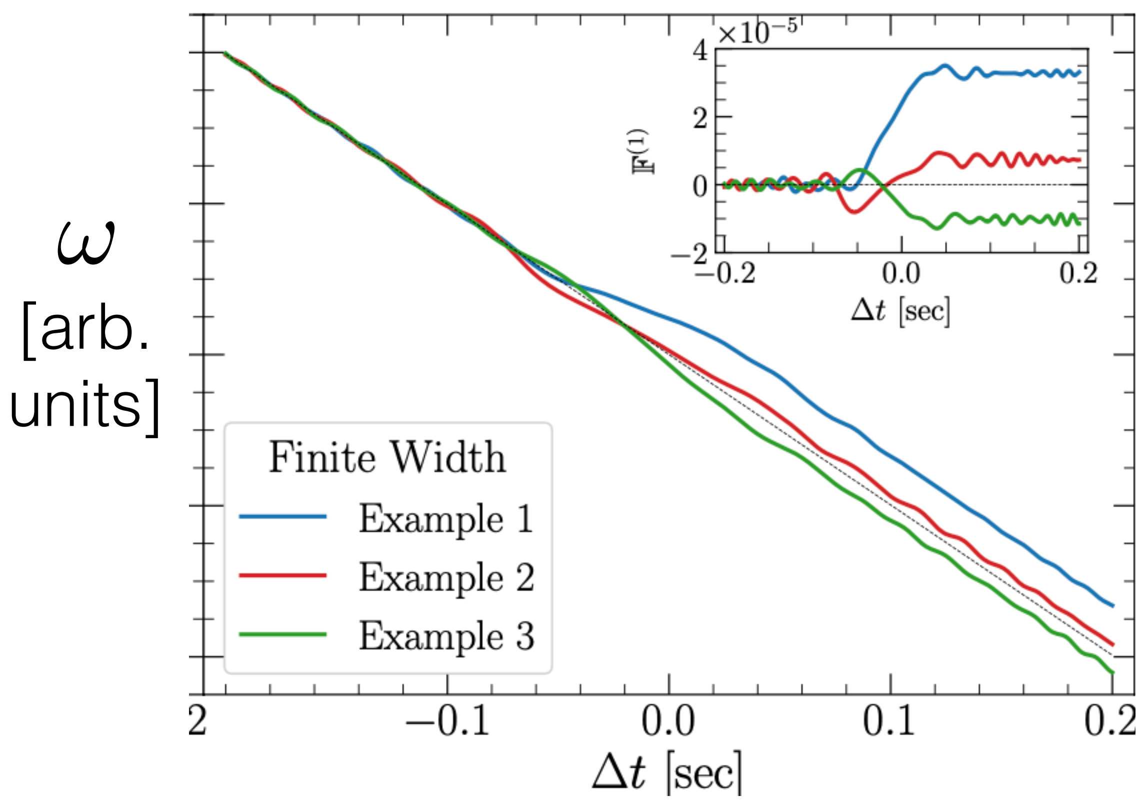
need
incredible
precision!

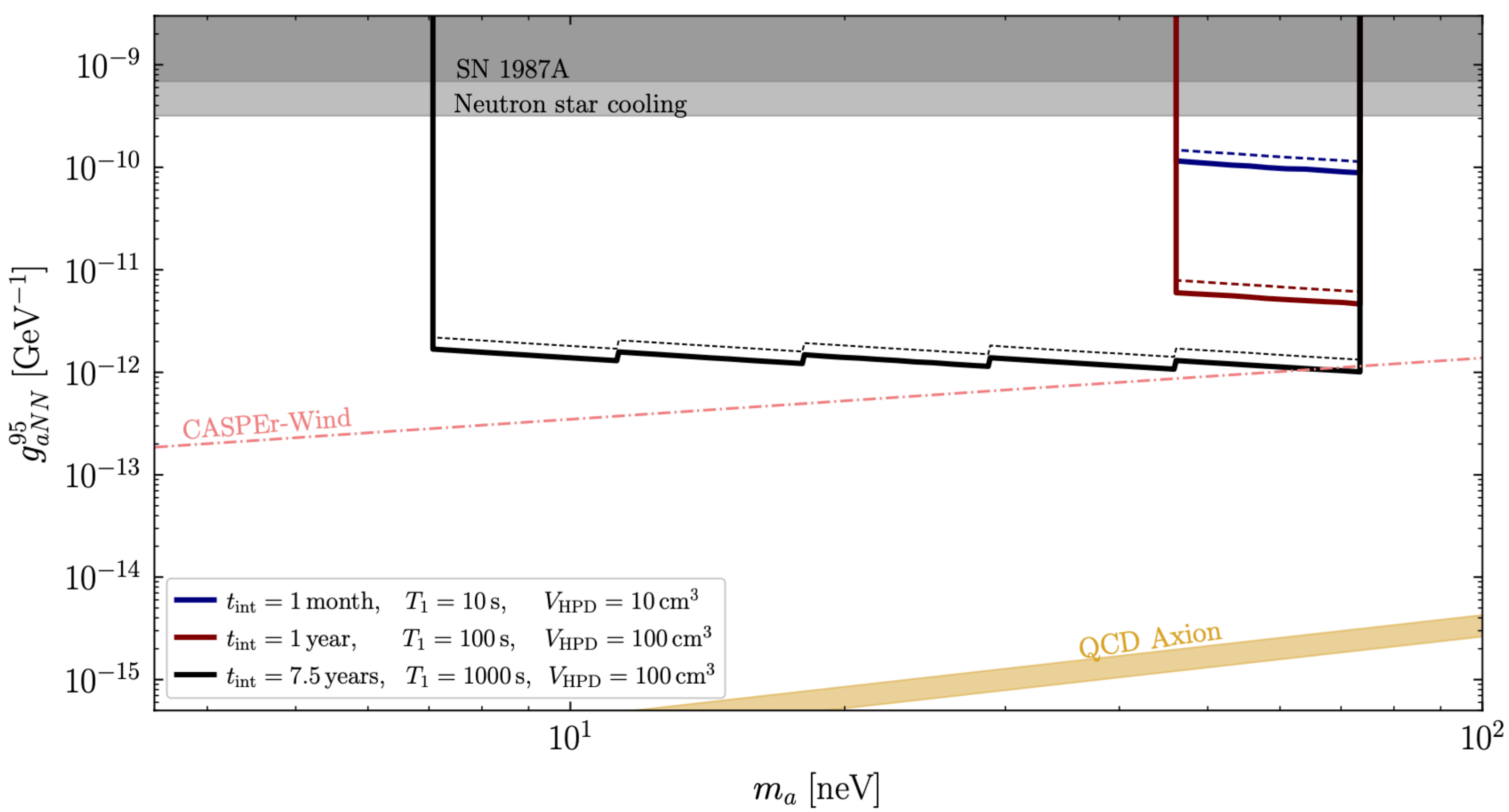


Application: axions + HPD

Use same optimal control scheme to measure a frequency drift:

$$H_{\dot{\omega}}(t) = A \sin(\omega t + \dot{\omega} t^2/2) \frac{\sigma_z}{2} \quad \rightarrow \quad \delta\dot{\omega} \approx \frac{\sqrt{5}\pi}{2AT^{5/2}\sqrt{T_q N_q}}$$





Frequency drift proportional to axion coupling: no information in the amplitude A, which only appears as an overall scale.

Robust to amplitude noise!