Intersection Theory for Fundamental Interactions

Mathematical methods from Scattering Amplitudes to modern Scientific Calculus

Pierpaolo Mastrolia

Loop the Loop:
Feynman calculus and its applications to Gravity and Particle Physics
MITP YoungST@RS workshop series

12.11.2024

In collaboration with: P. Benincasa, **G. Brunello**, S. Cacciatori, **V. Chestnov, G. Crisanti**, B. Eden, **W. Flieger, M. Giroux, M. Gottwald, H. Frellesvig,** S. Laporta, **M.K. Mandal,** *S. Matsubara-Heo*, S. Mizera, **T. Scherdin**, **S. Smith**, **F. Vazao,** *N. Takayama*

Physics and Astronomy Department
Galileo Galilei
University of Padova - Italy

INFN
Istituto Nazionale di Fisica Nucleare

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\,\partial_x^{(n-1)} f(x) + \ldots + p_1(x)\,\partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\,\partial_x^{(n-1)} f(x) + \ldots + p_1(x)\,\partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

- **Linear relations**

$$f_n(x) + a_{n-1}\, f_{n-1}(x) + \ldots + a_1\, f_1(x) + a_0\, f_0(x) = 0$$

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

$$\partial_x^{(n)} f(x) = -\, p_{n-1}(x)\, \partial_x^{(n-1)} f(x) - \ldots - p_1(x)\, \partial_x^{(1)} f(x) - p_0(x) f(x)$$

- **Linear relations**

$$f_n(x) + a_{n-1} f_{n-1}(x) + \ldots + a_1 f_1(x) + a_0 f_0(x) = 0$$

$$f_n(x) = -\, a_{n-1} f_{n-1}(x) - \ldots - a_1 f_1(x) - a_0 f_0(x)$$

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\,\partial_x^{(n-1)} f(x) + \ldots + p_1(x)\,\partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

$$\partial_x^{(n)} f(x) = -\,p_{n-1}(x)\,\partial_x^{(n-1)} f(x) - \ldots - p_1(x)\,\partial_x^{(1)} f(x) - p_0(x) f(x)$$

- **Linear relations**

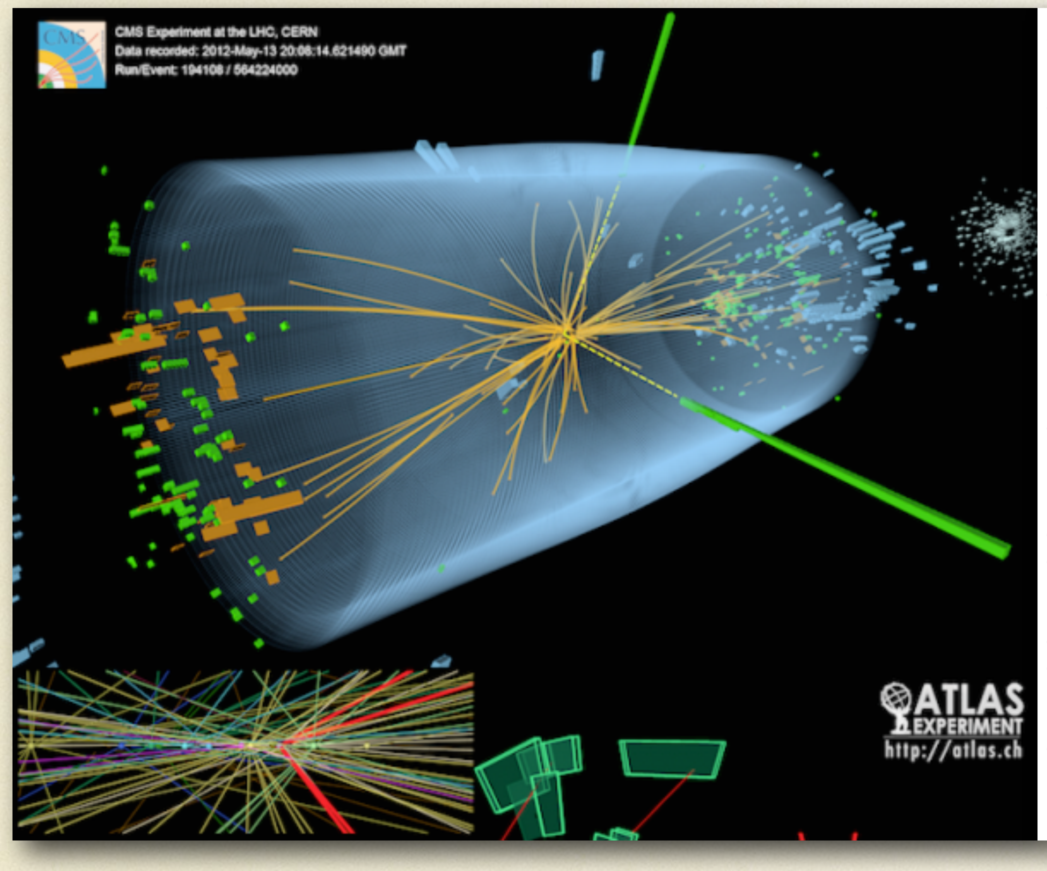
$$f_n(x) + a_{n-1}\, f_{n-1}(x) + \ldots + a_1\, f_1(x) + a_0\, f_0(x) = 0$$

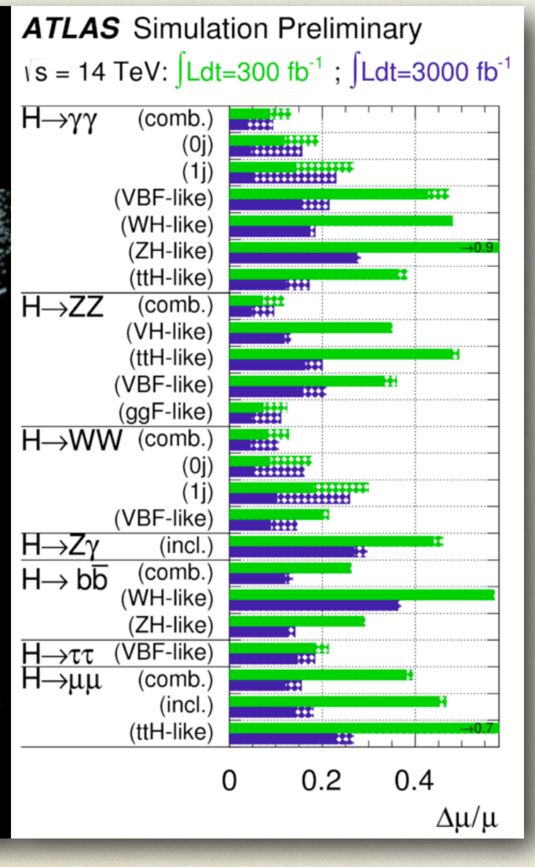
$$f_n(x) = -\,a_{n-1}\, f_{n-1}(x) - \ldots - a_1\, f_1(x) - a_0\, f_0(x)$$

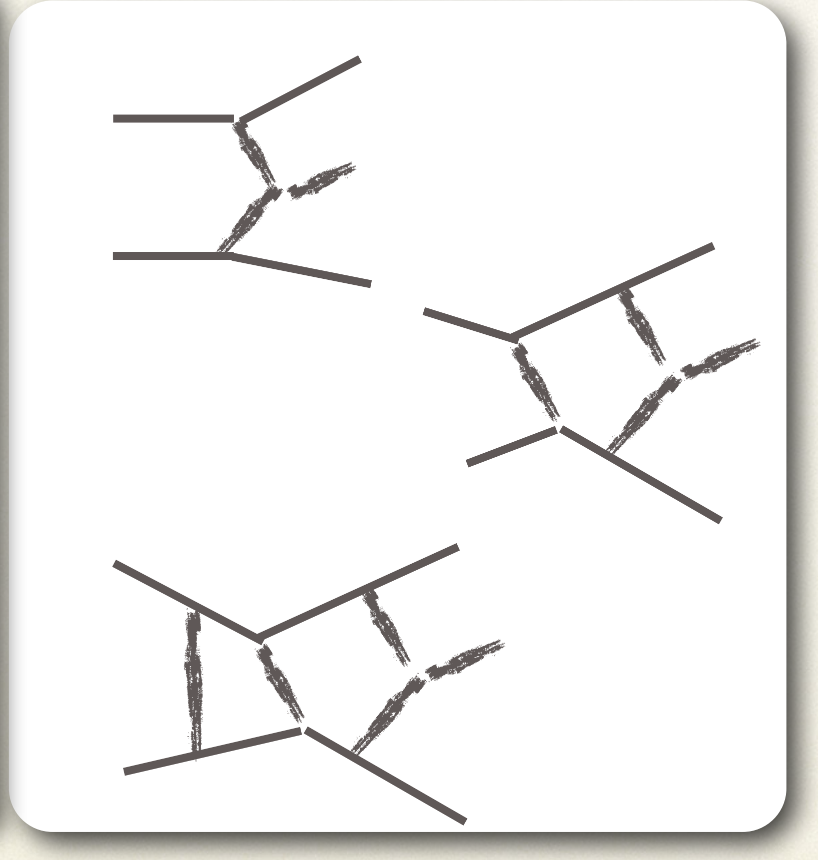
Decomposition formulas
in terms
of **n *independent* elements**

Impact of Scattering Amplitudes & Multiloop Calculus / Frontier of Theoretical Physics

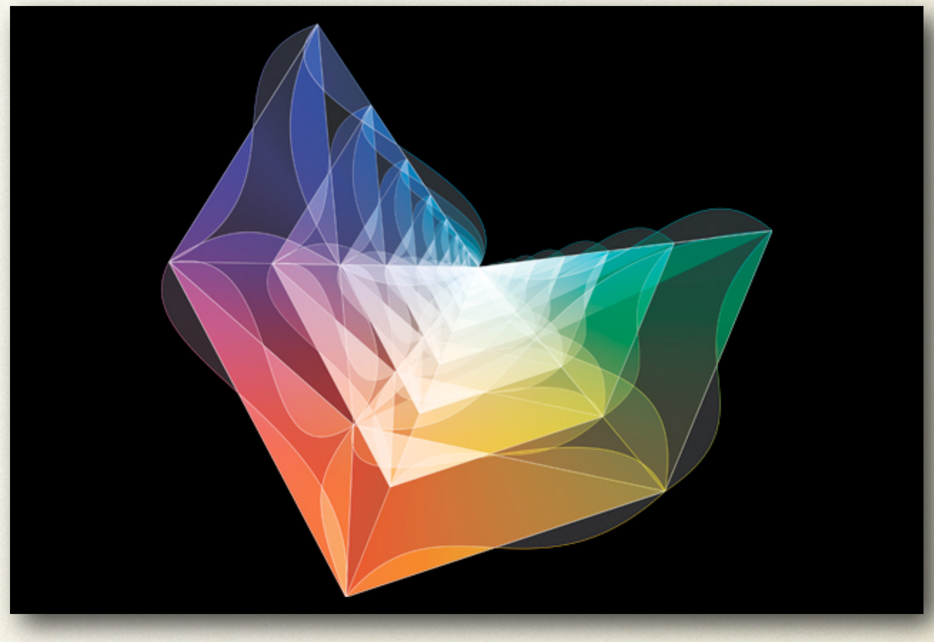
- **Collider Phenomenology**

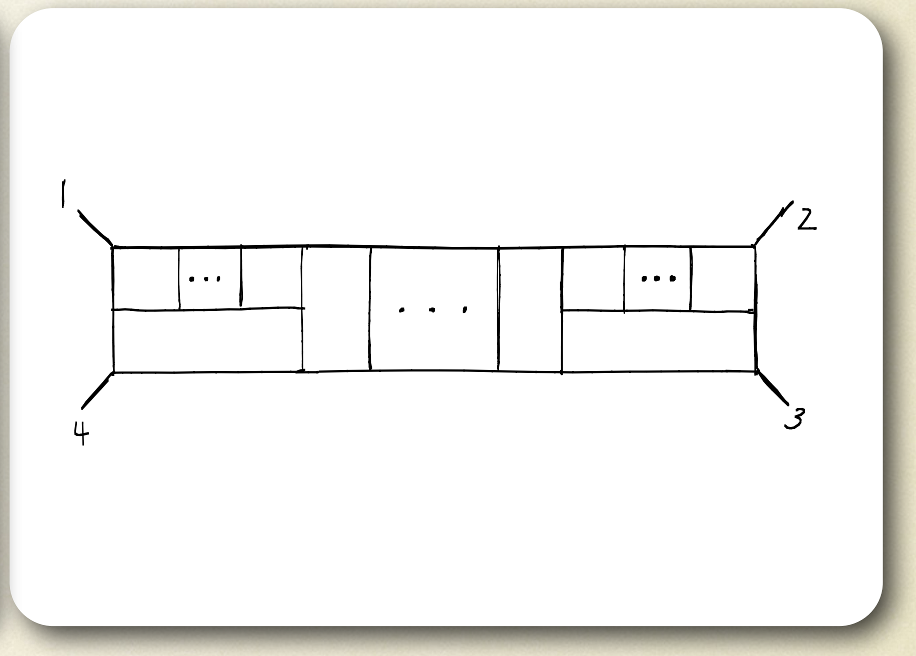




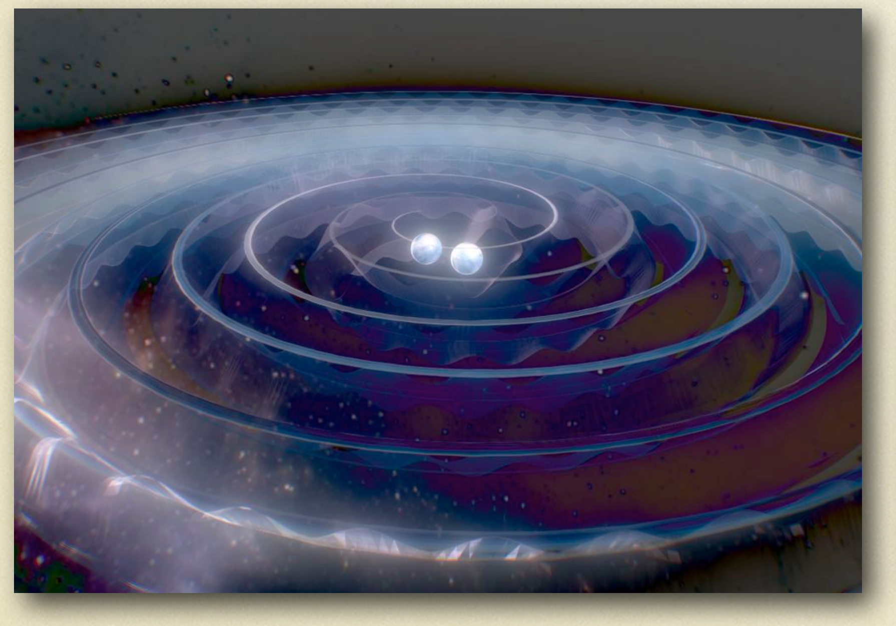


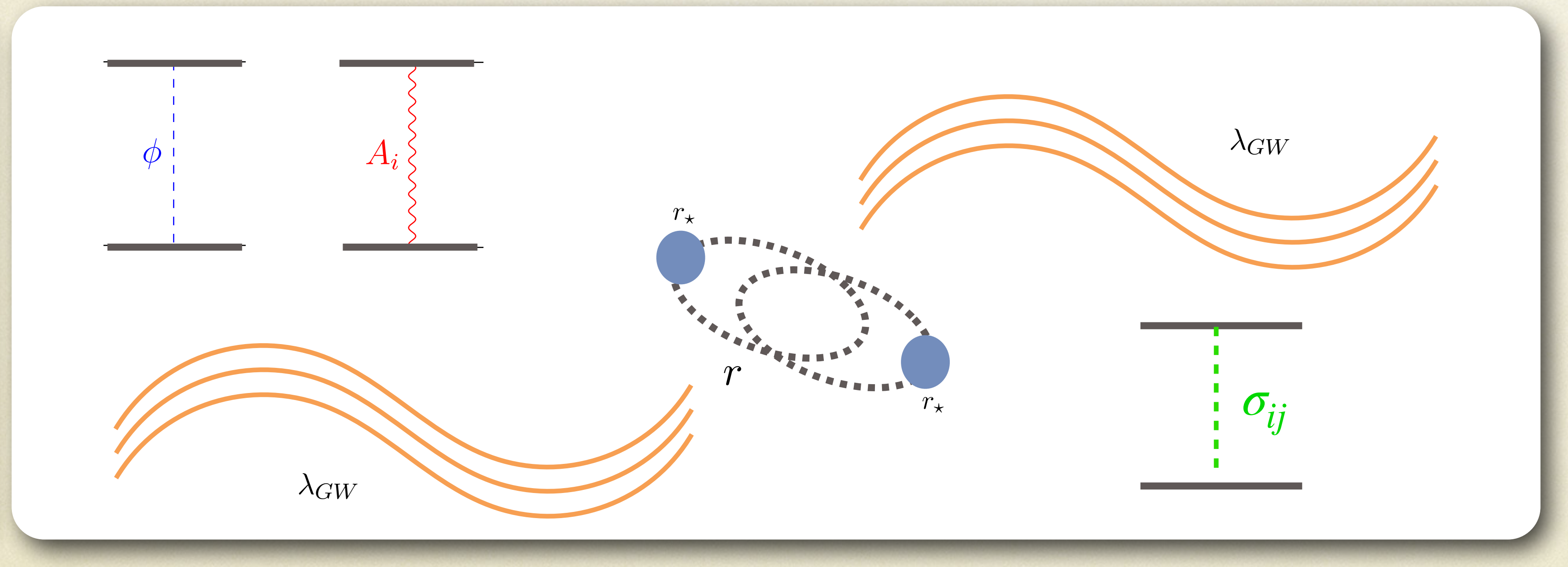
- **Geometry of Quantum Field Theory**



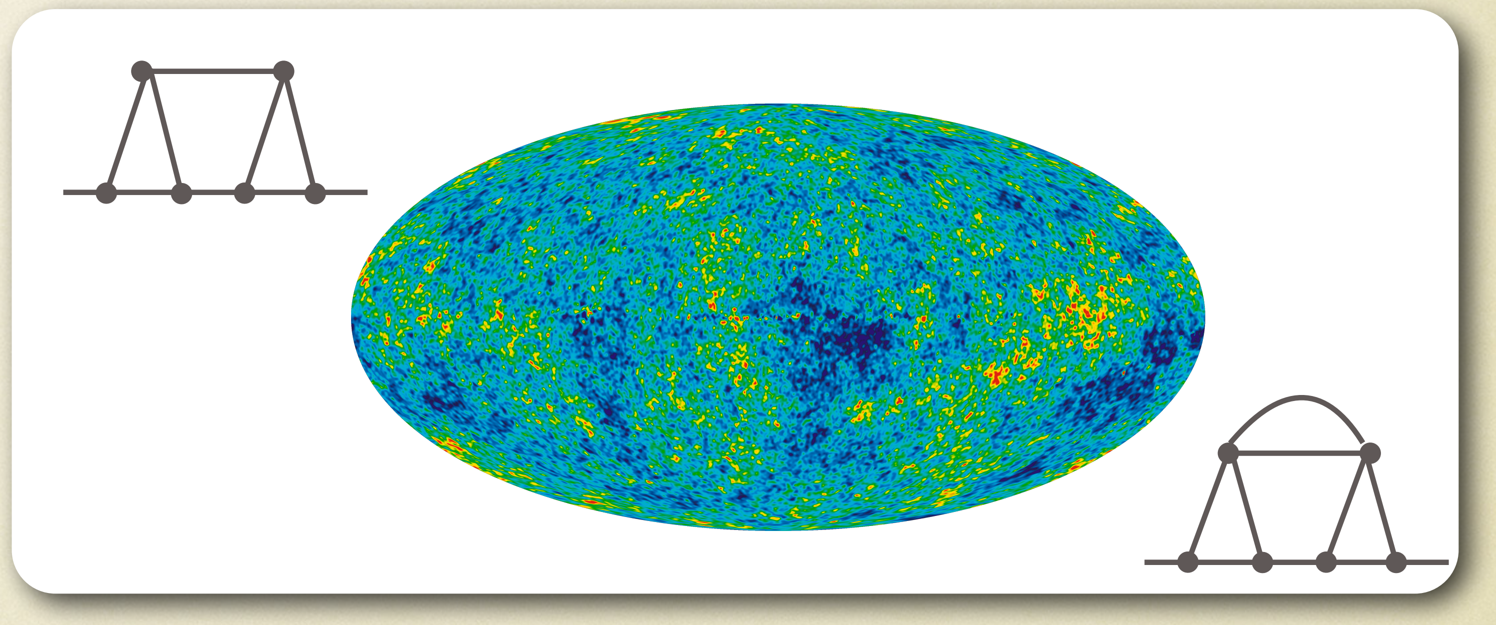


- **EFT Classical General Relativity**



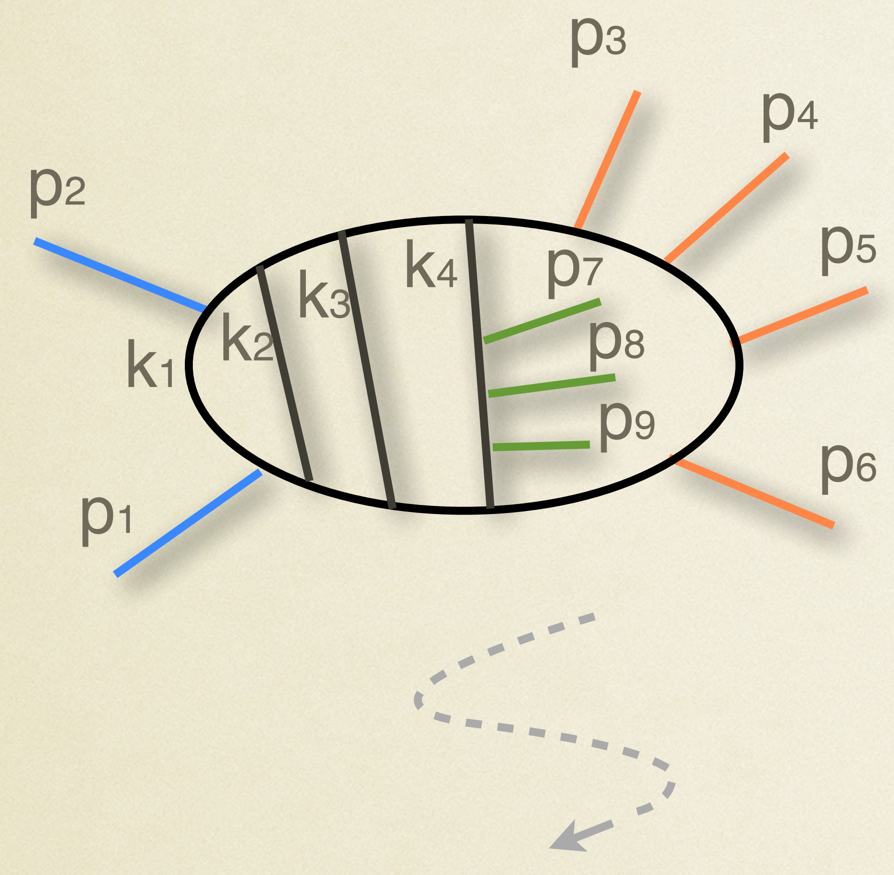


- **Cosmology**



Feynman Integrals

- **Momentum-space Representation**



N-denominator
generic Integral

$$= I^{[d]}_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

L loops, $E{+}1$ external momenta,

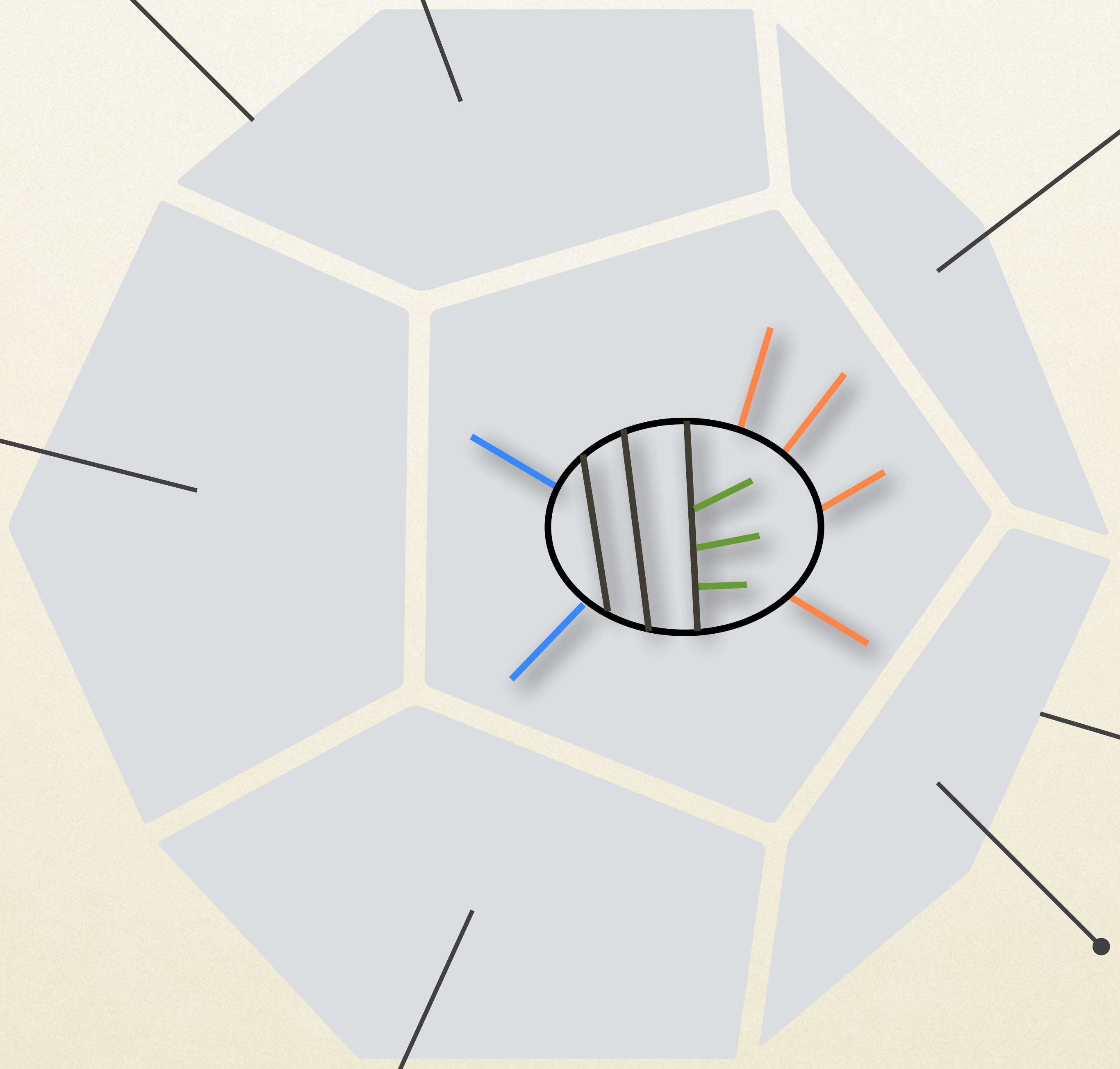
$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

$$D_n = (p_1 \pm p_2 \pm \ldots \pm k_1 \pm k_2 \pm \ldots)^2 - m_n^2$$

Feynman Integrals / (a few) Evaluation Methods



Integration-by-parts identities

Chetyrkin, Tkachov
Remiddi, Laporta; Laporta
Zhang, Larssen
………………..

Direct Integral Evaluation

Passarino, Veltman, t'Hooft, Veltman
Beneke, Smirnov, Tausk
Binoth, Heinrich,
Anastasiou, Melnikov
Weinzierl
Czakon
………………..

Unitarity-based and on-shell methods

Bern, Dixon, Dunbar, Kosower
Britto, Cachazo, Feng, Witten
Brandhuber, Spence, Travaglini
Britto, Buchbinder, Cachzao, Feng + P.M.
Glover, Badger & P.M.
Anastasiou, Britto, Kunszt & P.M.
Forde; Badger
Beger, Bern, Dixon, Forde, Kosower, Ita, Maitre
Ellis, Giele, Kunszt, Melnikov, Zanderighi
………………..

Integrand Decomposition & Polynomial division

Ossola Papadopoulos Pittau
Ossola Papadopoulos Pittau & P.M.
Bevilacqua, Garzelli, Papadopoulos, Pittau, Worek
Ossola, Reiter, Tramontano, & P.M.
Mirabella, Ossola, Peraro & P.M.
Ossola & P.M.
Gluza, Kajda, Kosower
Badger, Frellsvig, Zhang
Zhang
Peraro, Primo & P.M.
Abreu, Febres Cordero, Ita, De Laurentis
………………..

Difference Equations

Tarasov
Laporta
Lee
………………..

Differential Equations

Barucchi, Ponzano, Regge
Bern, Dixon, Kosower
Kotikov, Remiddi,
Gehrmann Remiddi
Bonciani, Remiddi, & P.M.
Czakon
Henn,
Papadopoulos,
Liu, Ma, Peking
………………..

What's next?

Feynman Integrals

- **Integration-by-parts Identities (IBPs)**

$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$v_\mu = v_\mu(p_i, k_j)$

arbitrary

- IBP equations
- Contiguity relations

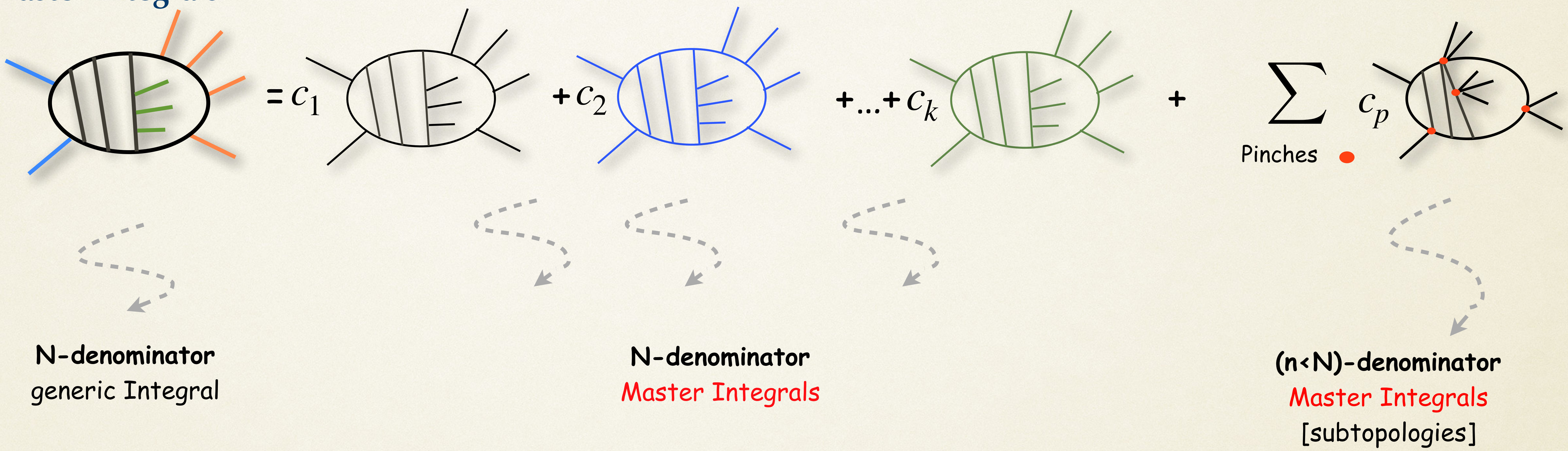
$$\sum_i b_i \, I^{[d]}_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$$

- **Generating an *overdimensioned (sparse) systems of linear equations***
- **Solutions:**

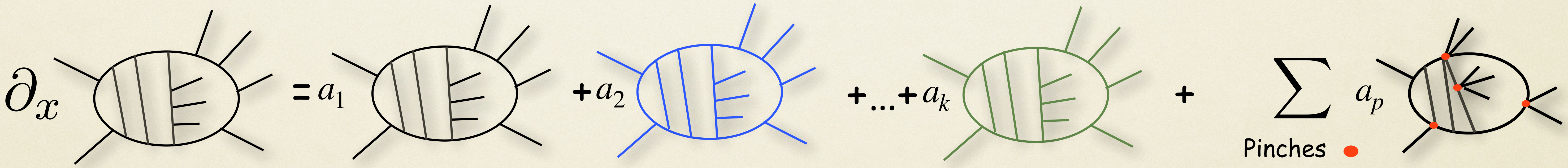
☑ Gauss' Elimination
☑ Groebner Bases
☑ Syzygy Equations
☑ **Finite Fields + Chinese Remainder Theorem + Rational Functions Reconstruction**

Linear relations for Feynman Integrals

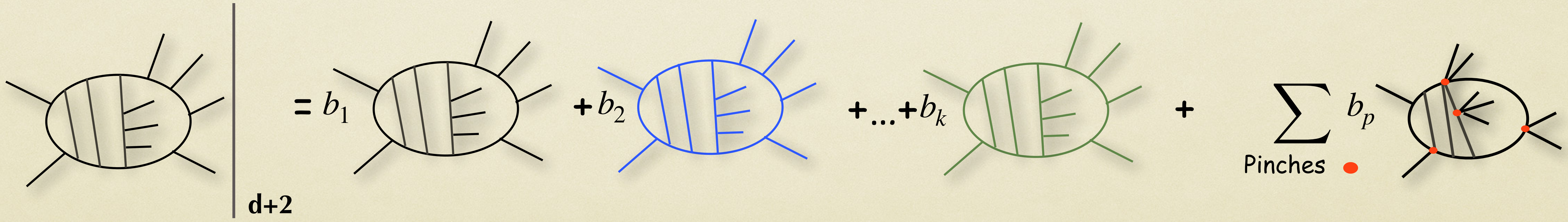
- Decomposition in terms of independent Master Integrals



- 1st order Differential Equations for MIs

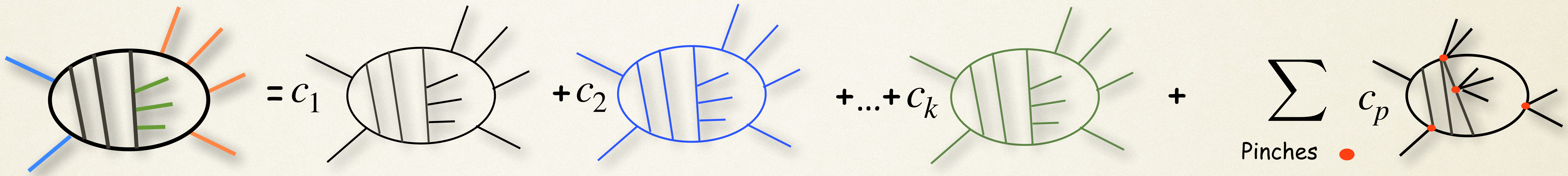


- Dimension-Shift relations and Gram determinant relations



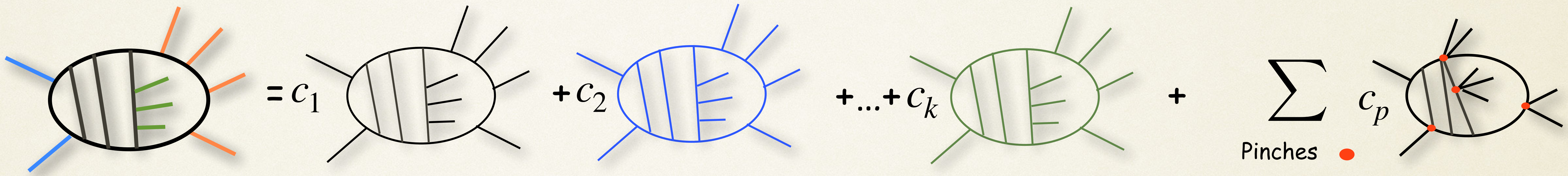
Linear relations for Feynman Integrals

- **Relations among Integrals in dim. reg.**

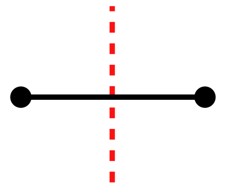


Linear relations for Feynman Integrals

- Relations among Integrals in dim. reg.

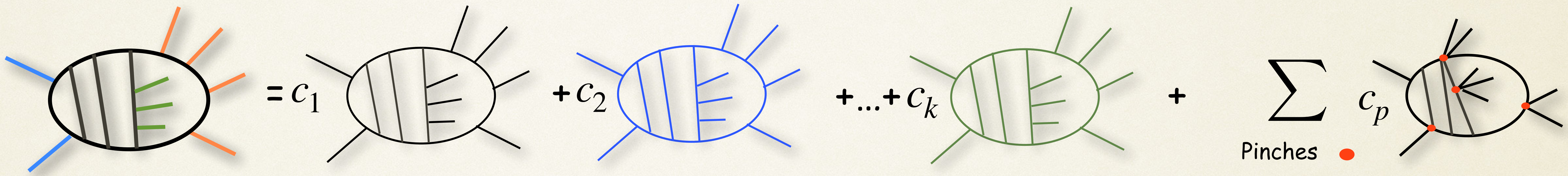


- Unitarity-based and on-shell methods $\frac{1}{D_i} \to \delta(D_i)$

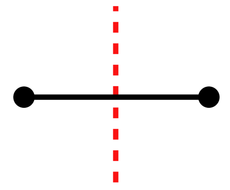


Linear relations for Feynman Integrals

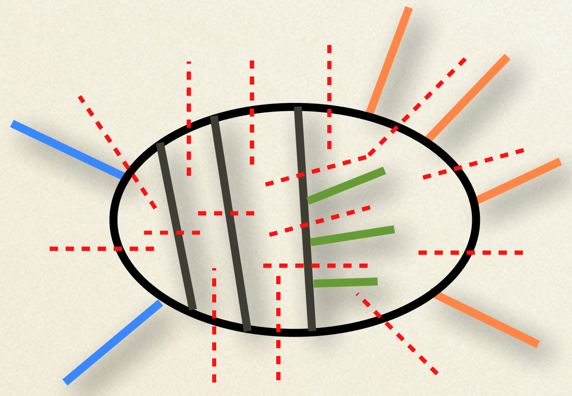
- **Relations among Integrals in dim. reg.**



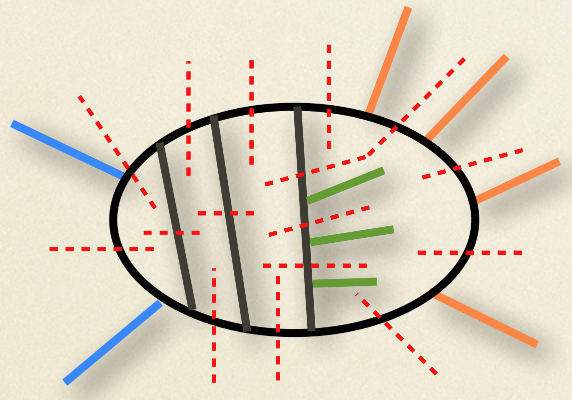
- **Unitarity-based and on-shell methods** $\frac{1}{D_i} \to \delta(D_i)$



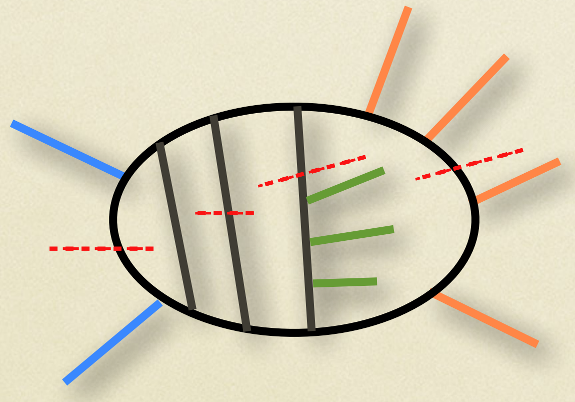
N-denominator cut



(N-1)—**denominator** cut



(#loop)—**denominator** cut

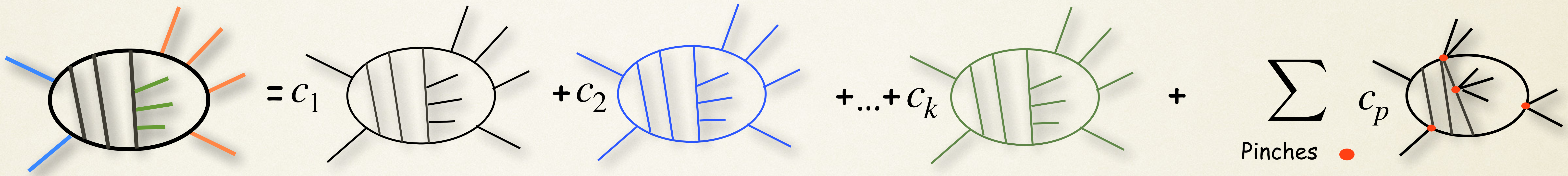


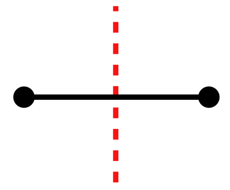
=

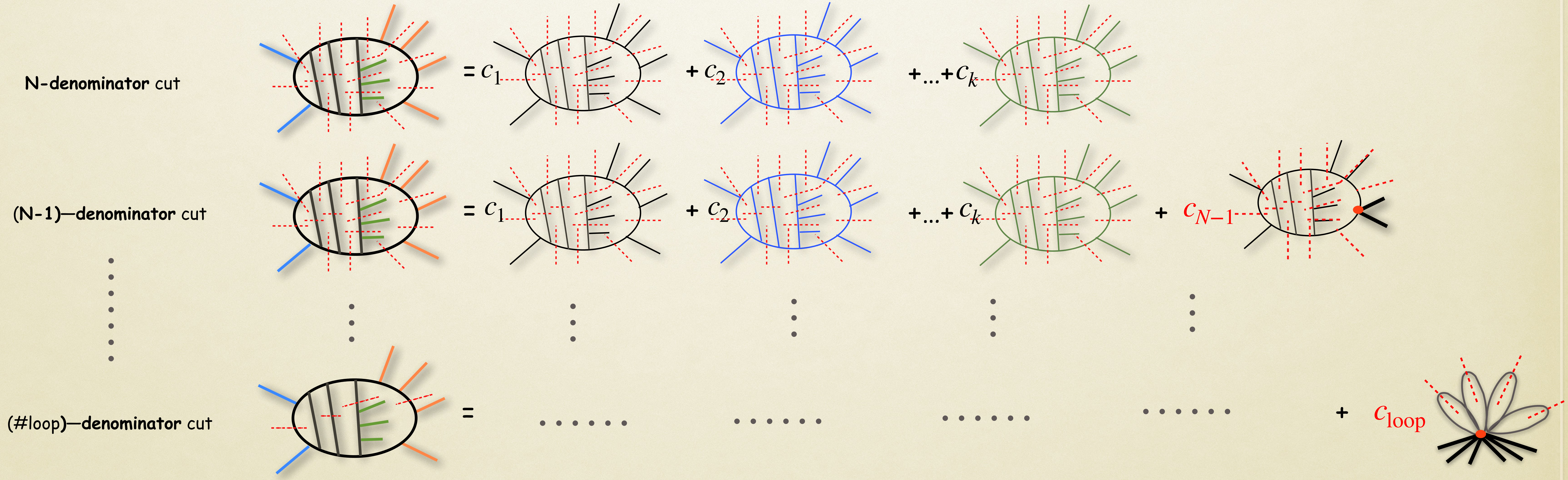
☑ Novel **integrand generation**: product of tree-amplitudes/diagrams; **complex momenta** across the cut

Linear relations for Feynman Integrals

- **Relations among Integrals in dim. reg.**



- **Unitarity-based and on-shell methods** $\frac{1}{D_i} \to \delta(D_i)$ 

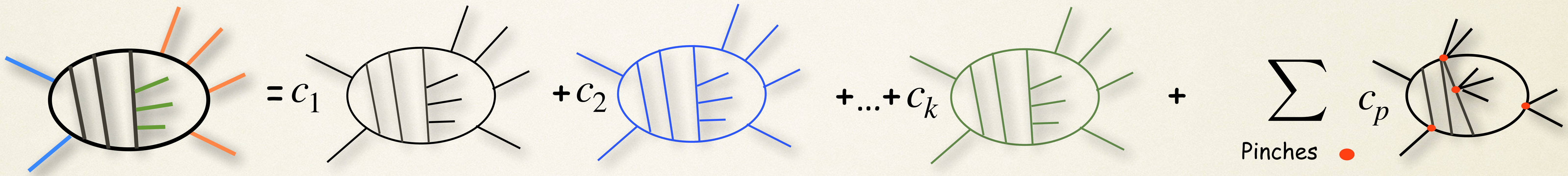


☑ Novel **integrand generation**: product of tree-amplitudes/diagrams; **complex momenta** across the cut

☑ Novel **complex-integration** techniques: (see 1loop 4ple-cut, 3ple-cut, 2ple-cut, …)

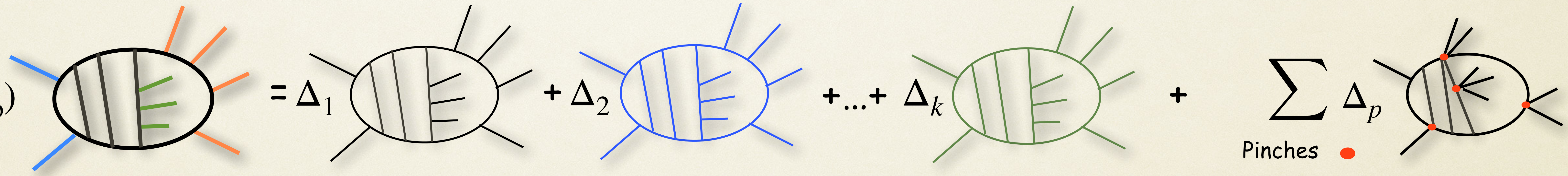
Linear relations for Feynman Integrands

- **Relations among Integrals in dim. reg.**



- **OPP Integrand Decomposition**

[integrand identity] $N(k_1, \ldots, k_{\text{loop}})$



- ☑ Δ_i are polynomials and $c_i \in \Delta_i$
- ☑ c_i determined by **polynomial fitting**

- (Block)-triangular system of linear equations:
  principle of polynomial identity: integration NOT required

- **Cuts vs Residues vs Remainders**

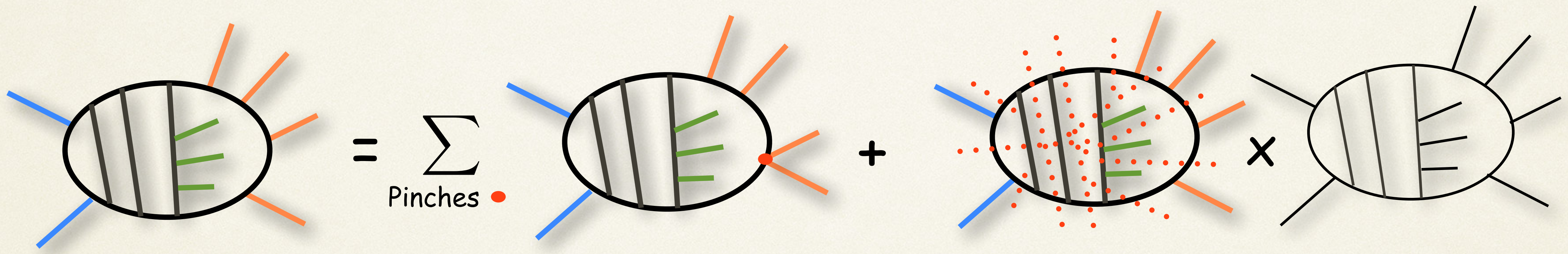
  - ☑ Δ_i , therefore c_i , determined by **polynomial division** (Δ_i are the **remainders**)

Polynomial Division & Integrand Recursion

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

- **Integrand Recursion**

$$\frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}\, \not{D}_{i_\kappa}}{D_{i_1}\cdots D_{i_{\kappa-1}}\not{D}_{i_\kappa} D_{i_{\kappa+1}}\cdots D_{i_n}} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1}i_{\kappa+1}i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} .$$



- **Ideal**

$$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n}\rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner basis**

$$\mathcal{J}_{i_1\ldots i_n} = \langle g_1, \ldots, g_m\rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Polynomial Division**

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(\mathbf{z})$$

- **Quotients**

$$\Gamma_{i_1\cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z}) = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1}i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) .$$

- **Remainder ~ Residue** $\Delta_{i_1\cdots i_n}(\mathbf{z})$
  - it contains **irreducible monomials** that generate the quotient space

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & **P.M.** (2012)

At any loop ℓ, loops we define *maximum cut* as the set of vanishing denominators

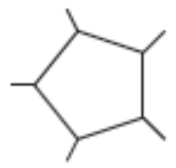
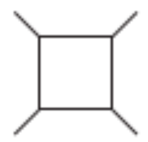
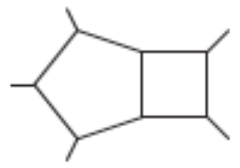
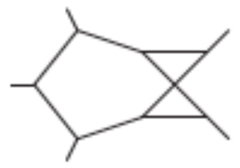
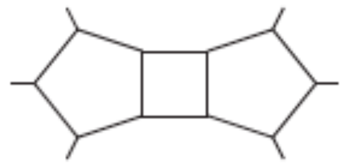
$$D_0 = D_1 = \ldots = 0$$

which constrains completely the components of the loop momenta. 0-dimensional

We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.

Then,

Theorem 4.1 (Maximum cut)**.** *The residue at the maximum-cut is a polynomial paramatrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^i$	4
	$\sum_{i=0}^{7} c_i z^i$	8		$\sum_{i=0}^{7} c_i z^i$	8

Projections and Discrete Fourier Transform

Britto, Feng, & **P.M.** (2008)

Ossola, Papadopoulos, Pittau, & **P.M.** (2008)

- **Polynomials**

$$\Delta_{n-1}[z] = c_0 + c_1\, z + \ldots + c_{n-1}\, z^{n-1} = \sum_{j=0}^{n-1} c_i\, z^j$$

- **Coefficients projection**

$$c_m = \frac{1}{n} \left(\sum_{k=0}^{n-1} z^{-m} \sum_{j=0}^{n-1} c_j\, z^j \right) \Bigg|_{z \to t^k,\; t \to e^{2\pi i/n}}$$

Projections and Discrete Fourier Transform

Britto, Feng, & **P.M.** (2008)

Ossola, Papadopoulos, Pittau, & **P.M.** (2008)

- **Polynomials**

$$\Delta_{n-1}[z] = c_0 + c_1\, z + \ldots + c_{n-1}\, z^{n-1} = \sum_{j=0}^{n-1} c_i\, z^j$$

- **Coefficients projection**

$$c_m = \frac{1}{n} \left(\sum_{k=0}^{n-1} z^{-m} \sum_{j=0}^{n-1} c_j\, z^j \right) \Bigg|_{z \to t^k,\, t \to e^{2\pi i/n}}$$

$$c_m = \frac{1}{n} \oint\!\!\!\!\!\!\sum_k t^{-m\,k} \sum_j c_j\, t^{j\,k} \qquad \bigcirc \Leftrightarrow t \to e^{2\pi i/n}$$

- *Discrete* version of a **double integral**, over a complex variable, along a closed path (unitary circle)

Stokes' Theorem for Double-Cuts

P.M. (2009)

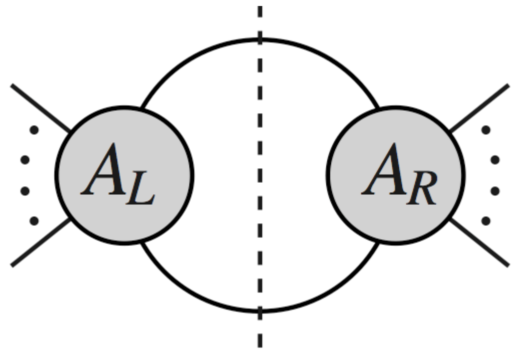
- **Generalised Cauchy Formula or Cauchy-Pompeiu Formula**

$$2\pi i \mathcal{F}(z_0) = \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz - \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} d\bar{z} \wedge dz.$$

- **Case-1** $\mathcal{F}$ is analytic, $\mathcal{F}_{\bar{z}} = 0$, $\mathcal{F}(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz$

  - Cauchy's residue theorem

- **Case-2** $\mathcal{F}|_{\partial D} = 0$ $\mathcal{F}(z_0) = \frac{1}{2\pi i} \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} dz \wedge d\bar{z}$



$$= \oint dz \int d\bar{z}\ f(z,\bar{z}) = \oint dz \int d\bar{z}\ F_{\bar{z}} = \oint dz\ F(z,\bar{z})$$

$$F(z,\bar{z}) = \int d\bar{z}\ f(z,\bar{z})$$

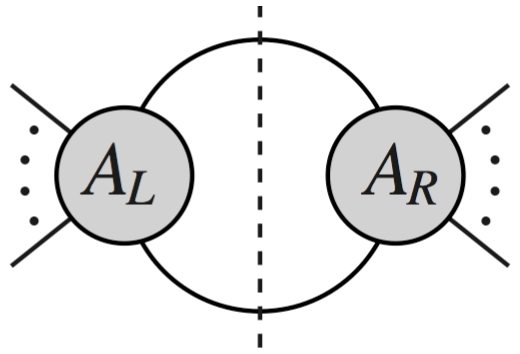
Stokes' Theorem for Double-Cuts

P.M. (2009)

- **Generalised Cauchy Formula or Cauchy-Pompeiu Formula**
$$2\pi i \mathcal{F}(z_0) = \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz - \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} d\bar{z} \wedge dz.$$

- **Case-1** $\mathcal{F}$ is analytic, $\mathcal{F}_{\bar{z}} = 0,$
$$\mathcal{F}(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mathcal{F}(z)}{z - z_0} dz$$
- Cauchy's residue theorem

- **Case-2** $\mathcal{F}|_{\partial D} = 0$
$$\mathcal{F}(z_0) = \frac{1}{2\pi i} \iint_D \frac{\mathcal{F}_{\bar{z}}}{z - z_0} dz \wedge d\bar{z}$$



$$= \oint dz \int d\bar{z}\ f(z, \bar{z}) = \oint dz \int d\bar{z}\ F_{\bar{z}} = \oint dz\ F(z, \bar{z})$$
$$F(z, \bar{z}) = \int d\bar{z}\ f(z, \bar{z})$$

- **Case-2 (multi-poles)**
$\mathcal{F}|_{\partial D} = 0$
$$\sum_j \iint_D \frac{\mathcal{F}^{(j)}_{\bar{z}}}{z - z_j} dz \wedge d\bar{z} = 2\pi i \sum_{j \in \text{poles}} \mathcal{F}^{(j)}(z_j)$$

due to the subtraction of a disk around each of the z-poles from the domain D.

- **Double integral**, over a complex variable, along a closed path (around all poles)

Novel Perspective on (Feynman) Calculus

Outline

- **Vector Space Structure of (Feynman, GKZ, Euler-Mellin, A-hypergeometric) twisted period Integrals**
    - De Rahm co-homology groups
    - Space dimensions, Linear and Quadratic relations
- **Intersection Numbers**
    - Computational Methods for n-forms:
        - iterative method / fibration-based approach
        - polynomial division and relative cohomology
        - Companion-tensor based method
    - D-modules and Pfaffians
- **Applications**
    - Hypergeometric functions
    - Feynman Integrals
    - Matrix elements in Quantum Mechanics
    - Green's functions and Wick's theorem
    - Kontsevich-Witten tau-function
    - Fourier integrals
    - Cosmological wave function integrals
- **Conclusions**

Based on:

- **PM**, Mizera
  Feynman Integral and Intersection Theory
  JHEP 1902 (2019) 139 [arXiv: 1810.03818]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
  JHEP 1095 (2019) 153 [arXiv: 1901.11510]
- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
  Vector Space of Feynman Integrals and Multivariate Intersection Numbers
  Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]
- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals by Multivariate Intersection Numbers.
  JHEP 03 (2021) 027 [arXiv 2008.04823]
- Chestnov, Gasparotto, Mandal, **PM**, Matsubara-Heo, Munch, Takayama
  Macaulay Matrix for Feynman Integrals: linear relations and intersection numbers.
  JHEP09 (2022) 187 [arXiv: 2204.12983]
- Cacciatori & **PM,**
  Intersection Numbers in Quantum Mechanics and Field Theory.
  2211.03729 [hep-th].
- ***Brunello, Chestnov, Crisanti***, Frellesvig, Mandal & **PM**
  Intersection Numbers, Polynomial Division & Relative Cohomology
  JHEP09(2024)015 [arXiv: 2401.01897]
- ***Brunello, Crisanti***, ***Giroux***, ***Smith*** & **PM,**
  Fourier Calculus from Intersection Theory
  Phys.Rev.D 109 (2024) 9, 094047 [arXiv: 2311.14432]
- ***Brunello, Chestnov,*** & **PM,**
  Intersection Numbers from Companion Tensor Algebra
  2408.16668 [hep-th].
- ***Benincasa, Brunello, Mandal, Vazão,*** & **PM,**
  On one-loop corrections to the Bunch-Davies wavefunction of the universe
  2408.16386 [hep-th].

What we have found

Vector Space Structure of *Feynman* [- *Euler-Mellin - GKZ - A-hypegeometric*] Integrals

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

Master Integral = basis

$\nu =$ dimension of the vector space

- **Projections**

$$c_i = I \cdot J_i \ , \qquad J_i \cdot J_j = \delta_{ij}$$

- **Completeness**

$$\sum_i J_i \, J_i = \mathbb{I}_{\nu \times \nu}$$

Vector Space Structure of *Feynman* [- *Euler-Mellin - GKZ - A-hypegeometric*] Integrals

- **Vector decomposition** $$I = \sum_{i=1}^{\nu} c_i \, J_i$$ Master Integral = basis

$\nu =$ dimension of the vector space

- **Projections** $$c_i = I \cdot J_i \; , \qquad J_i \cdot J_j = \delta_{ij}$$

- **Completeness** $$\sum_i J_i \, J_i = \mathbb{I}_{\nu \times \nu}$$

The two questions:

1) what is the vector space dimension ν ?

2) what is the *scalar product* "$\cdot$" between integrals ?

Integrals as a Pairing

Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\text{domain}} \text{integrand}\, d^m\mathbf{z}$$

Integrals as a Pairing

Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\text{domain}} \text{integrand}\, d^m\mathbf{z}$$

$$\text{integrand}\, d^m\mathbf{z} \equiv \Big(\text{multivalued f'n}\Big) \times \Big(\text{differential form}\Big)$$

$$= \int_{\text{domain}} \Big(\text{multivalued f'n}\Big)\Big(\text{differential form}\Big)$$

Integrals as a Pairing

Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\text{domain}} \text{integrand}\, d^m\mathbf{z}$$

$$\text{integrand}\, d^m\mathbf{z} \equiv \Big(\text{multivalued f'n}\Big) \times \Big(\text{differential form}\Big)$$

$$= \int_{\text{domain}} \Big(\text{multivalued f'n}\Big)\Big(\text{differential form}\Big)$$

$$= \boxed{\int_{\text{domain}} \text{multivalued f'n}} \equiv \boxed{\text{differential form}}$$

Pairing

The **domain** and the **diff. form** are elements of *certain* **vector spaces**

Integrals as a Pairing

Twisted Period Integrals

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\text{domain}} \text{integrand}\, d^m\mathbf{z}$$

$$\text{integrand}\, d^m\mathbf{z} \equiv \Big(\text{multivalued f'n}\Big) \times \Big(\text{differential form}\Big)$$

$$= \int_{\text{domain}} \Big(\text{multivalued f'n}\Big)\Big(\text{differential form}\Big)$$

$$= \int_{\text{domain}} \text{multivalued f'n} \equiv \text{differential form}$$

Pairing

The **domain** and the **diff. form** are elements of *certain* **vector spaces**

Important property:

$$\Big(\text{multivalued f'n}\Big)\Big|_{\partial(\text{domain})} = 0 \quad \Longrightarrow \quad \int_{domain} d\Big(\text{integrand}\Big) d^m\mathbf{z} = 0 = \int_{\partial(domain)} \Big(\text{integrand}\Big) d^m\mathbf{z}$$

Basics of Intersection Theory

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \; \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

$\varphi_m(\mathbf{z})$ is a differential m-form

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\mathcal{C}} u(\mathbf{z}) \; \varphi_m(\mathbf{z})$$

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

$\varphi_m(\mathbf{z})$ is a differential m-form

- **Gamma function**: $u = z^{s-1} e^{-z}$, $\mathcal{C} = (0, \infty)$, $\varphi_1 = \frac{dz}{z^{n_1}}$
- **Euler Beta function**: $u = z^a (z-1)^b$, $\mathcal{C} = (0,1)$, $\varphi_1 = \frac{dz}{z^{n_1}(z-1)^{n_2}}$
- **2F1 Hypergeometric**: $u = z^a (z-1)^b (z-1/x)^c$, $\mathcal{C} = (0,1)$, $\varphi_1 = \frac{dz}{z^{n_1}(z-1)^{n_2}(z-1/x)^{n_3}}$
- **… and many more**: …… …… ……

$n_i \in \mathbb{Z}$

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \; \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

$\varphi_m(\mathbf{z})$ is a differential m-form

- **The dawn of Integration by parts identities:**
  - **Equivalence Classes of DIFFERENTIAL FORMS**

    There could exist many forms φ_m that upon integration give the same result I
  - **Equivalence Classes of INTEGRATION CONTOURS**

    There could exist many contours $\mathcal{C}$ that do not alter the the result of I

Vector Space Structure of Twisted Period Integrals

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

- **Integral invariance** from the **vanishing of total differential**
- **Stokes' theorem** relating the invariance upon shifting the differential forms to the invariance upon contour deformation!

$$0 = \int_C d(u\,\varphi) = \int_{\partial C} u\,\varphi$$

$$\int_C u\,\varphi = \int_C u\,(\varphi + \nabla_\omega \phi) = \int_{C+\partial\Gamma} u\,\varphi$$

- **Covariant Derivative** $\nabla_\omega \equiv d + \omega\wedge \ \equiv u^{-1}\cdot d \cdot u$ $\quad \omega \equiv d\log u$

$$u \to u^{-1}$$

$$0 = \int_C d(u^{-1}\varphi) = \int_{\partial C} u^{-1}\varphi$$

$$\int_C u^{-1}\varphi = \int_C u^{-1}(\varphi + \nabla_{-\omega}\phi) = \int_{C+\partial\Gamma} u^{-1}\varphi$$

- **Dual Covariant Derivative** $\nabla_{-\omega} \equiv d - \omega\wedge \ \equiv u \cdot d \cdot u^{-1}$

De Rham Twisted Co-Homology Groups

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H_{\pm\omega}^m$ are **isomorphic** [same dimension] [same # of generators]

- **Cohomology group**

$$H_\omega^m(X) = \frac{\mathrm{Ker}(\nabla_\omega : \varphi_m \to \varphi_{m+1})}{\mathrm{Im}(\nabla_\omega : \varphi_{m-1} \to \varphi_m)}$$

Closed modulo exact m-forms

- **Dual Cohomology group**

$$H_{-\omega}^m(X) = \frac{\mathrm{Ker}(\nabla_{-\omega} : \varphi_m \to \varphi_{m+1})}{\mathrm{Im}(\nabla_{-\omega} : \varphi_{m-1} \to \varphi_m)}$$

Closed modulo exact dual m-forms

- **Homology group**

$$H_m^\omega(X) = \frac{\mathrm{Ker}(\partial \otimes u : \mathcal{C}_m \to \mathcal{C}_{m-1})}{\mathrm{Im}(\partial \otimes u : \mathcal{C}_{m+1} \to \mathcal{C}_m)}$$

m-cycles modulo boundaries

- **Dual Homology group**

$$H_m^{-\omega}(X) = \frac{\mathrm{Ker}(\partial \otimes u^{-1} : \mathcal{C}_m \to \mathcal{C}_{m-1})}{\mathrm{Im}(\partial \otimes u^{-1} : \mathcal{C}_{m+1} \to \mathcal{C}_m)}$$

Dual m-cycles modulo boundaries

De Rham Twisted Co-Homology Groups / **Elements**

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H^m_{\pm\omega}$ are **isomorphic** [same dimension] [same # of generators]

$$\langle\varphi_L| \equiv \varphi_L(\mathbf{z}) \in H^m_\omega$$

Closed modulo exact m-forms

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

Closed modulo exact dual m-forms

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^\omega$$

m-cycles modulo boundaries

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

Dual m-cycles modulo boundaries

De Rham Twisted Co-Homology Groups / **Pairing / Integrals**

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H^m_{\pm\omega}$ are **isomorphic** [same dimension] [same # of generators]

- **Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$\langle\, \varphi_L \mid \mathcal{C}_R \,] \;\equiv\; \int_{\mathcal{C}_R} u(\mathbf{z})\; \varphi_L(\mathbf{z}) \;=\; I$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^\omega \qquad [\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / **Pairing / Dual Integrals**

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H^m_{\pm\omega}$ are **isomorphic** [same dimension] [same # of generators]

- **Dual Integrals ::** pairings of cycles and co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_{\omega}$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$[\, \mathcal{C}_L \mid \varphi_R \,\rangle \;\equiv\; \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}\, \varphi_R(\mathbf{z}) \;=\; \tilde{I}$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

De Rham Twisted Co-Homology Groups / **Pairing / Homology Intersection Number**

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H^m_{\pm\omega}$ are **isomorphic** [same dimension] [same # of generators]

- **Intersection numbers for cycles ::** pairings of cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$\left[\, \mathcal{C}_{\mathrm{L}} \,|\, \mathcal{C}_{\mathrm{R}} \,\right] \equiv \text{intersection number}$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^\omega$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

- **Generalising Gauss' linking number**

De Rham Twisted Co-Homology Groups / **Pairing / Cohomology Intersection Number**

(dual) Homology groups $H_m^{\pm\omega}$ and (dual) Co-homology groups $H^m_{\pm\omega}$ are **isomorphic** [same dimension] [same # of generators]

- **Intersection numbers for co-cycles ::** pairings of co-cycles

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_{\omega}$$

$$|\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega}$$

$$\langle\, \varphi_{\mathrm{L}} \mid \varphi_{\mathrm{R}} \,\rangle \equiv \frac{1}{2\pi i}\int_{\mathcal{X}} \iota(\varphi_L) \wedge \varphi_R$$

$$|\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \in H_m^{\omega}$$

$$[\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \in H_m^{-\omega}$$

Identity Resolution

$$\dim H^n_{\pm\omega} = \dim H_n^{\pm\omega} \equiv \nu$$

- **Cohomology Space**

[vector space of differential forms]

Cohomology basis

$$\langle e_i| \in H^n_\omega$$

Dual Cohomology basis

$$|h_i\rangle \in H^n_{-\omega}$$

$$i = 1, \ldots, \nu$$

Identity resolution

$$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|$$

Metric matrix for Forms

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

Identity Resolution

$$\dim H^n_{\pm\omega} = \dim H_n^{\pm\omega} \equiv \nu$$

- **Cohomology Space**

[vector space of differential forms]

Cohomology basis

$\langle e_i | \in H^n_\omega$

Dual Cohomology basis

$| h_i \rangle \in H^n_{-\omega}$

$i = 1, \ldots, \nu$

Identity resolution

$$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|$$

Metric matrix for Forms

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

- **Homology Space**

[vector space of integration contours]

Homology basis

$|\gamma_i] \in H_n^\omega$

Dual Homology basis

$[\eta_i| \in H_n^{-\omega}$

$i = 1, \ldots, \nu$

Identity resolution

$$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] \left(\mathbf{H}^{-1}\right)_{ij} [\eta_j|$$

Metric Matrix for Contours

$$\mathbf{H}_{ij} \equiv [\eta_i | \gamma_j]$$

Linear Relations

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Master Integrals from Master Forms**

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}_R} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}_R] \,, \qquad i = 1, \ldots, \nu \,,$$

- **Integral Decomposition**

$$I = \int_{\mathcal{C}_R} u(\mathbf{z})\ \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} c_i\, J_i$$

- **Decomposition of Differential Forms**

$$\langle \varphi_L | \;=\; \langle \varphi_L | \mathbb{I}_c = \langle \varphi_L | \underbrace{\sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j |}_{=1}$$

Linear Relations / IBPs identity / Gauss contiguity relations

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Master Integrals from Master Forms**

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}_R} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}_R] \,, \qquad i = 1, \ldots, \nu \,,$$

- **Integral Decomposition**

$$I = \int_{\mathcal{C}_R} u(\mathbf{z})\ \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} \boxed{c_i} J_i \,.$$

- **Decomposition of Differential Forms**

- **Master Decomposition Formula**

$$\langle \varphi_L | \;=\; \langle \varphi_L | \mathbb{I}_c = \sum_{i=1}^{\nu} c_i \, \langle e_i | \,, \quad \text{with} \qquad \boxed{c_i} = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

coefficients depend on the basis choice
but **do not depend** on the dual basis choice

Quadratic Relations

Riemann Bilinear Relations

Riemann bilinear relations for periods of closed holomorphic (non-twisted) differentials forms

$$\langle \phi_L | \phi_R \rangle = \int_\Sigma \phi_L \wedge \phi_R \quad = \quad \sum_{i=1}^{g} \left(\int_{a_i} \phi_L \int_{b_i} \phi_R - \int_{b_i} \phi_L \int_{a_i} \phi_R \right)$$

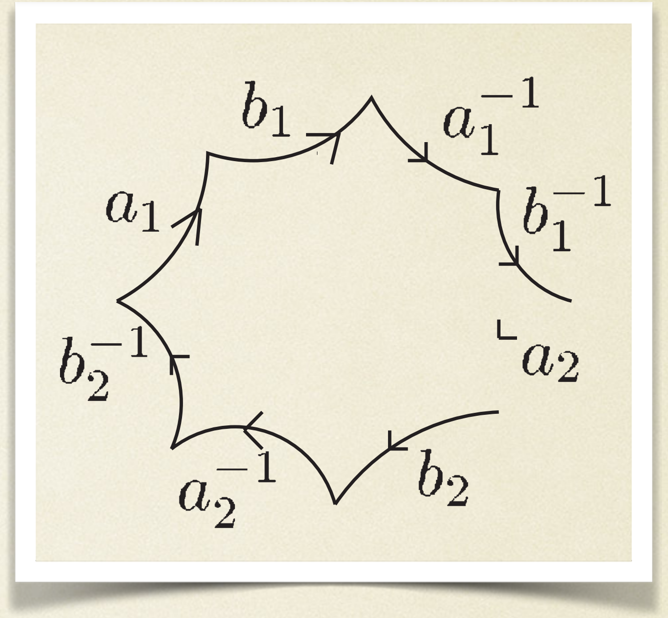
where Σ is an oriented Riemann surface of genus $g > 0$, built out of a $4g$-gon with edges $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$ (where the exponent ± 1 stands for clock/anticlockwise orientation) and gluing each edge with its inverse. The integration contours a_i and b_i, for $i = 1, \ldots g$, are a canonical bases of cycles, hence intersect *transversally*, *i.e.* their pairwise intersection numbers are: $a_i \cdot a_j = b_i \cdot b_j = 0$, and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$. Riemann bilinear relation can be cast as,

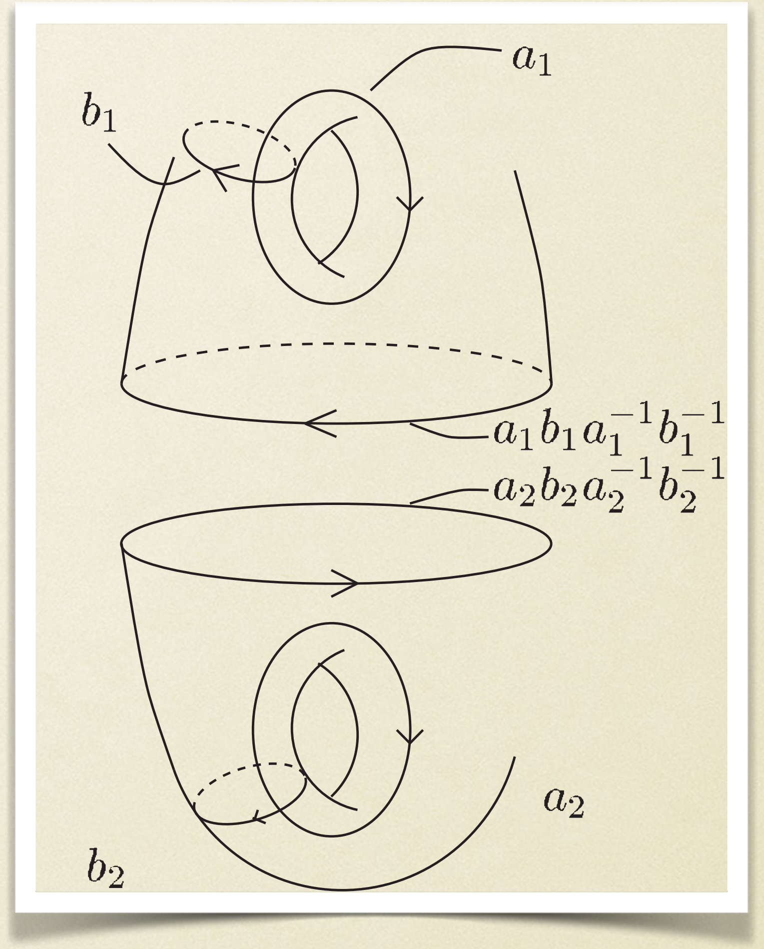
$$\langle \phi_L | \phi_R \rangle = \sum_{i,j}^{2g} \int_{\gamma_i} \phi_L \; (\mathbf{H}^{-1})_{ij} \int_{\gamma_j} \phi_R \, ,$$

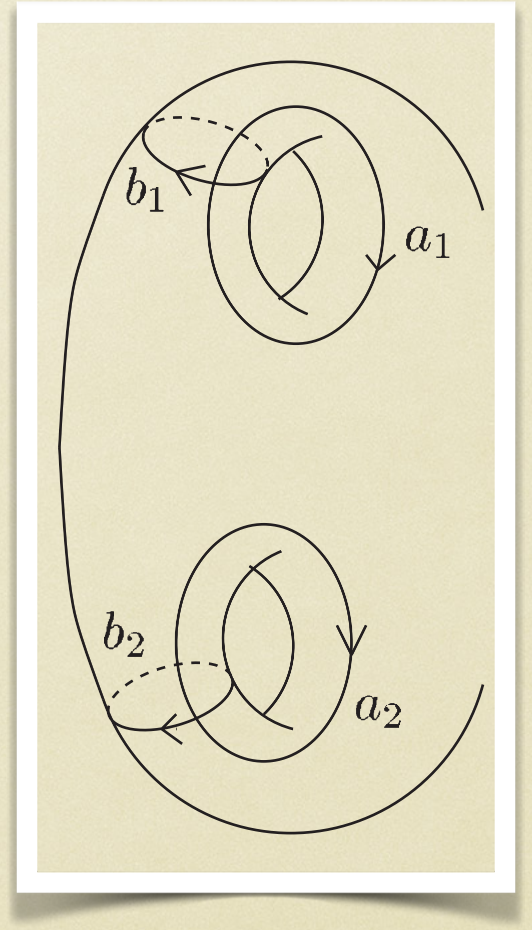
where $\{\gamma_i\}_{i=1,\ldots,g} = a_i$ and $\{\gamma_i\}_{i=g+1,\ldots,2g} = b_i$, and $\mathbf{H}_{ij} = [\gamma_i | \gamma_j]$, namely

$$\mathbf{H} = \begin{pmatrix} 0 & \mathbb{I}_{g\times g} \\ -\mathbb{I}_{g\times g} & 0 \end{pmatrix}, \qquad \text{yielding} \quad \mathbf{H}^{-1} = \begin{pmatrix} 0 & -\mathbb{I}_{g\times g} \\ \mathbb{I}_{g\times g} & 0 \end{pmatrix},$$

and $\mathbb{I}_{g\times g}$ is the identity matrix in the $(g \times g)$-space.







Pics: Jost, *Compact Riemann Surfaces*

Twisted Riemann Periods Relations (TRPR)

Cho, Matsumoto (1995)

$$\langle\varphi_L|\varphi_R\rangle \;=\; \langle\varphi_L|\mathbb{I}_h|\varphi_R\rangle = \sum_{i,j=1}^{\nu} \langle\varphi_L|\gamma_i]\left(\mathbf{H}^{-1}\right)_{ij}[\eta_j|\phi_R\rangle = \sum_{i,j}^{\nu}\int_{\gamma_i} u\,\varphi_L \left(\mathbf{H}^{-1}\right)_{ij}\int_{\eta_j} u^{-1}\varphi_R$$

$=1$

$$[C_L|C_R] \;=\; [C_L|\mathbb{I}_c|C_R] = \sum_{i,j=1}^{\nu} [C_L|h_i\rangle\left(\mathbf{C}^{-1}\right)_{ij}\langle e_j|C_R] = \sum_{i,j}^{\nu}\int_{C_L} u^{-1} h_i \left(\mathbf{C}^{-1}\right)_{ij}\int_{C_R} u\, e_j$$

$=1$

Generalising Riemann Bilinear Relations

Vector Space Structure of Feynman Integrals

Vector Space Dimensions / counting “holes”

Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

Number of **Master Integrals**

Betti numbers

Number of **Critical Points** Lee, Pomeranski (2013)

Relation with χ_E

Aluffi, Marcolli (2008)

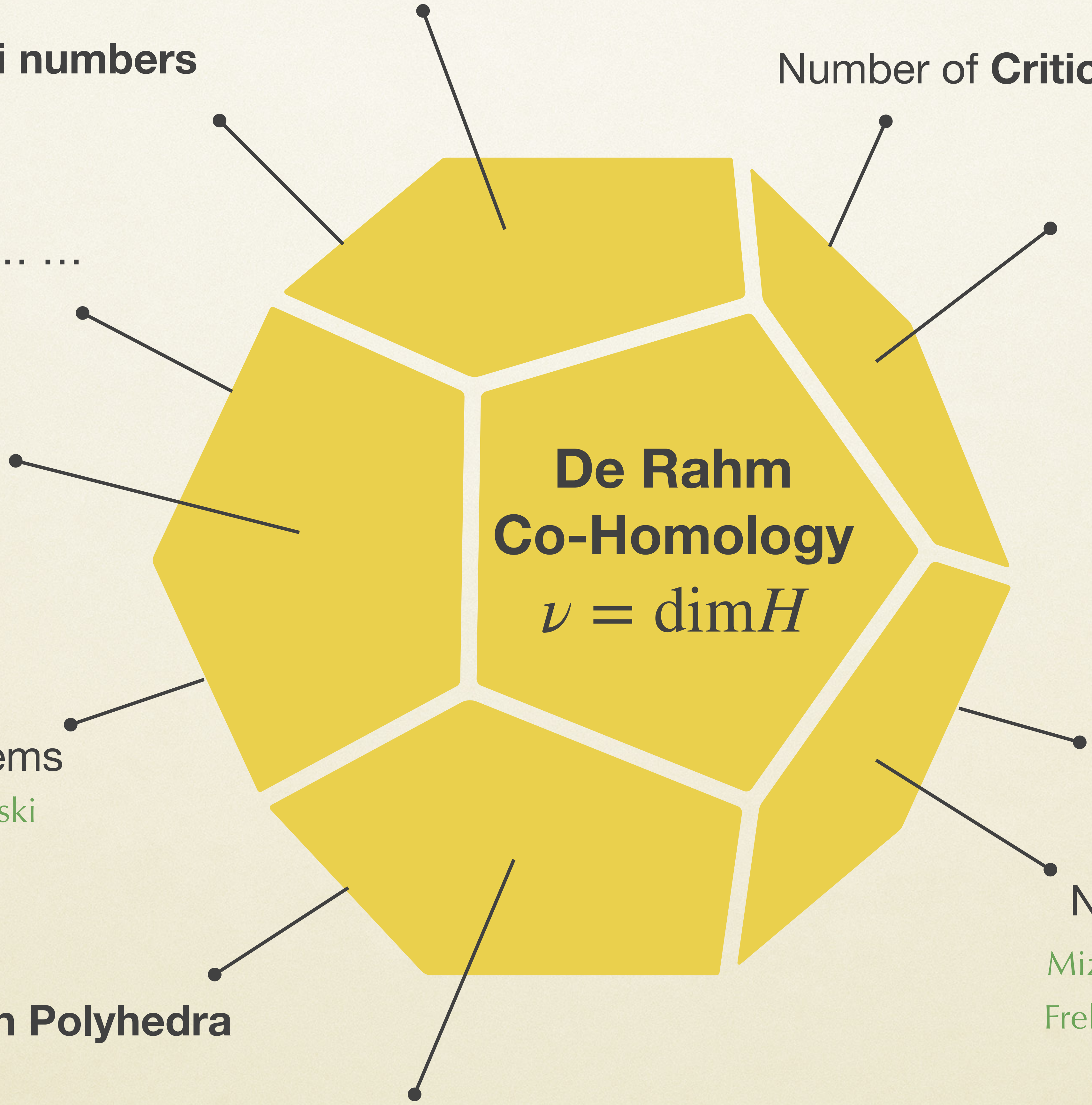
Bitoun, Bogner, Klausen, Panzer (2018)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

… … … … …

Maximum likelihood degree

Agostini, Brysiewicz, Fevola, Sturmfels, Tellen (2021)



Bosma, Sogaard, Zhang (2017) Primo, Tancredi (2017)

Number of **Independent Contours**

Holonomic rank of GKZ systems

Gelfand Kapranov Zelevinski

Number of **Independent Forms**

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Mixed volume of Newton Polyhedra

Bernstein-Khobaskii-Kushnirenko

Saito Sturmfels Takayama

Ideal saturation / **dimension of quotient space**

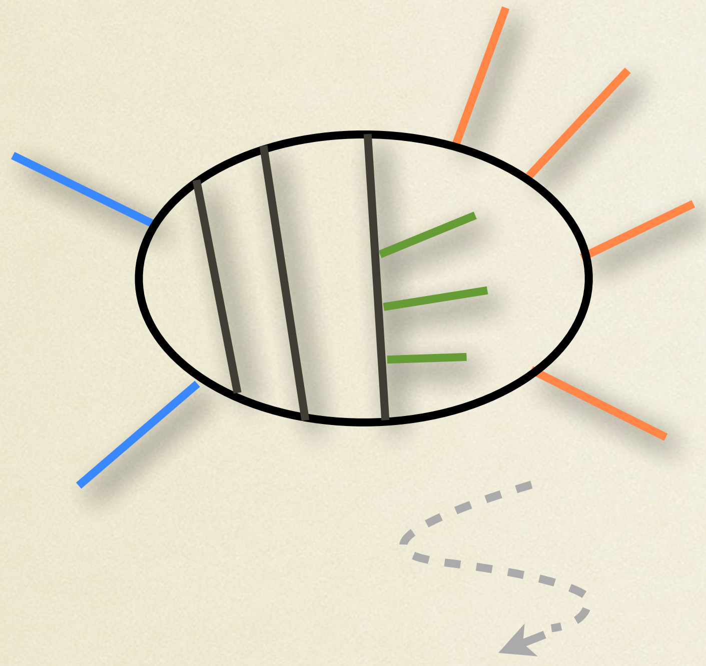
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2020)

Parametric Representation(s)

Mizera & **P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & **P.M.** (2019, 2020)

- **Upon a change of integration variables**



N-denominator
generic Integral

$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})$$

$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N \mathbf{z}$ **differential N-form**

$$d^N\mathbf{z} = dz_1 \wedge \ldots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$\mathcal{P}(\mathbf{z}) =$ **graph-Polynomial**

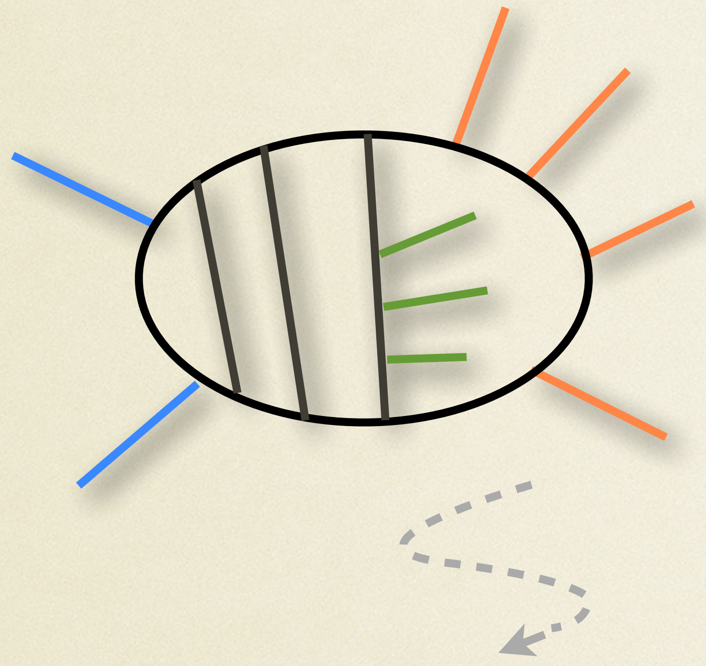
$\gamma(d) =$ **generic exponent**

Feynman Integrals :: *Baikov Representation*

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019, 2020)

- **Denominators as integration variables** Baikov (1996) Frellesvig and Papadopoulos (2017)



$$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$$

$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} B(\mathbf{z})^{\gamma} \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}}$$

$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \ldots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \ldots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z})$$

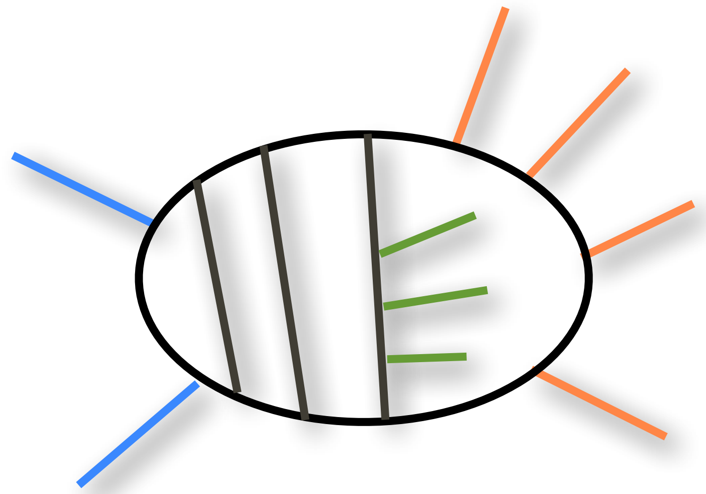
Gram determinant

$$\gamma \equiv (d - E - L - 1)/2$$

Vector Space of Feynman Integrals

Mizera & **P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & **P.M.** (2019, 2020)



$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} B(\mathbf{z})^\gamma \frac{d^n\mathbf{z}}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \qquad u = B^\gamma\,, \qquad \gamma \equiv (d-E-L-1)/2$$

- **Vector space dimension**

$$\omega \equiv \sum_{i=1}^{n} \hat{\omega}_i \, \mathrm{d}z_i = \mathrm{d}\log(u) \qquad \mathbb{Z}_\omega = \{\text{zeroes of } \omega\} \qquad \mathbb{P}_\omega = \{\text{poles of } \omega\} \cup \{\infty\}$$

$$\nu \equiv \dim(H^n_{\pm\omega}) = \dim(\mathbb{Z}_\omega) = (-1)^n(n+1-\chi(\mathbb{P}_\omega)) = \text{number of solutions of the system} \begin{cases} \omega_1 = 0 \\ \vdots \\ \omega_n = 0 \end{cases}$$

(Zero-dimensional)

- **(dual) bases choices:** Master Forms for Master Integrals

$$\langle e_i| \in H^n_\omega \qquad |h_i\rangle \in H^n_{-\omega} \qquad i = 1, \ldots, \nu$$

- **Decomposing Forms for Decomposing Integrals**

$$\langle\varphi| = c_1 \langle e_1| + c_2 \langle e_2| + c_3 \langle e_3| + \ldots + c_\nu \langle e_\nu| \qquad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1}\right)_{ji}$$

Four special applications:

i) Differential Equations / Pfaffian system

Mizera & P.M. (2018)
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- External Derivative

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C}] = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C}]$$

- External (connection) dLog-form

$$\nabla_{x,\sigma} \equiv \partial_x + \sigma \qquad \sigma = \partial_x \log u$$

- Derivative of Master Forms

$$\partial_x \langle e_i | = \langle \nabla_{x,\sigma} e_i | = \langle \nabla_{x,\sigma} e_i | h_k \rangle (C^{-1})_{kj} \langle e_j | = \mathbf{\Omega}_{ij} \langle e_j |$$

$\underbrace{| h_k \rangle (C^{-1})_{kj} \langle e_j |}_{=1}$

- System of DEQ for Master Forms

$$\partial_x \langle e_i | = \mathbf{\Omega}_{ij} \langle e_j | \, , \qquad \mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

An analogous System of DEQ can be derived for dual forms: $u \to u^{-1} \implies \nabla_{x,\sigma} \to \nabla_{x,-\sigma}$

ii) Differential Equations / Higher-Order DEQ

- **Generic Bases**

$$\begin{pmatrix} \langle e_1| \\ \langle e_2| \\ \langle e_3| \\ \vdots \\ \langle e_\nu| \end{pmatrix}$$

- **Decomposition**

$$\langle\varphi| = c_1\langle e_1| + c_2\langle e_2| + c_3\langle e_3| + \ldots + c_\nu\langle e_\nu|$$

- **Special Bases 1**

$$\begin{pmatrix} \langle e_i| \\ \partial_x\langle e_i| \\ \partial_x^2\langle e_i| \\ \vdots \\ \partial_x^{\nu-1}\langle e_i| \end{pmatrix}$$

- **Decomposition**

$$\partial_x^\nu\langle e_i| = a_{i,0}\langle e_i| + a_{i,1}\,\partial_x\langle e_i| + a_{i,2}\,\partial_x^2\langle e_i| + \ldots + a_{i,\nu-1}\,\partial_x^{\nu-1}\langle e_i|$$

- **Higher-order Diff.Eq. for the i-th Master Form**

$$\sum_{j=0}^{\nu} a_{i,j}\,\partial_x^j\langle e_i| = 0\ , \qquad (a_{i,\nu} \equiv -1)$$

ii) Differential Equations / Higher-Order DEQ

- **Generic Bases**

$$\begin{pmatrix} \langle e_1| \\ \langle e_2| \\ \langle e_3| \\ \vdots \\ \langle e_\nu| \end{pmatrix}$$

- **Decomposition**

$$\langle \varphi| = c_1 \langle e_1| + c_2 \langle e_2| + c_3 \langle e_3| + \ldots + c_\nu \langle e_\nu|$$

- **Special Bases 1**

$$\begin{pmatrix} \langle e_i| \\ \partial_x \langle e_i| \\ \partial_x^2 \langle e_i| \\ \vdots \\ \partial_x^{\nu-1} \langle e_i| \end{pmatrix}$$

- **Decomposition**

$$\partial_x^\nu \langle e_i| = a_{i,0} \langle e_i| + a_{i,1} \partial_x \langle e_i| + a_{i,2} \partial_x^2 \langle e_i| + \ldots + a_{i,\nu-1} \partial_x^{\nu-1} \langle e_i|$$

- **Higher-order Diff.Eq. for the i-th Master Integral**

$$\sum_{j=0}^{\nu} a_{i,j} \partial_x^j J_i = 0 \, , \qquad (a_{i,\nu} \equiv -1)$$

iii) Finite Difference Equation / Dimension-shift equation

Mizera & P.M. (2018)
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Generic Bases**

$$\begin{pmatrix} \langle e_1| \\ \langle e_2| \\ \langle e_3| \\ \vdots \\ \langle e_\nu| \end{pmatrix}$$

- **Decomposition**

$$\langle \varphi| = c_1 \langle e_1| + c_2 \langle e_2| + c_3 \langle e_3| + \ldots + c_\nu \langle e_\nu|$$

- **Special Bases 2**

$$\begin{pmatrix} \langle e_i| \\ \langle B\, e_i| \\ \langle B^2\, e_i| \\ \vdots \\ \langle B^{\nu-1} e_i| \end{pmatrix}$$

$$u = B^\gamma\,, \qquad \gamma \equiv (d-E-L-1)/2$$

$$J_i^{[d]} = \int_C u\, e_i = \langle e_i | C]$$

$$J_i^{[d+2j]} = \int_C u\, B^j\, e_i = \langle B^j\, e_i | C]$$

- **Decomposition**

$$\langle B^\nu e_i| = b_{i,0} \langle e_i| + b_{i,1} \langle B e_i| + b_{i,2} \langle B^2 e_i| + \ldots + b_{i,\nu-1} \langle B^{\nu-1} e_i|$$

- **Finite Difference Equation for the i-th Master Form**

$$\sum_{j=0}^{\nu} b_{i,j} \langle B^j e_i| = 0\ , \qquad (b_{i,\nu} \equiv -1)$$

iii) Finite Difference Equation / Dimension-shift equation

Mizera & P.M. (2018)
Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Generic Bases**

$$\begin{pmatrix} \langle e_1| \\ \langle e_2| \\ \langle e_3| \\ \vdots \\ \langle e_\nu| \end{pmatrix}$$

- **Decomposition**

$$\langle \varphi| = c_1 \langle e_1| + c_2 \langle e_2| + c_3 \langle e_3| + \ldots + c_\nu \langle e_\nu|$$

- **Special Bases 2**

$$\begin{pmatrix} \langle e_i| \\ \langle B\, e_i| \\ \langle B^2\, e_i| \\ \vdots \\ \langle B^{\nu-1} e_i| \end{pmatrix}$$

$$u = B^\gamma\,, \qquad \gamma \equiv (d-E-L-1)/2$$

$$J_i^{[d]} = \int_C u\, e_i = \langle e_i | C]$$

$$J_i^{[d+2j]} = \int_C u\, B^j\, e_i = \langle B^j\, e_i | C]$$

- **Decomposition**

$$\langle B^\nu e_i| = b_{i,0} \langle e_i| + b_{i,1} \langle B e_i| + b_{i,2} \langle B^2 e_i| + \ldots + b_{i,\nu-1} \langle B^{\nu-1} e_i|$$

- **Finite Difference Equation for the i-th Master Form Integral**

$$\sum_{j=0}^{\nu} b_{i,j}\, J_i^{[d+2j]} = 0\,, \qquad (b_{i,\nu} \equiv -1)$$

iv) Secondary Equation

Matsubara-Heo, Takayama (2019)

Weinzierl (2020)

Chestnov, Gasparotto, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

- **DEQ for forms**

$$\partial_x \langle e_i| = \Omega_{ij} \langle e_j| \qquad \Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$$

- **DEQ dual-forms**

$$\partial_x |h_i\rangle = \tilde{\Omega}_{ji} \, |h_j\rangle \qquad \tilde{\Omega}_{ji} = (\mathbf{C}^{-1})_{jk} \langle e_k | (\partial_x - \sigma_x) h_i \rangle$$

- **Secondary Equation for the Intersection Matrix**

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

$$\partial_x \mathbf{C} = \mathbf{\Omega}.\mathbf{C} + \mathbf{C}.\tilde{\mathbf{\Omega}} \,, \qquad \partial_x \mathbf{C}^{-1} = \tilde{\mathbf{\Omega}}.\mathbf{C}^{-1} - \mathbf{C}^{-1}.\mathbf{\Omega}$$

Intersection Numbers for 1-forms

Intersection Numbers for 1-forms

- **Zeroes and Poles of** ω

$$\omega \equiv d\log(u) = \gamma d\log(B) \qquad \nu = \text{number of critical points} \in Z(\omega) \qquad P(\omega) = \{\text{poles of } \omega, \text{including } \infty\}$$

- **Calculus and Differential Forms**

two closed forms $\varphi_1 \wedge \varphi_2$

$$\int_X \varphi_1 \wedge \varphi_2 = \int_X d\Omega = \int_{\partial X} \Omega = \sum_{p \in \text{Poles}} \text{Res}_{z=p}\big(\Omega\big) \qquad d\psi_1 = \varphi_1 \qquad \Omega \equiv \psi_1 \varphi_2$$

- **Intersection Number for twisted cocycles (1-form)** Cho, Matsumoto (1996)

$$\varphi_1 \equiv u\,\varphi_L\,, \qquad \varphi_2 \equiv u^{-1}\,\varphi_R \qquad \psi_1 \equiv u\,\psi_L$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X (u\varphi_L) \wedge (u^{-1}\varphi_R)$$

$$= \frac{1}{2\pi i} \sum_{z_i \in P(\omega)} \oint_{\gamma_i} \psi_i\,\varphi_R = \sum_{z_i \in P(\omega)} \text{Res}_{z=z_i}\big(\psi_i\,\varphi_R\big)$$

$$\nabla_\omega \psi_i = \varphi_L\,, \quad \text{for } z \to z_i \in P(\omega)$$

Intersection Numbers for 1-forms

- **Zeroes and Poles of** ω

$$\omega \equiv d\log(u) = \gamma d\log(B) \qquad \nu = \text{number of critical points} \in Z(\omega) \qquad P(\omega) = \{\text{poles of } \omega, \text{including } \infty\}$$

- **Calculus and Differential Forms**

two closed forms $\varphi_1 \wedge \varphi_2$

$$\int_X \varphi_1 \wedge \varphi_2 = \int_X d\Omega = \int_{\partial X} \Omega = \sum_{p \in \text{Poles}} \text{Res}_{z=p}\Big(\Omega\Big) \qquad d\psi_1 = \varphi_1 \qquad \Omega \equiv \psi_1\, \varphi_2$$

- **Intersection Number for twisted cocycles (1-form)** Cho, Matsumoto (1996)

$$\varphi_1 \equiv u\, \varphi_L \;, \qquad \varphi_2 \equiv u^{-1}\, \varphi_R \qquad \psi_1 \equiv u\, \psi_L$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X (u\varphi_L) \wedge (u^{-1}\varphi_R)$$

$$= \frac{1}{2\pi i} \sum_{z_i \in P(\omega)} \oint_{\gamma_i} \psi_i\, \varphi_R = \sum_{z_i \in P(\omega)} \text{Res}_{z=z_i}\Big(\psi_i\, \varphi_R\Big)$$

$$\nabla_\omega \psi_i = \varphi_L \;, \quad \text{for } z \to z_i \in P(\omega)$$

$$c_m = \frac{1}{n} \sum_k t^{-mk} \sum_j c_j t^{jk}$$

$$\sum_j \iint_D \frac{\mathcal{F}^{(j)}_{\bar{z}}}{z - z_j} dz \wedge d\bar{z} = 2\pi i \sum_{j \in \text{poles}} \mathcal{F}^{(j)}(z_j)$$

Intersection Numbers for n-forms :: Iterative Method

Nested Integrations / Fibration-based approach

- **Multivariate integral decomposition**

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \int dz_1 \, f(z_n, \ldots, z_3, z_2, z_1)$$

$$J_i \equiv \int dz_n \ldots \int dz_3 \int dz_2 \int dz_1 \, f_i(z_n, \ldots, z_1)$$

[master integrals]

- **Fibrations: decompositions' tower**

$$I = \int dz_n \ldots \int dz_3 \int dz_2 \underbrace{\int dz_1 \, f(z_n, \ldots, z_3, z_2, z_1)}_{\exists \ \nu^{(1)} \ \text{master integrals in } z_1}$$

$$I = \int dz_n \ldots \int dz_3 \underbrace{\int dz_2 \sum_{i_1=1}^{\nu^{(1)}} c_{i_1}(z_n, \ldots, z_3, z_2) \, J_{i_1}(z_n, \ldots, z_3, z_2)}_{\exists \ \nu^{(2)} \ \text{master integrals in } z_2}$$

$$I = \int dz_n \ldots \underbrace{\int dz_3 \sum_{i_2=1}^{\nu^{(2)}} c_{i_2}(z_n, \ldots, z_3) \, J_{i_2}(z_n, \ldots, z_3)}_{\exists \ \nu^{(3)} \ \text{master integrals in } z_3}$$

$$\vdots$$

$$I = \underbrace{\int dz_n \sum_{i_n=1}^{\nu^{(n-1)}} c_{i_n}(z_n) \, J_{i_n}(z_n)}_{\exists \ \nu \ \text{master integrals in } z_n}$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

Intersection Numbers for n-forms (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- by *Induction*:
- (n-1)-form Vector Space: known!

$$\nu_{\mathbf{n-1}} \qquad \langle e_i^{(\mathbf{n-1})}| \qquad |h_i^{(\mathbf{n-1})}\rangle \qquad \left(\mathbf{C}_{(\mathbf{n-1})}\right)_{ij} \equiv \langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

- n-form decomposition: n = (n-1) + (n)

$$\langle \varphi_L^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle e_i^{(\mathbf{n-1})}| \wedge \langle \varphi_{L,i}^{(n)}| \,, \qquad \langle \varphi_{L,i}^{(n)}| = \langle \varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle \left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ji} \,, \qquad \langle \varphi_{L,i}^{(n)}| (C_{(\mathbf{n-1})})_{ij} = \langle \varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle$$

$$|\varphi_R^{(\mathbf{n})}\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |h_i^{(\mathbf{n-1})}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle \,, \qquad |\varphi_{R,i}^{(n)}\rangle = \left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ij} \langle e_j^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle \,, \qquad (C_{(\mathbf{n-1})})_{ij} |\varphi_{R,j}^{(n)}\rangle = \langle e_i^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle$$

Intersection Numbers for n-forms :: Recursive Formula

$$\begin{aligned}\langle \varphi_L^{(\mathbf{n})}|\varphi_R^{(\mathbf{n})}\rangle &= \sum_{i,j} \langle \varphi_L^{(\mathbf{n})}|h_j^{(\mathbf{n-1})}\rangle (C_{(\mathbf{n-1})})^{-1}_{ji} \langle e_i^{(\mathbf{n-1})}|\varphi_R^{(\mathbf{n})}\rangle \\ &= \sum_{i,j} \langle \varphi_{L,i}^{(n)}|(C_{(\mathbf{n-1})})_{ij}\, \varphi_{R,j}^{(n)}\rangle\end{aligned}$$

=1

$$\partial_{z_n} \psi_i^{(n)} + \psi_j^{(n)}\, \hat{\mathbf{\Omega}}_{ji}^{(n)} = \hat{\varphi}_{L,i}^{(n)} \,,$$

$\hat{\mathbf{\Omega}}^{(n)}$ is a $\nu_{\mathbf{n-1}} \times \nu_{\mathbf{n-1}}$ matrix, whose entries are given by $\hat{\mathbf{\Omega}}_{ji}^{(n)} = \langle (\partial_{z_n} + \hat{\omega}_n) e_j^{(\mathbf{n-1})}|h_k^{(\mathbf{n-1})}\rangle \left(\mathbf{C}_{(\mathbf{n-1})}^{-1}\right)_{ki}$

Intersection Numbers for n-forms (I)

Ohara (1998) Mizera (2019)
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

- **Property of Intersection Number**

invariance under differential forms redefinition within the same equivalence classes,

$$\langle \varphi_L | \varphi_R \rangle = \langle \varphi'_L | \varphi'_R \rangle \ , \qquad \varphi'_L = \varphi_L + \nabla_\omega \xi_L \ , \qquad \varphi'_R = \varphi_R + \nabla_{-\omega} \xi_R$$

- **Global Residue Thm** Weinzierl (2020)

choose ξ_L and ξ_R, to build φ'_L and φ'_R that contain only simple poles. and if $\hat{\Omega}^{(n)}$ is reduced to Fuchsian form

$\bigtriangledown$

the computation of multivariate intesection number can benefit of the evaluation of intersection numbers for dlog forms at each step of the iteration.

- **Special dual basis choice** CaronHuot Pokraka (2019-2021)

Relative Dirac-delta basis elements trivialise the evaluation of the intersection numbers

- **Multi-pole ansatz** Fontana Peraro (2023)

Solving $\nabla_\omega \psi = \varphi_L$, bypassing the pole factorisation, and using FF reconstruction methods.
(avoiding irrational functions which would disappear in the intersection numbers)

Contiguity relations & Differential Equations of Special Functions

- ☑ Gamma Functions
- ☑ Beta Functions
- ☑ Hypergeometric ${}_2F_1$
- ☑ Appel F_D
- ☑ Lauricella functions
- ☑ Hypergeometric ${}_3F_2$
- ☑ Euler-Mellin integrals

Lauricella F_D Functions

$$\beta(a, c-a)\, F_D(a, b_1, b_2, \ldots, b_m, c; x_1, \ldots, x_m) = \int_{\mathcal{C}} u\,\varphi = \ {}_{\omega}\langle \varphi | \mathcal{C}]$$

$$u = z^{a-1}\,(1-z)^{-a+c-1}\,\prod_{i=1}^{m}(1-x_i z)^{-b_i}\,,$$

$$\mathcal{C} = [0,1], \qquad \varphi = dz\,, \qquad \omega = d\log(u),$$

$$\nu = m+1, \qquad \mathcal{P} = \left\{0, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}, 1, \infty\right\}$$

$\nu = \dim H^1_{\pm\omega}$ = [number of P-poles - 2] = [number of P-poles - (1+1)]

Feynman Integrals Decomposition

Intersection Numbers for 1-forms (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal **& P.M.** (2023)

- **Polynomial Division** Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\mathrm{Res}_{\langle B \rangle}(g) \; - \; \mathrm{Res}_{z=\infty}(g)$$

$$g \; = \; \psi_R \, \varphi_L$$

$$\Big\lfloor \partial_z \psi_R(z,\beta) \; + \; \partial_\beta \psi_R(z,\beta)\, \partial_z B(z) - \omega\, \psi_R(z,\beta) - \varphi_R \Big\rfloor_{\mathcal{B}} \; = \; 0\,,$$

$$\psi_R = \sum_{i=min}^{max} \sum_{j=0}^{\kappa-1} \psi_{R,ij}\, z^j\, \beta^i \qquad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by **polynomial division** modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Simultaneous Residue at all zeros of B, hence avoiding algebraic extension and explicit polynomial factorisation

Intersection Numbers for 1-forms (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal **& P.M.** (2023)

- **Polynomial Division** Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\mathrm{Res}_{\langle B \rangle}(g) \; - \; \mathrm{Res}_{z=\infty}(g)$$

$$g \; = \; \psi_R \, \varphi_L$$

$$\Big[\partial_z \psi_R(z,\beta) \; + \; \partial_\beta \psi_R(z,\beta) \, \partial_z B(z) - \omega \, \psi_R(z,\beta) - \varphi_R \Big]_{\mathcal{B}} \; = \; 0 \, ,$$

$$\psi_R = \sum_{i=min}^{max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} \, z^j \, \beta^i \qquad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by polynomial division modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Simultaneous Residue at all zeros of B, hence avoiding algebraic extension and explicit polynomial factorisation

- **Delta-bases** Caron-Huot and Pokraka (2021)

$$\delta_z := \frac{u(z)}{u(0)} \, \mathrm{d}\theta_{z,0} \, , \qquad \langle \varphi_L \mid \delta_z \rangle := \frac{-1}{2\pi i} \int_{\mathcal{X}} \varphi_L \wedge \delta_z = \mathrm{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi_L \right)$$

Intersection Numbers for 1-forms (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal **& P.M.** (2023)

- **Polynomial Division** Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\mathrm{Res}_{\langle B \rangle}(g) \; - \; \mathrm{Res}_{z=\infty}(g)$$

$$g \; = \; \psi_R \, \varphi_L$$

$$\Big[\partial_z \psi_R(z,\beta) \; + \; \partial_\beta \psi_R(z,\beta)\, \partial_z B(z) - \omega\, \psi_R(z,\beta) - \varphi_R \Big]_{\mathcal{B}} \; = \; 0 \, ,$$

$$\psi_R = \sum_{i=min}^{max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} \, z^j \, \beta^i \qquad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by polynomial division modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Simultaneous Residue at all zeros of B, hence avoiding algebraic extension and explicit polynomial factorisation

- **Delta-bases** Caron-Huot and Pokraka (2021)

$$\delta_z := \frac{u(z)}{u(0)} \, \mathrm{d}\theta_{z,0} \, , \qquad \langle \varphi_L \, | \, \delta_z \rangle := \frac{-1}{2\pi i} \int_{\mathcal{X}} \varphi_L \wedge \delta_z = \mathrm{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi_L \right)$$

- **Ordinary Cohomology *vs* Relative Cohomology**

☑ Evanescent regulator limit

$$c_i = \lim_{\rho \to 0} \sum_{j=1}^{\nu} \langle \varphi_L \, | \, h_j \rangle \, \mathbf{C}^{-1}_{ji} \;\; = \sum_{j=1}^{\nu} \langle \varphi_L \, | \, h_j \rangle_{\mathrm{LT}} \, (\mathbf{C}^{-1}_{\mathrm{LT}})_{ji}$$

$$h_j \; \sim \; z^\tau \;\; \text{with} \;\; \tau \; < \; 0, \;\; \text{around} \;\; z \; = \; 0$$

$$\langle \eta \, | \, h_j \rangle_{\mathrm{LT}} \; = \langle \eta \, | \, \delta_z^{(-\tau)} \rangle \qquad \delta_z^{(k)} \sim \frac{\partial_k^{(k-1)} u(z)}{u(0)} \, \mathrm{d}\theta$$

Intersection Numbers for 1-forms (II)

Brunello, Chestnov, Crisanti, Frellesvig, Mandal **& P.M.** (2023)

- **Polynomial Division** Fontana Peraro (2023)

$$\langle \varphi_L | \varphi_R \rangle = -\mathrm{Res}_{\langle B \rangle}(g) - \mathrm{Res}_{z=\infty}(g)$$

$$g = \psi_R \, \varphi_L$$

$$\Big[\partial_z \psi_R(z,\beta) + \partial_\beta \psi_R(z,\beta) \, \partial_z B(z) - \omega \, \psi_R(z,\beta) - \varphi_R \Big]_{\mathcal{B}} = 0 \,,$$

$$\psi_R = \sum_{i=min}^{max} \sum_{j=0}^{\kappa-1} \psi_{R,ij} \, z^j \, \beta^i \qquad \beta = B(z)$$

where κ and ℓ_c are the degree and the leading coefficient of B

☑ Series expansion by polynomial division modulo $\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle$

☑ Simultaneous Residue at all zeros of B, hence avoiding algebraic extension and explicit polynomial factorisation

- **Delta-bases** Caron-Huot and Pokraka (2021)

$$\delta_z := \frac{u(z)}{u(0)} \, d\theta_{z,0} \,, \qquad \langle \varphi_L \mid \delta_z \rangle := \frac{-1}{2\pi i} \int_{\mathcal{X}} \varphi_L \wedge \delta_z = \mathrm{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi_L \right)$$

- **Ordinary Cohomology *vs* Relative Cohomology**

☑ Evanescent regulator limit

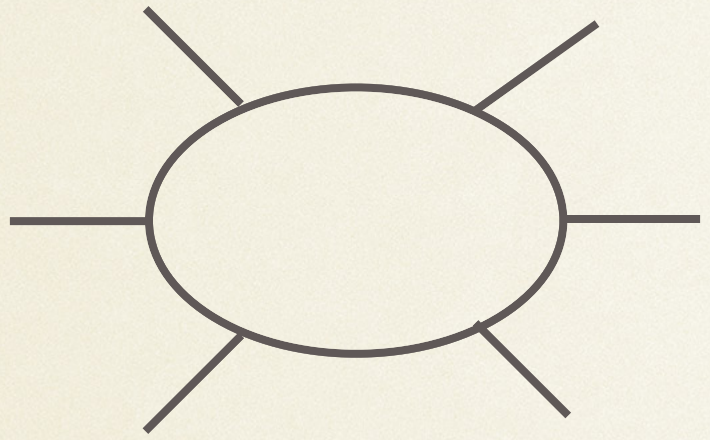
$$c_i = \lim_{\rho \to 0} \sum_{j=1}^{\nu} \langle \varphi_L \mid h_j \rangle \, \mathbf{C}^{-1}_{ji} = \sum_{j=1}^{\nu} \langle \varphi_L \mid h_j \rangle_{\mathrm{LT}} \, (\mathbf{C}^{-1}_{\mathrm{LT}})_{ji}$$

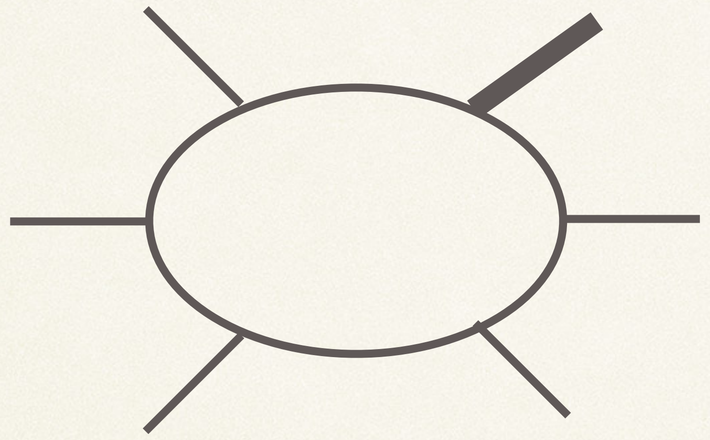
Simplifying Intersection Numbers for n-forms

Complete decomposition @ 1- & 2-Loop

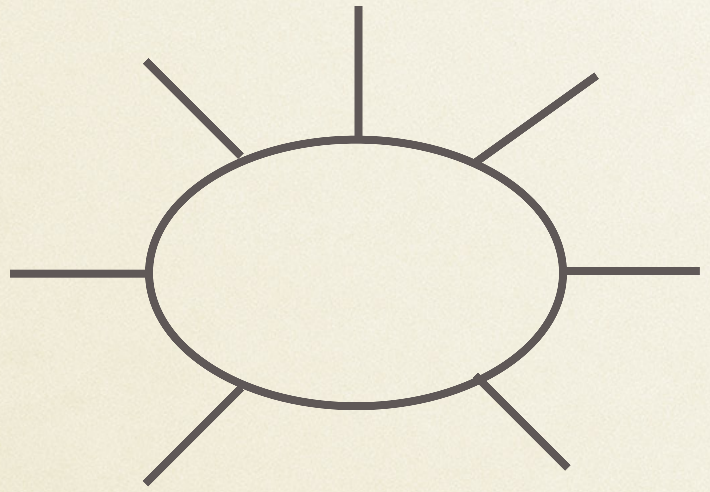
Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M. (2023)

☑ **1-Loop 6-point**

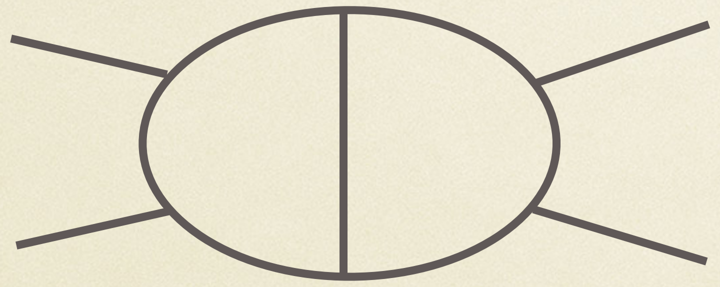




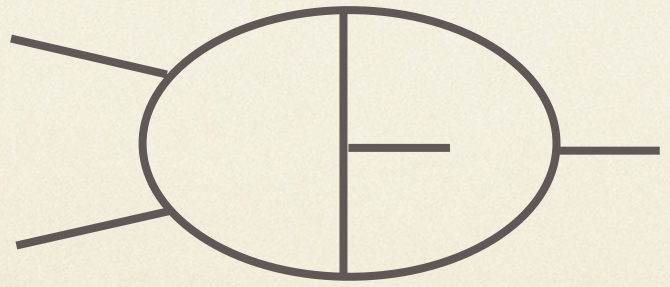
☑ **1-Loop 7-point**



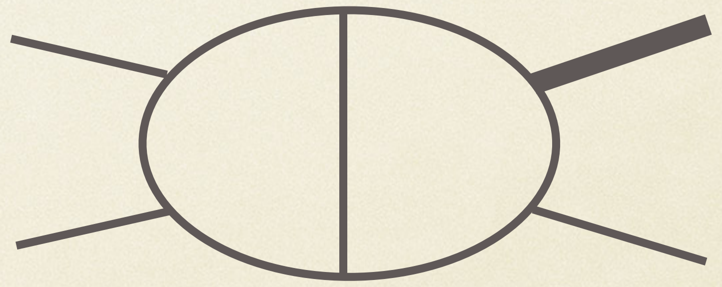
☑ **2-loop 4-point**



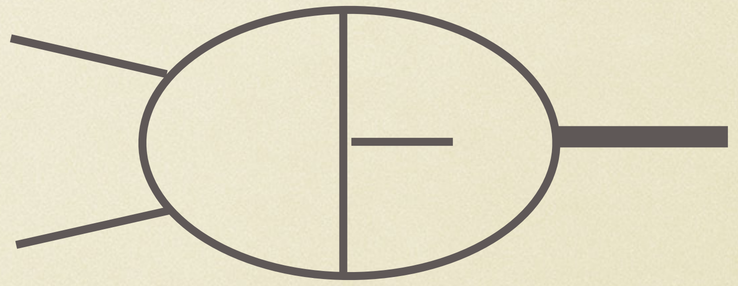
planar diagram



non-planar diagram



planar diagram



non-planar diagram

Orthogonal Bases for quadratic twists

Crisanti, Smith (2024)

- **Quadratic polynomial in the twist**

$$u(\mathbf{z}) = b(\mathbf{z})^\gamma \qquad \text{for } b(\mathbf{z}) \text{ quadratic}$$

- **Master Decomposition Formula**

$$\langle \varphi_L | = \langle \varphi_L | \mathbb{I}_c = \sum_{i=1}^{\nu} c_i \langle e_i | , \quad \text{with} \quad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \qquad \mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

coefficients depend on the basis choice
but **do not depend** on the **dual basis** choice

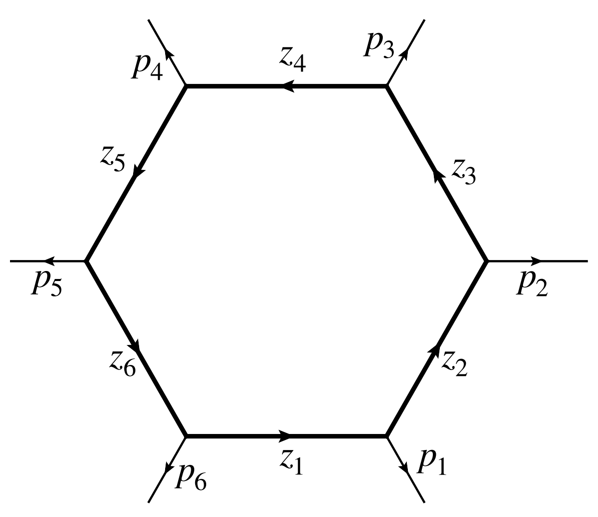
- **Special Dual Bases** — Combination of $\delta-$**forms** and **inverse powers of** $b(\mathbf{z})$ → **Diagonal C matrix**

- **Intersection Numbers and Resultants**

- **1-loop Feynman integrals**

Quadratic Baikov polynomial $b(\mathbf{z})$

- ☑ Bubbles
- ☑ Triangles
- ☑ Boxes
- ☑ Pentagons
- ☑ Hexagons



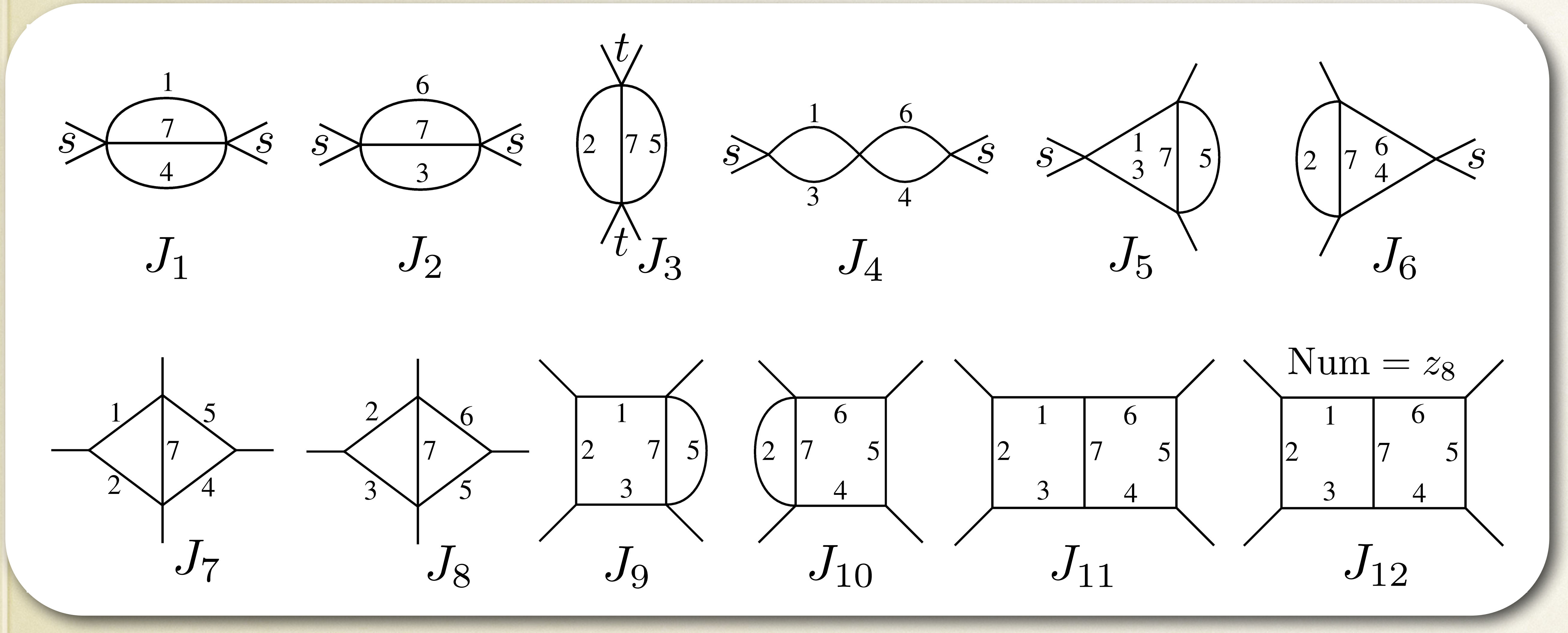
Cut $\{z_6\}$ $\qquad \nu = 32$ Master Integrals

$$e = \left\{1, \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_5}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_1 z_5}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_2 z_5}, \frac{1}{z_3 z_4}, \frac{1}{z_3 z_5}, \frac{1}{z_4 z_5}, \frac{1}{z_1 z_2 z_3}, \frac{1}{z_1 z_2 z_4}, \frac{1}{z_1 z_2 z_5}, \frac{1}{z_1 z_3 z_4}, \frac{1}{z_1 z_3 z_5}, \frac{1}{z_1 z_4 z_5}, \frac{1}{z_2 z_3 z_4}, \frac{1}{z_2 z_3 z_5}, \frac{1}{z_2 z_4 z_5}, \frac{1}{z_3 z_4 z_5}, \frac{1}{z_1 z_2 z_3 z_4}, \frac{1}{z_1 z_2 z_3 z_5}, \frac{1}{z_1 z_2 z_4 z_5}, \frac{1}{z_1 z_3 z_4 z_5}, \frac{1}{z_2 z_3 z_4 z_5}, \frac{1}{z_1 z_2 z_3 z_4 z_5}\right\}$$

$$h = \left\{\frac{1}{b^5}, \frac{\delta_1}{b^4}, \frac{\delta_2}{b^4}, \frac{\delta_3}{b^4}, \frac{\delta_4}{b^4}, \frac{\delta_5}{b^4}, \frac{\delta_{12}}{b^3}, \frac{\delta_{13}}{b^3}, \frac{\delta_{14}}{b^3}, \frac{\delta_{15}}{b^3}, \frac{\delta_{23}}{b^3}, \frac{\delta_{24}}{b^3}, \frac{\delta_{25}}{b^3}, \frac{\delta_{34}}{b^3}, \frac{\delta_{35}}{b^3}, \frac{\delta_{45}}{b^3}, \frac{\delta_{123}}{b^2}, \frac{\delta_{124}}{b^2}, \frac{\delta_{125}}{b^2}, \frac{\delta_{134}}{b^2}, \frac{\delta_{135}}{b^2}, \frac{\delta_{145}}{b^2}, \frac{\delta_{234}}{b^2}, \frac{\delta_{235}}{b^2}, \frac{\delta_{245}}{b^2}, \frac{\delta_{345}}{b^2}, \frac{\delta_{1234}}{b}, \frac{\delta_{1235}}{b}, \frac{\delta_{1245}}{b}, \frac{\delta_{1345}}{b}, \frac{\delta_{2345}}{b}, \delta_{12345}\right\}$$

Complete decomposition @ Planar double-box integral

Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M.



$z_1 = k_1^2\,, \quad z_2 = (k_1 - p_1)^2, \quad z_3 = (k_1 - p_1 - p_2)^2, \quad z_4 = (k_2 - p_1 - p_2)^2, \quad z_5 = (k_2 + p_4)^2,$

$z_6 = k_2^2\,, \quad z_7 = (k_1 - k_2)^2, \quad z_8 = (k_1 + p_4)^2, \quad z_9 = (k_2 - p_1)^2$

$p_i^2 = 0\,, \quad s = (p_1 + p_2)^2\,, \quad t = (p_1 + p_4)^2\,, \quad s + t + u = 0$

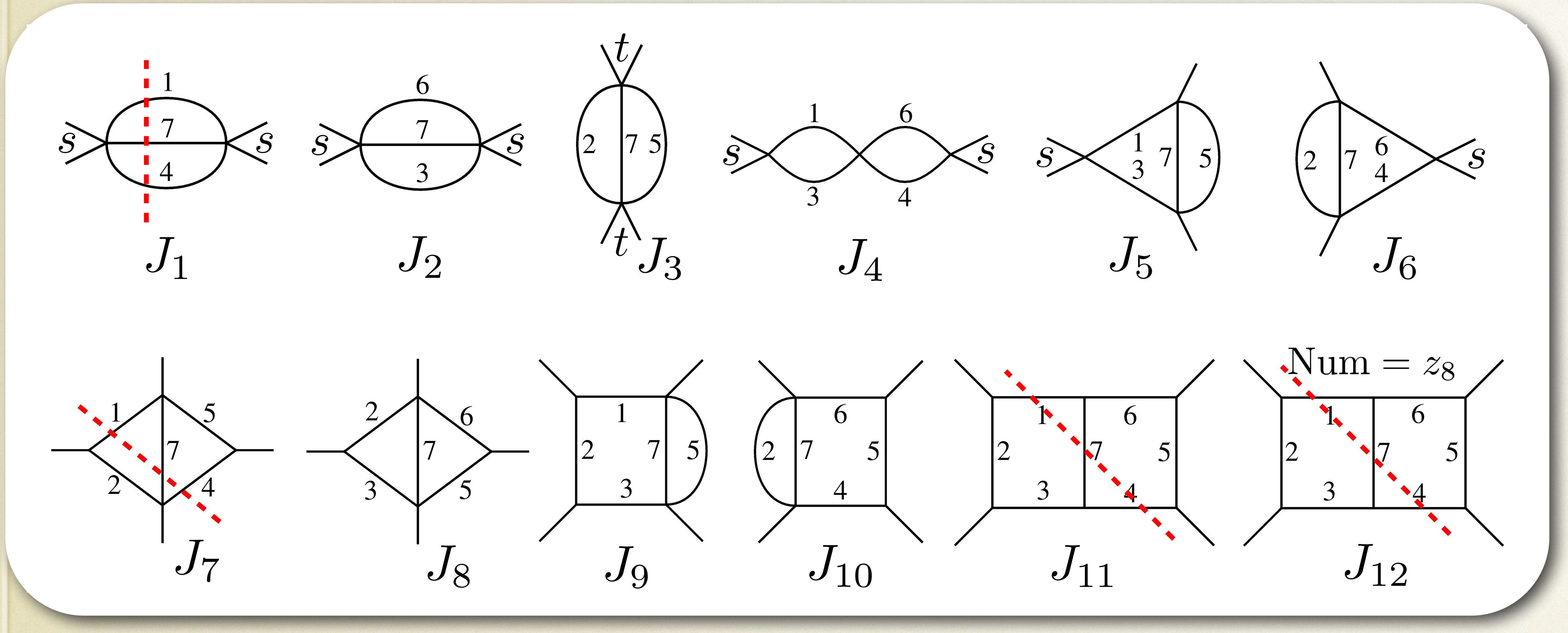
☑ **intersection numbers of (up to) 6-forms (instead of 9-forms)**

$$\text{spanning cuts} = \text{maximal cuts of } \{J_1, \ldots, J_6\}$$

$$I = \sum_{i=1}^{12} c_i \, J_i$$

Complete decomposition @ Planar double-box integral

Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M.



$z_1 = k_1^2\,,\quad z_2 = (k_1-p_1)^2,\quad z_3 = (k_1-p_1-p_2)^2,\quad z_4 = (k_2-p_1-p_2)^2,\quad z_5 = (k_2+p_4)^2,$

$z_6 = k_2^2\,,\quad z_7 = (k_1-k_2)^2,\quad z_8 = (k_1+p_4)^2,\quad z_9 = (k_2-p_1)^2$

$p_i^2 = 0\,,\quad s = (p_1+p_2)^2\,,\quad t = (p_1+p_4)^2\,,\quad s+t+u = 0$

☑ **intersection numbers of (up to) 6-forms (instead of 9-forms)**

spanning cuts $=$ maximal cuts of $\{J_1, \ldots, J_6\}$

Cut 147, maximal cut of J_1

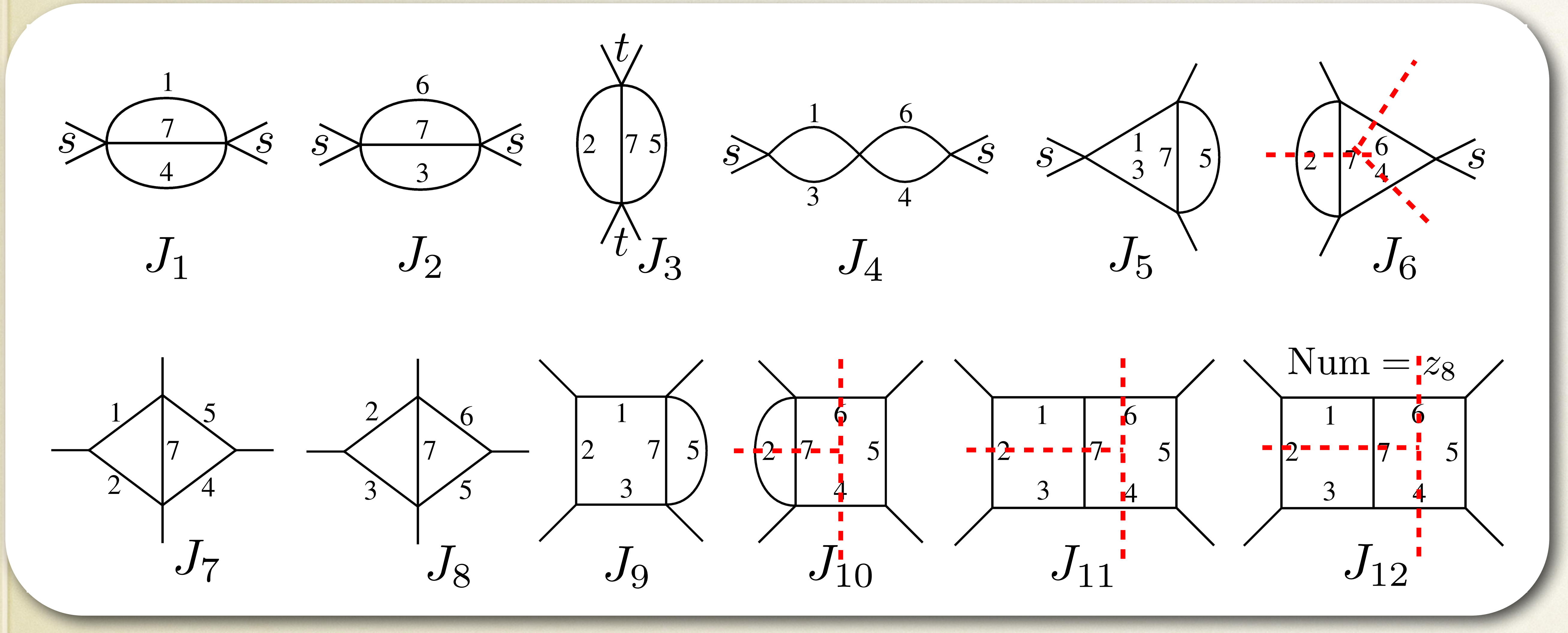
$$\nu^{(9)} = 1\,,\quad \nu^{(59)} = 2\,,\quad \nu^{(659)} = 2\,,\quad \nu^{(2659)} = 4\,,\quad \nu^{(82659)} = 5\,,\quad \nu^{(382659)} = 4$$

$$e^{(9)} = \{1\}\,,\quad e^{(59)} = \left\{1, \tfrac{1}{z_5}\right\}\,,\quad e^{(659)} = \left\{1, \tfrac{1}{z_5 z_6}\right\}\,,\quad e^{(2659)} = \left\{1, \tfrac{1}{z_2}, \tfrac{1}{z_5 z_6}, \tfrac{1}{z_2 z_5 z_6}\right\}\,,\quad e^{(82659)} = \left\{1, \tfrac{1}{z_5}, \tfrac{1}{z_2 z_5}, \tfrac{1}{z_2 z_5 z_6}, \tfrac{z_8}{z_2 z_5 z_6},\right\}\quad e^{(382659)} = \left\{1, \tfrac{1}{z_2 z_5}, \tfrac{1}{z_2 z_3 z_5 z_6}, \tfrac{z_8}{z_2 z_3 z_5 z_6}\right\}$$

$$h^{(9)} = \{1\}\,,\quad h^{(59)} = \{1, \delta_5\}\,,\quad h^{(659)} = \{1, \delta_{56}\}\,,\quad h^{(2659)} = \{1, \delta_2, \delta_{56}, \delta_{256}\}\,,\quad h^{(82659)} = \{1, \delta_5, \delta_{25}, \delta_{256}, z_8\delta_{256}\}\,,\quad h^{(382659)} = \{1, \delta_{25}, \delta_{2356}, z_8\delta_{2356}\}$$

Complete decomposition @ Planar double-box integral

Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M.



$z_1 = k_1^2\,, \quad z_2 = (k_1-p_1)^2, \quad z_3 = (k_1-p_1-p_2)^2, \quad z_4 = (k_2-p_1-p_2)^2, \quad z_5 = (k_2+p_4)^2,$

$z_6 = k_2^2\,, \quad z_7 = (k_1-k_2)^2, \quad z_8 = (k_1+p_4)^2, \quad z_9 = (k_2-p_1)^2$

$p_i^2 = 0\,, \quad s = (p_1+p_2)^2\,, \quad t = (p_1+p_4)^2\,, \quad s+t+u = 0$

☑ **intersection numbers of (up to) 6-forms (instead of 9-forms)**

spanning cuts $=$ maximal cuts of $\{J_1, \ldots, J_6\}$

Cut 147, maximal cut of J_1

Cut 367, maximal cut of J_2

$\cdots$

Cut 2467, maximal cut of J_6

$$\nu^{(8)} = 1\,, \quad \nu^{(58)} = 2\,, \quad \nu^{(358)} = 4\,, \quad \nu^{(1358)} = 4\,, \quad \boxed{\nu^{(91358)} = 4}$$

$$e^{(8)} = \{1\}\,, \quad e^{(58)} = \left\{1, \frac{1}{z_5}\right\}, \quad e^{(358)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_5}, \frac{1}{z_3 z_5}\right\}, \quad e^{(1358)} = \left\{1, \frac{1}{z_5}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_3 z_5}\right\}, \quad \boxed{e^{(91358)} = \left\{1, \frac{1}{z_5}, \frac{1}{z_1 z_3 z_5}, \frac{z_8}{z_1 z_3 z_5}\right\}}$$

$$h^{(8)} = \{1\}\,, \quad h^{(58)} = \{1, \delta_5\}\,, \quad h^{(358)} = \{1, \delta_3, \delta_5, \delta_{35}\}\,, \quad h^{(1358)} = \{1, \delta_5, \delta_{13}, \delta_{135}\}\,, \quad \boxed{h^{(91358)} = \{1, \delta_5, \delta_{135}, z_8\delta_{135}\}}$$

Intersection Numbers for 1-forms (III)

- **Polynomial Division** $Q = C[\mathbf{z}]/I$ [quotient space] **b** = monomial basis of Q

- **Companion Tensor Algebra** multiplication map $Q \rightarrow Q$ $T_i : \ f \rightarrow x_i f$

[Rational function] [Matrix]

$$f(z_1, z_2, \ldots) = \frac{n(z_1, z_2, \ldots)}{d(z_1, z_2, \ldots)} \rightarrow f(z_1, z_2, \ldots) \bmod I \rightarrow f(T_1, T_2, \ldots) = n(T_1, T_2, \ldots) \, . \, d(T_1, T_2, \ldots))^{-1}$$

- **Example** $I := \langle xy - z, yz - x, zx - y \rangle$ $\mathbf{b} = \{1, y, yz, z, z^2\}$

Huang, Feng, He (2015) (1998)

$x \, . \, \mathbf{b} = \{yz, z, z^2, y, yz\}$, $y \, . \, \mathbf{b} = \{y, z^2, z, yz, y\}$, $z \, . \, \mathbf{b} = \{z, yz, y, z^2, z\}$

$$T_x = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_z = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Intersection Numbers for 1-forms (III)

Brunello, Chestnov, & P.M. (2024)

- **Polynomial ideal**

$$\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle = \langle b_0 - \beta + z\, b_1 + \ldots + z^{\kappa-1}\, b_{\kappa-1} + z^{\kappa} \rangle$$

$$\langle \varphi \,|\, \varphi^\vee \rangle + \mathrm{Res}_{\langle \mathcal{B} \rangle} \Big(\varphi\, \psi \Big) = 0\,,$$

$$\Big[\widehat{\nabla}_{-\omega}\, \psi - \widehat{\varphi}^\vee \Big]_{\langle \mathcal{B} \rangle} = 0\,,$$

$$\widehat{\nabla}_{-\omega} \equiv (\partial_z \mathcal{B})\, \partial_\beta - \widehat{\omega} \, + \partial_z$$

$$\psi(\beta, z) = \sum_{a=0}^{\kappa-1} \sum_{n \in \mathbb{Z}} z^a\, \beta^n\, \psi_{an}$$

- **Companion Tensor Algebra**

$$\langle \varphi \,|\, \varphi^\vee \rangle + R \cdot \mathcal{T}_\varphi \cdot \psi = 0\,,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \cdot \psi - \widehat{\varphi}^\vee = 0\,,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} - \mathcal{T}_{\widehat{\omega}} + \mathcal{T}_{\partial_z}$$

$$\psi_i^{(m)} = \sum_{a\, n} z^a\, \beta^n\, \psi_{ian}$$

- **Three vector spaces**

$$\psi^{(m)} \in \mathbb{K}^\nu \otimes \mathcal{Q} \otimes \mathcal{L}$$

$\mathbb{K}^\nu$ Vector space of ν-dimensional vectors labeled by the first index $i = 1, \ldots, \nu$

$\mathcal{Q} = \mathrm{Span}_{\mathbb{K}}(1, \ldots, z^{\kappa-1})\,, \quad \kappa := \deg(\mathcal{B}(z))$

$\mathcal{L} = \mathrm{Span}_{\mathbb{K}}(\ldots,\, \beta^{-1},\, \beta^0,\, \beta^1,\, \ldots)$

Intersection Numbers for 1-forms (III)

Brunello, Chestnov, & **P.M.** (2024)

- **Companion Tensor Algebra**

$$\langle \varphi \,|\, \varphi^\vee \rangle + R \cdot \mathcal{T}_\varphi \cdot \psi = 0\,,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \cdot \psi - \widehat{\varphi}^\vee = 0\,,$$

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} - \mathcal{T}_{\widehat{\omega}} + \mathcal{T}_{\partial_z}$$

$$\psi^{(m)} \in \mathbb{K}^\nu \otimes \mathcal{Q} \otimes \mathcal{L}$$

- **Companion Tensor Representation**

$$\begin{aligned}
z &\rightsquigarrow \mathcal{T}_z = \mathbb{1} \otimes Q_{z,0} + L_\beta \otimes Q_{z,1}\,,\\
\partial_z &\rightsquigarrow \mathcal{T}_{\partial_z} = \mathbb{1} \otimes Q_{\partial_z}\,,\\
\beta &\rightsquigarrow \mathcal{T}_\beta = L_\beta \otimes \mathbb{1}\,,\\
\partial_\beta &\rightsquigarrow \mathcal{T}_{\partial_\beta} = L_{\partial_\beta} \otimes \mathbb{1}\,,\\
\mathrm{Res}_{\langle \mathcal{B} \rangle} &\rightsquigarrow R = E_{\kappa-1} \otimes E_{-1}\,, = \Big[\, 0 \cdots 0 \;\; \underset{\uparrow \atop |\mu|\,\kappa}{1} \;\; 0 \cdots 0 \,\Big]\,,
\end{aligned}$$

$$f(z,\beta)\Big|_{\beta \to 0} = \sum_{a\,n} z^a\, \beta^n\, f_{an} \;\rightsquigarrow\; \mathcal{T}_f = \sum_{a\,n} (\mathcal{T}_z)^a \cdot (\mathcal{T}_\beta)^n\, f_{an} = \sum_{a\,n} \mathbb{1} \otimes (Q_{z,0} + L_\beta \otimes Q_{z,1})^a \cdot (L_\beta \otimes \mathbb{1})^n\, f_{an}\,.$$

- **Q-space operators**

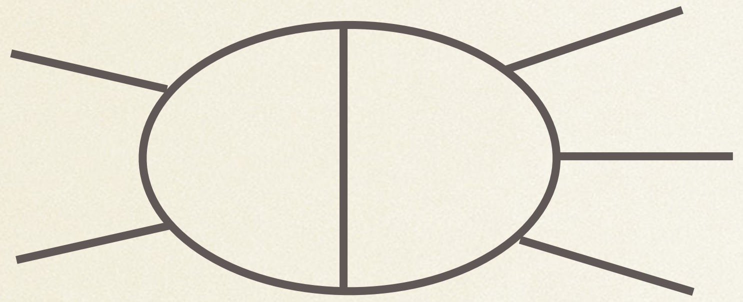
$$Q_z := \begin{bmatrix} 0 & & & & -b_0+\beta \\ 1 & 0 & & & -b_1 \\ & 1 & 0 & & -b_2 \\ & & \ddots & \ddots & \vdots \\ & & 1 & 0 & -b_{\kappa-2} \\ & & & 1 & -b_{\kappa-1} \end{bmatrix}_{\kappa \times \kappa}
\qquad
Q_{\partial_z} := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \\ & & & 0 & \kappa-1 \\ & & & & 0 \end{bmatrix}_{\kappa \times \kappa}$$

- **L-space operators**

$$L_\beta := \begin{matrix} \mu+1 \to \\ \vdots \\ -1 \to \\ 0 \to \\ 1 \to \\ \vdots \\ -\mu^\vee \to \end{matrix}
\begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \\ & & & & \ddots & \ddots \\ & & & & 1 & 0 \end{bmatrix}_{(-\mu^\vee-\mu+1) \times (-\mu^\vee-\mu+1)}
\qquad
L_{\partial_\beta} := \begin{matrix} \mu+1 \to \\ \vdots \\ -1 \to \\ 0 \to \\ 1 \to \\ \vdots \\ -\mu^\vee \to \end{matrix}
\begin{bmatrix} 0 & \mu & & & & \\ & 0 & \ddots & & & \\ & & 0 & 0 & & \\ & & & 0 & 1 & \\ & & & & 0 & 2 \\ & & & & & \ddots \\ & & & & 0 & -\mu^\vee \\ & & & & & 0 \end{bmatrix}_{(-\mu^\vee-\mu+1) \times (-\mu^\vee-\mu+1)}$$

Complete decomposition @ 1- & 2-Loop

☑ **2-loop 5-point**

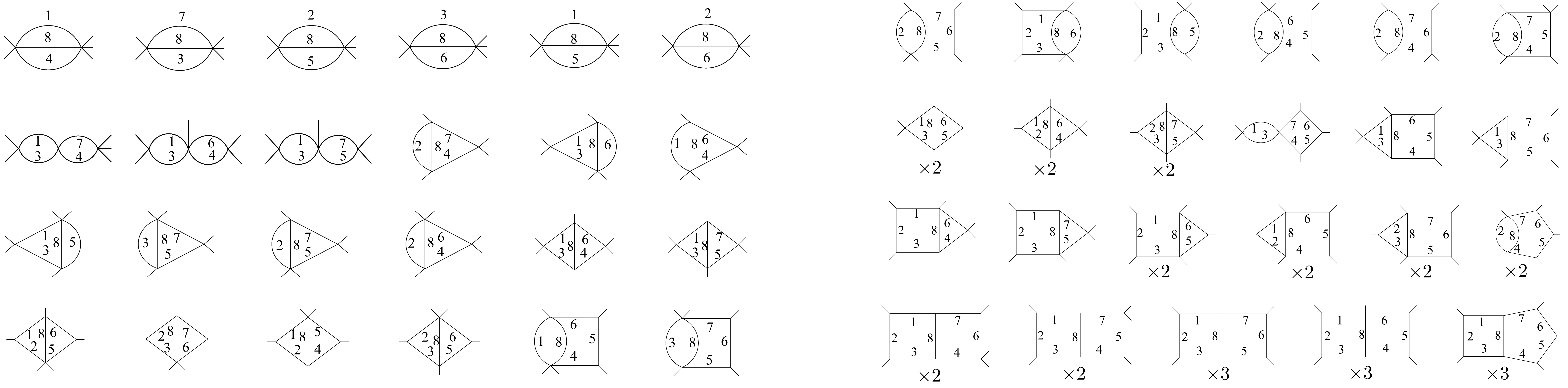


Brunello, **Chestnov**, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M. (2023)

Brunello, **Chestnov**, & P.M. (2024)

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int \mathrm{d}^{11} z \; u(\mathbf{z}) \frac{z_9^{-a_9} \, z_{10}^{-a_{10}} \, z_{11}^{-a_{11}}}{z_1^{a_1} \, z_2^{a_2} \, z_3^{a_3} \, z_4^{a_4} \, z_5^{a_5} \, z_6^{a_6} \, z_7^{a_7} \, z_8^{a_8}}$$

- **62 MIs and 47 sectors**



☑ decomposition up to rank-15

Intersection Numbers for n-forms :: nPDE

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

Intersection Numbers for n-forms (IV)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \,|\, \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\,\varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Proof.

$$\eta := \bar{h}_1 \ldots \bar{h}_n \, (u\,\psi) \, (u^{-1} \varphi_R^{(\mathbf{n})}) \qquad \mathrm{d}_{z_1} \ldots \mathrm{d}_{z_n} \eta = (u\,\varphi_{L,c}) \wedge (u^{-1}\, \varphi_R) \,,$$

$$\bar{h}_i := 1 - h_i$$

$$h_i \equiv h(z_i) := \begin{cases} 1 & \text{for } |z_i| < \epsilon \,, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{L,c} := \bar{h}_1 \ldots \bar{h}_n \, \varphi_L + \ldots + (-1)^n \, \psi \, dh_1 \wedge \ldots \wedge dh_n \equiv \nabla_{\omega_1} \ldots \nabla_{\omega_n} (\bar{h}_1 \ldots \bar{h}_n \psi)$$

$$\int_X (u\,\varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \int_{D_p} \mathrm{d}_{z_1} \ldots \mathrm{d}_{z_n} \eta \quad = (-1)^n \sum_{p \in \mathbb{P}_\omega} \int_{D_p} (u\,\psi) \, dh_1 \wedge \ldots \wedge dh_n \wedge (u^{-1} \varphi_R^{(\mathbf{n})})$$

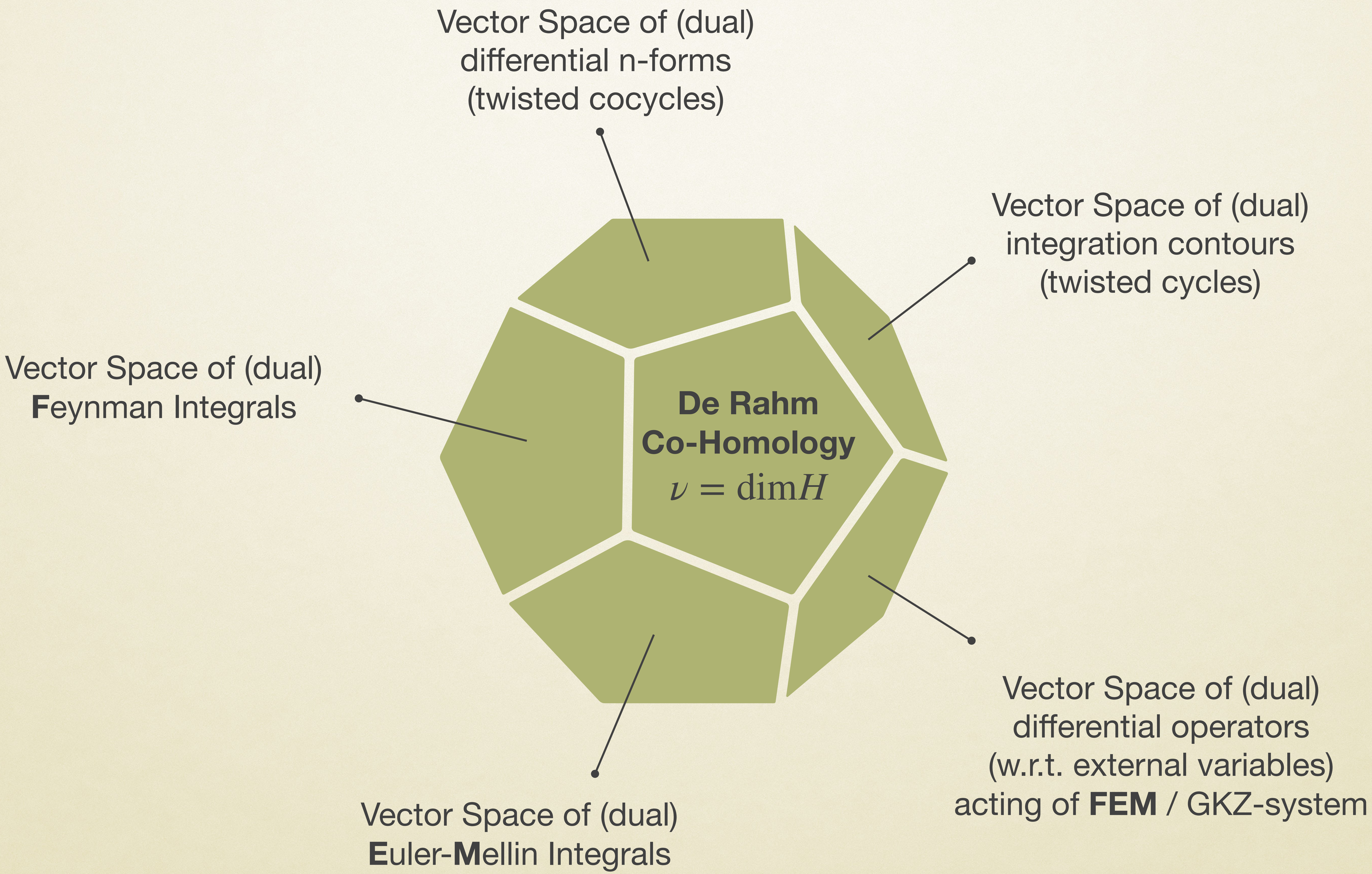
$$= \sum_{p \in \mathbb{P}_\omega} \int_{\circlearrowleft_1 \wedge \ldots \wedge \circlearrowleft_n} \psi\, \varphi_R^{(\mathbf{n})} \quad = (2\pi \mathrm{i})^n \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- [x] It avoids fibrations
- [ ] It requires the knowledge of the poles' position: ok for hyperplane arrangement
- [ ] It requires blow-ups

Intersection Numbers for n-forms: Pfaffian systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

De Rham Thm & Vector Spaces *Isomorphism*



GKZ Hypergeometric Systems

- **Euler-Mellin Integral / A-Hypergeometric function** $$f_\Gamma(z) = \int_\Gamma g(z;x)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n} \, \frac{\mathrm{d}x}{x}$$

Bernstein, Saito, Sturmfels, Takayama, Matsubara-Heo, Agostini, Fevola, Sattelberger, Tellen, De La Crux,…

$$\frac{\mathrm{d}x}{x} := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \qquad u(\mathbf{x}) = g(z,x)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n}$$

$$g(z;x) = \sum_{i=1}^{N} z_i\, x^{\alpha_i} \qquad x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}} \qquad A = \begin{pmatrix} a_1 & \ldots & a_N \end{pmatrix} \quad (n+1)\times N \text{ matrix} \qquad a_i := (1, \alpha_i)$$

$$\mathrm{Ker}(A) = \{u = (u_1, \ldots, u_N) \in \mathbb{Z}^N \,|\, \sum_{j=1}^{N} u_j\, a_j = \mathbf{0}\}$$

- **Gelfand-Kapranov-Zelevinsky (GKZ) system of PDEs**

$$E_j\, f_\Gamma(z) = 0 \,,$$
$$\Box_u\, f_\Gamma(z) = 0 \,,$$

- **Generators**

$$E_j = \sum_{i=1}^{N} a_{j,i}\, z_i \frac{\partial}{\partial z_i} - \beta_j \,, \qquad j = 1, \ldots, n+1$$

$$\Box_u = \prod_{u_i > 0} \left(\frac{\partial}{\partial z_i}\right)^{u_i} - \prod_{u_i < 0} \left(\frac{\partial}{\partial z_i}\right)^{-u_i} \,, \qquad \forall u \in \mathrm{Ker}(A) \,.$$

GKZ D-Module and De Rham Cohomolgy group

- **Weyl Algebra:** E_j $\Box_u$ can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \ldots, z_N]\langle \partial_1, \ldots, \partial_N \rangle \quad , \quad [\partial_i, \partial_j] = 0 \quad , \quad [\partial_i, z_j] = \delta_{ij}$$

GKZ system as the left $\mathcal{D}_N$-module $\mathcal{D}_N / H_A(\beta)$

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \mathrm{Ker}(A)} \mathcal{D}_N \cdot \Box_u$$

- **Standard Monomials** $\mathrm{Std} := \{\partial^k\}$ found by Groebner basis Hibi, Nishiyama, Takayama (2017)

The holonomic rank equals the number of independent solutions to the system of PDEs

$$r = n! \cdot \mathrm{vol}(\Delta_A)$$

- **Isomorphism**

$$\mathcal{D}_N / H_A(\beta) \simeq \mathbb{H}^n$$

GKZ D-module — nth-Cohomology group

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module**

Gelf'and Kapranov Zelevinsky)1990)

Euler-Mellin Integrals

Differential Operators (w.r.t. external variables)

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module** — Gelf'and Kapranov Zelevinsky (1990)

- nth-Cohomology group → **Euler-Mellin Integrals**
- GKZ D-module → **Differential Operators (w.r.t. external variables)**

Pfaffian Systems: for Master Integrals (alias Master forms)

$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij} \langle e_j| \qquad \mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

- **Pfaffian Matrix**

$\langle e_i|$, $\langle e_j|$: **Basis of the Cohomology group**

$\mathbf{\Omega}$: **Integral decomposition (IBP/InterX)**

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module**

Gelf'and Kapranov Zelevinsky)1990)

Euler-Mellin Integrals — **Differential Operators (w.r.t. external variables)**

Pfaffian Systems: for Master Integrals (alias Master forms) & for D-operators (alias Std mon's)

$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij} \langle e_j|$$

$$\mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

- **Pfaffian Matrix**

Basis of the D-Operators

Macaulay Matrix method

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Chestnov,,, Munch, Matsubara-Heo, Takayama **& P.M.** (2023)

D-modules of Feynman Integrals

Chestnov, Flieger, Matsubara-Heo, Takayama **& P.M.** (in progress)

- **GKZ-Euler integrals obey systems of differential equations**
- **Annihilators of GKZ Euler-integrals are known**
- **Annihilators can be used to derive the generators of the D-module via Macaulay matrix method**
- **GKZ Theorem: de Rham isomorphism b/ween D-module and Euler integral space**

- **Feynman integrals obey systems of differential equations**
- **Feynman integrals are restrictions of GKZ Euler-integrals**
- **Griffiths' theorem and Annihilators for Feynman integrals:** generalising Lorentz invariance and homogeneity relations
- Conjecture**: de Rham isomorphism b/ween (restricted) D-module and Feynman integral space**

Intersections Numbers beyond Feynman Integrals

Intersections Numbers @ QM and QFT

Cacciatori **& P.M.** (2022)

Orthogonal Polynomials and Matrix Elements in QM

Case i) $$I_{nm} \equiv \int_\Gamma P_n(z)\, P_m(z)\, f(z)\, dz\,,$$

Case ii) $$I_{nm} \equiv \langle n|\mathscr{O}|m\rangle = \int_\Gamma \psi_n^*(z)\, \mathscr{O}(z)\, \psi_m(z)\, f(z)\, dz$$

- **Master Decomposition formula**

For the considered cases, we obtain: $$\varphi = c_1\, e_1\,,$$ in terms of just one basic form, $e_1 = dz$

corresponding to: $$I_{nm} = c_1\, E_1$$ (one master integral)

i) Orthogonal Polynomials

Laguerre $L_n^{(\rho)}$, Legendre P_n, Tchebyshev T_n, Gegenbauer $C_n^{(\rho)}$, and Hermite H_n polynomials:

$$I_{nm} \equiv \int_\Gamma \mu\, P_n P_m\, dz = f_n\, \delta_{nm} \;= \int_\Gamma \mu\, \varphi = c_1\, E_1$$

$$\varphi \equiv P_n\, P_m\, dz$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$L_n^{(\rho)}$	$z^\rho \exp(-z)$	1	1	ρ	–	$\Gamma(1+\rho)$	$(\rho+1)(\rho+2)\cdots(\rho+n)/n!$
P_n	$(z^2-1)^\rho$	1	1	$2\rho/(4\rho^2-1)$	0	2	$1/(2n+1)$
T_n	$(1-z^2)^\rho$	1	1	$2\rho/(4\rho^2-1)$	$-1/2$	π	$1/2$
$C_n^{(\rho)}$	$(1-z^2)^{\rho-1/2}$	1	1	$(1-2\rho)/(4\rho(\rho-1))$	–	$\sqrt{\pi}\Gamma(1/2+\rho)/\Gamma(1+\rho)$	$\rho(2\rho(2\rho+1)\cdots(2\rho+n-1))/((n+\rho)n!)$
H_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	$\text{diagonal}(1/2, 1/\rho)$	0	$\sqrt{\pi}$	$2^n n!$

Let us observe that, in the case of Hermite polynomials, $\nu = 2$, yielding $\varphi = c_1\, e_1 + c_2\, e_2$, but $c_2 = 0$, due to the adopted basis

ii) Matrix Elements in QM

Harmonic Oscillator. (for unitary mass and pulsation, $m = 1 = \omega$)

$$\langle z|n\rangle = \psi_n(z) = e^{-\frac{z^2}{2}}\, W_n(z) \ , \quad \text{with} \quad W_n(z) \equiv N_n H_n(z) \ , \qquad N_n \equiv 1/\sqrt{(2^n n! \sqrt{\pi})}$$

- **Position operator**

$$\langle m|z^k|n\rangle = \int_{-\infty}^{\infty} dz\, \psi_m(z)\, z^k\, \psi_n(z) = \int_\Gamma \mu\, \varphi = c_1 E_1 \ , \quad \text{with} \quad \mu \equiv e^{-z^2} \ , \quad \text{and} \quad \varphi \equiv W_m(z)\, z^k\, W_n(z)\, dz.$$

Type	u	ν	$\hat{e}_i$	C-matrix	ρ_0	E_1
W_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	diagonal$(1/2, 1/\rho)$	0	$\sqrt{\pi}$

$$\begin{aligned}
\langle n|m\rangle &= \delta_{nm} \ , \\
\langle n|z^{2k+1}|n\rangle &= 0 \ , \\
\langle n|z^4|n\rangle &= \frac{3}{4}(2n^2 + 2n + 1) \ , \\
\langle n|z^3|n-3\rangle &= \sqrt{n(n-1)(n-2)/8} \ , \\
\langle n|z^3|n-1\rangle &= \sqrt{9n^3/8} \ .
\end{aligned}$$

- **Hamiltonian operator**

$$\langle n|H|n\rangle = (n + 1/2) \qquad H \equiv (1/2)(-\nabla^2 + z^2) \qquad \varphi = \sum_{k=0}^{n} b_k\, z^{2k}$$

ii) Matrix Elements in QM

Hydrogen Atom. (for unitary Bohr radius $a_0 = 1$)

$$\langle z|n,\ell\rangle = R_{n,\ell}(z) = e^{-\frac{z}{n}}\,W_{n,\ell}(z)\,, \quad \text{with} \quad W_{n,\ell}(z) \equiv N_{n\ell}\left(\frac{2z}{n}\right)^{\ell} L^{2\ell+1}_{(n-\ell-1)}\left(\frac{2z}{n}\right) \qquad N_{n\ell} = (2/n)^{3/2}\sqrt{(n-\ell-1)!/(2n\,(n+\ell)!)}$$

- **Position operator**

$$\langle n_1,\ell|z^k|n_2,\ell\rangle = \int_0^\infty dz\,z^2\,R_{n_1,\ell}(z)\,z^k\,R_{n_2,\ell}(z) = \int_\Gamma \mu\,\varphi = c_1\,E_1\,, \quad \text{with} \quad \mu \equiv z^2\,e^{-z\left(\frac{1}{n_1}+\frac{1}{n_2}\right)},\ \text{and}\ \varphi \equiv W_{n_1,\ell}(z)\,z^k\,W_{n_2,\ell}(z)$$

Type	u	ν	$\hat{e}_i$	C-matrix	ρ_0	E_1
$W_{n,\ell}$	$z^{\rho+2}\exp(-z(n_1+n_2)/(n_1 n_2))$	1	1	$(n_1 n_2/(n_1+n_2))^2(2+\rho)$	0	$2(n_1 n_2/(n_1+n_2))^3$

$$\langle n_1,\ell|n_2,\ell\rangle = \delta_{n_1 n_2}\,,$$

$$\langle n,\ell|z|n,\ell\rangle = \frac{1}{2}[3n^2 - \ell(\ell+1)]\,,$$

$$\langle n,\ell|z^{-1}|n,\ell\rangle = \frac{1}{n^2}\,,$$

$$\langle n,\ell|z^{-2}|n,\ell\rangle = \frac{2}{n^3(2\ell+1)}\,,$$

$$\langle n,\ell|z^{-3}|n,\ell\rangle = \frac{2}{n^3\ell(\ell+1)(2\ell+1)}$$

Green's Function and Kontsevich-Witten tau-function

Case iii) $$G_n \equiv \frac{\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, \exp[-S_E]}{\int \mathscr{D}\phi\, \exp[-S_E]}$$ Weinzierl (2020)

Case iv) $$Z_{KW} \equiv \frac{\int d\Phi \exp\left[-\mathrm{Tr}\left(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2\right)\right]}{\int d\Phi \exp\left[-\mathrm{Tr}\left(\frac{\Lambda}{2}\Phi^2\right)\right]}$$ Cacciatori & **P.M.**(2022)

$$c_1 = \frac{\int_\Gamma \mu\, \varphi}{\int_\Gamma \mu\, e_1}\,,$$ equivalently rewritten as $$\int_\Gamma \mu\, \varphi = c_1\, E_1$$

- **Master Decomposition formula**

- **Toy models univariate integrals**

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $\quad S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2)\,\phi^2(x)$, and $S_1 = \phi^4(x)$

$$\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{-S_E} = G_n \int \mathscr{D}\phi\; e^{-S_E}$$

$$\int_\Gamma \mu\,\varphi = G_n E_1\,, \quad \text{with} \quad \mu \equiv e^{-S_E}\,, \quad \varphi \equiv \phi(x_1)\cdots\phi(x_n)\,\mathscr{D}\phi\,, \quad E_1 \equiv \int_\Gamma \mu\, e_1\,, \quad \text{and} \quad e_1 \equiv \mathscr{D}\phi$$

Free theory. The n-point Green's function $G_n^{(0)}$ $\quad \phi(x) \equiv z \quad$ $\mu \equiv e^{-S_0} \quad \varphi = z^n\, dz$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma z^2/2)$	2	$1, 1/z$	diagonal$(1/\gamma, 1/\rho)$	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

- **2-point function: the propagator** $\quad G_2^{(0)} = 1/\gamma$

Perturbation Theory. The n-point correlation function G_n in the full theory can be computed perturbatively, in the small coupling limit, $\varepsilon \to 0$, and expressed in terms of $G_n^{(0)}$. For example, the determination of the next-to-leading order (NLO) corrections to the 2-point function, proceeds as follows,

$$G_2 = \frac{\int dz\; z^2\; e^{-S_0-\epsilon S_1}}{\int dz\; e^{-S_0-\epsilon S_1}} = \frac{\int dz\; z^2\; e^{-S_0}(1-\epsilon S_1+\ldots)}{\int dz\; e^{-S_0}(1-\epsilon S_1+\ldots)} = \left(G_2^{(0)} - \epsilon\, G_6^{(0)} + \ldots\right)\left(1 + \epsilon\, G_4^{(0)} + \ldots\right) = G_2^{(0)} + \epsilon\left(G_2^{(0)} G_4^{(0)} - G_6^{(0)}\right) + \mathcal{O}(\epsilon^2)$$

$$= \frac{1}{\gamma}\left(1 - 12\epsilon\frac{1}{\gamma^2}\right) + \mathcal{O}(\epsilon^2)$$

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2)\,\phi^2(x)$, and $S_1 = \phi^4(x)$

Exact theory. $\phi(x) \equiv z$ $\mu \equiv e^{-S_E}$ $\varphi = z^n\, dz$

$$u \equiv z^\rho\, \mu \qquad \nu = 4,$$

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\} = \{1, 1/z, z, z^2\},$$

$$\{\hat{h}_i\}_{i=1}^4 = \{\hat{e}_i\}_{i=1}^4,$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4\gamma} \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{1}{4\gamma} & 0 \\ \frac{1}{4\gamma} & 0 & 0 & -\frac{\gamma}{16\epsilon^2} \end{pmatrix}$$

For instance, let us consider the decomposition:

$$\varphi = z^4\, dz = c_1\, e_1 + c_2\, e_2 + c_3\, e_3 + c_4\, e_4 \qquad c_1 = \frac{1}{4\epsilon}, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = -\frac{\gamma}{4\epsilon}$$

$$\int_\Gamma dz\, z^4\, e^{-S_E} = c_1 \int_\Gamma dz\, e^{-S_E} + c_4 \int_\Gamma dz\, z^2\, e^{-S_E} \qquad G_4 = c_1 + c_4 G_2 \qquad G_2 = \frac{1}{\gamma}\Big(1 - 4\epsilon G_4\Big)$$

ii) Kontsevich-Witten tau-function

$$Z_{KW} \equiv \frac{\int d\Phi \exp\Big[-\mathrm{Tr}\Big(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2\Big)\Big]}{\int d\Phi \exp\Big[-\mathrm{Tr}\Big(\frac{\Lambda}{2}\Phi^2\Big)\Big]}$$

- **Univariate Model** Itzykson-Zuber (1992)

$Z_{KW} = \sum_{n=0}^{\infty} Z_{KW}^{(n)}$. $\qquad \int_\Gamma \mu\,\varphi = c_1\,E_1 \qquad c_1 = Z_{KW}^{(n)}$

$\varphi \equiv N_n\, z^{6n}, \qquad N_n \equiv \varepsilon^{2n} \qquad \varepsilon \equiv i/(3!)(\Lambda/2)^{-3/2}$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$Z_{KW}^{(n)}$	$z^\rho \exp(-z^2)$	2	$1, 1/z$	diagonal$(1/2, 1/\rho)$	0	not needed	$(-2/9)^n\,(\Lambda^{-3n}/(2n)!)\,\prod_{j=0}^{3n-1}(j+1/2)$

Intersection Numbers @ Fourier Integrals

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

Fourier integrals from Intersection Theory

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

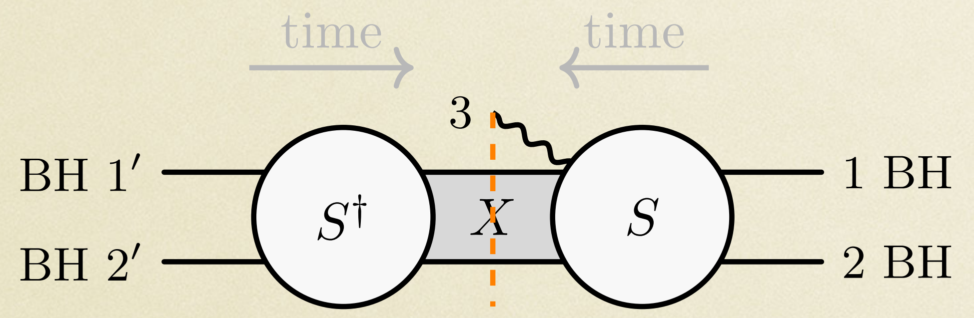
- **Fourier integrals in Baikov representation as twisted periods**

$$\tilde{f}(\{x_i\}) = \int f(\{q_i\}) \prod_{j=1}^{L} \mathrm{e}^{i q_j \cdot x_j} \frac{\mathrm{d}^{\mathrm{D}} q_j}{(2\pi)^{\mathrm{D}/2}} = \int_{C_R} u(\mathbf{z})\, \varphi_L(\mathbf{z}) \qquad u(\mathbf{z}) = \kappa\ \mathrm{e}^{i g(\mathbf{z})} B(\mathbf{z})^{\frac{\mathrm{D}-L-E-1}{2}}$$

- **Application-1: Feynman propagator in position-space**

$$I_n = \int_{\mathcal{M}} \text{đ}^{\mathrm{D}} q \frac{\mathrm{e}^{i q \cdot x}}{(q^2 + m^2 - i\varepsilon)^n}$$

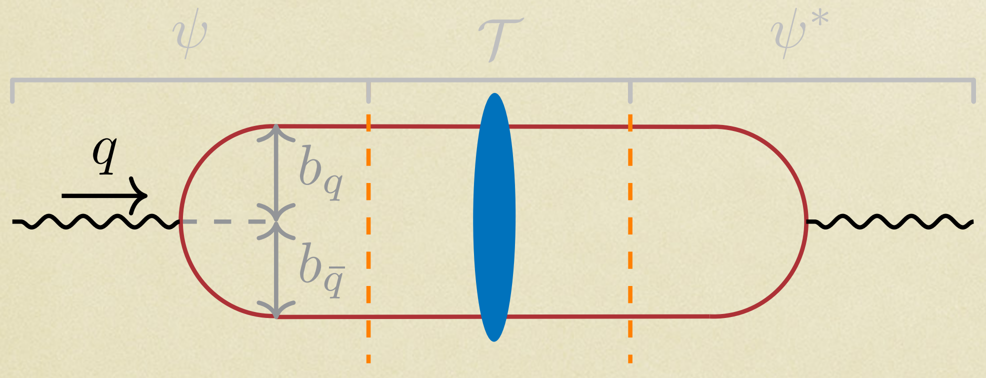
- **Application-2: Spectral gravitation wave form in KMOC formalism**

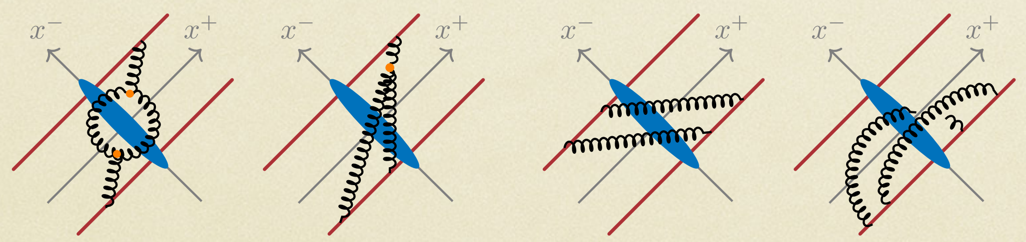


$\mathrm{Exp}_3 = {}_{\mathrm{in}}\langle 2'1' | S^\dagger a_3 S | 12 \rangle_{\mathrm{in}}$

$$\mathcal{I}^{\boldsymbol{\nu}_{2m}}_{\beta_1 \beta_2} = \int_{\mathcal{M}} \text{đ}^{\mathrm{D}} q \frac{\delta(u_1 \cdot q)\delta(u_2 \cdot (q-k))\, q^{\nu_1} \ldots q^{\nu_{2m}} \mathrm{e}^{-i q \cdot b}}{[q^2 - i\varepsilon]^{\beta_1} [(q-k)^2 - i\varepsilon]^{\beta_2}}$$

- **Application-3: QCD Color Dipole Scattering and Balitski-Kovchegov Equations**





$$I^{ij} = \int_{\mathbb{R}^{2\mathrm{D}}} \text{đ}^{\mathrm{D}} q_1 \text{đ}^{\mathrm{D}} q_2 \frac{N_I^{ij}(q_1, q_2)\, \mathrm{e}^{i(q_1 \cdot x_1 + q_2 \cdot x_2)}}{q_1^2 (q_1^2 \tau + q_2^2)}$$

$$G^{ij} = \int_{\mathbb{R}^{2\mathrm{D}}} \text{đ}^{\mathrm{D}} q_1 \text{đ}^{\mathrm{D}} q_2 \frac{N_G^{ij}(q_1, q_2)\, \mathrm{e}^{i(q_1 \cdot x_1 + q_2 \cdot x_2)}}{(q_1 + q_2)^2 (q_1^2 \tau + q_2^2)}$$

$$N_I^{ij} = q_1^i q_2^j\,,$$

$$N_G^{ij} = \delta^{ij}(q_1^2 - q_2^2) - \frac{2 q_1^i (q_1 + q_2)^j}{u} + \frac{2 (q_1 + q_2)^i q_2^j}{u\tau}$$

Intersection Numbers @ Cosmological Integrals

Cosmological wavefunctions

Arkani-Hamed, Benincasa, Postnikov

Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel

Benincasa, Vazao

- **Toy-model:** conformally coupled scalar field (with polynomial self-interactions),

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \sum_{p>2}\frac{\lambda_p}{p!}\phi^p\right]$$

- **Goal:** correlation functions in an FRW cosmology $a(\eta) = (\eta/\eta_0)^{-(1+\varepsilon)}$

$$\Psi_{\text{FRW}}(E_v, E_I) = \int_0^\infty \prod_v d\omega_v \left(\prod_v \omega_v\right)^\varepsilon \Psi_{\text{flat}}(E_v + \omega_v, E_I)$$

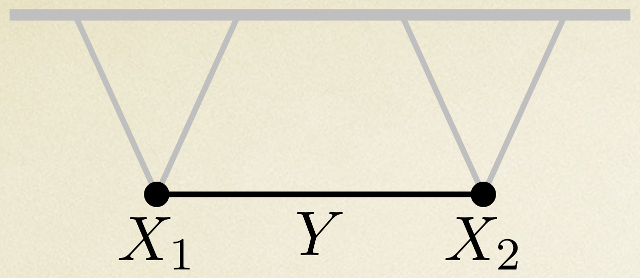
rational function of E_v and E_I
("energies" associated with the vertices and the internal edges)

- **Twisted period integrals**

$$I(C, D; n; \varepsilon) = \int_0^\infty dx_1 \cdots dx_m\, P(x) \prod_I (C_{Ij}x_j + D_I)^{-n_I+\varepsilon_I}$$

The cosmological wavefunction satisfies a differential equation, which governs how it changes as the external kinematics are varied.

Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

Brunello & P.M. (2023) [unpublished]

- **Twisted Period Integrals**

$$I = \int_{\mathcal{C}} u(z_1, z_2)\varphi(z_1, z_2) \qquad u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \qquad D_1 = (z_1+y_1+1)\,, \quad D_2 = (z_2+y_2+1)\,, \quad D_3 = (z_1+z_2+y_1+y_2)$$

γ is a regulator

$$\omega = d\log(u) = \omega_1 dz_1 + \omega_2 dz_2 \qquad \omega_1 = \frac{\gamma(2y_1+y_2+2z_1+z_2+1)}{(y_1+z_1+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_1} \qquad \omega_2 = \frac{\gamma(y_1+2y_2+z_1+2z_2+1)}{(y_2+z_2+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_2}$$

- **Number of MIs = dimH and bases choice**

$$\omega_2 = 0 \qquad \nu_2 = 2 \qquad e^{(2)} = h^{(2)} = \left\{\frac{1}{D_1}, \frac{1}{D_2}\right\}$$

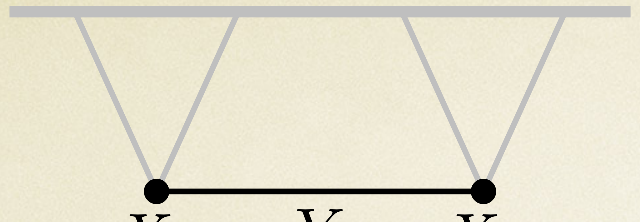
- **2 MIs in the internal layer**

$$\begin{cases}\omega_1 = 0\\ \omega_2 = 0\end{cases} \qquad \nu_{21} = 4 \qquad e^{(21)} = h^{(21)} = \left\{\frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3}\right\}$$

- **4 MIs in the external layer**

- **Intersection Matrix**

$$C = \begin{pmatrix} \frac{(\gamma+\epsilon)^2}{\gamma(\gamma^2-1)\epsilon^2(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & \frac{1}{\gamma\epsilon-\gamma^2\epsilon} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{1}{\gamma^2} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ -\frac{1}{\gamma^2\epsilon+\gamma\epsilon} & \frac{1}{\gamma^2} & \frac{1}{\gamma^2} & \frac{3}{\gamma^2} \end{pmatrix}$$



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

Brunello & P.M. (2023) [unpublished]

- **4 MIs**

$$e^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

- **System of Differential Equations**

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

- **Master Decomposition Formula**

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$$

after taking the limit $\gamma \to 0$:

- **Canonical system**

$$\Omega_{y_1} = \begin{pmatrix} \frac{2\epsilon}{y_1+y_2+1} & 0 & 0 & 0 \\ -\frac{\epsilon}{y_1+1} & \frac{\epsilon}{y_1+1} & 0 & 0 \\ \frac{\epsilon}{y_1} & 0 & \frac{\epsilon}{y_1} & 0 \\ \frac{\epsilon}{y_1(y_1+1)} & 0 & \frac{\epsilon}{y_1(y_1+1)} & \frac{\epsilon}{y_1+1} \end{pmatrix}$$

$$\Omega_{y_2} = \begin{pmatrix} \frac{2\epsilon}{y_1+y_2+1} & 0 & 0 & 0 \\ \frac{\epsilon}{y_2} & \frac{\epsilon}{y_2} & 0 & 0 \\ -\frac{\epsilon}{y_2+1} & 0 & \frac{\epsilon}{y_2+1} & 0 \\ \frac{\epsilon}{y_2(y_2+1)} & \frac{\epsilon}{y_2(y_2+1)} & 0 & \frac{\epsilon}{y_2+1} \end{pmatrix}$$

☑ **Cohomology-based methods for cosmological correlations @ tree level** Pokraka et al. (2023) Gasparotto, Mazloumi, Xu (2024)

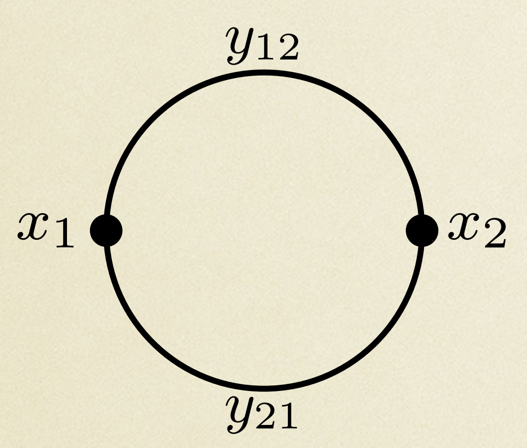
☑ **Differential Equations for cosmological correlations @ tree level** Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel (2023)

Cosmological Integrals @ 1-loop

Benincasa, Brunello, Mandal, Vazão, & PM (2024)

- Mapping cosmological integrals to **QFT-like integrals in momentum space**, with **semi-integer denominator powers**
- From momentum-space to **Baikov representation** to cast them as **twisted period integrals**

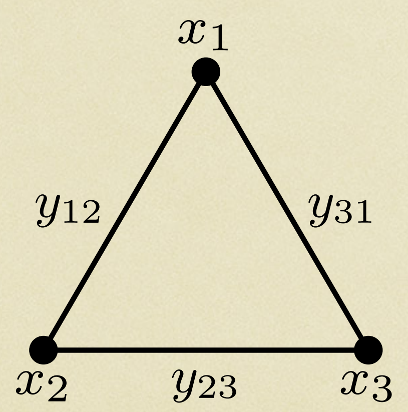
Two-site graph



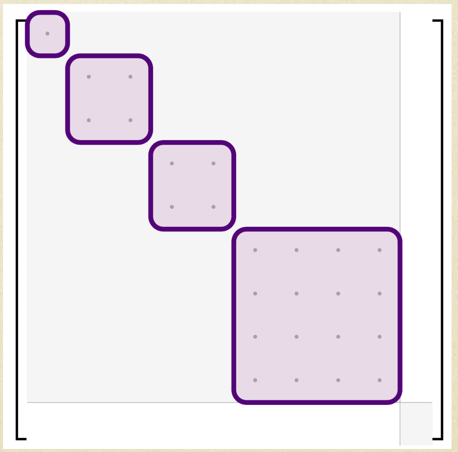
- ☑ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ☑ Linear algebra from **Intersection Theory**
- ☑ (y-integration) **Canonical Differential Equations for $\nu = 6$ MIs:** polylog structure
- ☑ (y-integration) **Analytic solution**
- ☑ **Site-weight** x-integration: Mellin Transform and **Method of Brackets**
- ☑ **Analytic solution:** back of a envelope result

$$
\begin{aligned}
\mathcal{I}_{(2,1)} = {} & \frac{2^{-3-2\alpha}\pi^{3/2}(X_1+X_2)^{1+2\alpha}\csc(\pi\alpha)^2\Gamma\left(-\frac{1}{2}-\alpha\right)}{\Gamma[-\alpha]}\left(2-\frac{1}{\epsilon}-\log\left(4\pi e^{\gamma_E}P^2\right)\right) \\
& + \frac{\pi^{3/2}\csc^2(\pi\alpha)}{8(\alpha+1)^2 P}\left[-4\sqrt{\pi}\left((P+X_1)^{\alpha+1}-2\,(X_1-P)^{\alpha+1}\right)(P+X_2)^{\alpha+1}\right. \\
& \left. - \frac{4^{-\alpha}\Gamma\left(-\alpha-\frac{1}{2}\right)(X_1+X_2)^{2\alpha+2}}{\Gamma(-\alpha)}\,{}_2F_1\left(1,-2(\alpha+1);-\alpha;\frac{P+X_1}{X_1+X_2}\right)\right] \\
& + \frac{\pi^2\csc(\pi\alpha)\csc(2\pi\alpha)(P+X_1)^\alpha}{4\alpha+2}\left[-2(P+X_1)\left((P-X_2)^\alpha+(-1)^\alpha(P+X_2)^\alpha\right)\right. \\
& + (-1)^\alpha(X_1-X_2)(P+X_1)^\alpha\,{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1-X_2}{P+X_1}\right) \\
& \left. + (X_1+X_2)(P+X_1)^\alpha\,{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1+X_2}{P+X_1}\right)\right] \\
& - \frac{\pi^{5/2}4^{-\alpha-1}\csc(\pi\alpha)\csc(2\pi\alpha)}{\Gamma(-\alpha)\Gamma\left(\alpha+\frac{3}{2}\right)(P+X_1)}\left[(-1)^\alpha(X_1-X_2)^{2\alpha+2}\,{}_3F_2\left(1,1,\alpha+2;2,2\alpha+3;\frac{X_1-X_2}{P+X_1}\right)\right. \\
& \left. + (X_1+X_2)^{2\alpha+2}\,{}_3F_2\left(1,1,\alpha+2;2,2\alpha+3;\frac{X_1+X_2}{P+X_1}\right)\right] \\
& + \frac{\pi^{5/2}2^{-2\alpha-1}\csc(\pi\alpha)\csc(2\pi\alpha)\left((-1)^\alpha(X_1-X_2)^{2\alpha+1}+(X_1+X_2)^{2\alpha+1}\right)}{\Gamma(-\alpha)\Gamma\left(\alpha+\frac{3}{2}\right)}\log\left(\frac{P+X_1}{P}\right) \\
& + (X_1 \leftrightarrow X_2).
\end{aligned}
$$

Three-site graph



- ☑ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ☑ Linear algebra from **Intersection Theory**
- ☑ (y-integration) **Differential Equations for $\nu = 41$ MIs**: polylog and **elliptic** structure



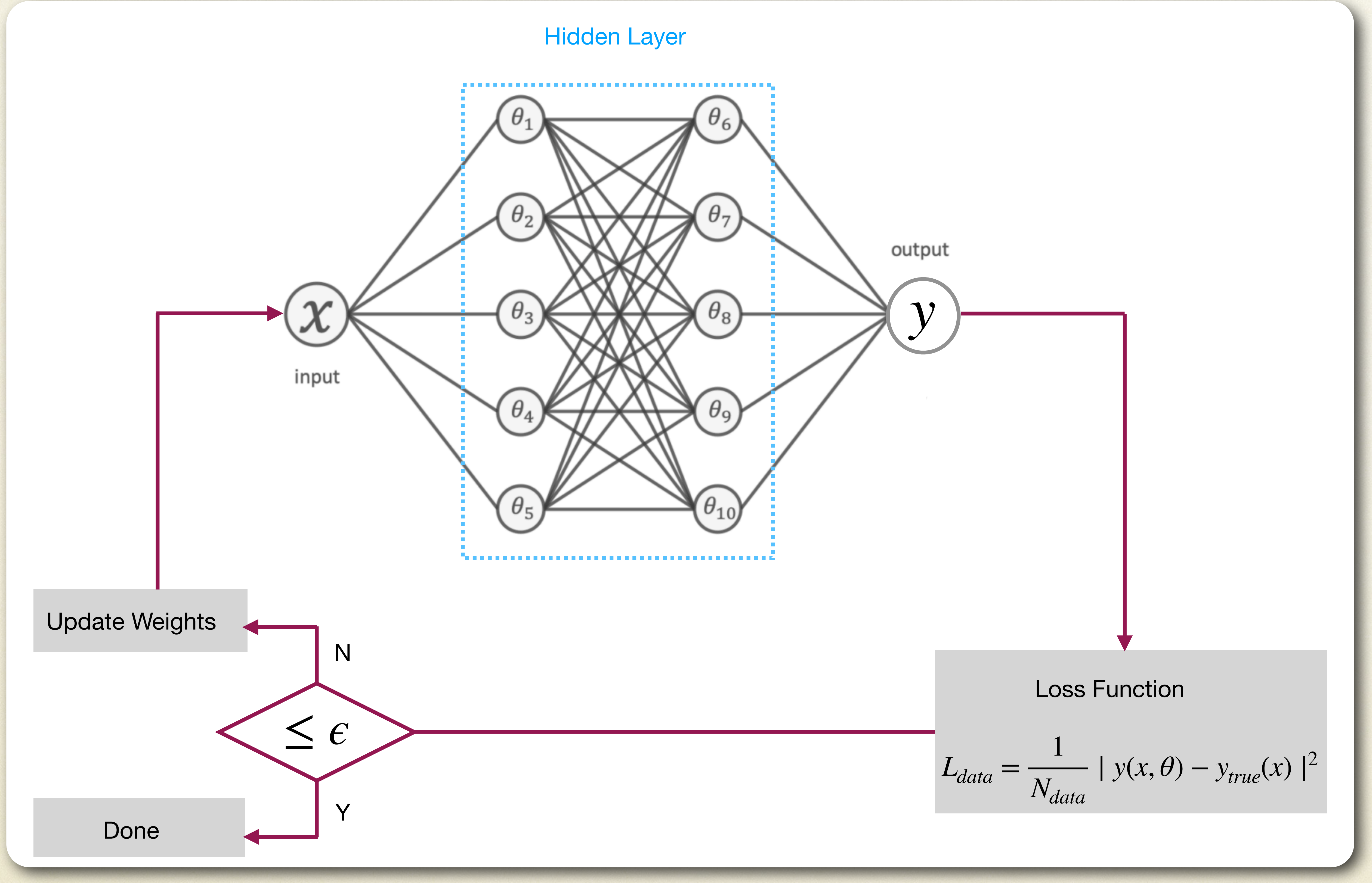
DEQ:
structure of the
elliptic sector
(4x4)-block

Differential Equations and Neural Networks

Calisto, Moodie, Zoia (2023)

Boni, Mandal, & PM (2024)

Artificial Neural Network



- **Universal approximator** of $y(\mathbf{x})$: it take in input $\mathbf{x}$, and estimating $y(\mathbf{x})$ by adjusting the values of its internal parameters θ

$$\mathbf{x} \rightarrow y_{nn}(\mathbf{x}, \theta) \qquad\qquad |\, y_{nn}(\mathbf{x}, \theta) - \mathbf{y}(\mathbf{x}) \,| < \epsilon$$

- **NN** updates its parameter during several iterations "**epochs**" by **supervised learning**, namely requiring a pre-generated dataset $(\mathbf{x}_{\text{data}}, y_{\text{data}})$

- A **LOSS functions** measure the accuracy between the *prediction* and the *actual data*:

$$L_{\text{data}} = \frac{1}{N_{\text{data}}} \,|\, y_{nn}(\mathbf{x}_{\text{data}}, \theta) - y(\mathbf{x}_{\text{data}}) \,|^2$$

- **Gradient descent** algorithm controls the convergence towards a minimum

Physics Informed Neural Network

- Including Physical laws in the **training** process:

$$PDE(\mathbf{x}) = \sum_k g_k(x_i, \partial_i) y(\mathbf{x}) = 0$$

$$L_{PDE} = \frac{1}{N_{\text{coll}}} \sum_{\mathbf{x} \in \mathbf{x}_{\text{coll}}} |PDE(\mathbf{x}) - 0|^2$$

Standard NN	Physics Informed NN
• Requires large amounts of data for supervised learning. • The data must be generated by employing other methods, analytical or numerical. • Fails to generalize the behaviour outside of the training range, especially for the solutions of DEs.	• Embeds knowledge of the system through PDEs and/or ODEs during training. • Does not require a large dataset. • Can predict the solution of the DEs beyond the training range (unsupervised learning).

- More stringent **LOSS**

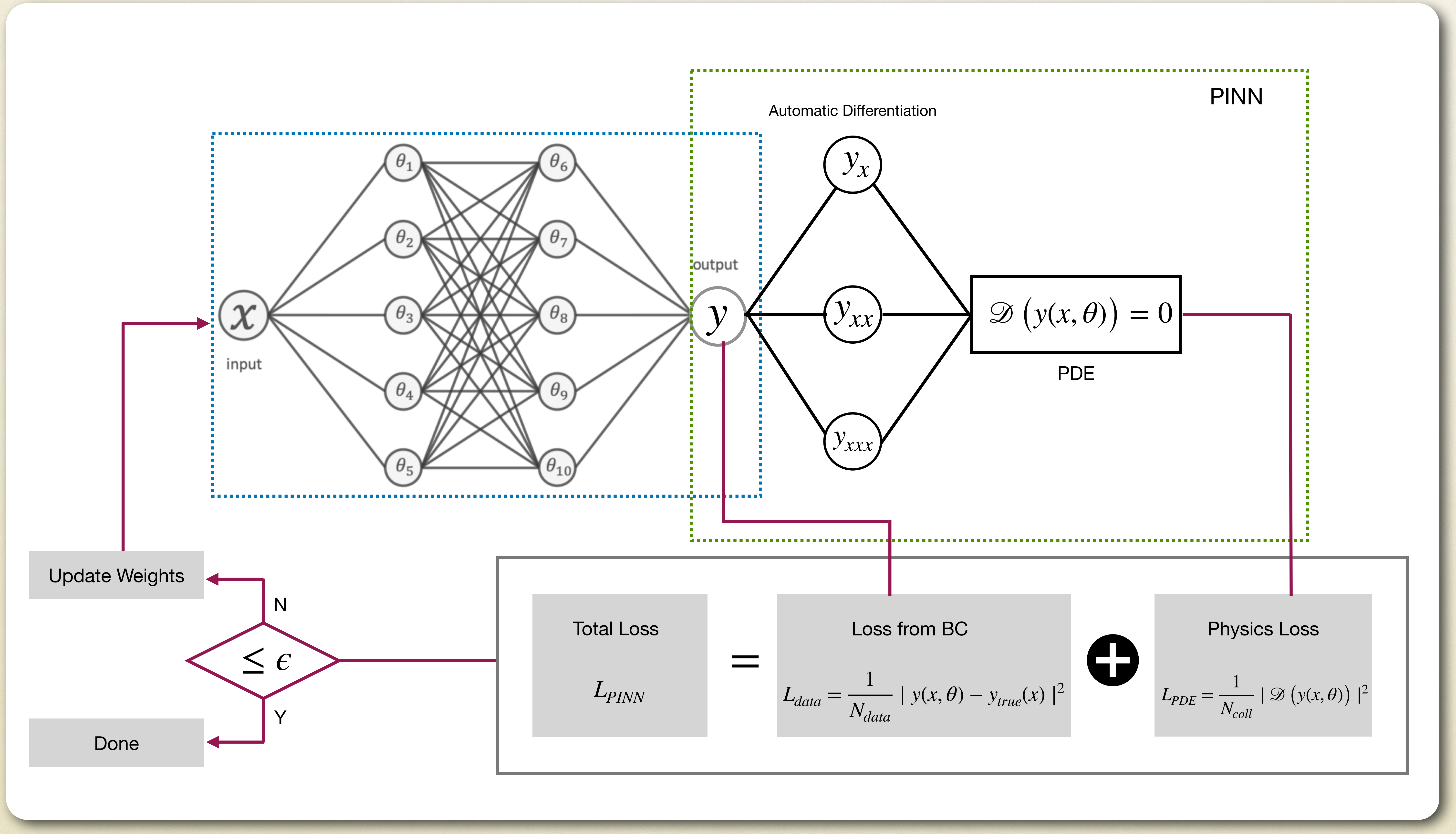
$$L_{PINN} = \lambda_{\text{data}} L_{\text{data}} + \lambda_{PDE} L_{PDE}$$

Physics Informed Neural Network

- Including Physical laws in the **training** process:

$$PDE(\mathbf{x}) = \sum_k g_k(x_i, \partial_i) y(\mathbf{x}) = 0$$

$$L_{PDE} = \frac{1}{N_{\text{coll}}} \sum_{\mathbf{x} \in \mathbf{x}_{\text{coll}}} |PDE(\mathbf{x}) - 0|^2$$



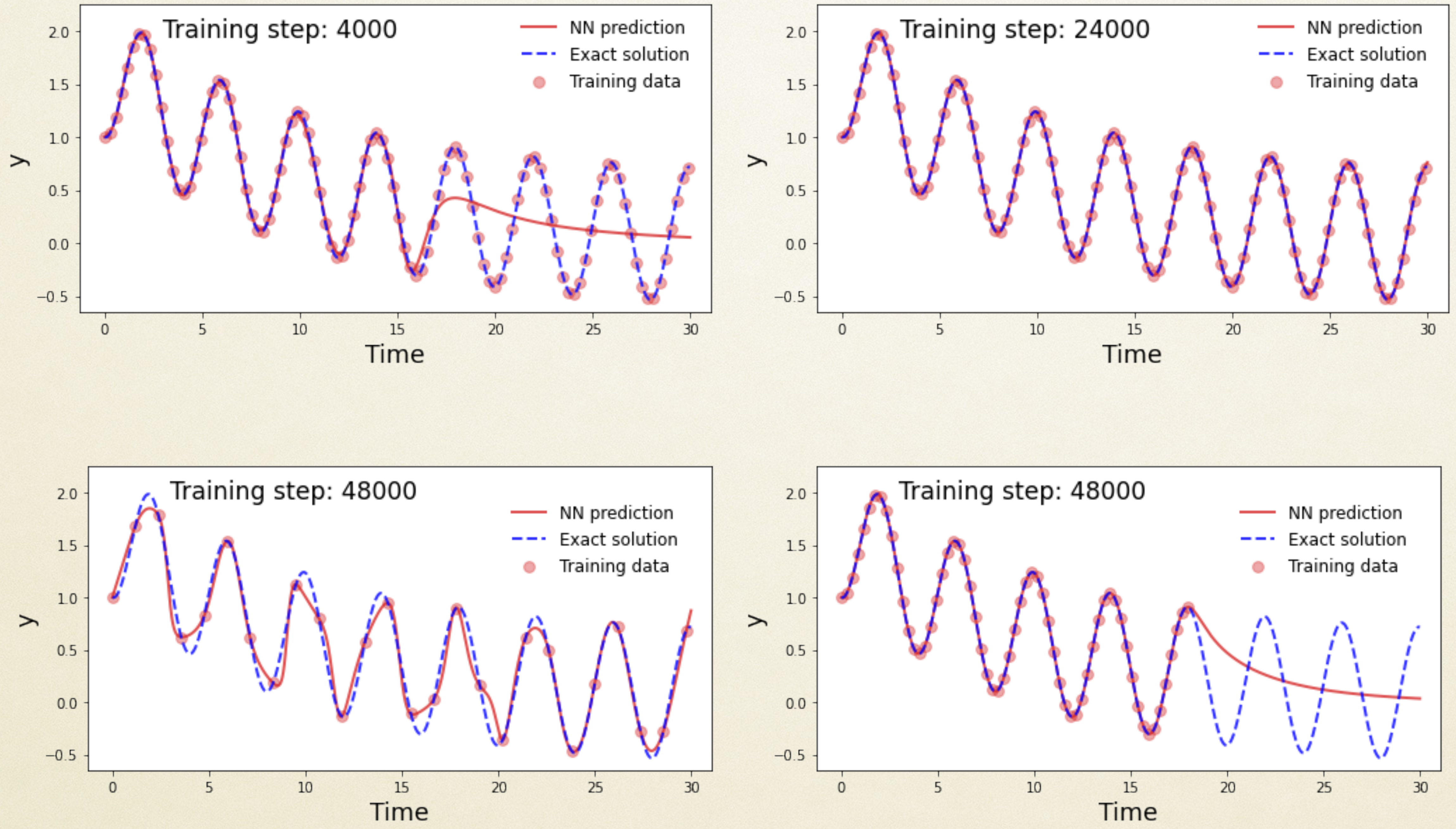
$$L_{PINN} = \lambda_{\text{data}} L_{\text{data}} + \lambda_{PDE} L_{PDE}$$

Physics Informed Neural Network

- **Harmonic Oscillator**

$$\frac{\partial^2 y}{\partial t^2} + \omega^2 y = 0$$

STANDARD NN

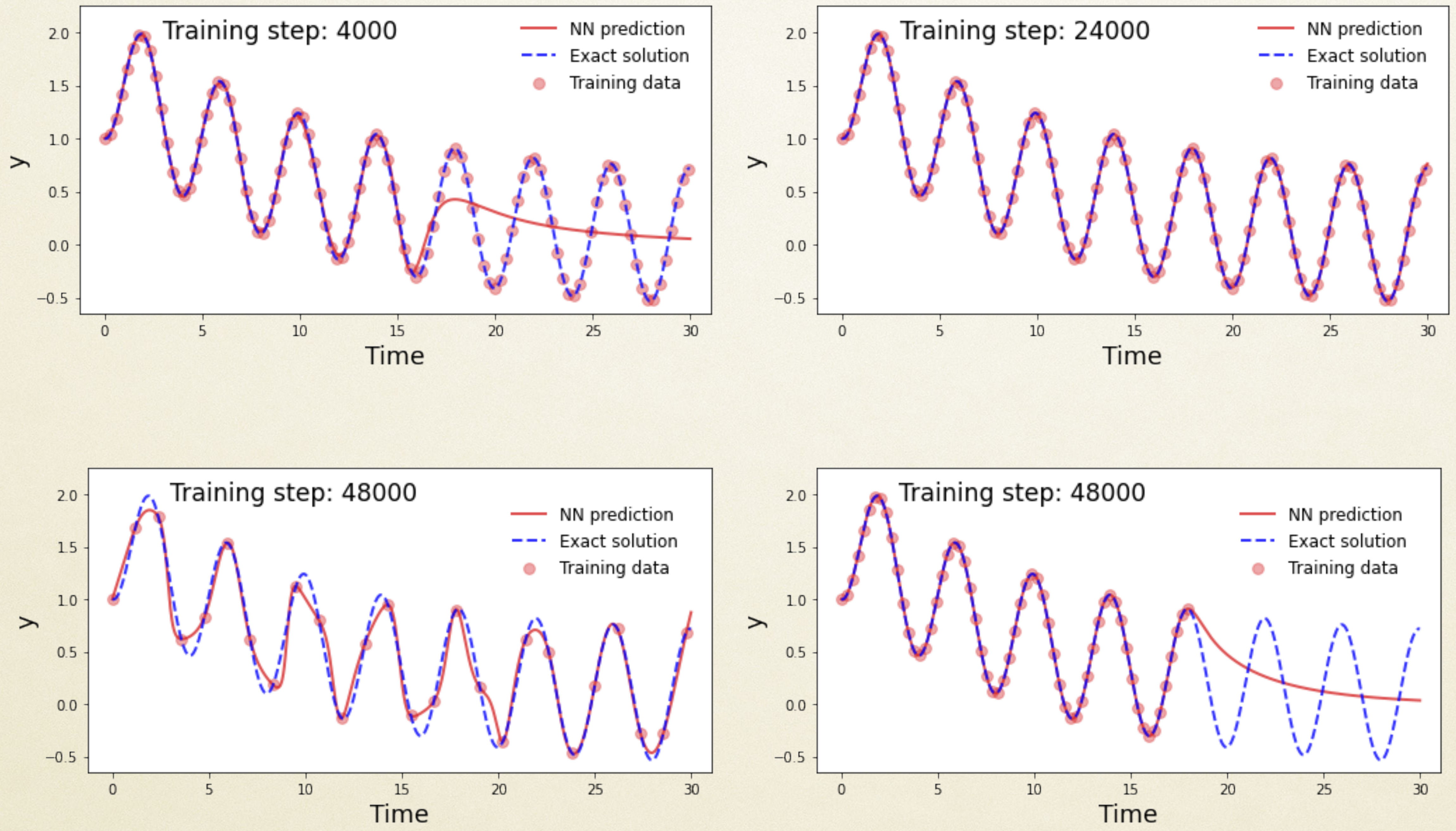


Physics Informed Neural Network

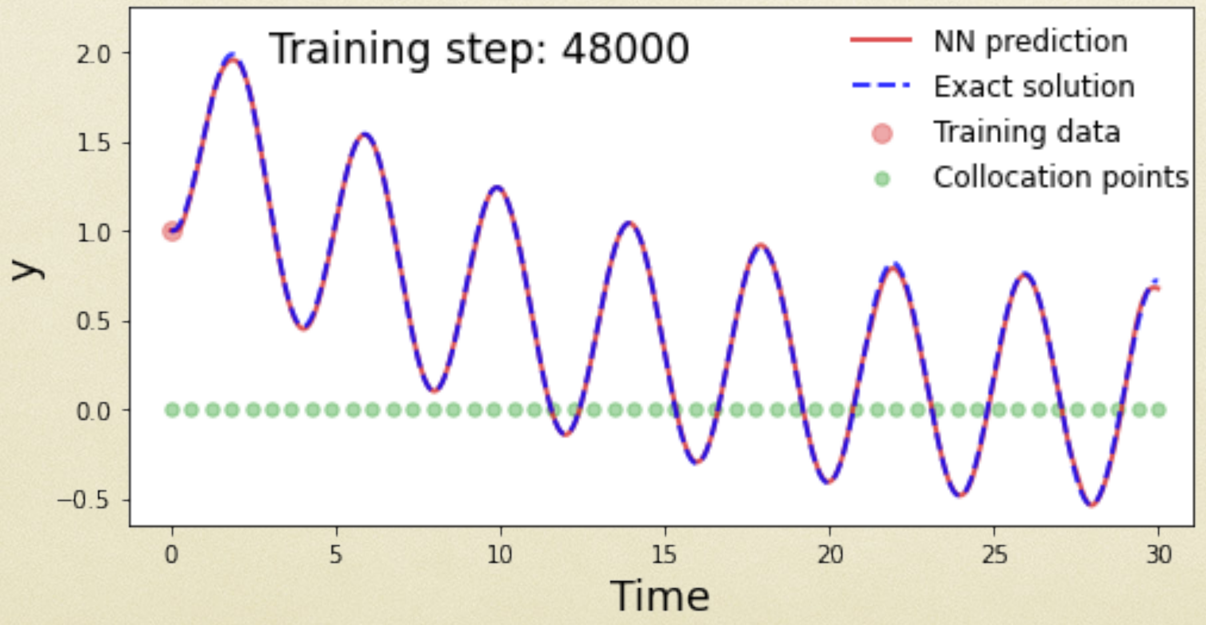
- **Harmonic Oscillator**

$$\frac{\partial^2 y}{\partial t^2} + \omega^2 y = 0$$

STANDARD NN



PINN



Physics Informed Neural Network

Boni, Mandal & P.M. (2024)

- **Hypergeometric 2F1**

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\,\frac{dy}{dx} - ab\,y = 0$$

$$\begin{aligned} M_1(x;a,b,c) &= \beta(b-1,c-b-1)\ {}_2F_1(a,b-1,c-2,x) \\ M_2(x;a,b,c) &= \beta(b,c-b-1)\,(x-1)\ {}_2F_1(a+1,b,c-1,x) \end{aligned}$$

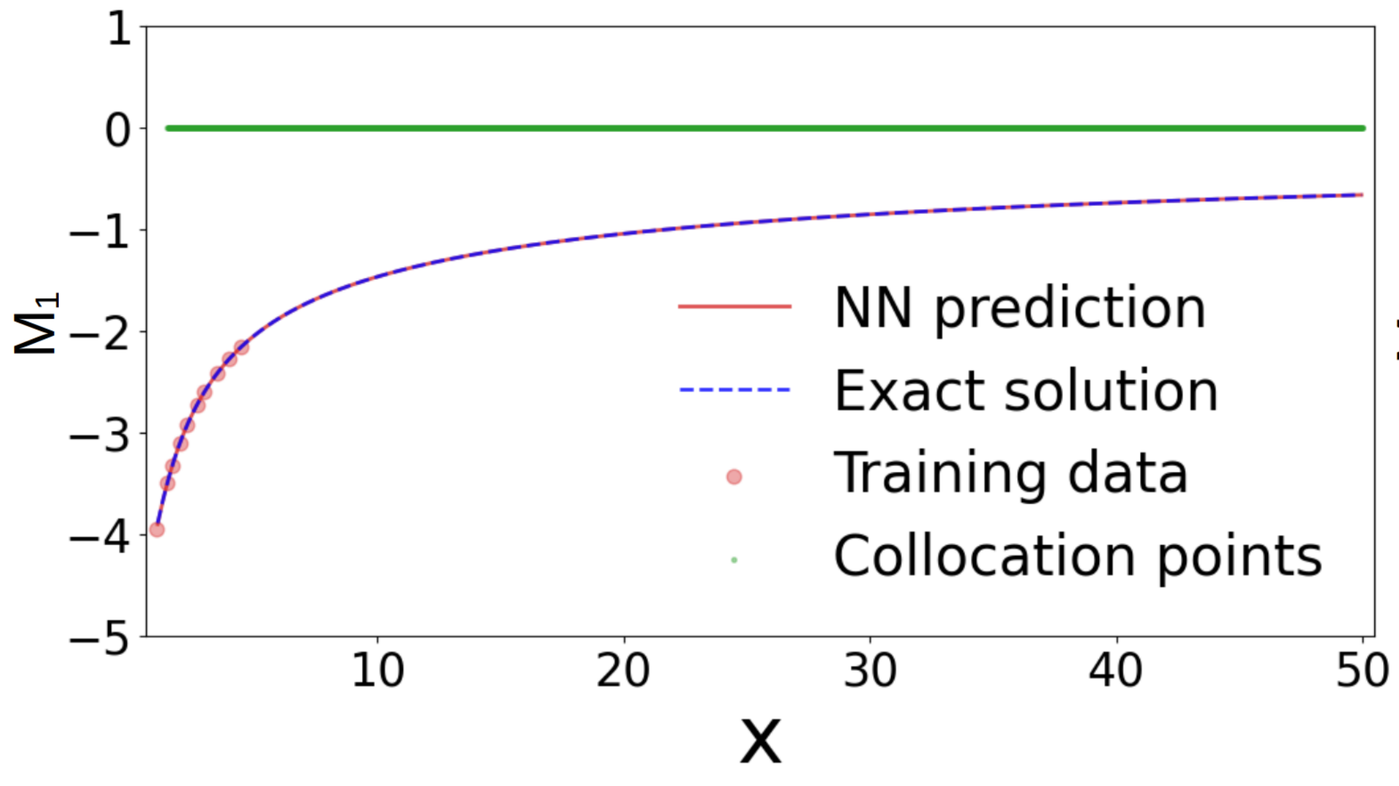
$$\frac{d}{dx}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \mathbb{A}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \qquad \mathbb{A} = \begin{pmatrix} 0 & \frac{a}{x-1} \\ \frac{1-b}{x} & \frac{-2+c-(-1+a+b)x}{(x-1)x} \end{pmatrix}$$

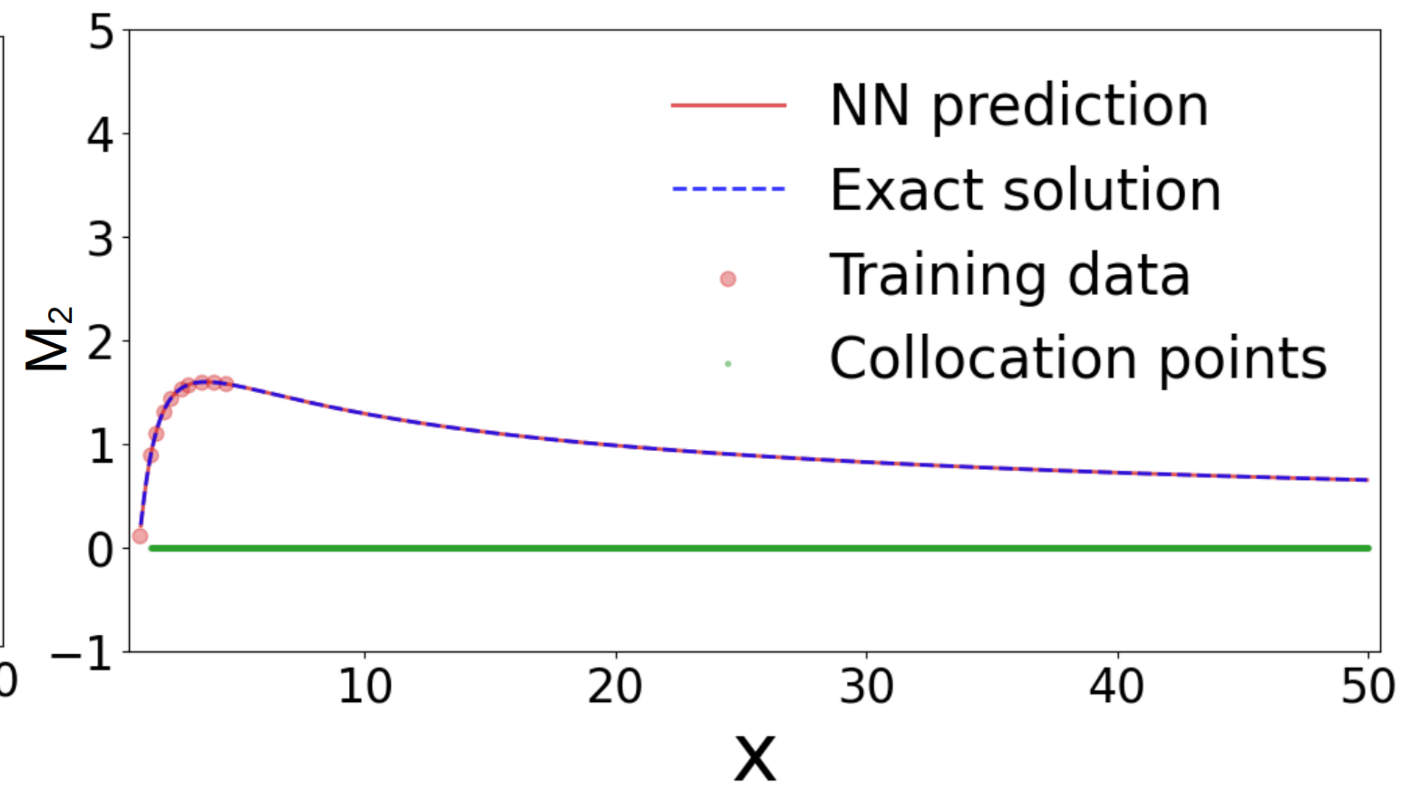
- **Elliptic integrals**

$a = 1/2,\ b = 1/2,\ c = 4$

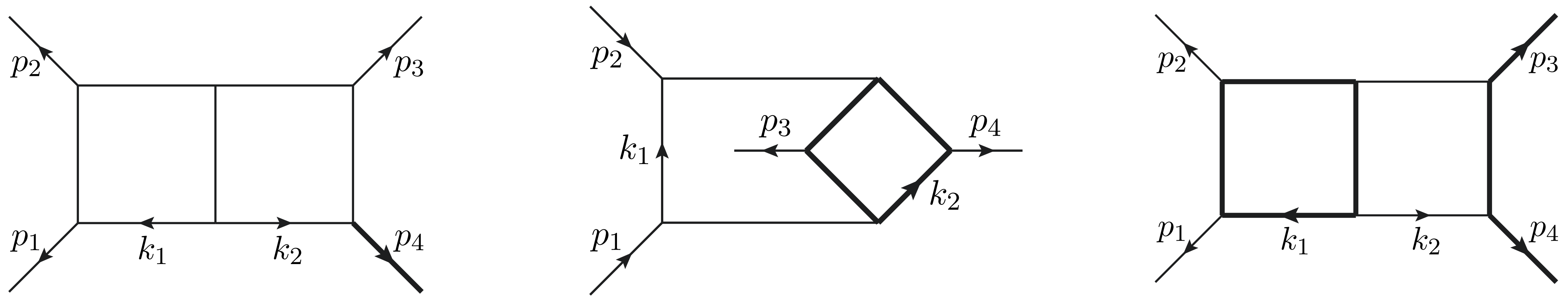
$$M_1(1/2,1/2,4;x) = \frac{4}{3\pi x}\Big((x+1)E(x) + (x-1)K(x)\Big)$$

$$M_2(1/2,1/2,4;x) = \frac{16}{3\pi x^2}\Big((2-x)E(x) + 2(x-1)K(x)\Big)$$

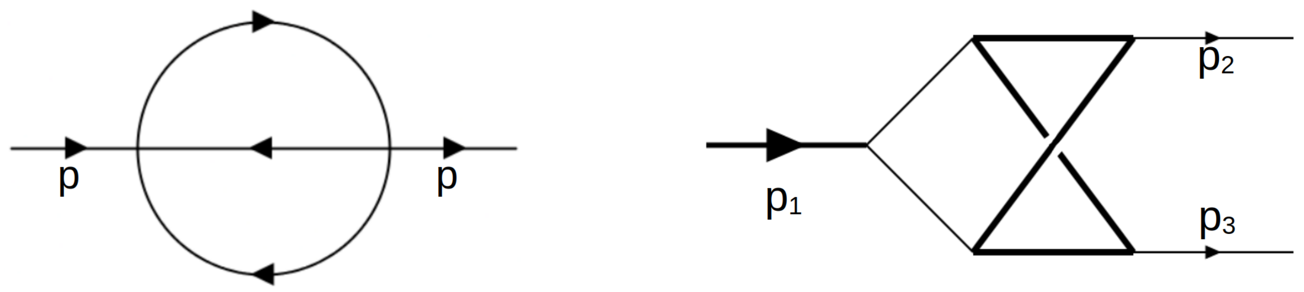




Physics Informed Neural Network @ Feynman Integrals



Calisto, Moodie, Zoia (2023)



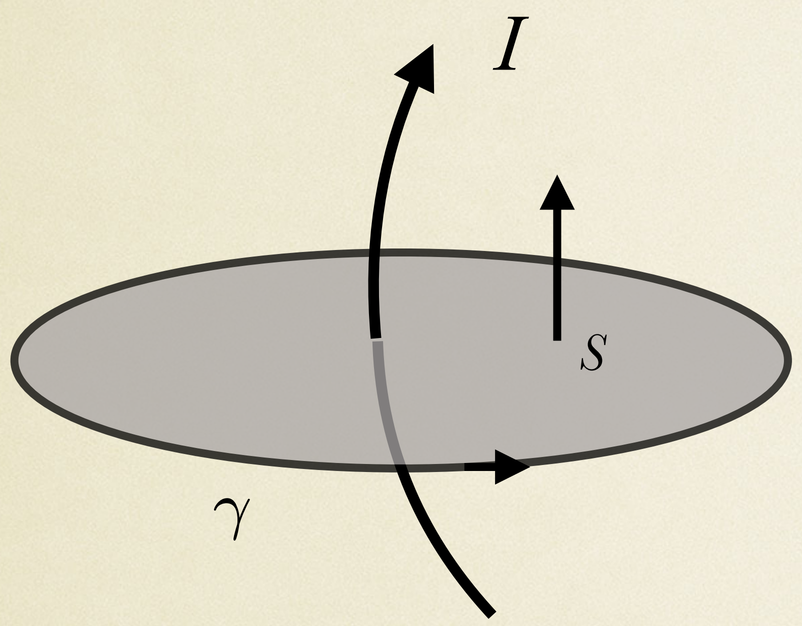
Boni, Mandal & P.M. (2024)

- **Improved LOSS** including **higher-order derivatives**
- **PINNS & Finite Difference Equations:** (work in progress)
  - **Feynman Integrals** (propagator powers, dimension-shift)
  - **Euler-Mellin Integrals**

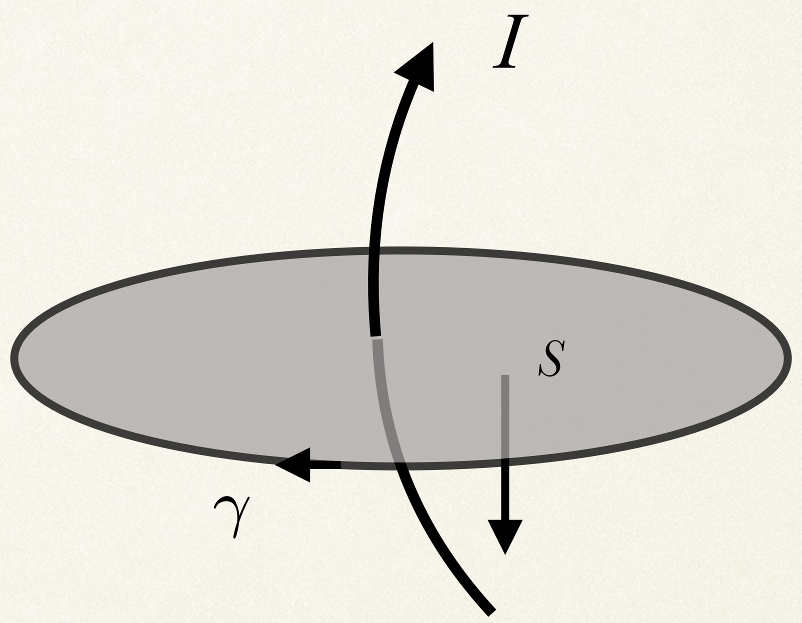
To Conclude:

Ampere's Law

Cacciatori & P.M.

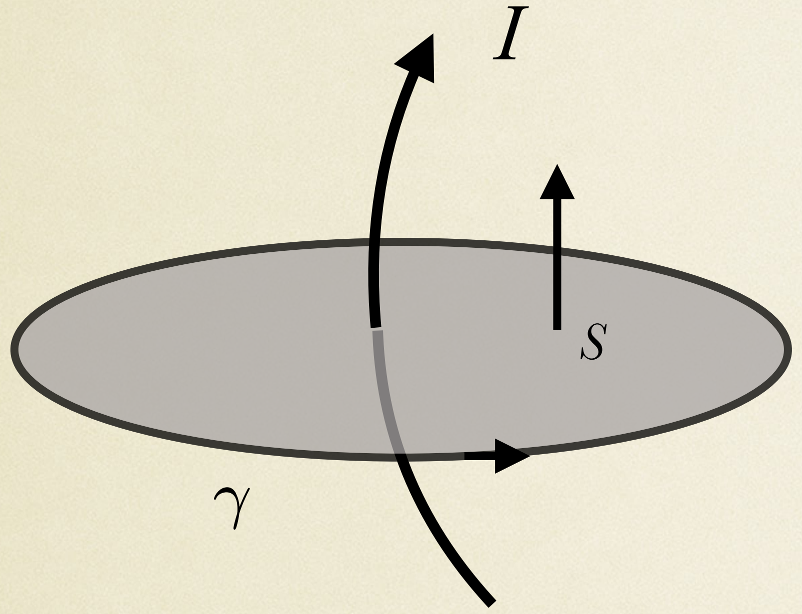


$$\oint_\gamma \mathbf{B} \cdot d\vec{\ell} = \mu_0 I.$$

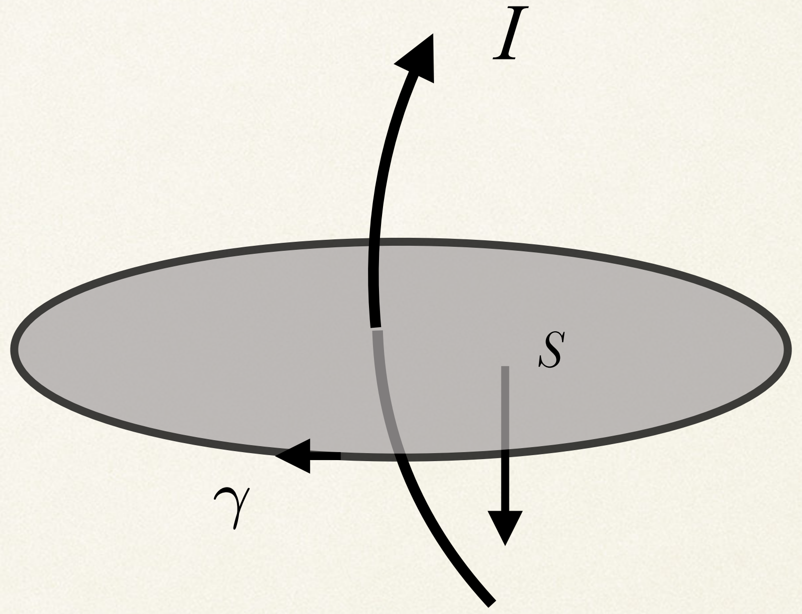


$$\oint_\gamma \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I.$$

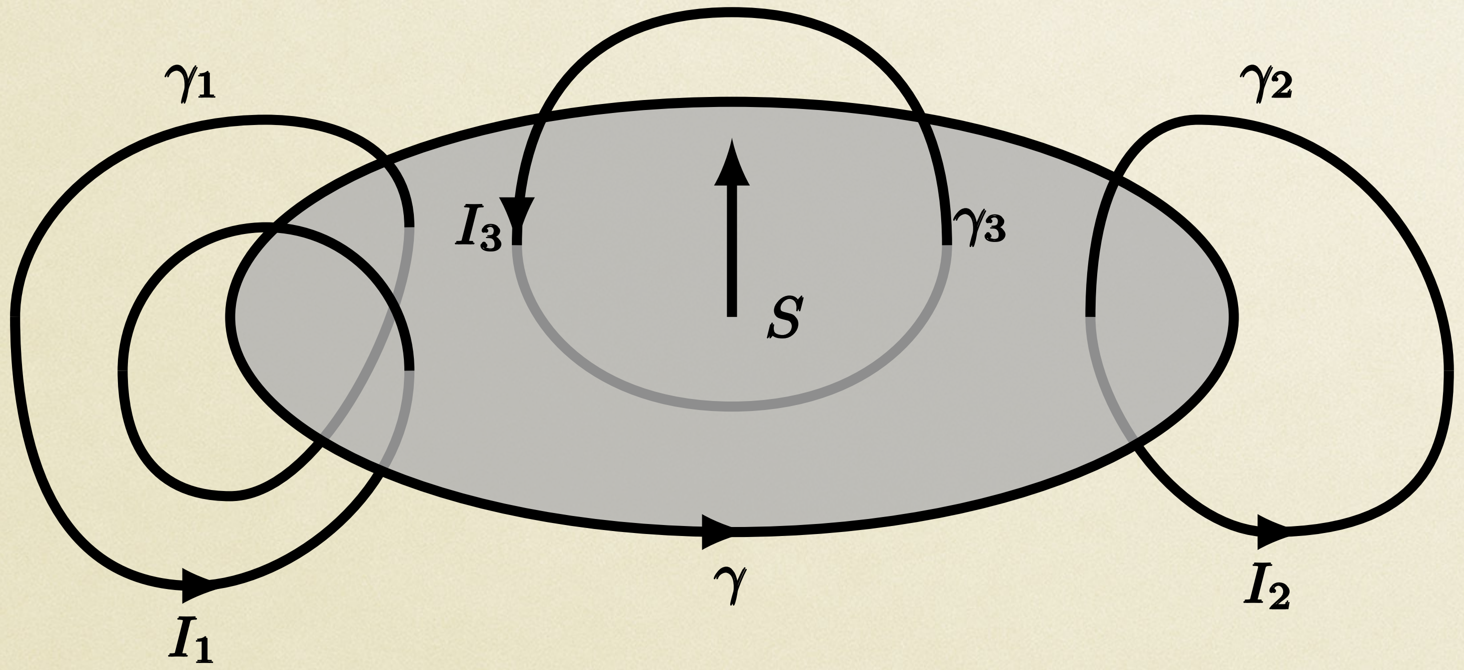
Ampere's Law



$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = \mu_0 I.$$

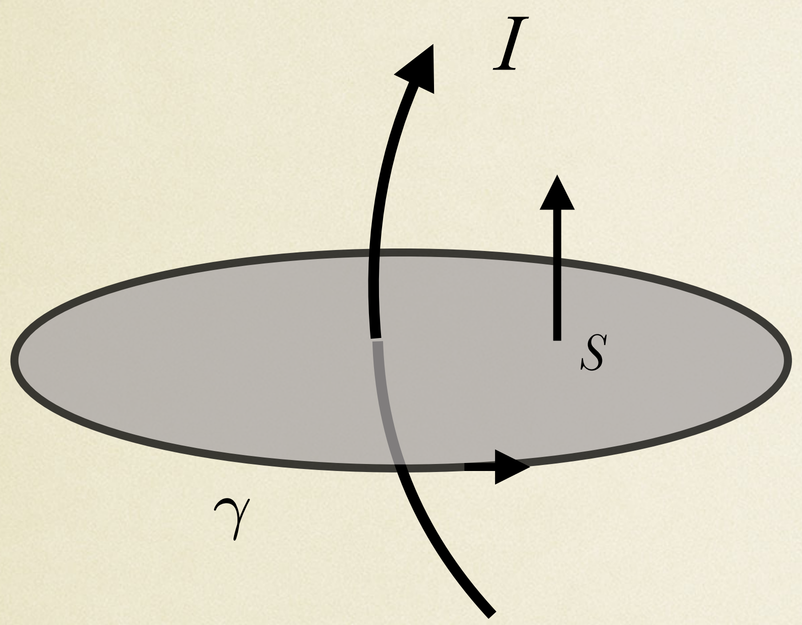


$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = -\mu_0 I.$$

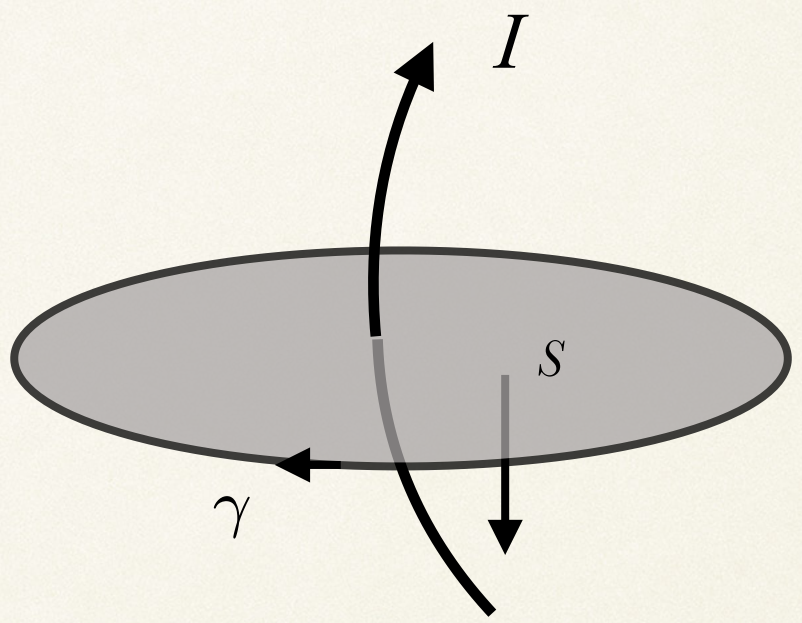


$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = \ ?$$

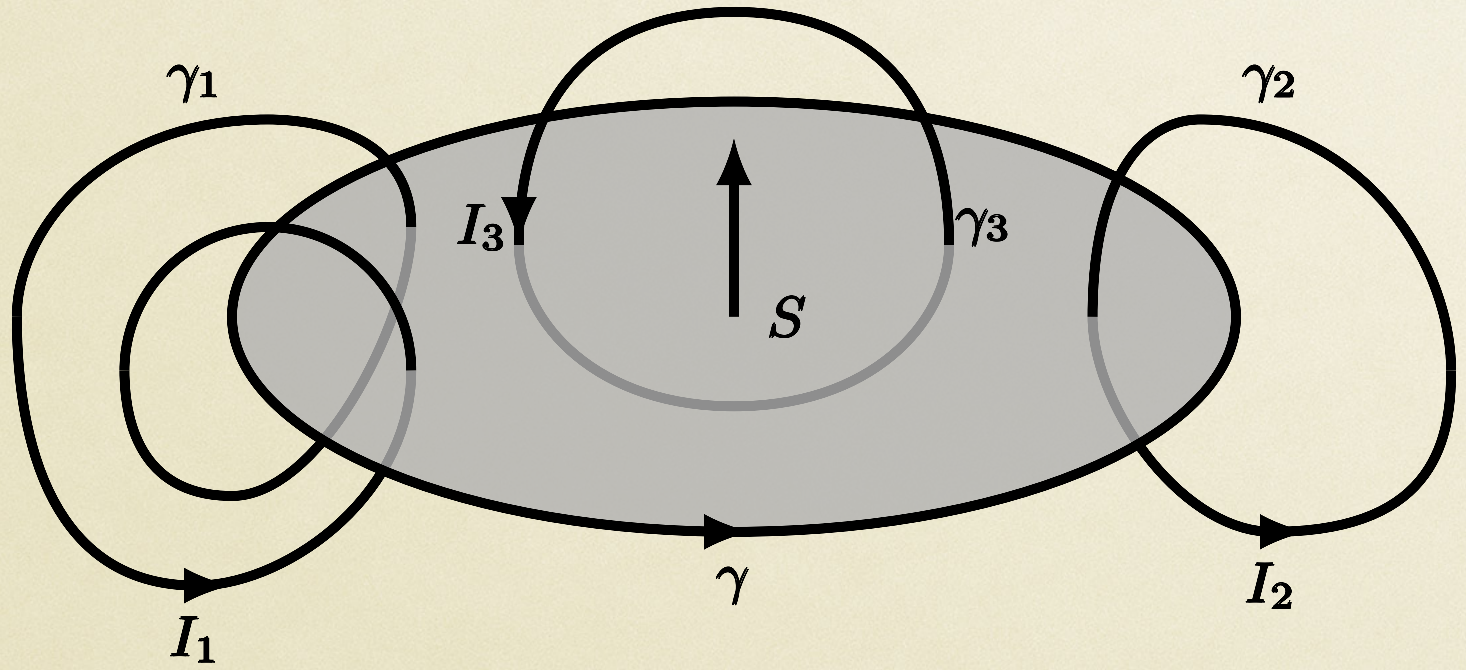
Ampere's Law



$$\oint_\gamma \mathbf{B} \cdot d\vec{\ell} = \mu_0 I$$



$$\oint_\gamma \mathbf{B} \cdot d\vec{\ell} = -\mu_0 I$$



$\text{Link}(\gamma_1, \gamma) = +2$, $\text{Link}(\gamma_2, \gamma) = -1$, and $\text{Link}(\gamma_3, \gamma) = 0$

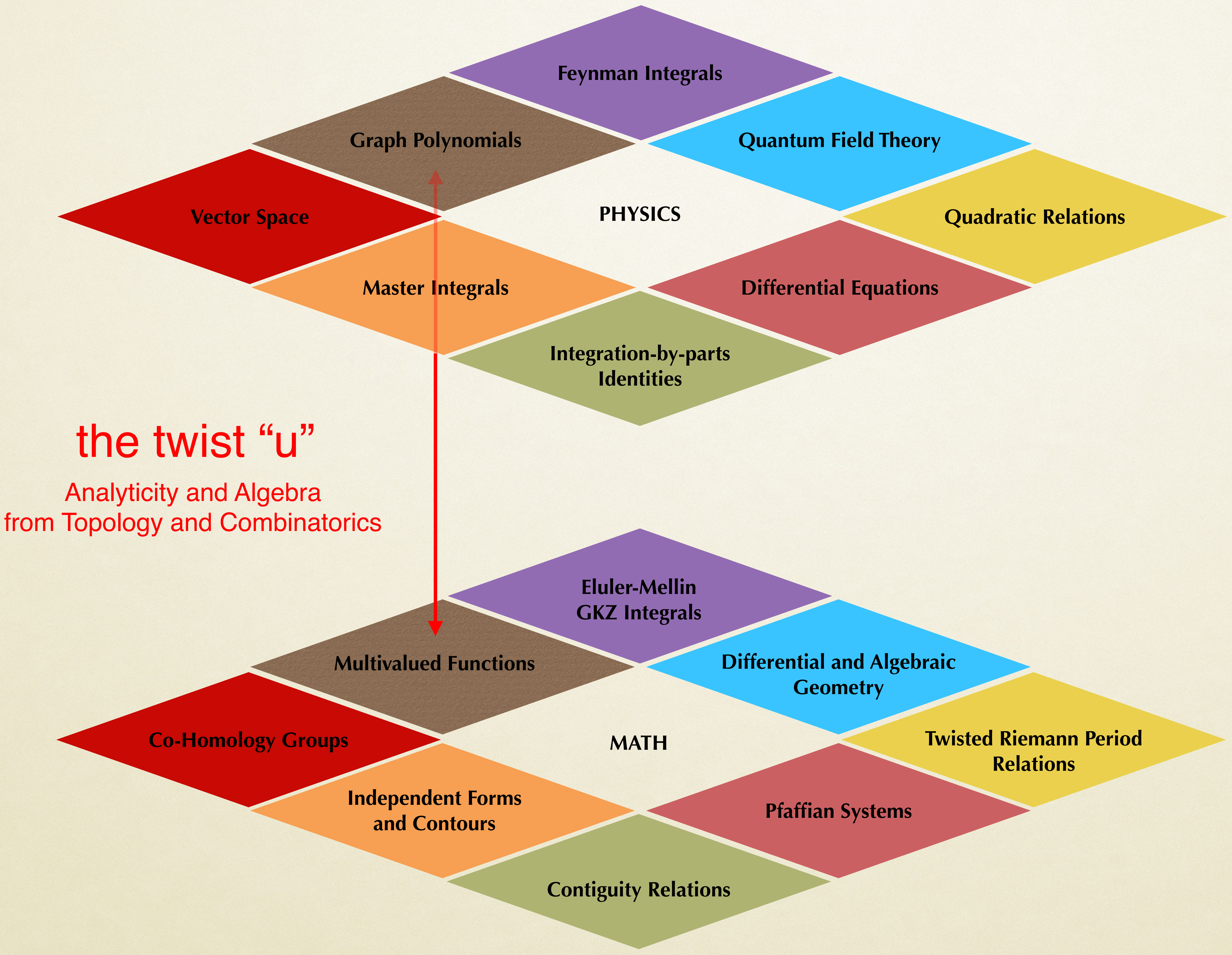
- **Integral decomposition by geometry**

$$\oint_\gamma \mathbf{B} \cdot d\vec{\ell} = \sum_k (\pm n_k) \oint_{\gamma_k} \mathbf{B} \cdot d\vec{\ell} = \mu_0 \sum_k (\pm n_k) I_k$$

Master Contributions

Gauss' Linking Number

$$n_k = \text{Link}(\gamma_k, \gamma)$$



Quantum Field Theory

Twisted de Rham Theory

Summary

- ***The ubiquitous De Rahm Theory***
  - Intersection Theory for Twisted de Rham co-homology
  - Analyticity & Unitarity vs Differential and Algebraic Geometry, Topology, Number Theory, Combinatorics, Statistics

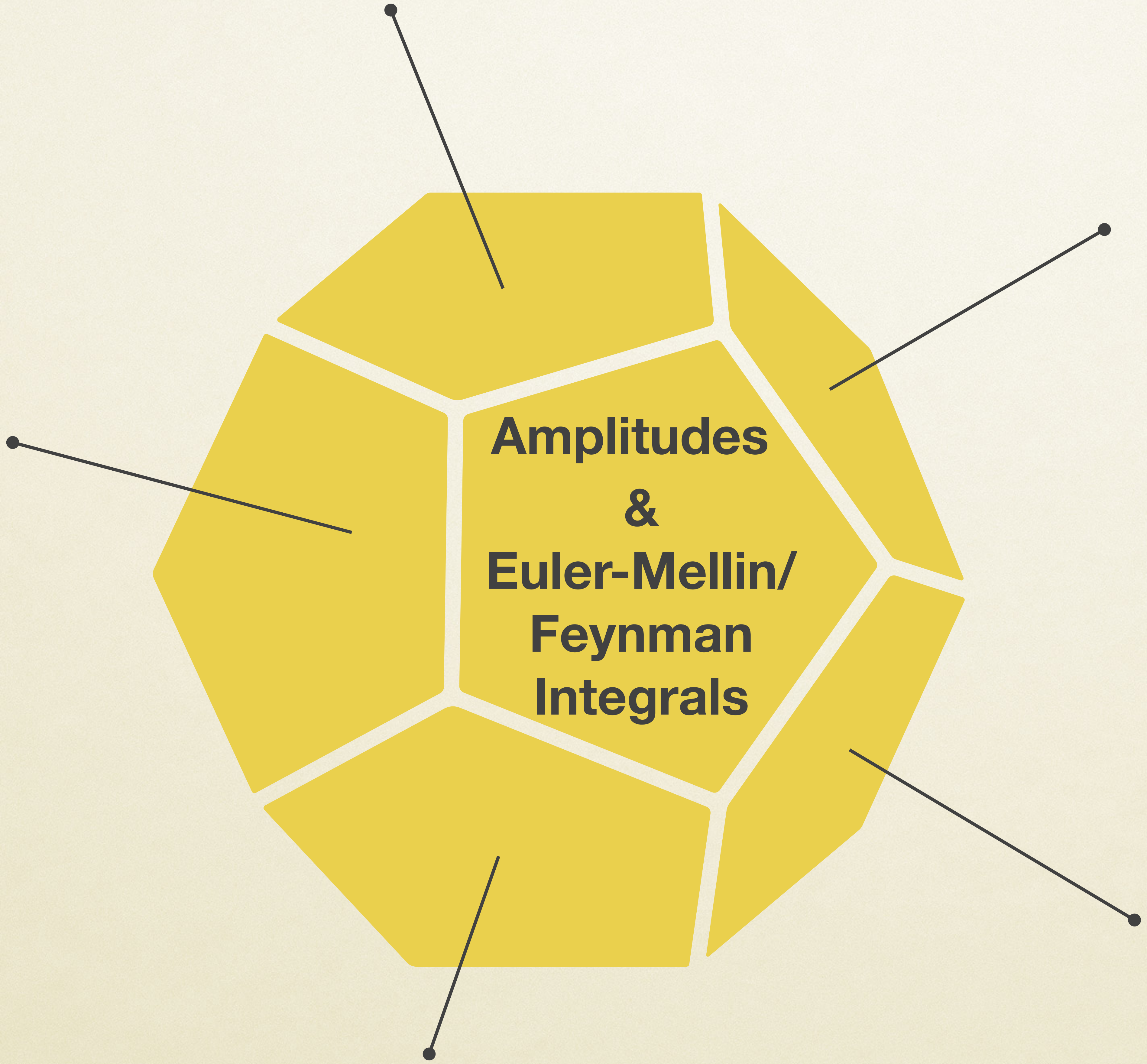
- **Novel Concepts: Vector Space Structures**
  - Vector-space dimensions = dimension of co-homology group = *counting holes* = number of independent Integrals
  - Intersection Numbers ~ **Scalar Product** for Feynman (Twisted Period) Integrals
- **New Methods for Multivariate Intersection number**
  - Relation between Ordinary and Relative Cohomogy
  - Fibration-based method and Companion Tensor Algebra

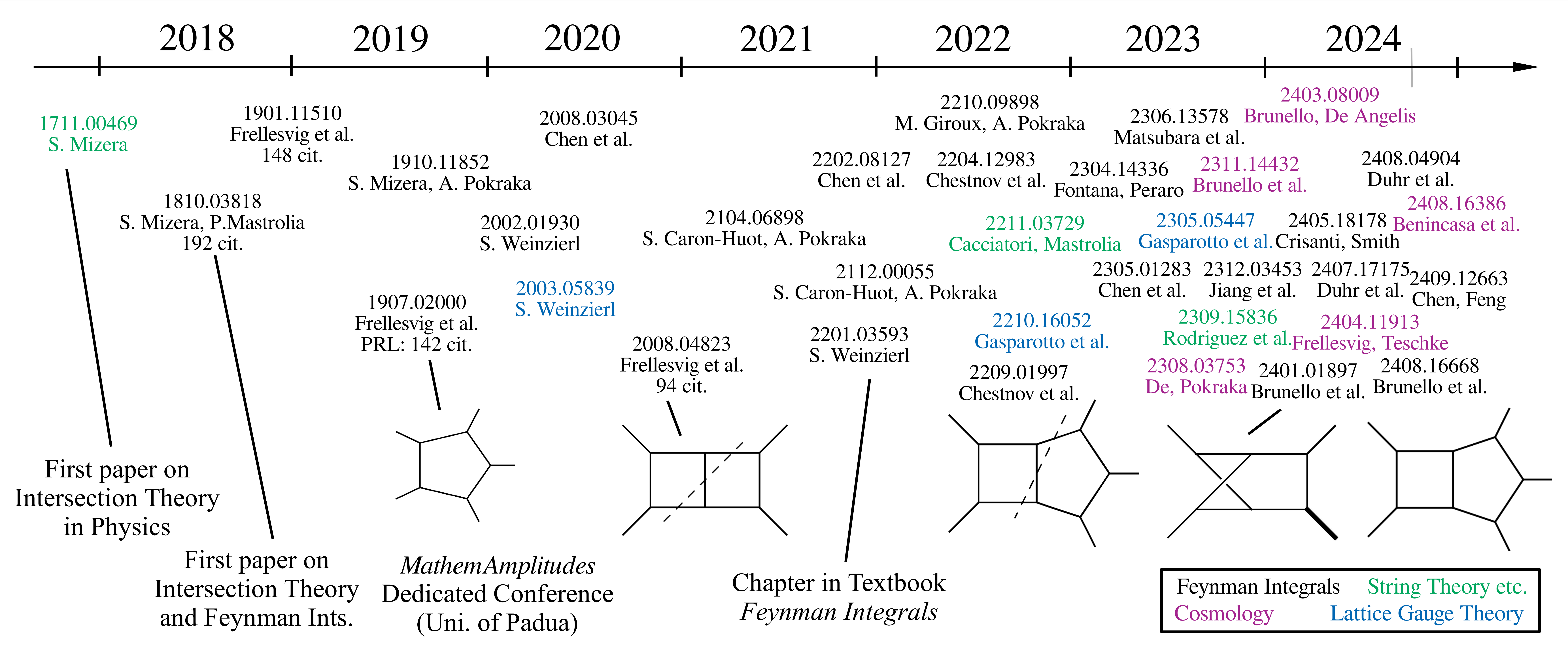
- **General algorithm for Physics and Math applications**
  - key: Co-Homology Group Isomorphisms
  - Triggering interdisciplinarity: interwinement between Fundamental Physics, Geometry and Statistics: fluxes ~ period integrals ~ statistical moments
    - Feynman Integrals
    - D-modules & GKZ theory
    - QM Matrix Elements
    - Correlator functions in Cosmology
    - (Gluing methods in N=4 SYM)
    - Euler-Mellin Integrals
    - Orthogonal polynomials
    - Correlator functions in QFT
    - Fourier integrals
    - … … …
  - Related recent interesting applications

Loebbert, Stawiński (2024) Hang (2024) Chen, Feng (2024)

Duhr, Porkert, Semper, Stawiński (2024) Lu, Wang, Lin Yang (2024) Chen, Feng, Tao (2024) **(Eden, Crisanti Gottwald, Scherdin,** & PM (tomorrow)).

Scattering Amplitudes & Multiloop Calculus: interdisciplinary toolbox





courtesy: **Hjalte Frellesvig**

MathemAmplitudes 2025

Co-homology and Combinatorics of GKZ Systems, Euler-Mellin-Feynman Integrals, and Scattering Amplitudes.

September 22-26, 2025
MITP Mainz

Bringing together mathematicians and theoretical physicists with interdisciplinary expertise, the workshop will cover a broad range of topics, including Differential and Algebraic Geometry, Number Theory, Combinatorics, Statistics, Feynman Integrals, and Scattering Amplitudes.

Mainz Institute for Theoretical Physics

Organizers:
Claudia Fevola, Federico Gasparotto, Manoj K. Mandal, Sebastian Mizera, and Pierpaolo Mastrolia

Definition. *Physics is a part of mathematics devoted to the calculation of integrals of the form $\int g(x)e^{f(x)}dx$. Different branches of physics are distinguished by the range of the variable x and by the names used for $f(x)$, $g(x)$ and for the integral.* [...]

Schwarz, Shapiro (2018)

Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form

$$I(g) = \int g(x)e^{-f(x)}dx$$

[...] If f can be represented as $f_0 + \lambda V$ where f_0 is a negative quadratic form, then the integral $\int g(x)e^{f(x)}\,dx$ can be calculated in the framework of perturbation theory with respect to the formal parameter λ. We will fix f and consider the integral as a functional $I(g)$ taking values in $\mathbb{R}[[\lambda]]$. It is easy to derive from the relation

$$\int \partial_a(h(x)e^{f(x)})dx = 0$$

that the functional $I(g)$ vanishes in the case when g has the form

$$g = \partial_a h + (\partial_a f)h.$$

☑ Addressing a common math problem might be useful to make progress in different disciplines

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Acknowledgements

For collaboration and/or discussions:
S. Mizera,
H. Frellesvig, M. Mandal, F. Gasparotto, L. Mattiazzi, S. Laporta,
K. Matsumoto, Y. Goto, S. Matsubara-Heo, N. Takayama,
S. Cacciatori, T. Damour, M. Kontsevich, D. Bini,
V. Chestnov, H. Munch, T. Peraro, G. Fontana,
G. Brunello, G. Crisanti, S. Smith, M. Giroux, W. Flieger,
W. J. Torres, J. Ronca, R. Patil, M. Bigazzi, G. Boni,
J. Henn, B. Sturmfels,
G. Pimentel, P. Benincasa, F. Vazao, B. Eden,
C. Fevola, M. Bertola
B. Eden, M. Gottwald, T. Scherdin

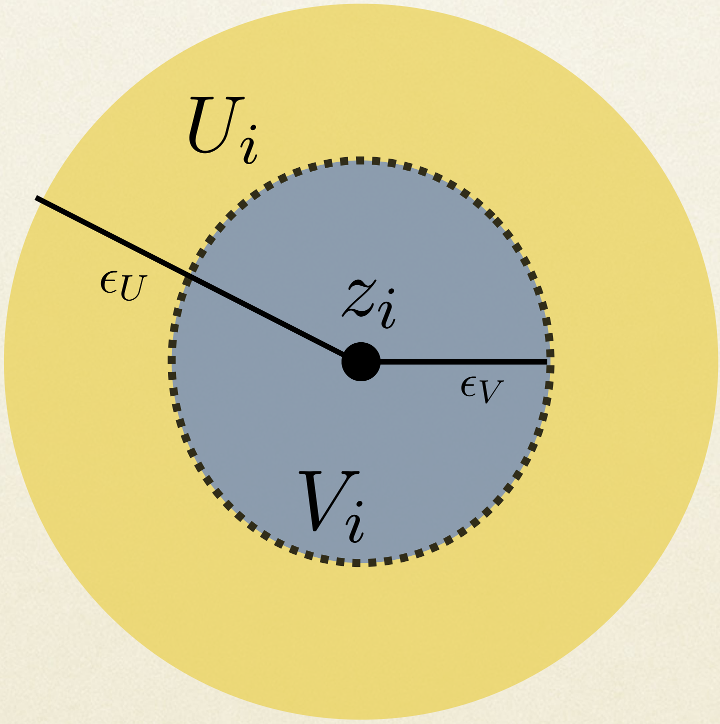
Extra Slides

Univariate Intersection Number

Matsumoto (1996)
Mizera (2017)

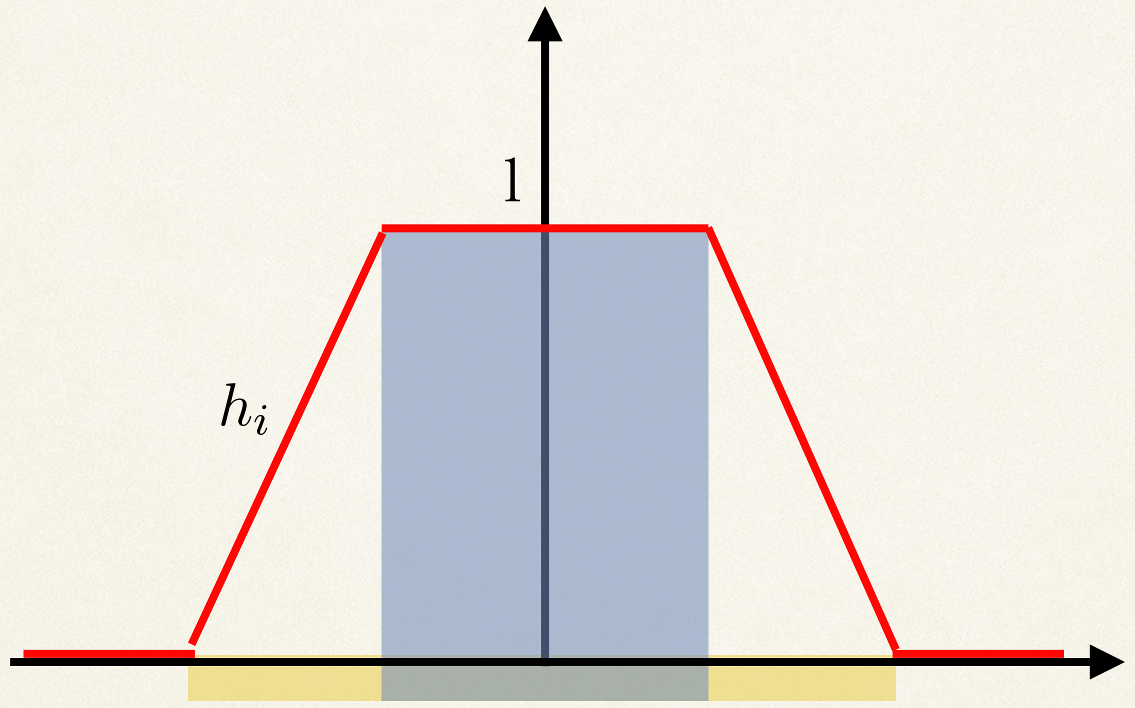
1. Regularized Forms

Logarithmic twisted cocycles φ_L can have simple poles only at z_i's. To construct φ_L^c with compact support, we must find a cocycle in the sme cohomology class, which vanishes in a small tubular neighborhood around each z_i.
Let's divide the space $X = \mathbb{CP}^1 \setminus \cup_{i=1}^k \{z = z_i\}$, into regions:



where V_i and U_i are discs centered in z_i with small radii $0 < \epsilon_V < \epsilon_U$. For convenience, let us define the annulus $D_i = U_i \setminus V_i$.

We introduce the regulating function



$$h_i = h_i(z,\bar z) \equiv \begin{cases} 0, & \text{on } U_i \\ 0 < h_i < 1, & \text{on } D_i = U_i \setminus V_i \\ 1, & \text{on } V_i \end{cases} \tag{1.1}$$

and define

$$\varphi_L^c \equiv \varphi_L - \sum_{z_i \in \mathcal{P}_\omega} \nabla_\omega (h_i \psi_i) \tag{1.2}$$

For notation ease, we omit the sum over the poles of ω, and restore it at the end. Observe that,

$$\nabla_\omega (h_i\psi_i) = (d+\omega)(h_i\psi_i) = \psi_i(dh_i) + h_i(d\psi_i) + h_i\omega\psi_i = \psi_i(dh_i) + h_i \nabla_\omega \psi_i \tag{1.3}$$

Therefore,

$$\varphi_L^c \equiv \varphi_L - (\psi_i(dh_i) + h_i \nabla_\omega \psi_i) \tag{1.4}$$

Iff

$$\nabla_\omega \psi_i = \varphi_L \ , \qquad \text{for } z \to z_i, \qquad \text{namely on } U_i \setminus \{z_i\} \tag{1.5}$$

then

$$\varphi_L^c \equiv \begin{cases} 0, & \text{on } V_i \\ \varphi_L - (\psi_i(dh_i) + h_i \varphi_L) \ , & \text{on } D_i = U_i \setminus V_i \\ \varphi_L \ , & \text{on } X \setminus U_i \end{cases} \tag{1.6}$$

hence φ_L^c has *compact support*, because $\varphi_L^c = 0$ on $\cup_{i=1}^k V_i$.

Let us consider the following two identities:

1. Since $\varphi_L^c = \varphi_L$,on $X \setminus U_i$,

$$\int_{X\setminus U_i} \varphi_L^c \wedge \varphi_R = 0 \tag{1.7}$$

2. Useful identity

$$d(h_i\psi_i\varphi_R) = d(h_i\psi_i) \wedge \varphi_R + h_i\psi_i \wedge \underbrace{d\varphi_R}_{=0} = \psi_i dh_i \wedge \varphi_R + \underbrace{h_i \underbrace{d\psi_i}_{\varphi_L - \psi_i\omega} \wedge \varphi_R}_{=0}$$
$$= \psi_i \, dh_i \wedge \varphi_R \tag{1.8}$$

We are now in the position for defining the *intersection number* as the pairing

$$\begin{aligned}
\langle \varphi_L | \varphi_R \rangle &= \int_X \varphi_L^c \wedge \varphi_R \\
&= \underbrace{\int_{V_i} \underbrace{\varphi_L^c}_{=0} \wedge \varphi_R}_{=0} + \int_{X\setminus V_i} \varphi_L^c \wedge \varphi_R \\
&= (\text{by adding and subtracting } D_i) \\
&= \int_{D_i} \varphi_L^c \wedge \varphi_R + \underbrace{\int_{X\setminus U_i} \varphi_L^c \wedge \varphi_R}_{=0} \\
&= \int_{D_i} (\varphi_L - \psi_i(dh_i) - h_i\varphi_L) \wedge \varphi_R \\
&= -\int_{D_i} \psi_i(dh_i) \wedge \varphi_R \\
&= -\int_{D_i} d(h_i\psi_i\varphi_R) = -\int_{\partial D_i} h_i\psi_i\varphi_R \\
&= -\int_{\partial U_i} \underbrace{h_i}_{=0} \psi_i\varphi_R + \int_{\partial V_i} \underbrace{h_i}_{=1} \psi_i\varphi_R \\
&= \int_{\partial V_i} \psi_i\varphi_R = \oint \psi_i\varphi_R = (2\pi i)\mathrm{Res}_{z=z_i}\Big\{\psi_i\varphi_R\Big\} \ .
\end{aligned} \tag{1.9}$$

Lt us recap our result, which, after reinserting the sum over poles, reads as:

$$\langle \varphi_L | \varphi_R \rangle = (2\pi i) \sum_i \mathrm{Res}_{z=z_i}\Big\{\psi_i\varphi_R\Big\} \ , \tag{1.10}$$

where ψ_i is the solution of the differential equation $\nabla_\omega \psi = \varphi_L$ areound the pole $z = z_i$.

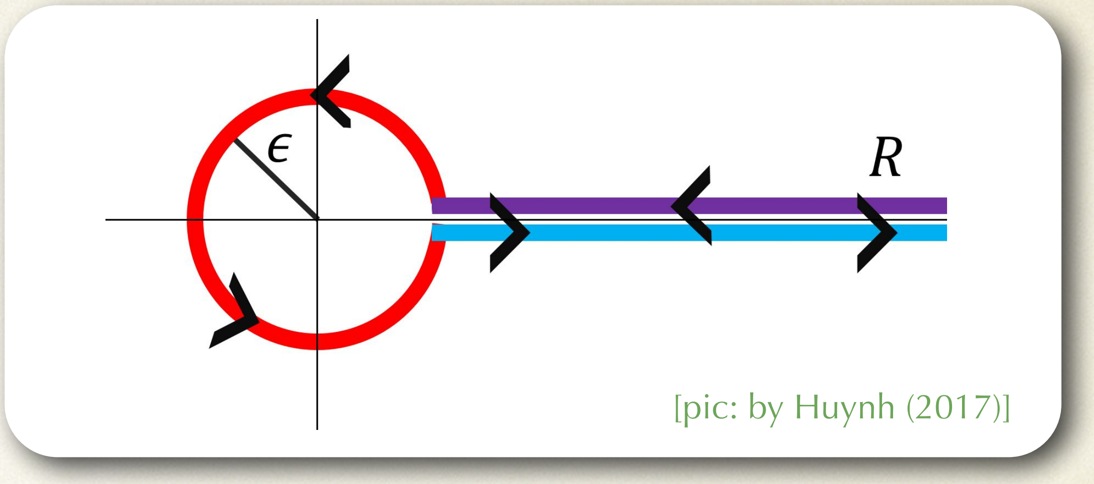
1.1 Alternative definition

Mizera (2019)

$\epsilon_V = \epsilon_U \equiv \epsilon$

$$h_i = h_i(z,\bar z) \equiv \theta(|z - z_i|^2 - \epsilon^2)$$

Regularised Twisted Cycles



- **Analytic Continuation of Gamma Function** [Hankel Contour]

$$\Gamma(s) = \frac{1}{e^{2\pi i s} - 1} \oint_C t^{s-1} e^{-t} \mathrm{d}t.$$

Due to the branch cut and the denominator of $e^{2\pi i s} - 1$, *this representation is defined for complex numbers* s *where* $s \notin \mathbb{R}_{\geq 0} \cup \mathbb{Z}_{<0}$.

$$\oint_C t^{s-1} e^{-t} \mathrm{d}t = \lim_{R\to\infty, \epsilon\to 0} \left(\int_R^\epsilon x^{s-1} e^{-x} \mathrm{d}x + \int_0^{2\pi} \left(\epsilon e^{i\theta}\right)^{s-1} e^{-\epsilon e^{i\theta}} i\epsilon e^{i\theta} \mathrm{d}\theta + \int_\epsilon^R \left(x e^{2\pi i}\right)^{s-1} e^{-x e^{2\pi i}} \mathrm{d}x \right) = \lim_{R\to\infty} \left(-\int_0^R x^{s-1} e^{-x} \mathrm{d}x + 0 + e^{2\pi i s} e^{-2\pi i} \int_0^R x^{s-1} e^{-x} \mathrm{d}x \right) = \left(e^{2\pi i s} - 1\right) \Gamma(s).$$

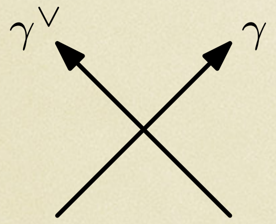
[purple line] [red circle] [blu line]

- **Analytic Continuation of Beta Function** Yoshida; Matsumoto; Mizera [1706.08527]

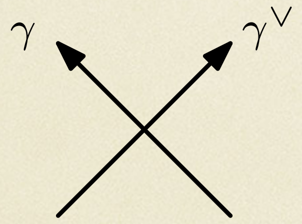
$$\oint_\gamma z^s (1-z)^t \varphi(z) = \left(1 - e^{2\pi i t} + e^{2\pi i (s+t)} - e^{2\pi i s}\right) \int_0^1 z^s (1-z)^t \varphi(z),$$

$$\mathrm{reg}\,\overrightarrow{(0,1)} := \frac{S(\varepsilon, 0)}{e^{2\pi i s} - 1} + \overrightarrow{(\varepsilon, 1-\varepsilon)} - \frac{S(1-\varepsilon, 1)}{e^{2\pi i t} - 1} =$$

- **Intersection Numbers for Twisted cycles:** graphical rules :: **signs**, **circles** and **monodromy factors** ::



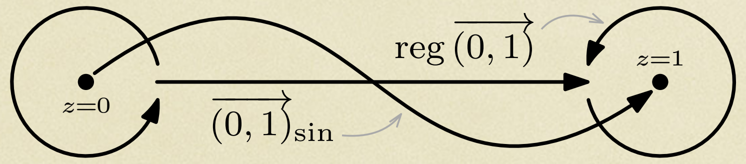
$= +1$ or



$= -1.$

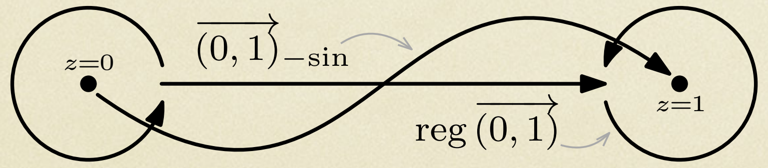
$$\mathsf{C}(1234) = \{0 < z_2 < 1\} \otimes z_2^{s_{12}} (1-z_2)^{s_{23}} = \overrightarrow{(0,1)} \otimes z^s (1-z)^t,$$

$\langle \mathsf{C}(1234), \mathsf{C}(1234) \rangle =$



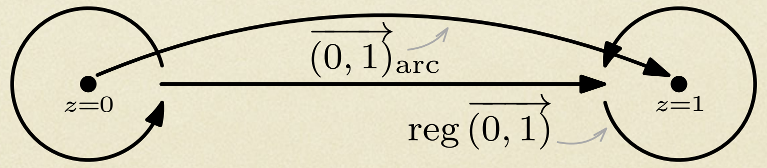
$= -\frac{1}{e^{2\pi i s} - 1} - 1 - \frac{1}{e^{2\pi i t} - 1}.$

$=$



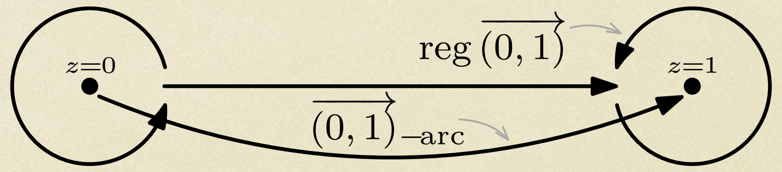
$= -\frac{e^{2\pi i s}}{e^{2\pi i s} - 1} + 1 - \frac{e^{2\pi i t}}{e^{2\pi i t} - 1}.$

$=$



$= -\frac{1}{e^{2\pi i s} - 1} - \frac{e^{2\pi i t}}{e^{2\pi i t} - 1},$

$=$



$= -\frac{e^{2\pi i s}}{e^{2\pi i s} - 1} - \frac{1}{e^{2\pi i t} - 1}.$

$$= \frac{i}{2} \left(\frac{1}{\tan \pi s} + \frac{1}{\tan \pi t} \right)$$

Intersection Numbers for *Logarithmic* n-forms

Matsumoto (1998), Mizera (2017)

If $\langle\varphi_L|$ and $\langle\varphi_R|$ are dLog n-forms (hence contain only simple poles)

$$\langle\varphi_L|\varphi_R\rangle = \int dz_1 \cdots dz_n \, \delta(\omega_1)\cdots\delta(\omega_n) \; \hat{\varphi}_L \, \hat{\varphi}_R =$$

$$= \sum_{(z_1^*,\ldots,z_n^*)} \det{}^{-1} \begin{bmatrix} \frac{\partial\omega_1}{\partial z_1} & \cdots & \frac{\partial\omega_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\omega_n}{\partial z_1} & \cdots & \frac{\partial\omega_n}{\partial z_n} \end{bmatrix} \widehat{\varphi}_L \, \widehat{\varphi}_R \Bigg|_{(z_1,\ldots,z_n)=(z_1^*,\ldots z_n^*)}$$

[Global Residue Theorem]

$(z_1^*,\ldots z_n^*)$ *critical points*, namely the solutions of the system $\quad \omega_i = 0, \quad i = 1,\ldots n.$

In the 1-variate case: $\quad \langle\varphi_L|\varphi_R\rangle = \mathrm{Res}_{z\in\mathcal{P}_{\omega_1}}\left(\frac{\hat{\varphi}_L\,\hat{\varphi}_R}{\omega}\right) = \int dz_1\,\delta(\omega_1)\,\hat{\varphi}_L\,\hat{\varphi}_R = \sum_{(z_1^*)} \frac{\hat{\varphi}_L\,\hat{\varphi}_R}{\partial\omega_1/\partial z_1}$

[Residue Theorem]

- **Efficiently implemented also *via* Companion Matrix** credit **Salvatori**

Quadratic Relations

Twisted Riemann Periods Relations (TRPR)

- **Completeness for forms**

$$\sum_{i,j=1}^{\nu} |e_j\rangle \, (\mathbf{C}^{-1})_{ji} \, \langle e_i| = \mathbb{I}_c \qquad \mathbf{C}_{ij} \equiv \langle e_i | e_j \rangle$$

- **Completeness for contours**

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j] \, (\mathbf{H}^{-1})_{ji} \, [\mathcal{C}_i| = \mathbb{I}_h \qquad \mathbf{H}_{ij} \equiv [\mathcal{C}_i | \mathcal{C}_j]$$

- **Riemann Twisted Period Relations** Cho, Matsumoto (1995)

$$\langle \, \varphi_{\rm L} \, | \, \varphi_{\rm R} \, \rangle = \sum_{i,j} \langle \, \varphi_{\rm L} \, | \, \mathcal{C}_{{\rm R},j} \,] \, [\, \mathcal{C}_{{\rm L},j} \, | \, \mathcal{C}_{{\rm R},i} \,]^{-1} \, [\, \mathcal{C}_{{\rm L},i} \, | \, \varphi_{\rm R} \, \rangle$$

$$[\, \mathcal{C}_{\rm L} \, | \, \mathcal{C}_{\rm R} \,] = \sum_{i,j} [\, \mathcal{C}_{\rm L} \, | \, \varphi_{{\rm R},j} \, \rangle \, \langle \, \varphi_{{\rm L},j} \, | \, \varphi_{{\rm R},i} \, \rangle^{-1} \, \langle \, \varphi_{\rm L} \, | \, \mathcal{C}_{\rm R} \,]$$

TRPR for Gauss Hypergeometric Function

Cho, Matsumoto (1995)

$$u = t^{\alpha}(1-t)^{\gamma-\alpha}(1-xt)^{-\beta}, \qquad \varphi_1 = \left(\frac{dt}{t-x_1} - \frac{dt}{t-x_2}\right) = \frac{dt}{t(1-t)}, \ \varphi_3 = \left(\frac{dt}{t-x_3} - \frac{dt}{t-x_4}\right) = \frac{-xdt}{1-xt},$$

$$P^+ = \begin{pmatrix} \int_0^1 u\varphi_1 & \int_{1/x}^{\infty} u\,\varphi_1 \\ \int_0^1 u\varphi_3 & \int_{1/x}^{\infty} u\,\varphi_3 \end{pmatrix}, \ P^- = \begin{pmatrix} \int_0^1 u^{-1}\varphi_1 & \int_{1/x}^{\infty} u^{-1}\varphi_1 \\ \int_0^1 u^{-1}\varphi_3 & \int_{1/x}^{\infty} u^{-1}\varphi_3 \end{pmatrix}, \qquad I_{ch} = 2\pi i \begin{pmatrix} 1/\alpha + 1/(\gamma-\alpha) & 0 \\ 0 & -1/\beta + 1/(\beta-\gamma) \end{pmatrix} \qquad I_h = -\begin{pmatrix} d_{12}/d_1 d_2 & 0 \\ 0 & d_{30}/d_3 d_0 \end{pmatrix},$$

$$c_{jk\dots} = c_j c_k \cdots, \ d_{jk\dots} = c_j c_k \cdots - 1$$

$$c_j = \exp 2\pi i \alpha_j$$

$$\int_0^1 u\,\varphi_1 = B(\alpha, \gamma-\alpha) F(\alpha, \beta, \gamma; x),$$

$$\int_{1/x}^{\infty} u\,\varphi_1 = -(-1)^{\gamma-\alpha-\beta} x^{1-\gamma} B(\beta-\gamma+1, -\beta+1) \times F(\beta-\gamma+1, \alpha-\gamma+1, 2-\gamma; x),$$

- **Riemann Twisted Period Relations**

$$P^{+}\, {}^{t}I_h^{-1}\, {}^{t}P^{-} = I_{ch}$$

(1,2)- component $\quad F(\alpha, \beta, \gamma; x) F(1-\alpha, 1-\beta, 2-\gamma; x) = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x) F(\gamma-\alpha, \gamma-\beta, \gamma; x)$

(1, 1)-component $\quad F(\alpha, \beta, \gamma; x) F(-a, -\beta, -\gamma; x) - 1 = \dfrac{\alpha\beta(\gamma-\alpha)(\gamma-\beta)}{\gamma^2(\gamma+1)(\gamma-1)} F(\beta-\gamma+1, \alpha-\gamma+1, -\gamma+2; x) \times F(\gamma-\beta+1, \gamma-\alpha+1, \gamma+2; x).$

Elliot's Identity from Intersections

Matsumoto **& P.M.**

The complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2\phi}} \qquad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\phi}\, d\phi$$

$r \in (0, 1)$

- **Legendre Identity**

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

$$\mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r')$$

$$r^2 + r'^2 = 1$$

Elliot's Identity from Intersections

Matsumoto **& P.M.**

The complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2\phi}} \qquad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\phi}\, d\phi$$

$r \in (0, 1)$

- **Legendre Identity**

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

$$\mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r')$$

$$r^2 + r'^2 = 1$$

- **Elliot's Identity and Hypergeometric Functions**

Balasubramanian, Naik, Ponnusamy, Vuorinen (2001)

$$\begin{aligned}
&F(\tfrac{1}{2}+\lambda, -\tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\
&+F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(-\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\
&-F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\
&=\frac{\Gamma(1+\lambda+\mu)\Gamma(1+\mu+\nu)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}.
\end{aligned}$$

the choice $\lambda = \mu = \nu = 0$ gives the Legendre relation.

Elliot's Identity from Intersections

Matsumoto **& P.M.**

- *Hypothesys*: **too close to RTPR to be accidental**
- *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \Big(\frac{1}{t} - \frac{1}{t-1/r}\Big)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \Big(\frac{1}{t} - \frac{1}{t-1}\Big)dt.$$

$$\gamma = (0,1)\otimes u(t) \text{ and } \delta = (-\infty,0)\otimes 1/u(t)$$

- *Twisted Riemann Period Relation*

$${}^t\Pi_\omega \, {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\Big(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\Big)\, {}^tH_c^{-1} \begin{pmatrix} \int_{-\infty}^0 \frac{1}{u(t)}\psi_1 \\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2 \end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}.$$

$$\Big(F(\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r), F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r)\Big)\cdot\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\ F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r)\end{pmatrix} = \frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

Elliot's Identity from Intersections

Matsumoto & P.M.

- *Hypothesys*: **too close to RTPR to be accidental**
- *Proof*

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \Big(\frac{1}{t} - \frac{1}{t-1/r}\Big)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \Big(\frac{1}{t} - \frac{1}{t-1}\Big)dt.$$

$$\gamma = (0,1)\otimes u(t) \text{ and } \delta = (-\infty,0)\otimes 1/u(t)$$

- *Twisted Riemann Period Relation*

$${}^t\Pi_\omega \, {}^tH_c^{-1} \Pi_{-\omega} = H_h.$$

$$\Big(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\Big)\, {}^tH_c^{-1} \begin{pmatrix} \int_{-\infty}^0 \frac{1}{u(t)}\psi_1 \\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2 \end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}.$$

$$\Big(F(\tfrac{1}{2}+\lambda, -\tfrac{1}{2}-\nu, 1+\lambda+\mu; r), F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu, 1+\lambda+\mu; r)\Big)\cdot\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\ F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \end{pmatrix} = \frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

- **Quadratic relations for Feynman Integrals**

Broadhurst, Roberts (2018)
Lee, Pomeranski (2019)

$$\mathsf{P}_k^{\mathrm{BR}} \cdot \mathsf{D}_k^{\mathrm{BR}} \cdot {}^t\mathsf{P}_k^{\mathrm{BR}} = \mathsf{B}_k^{\mathrm{BR}}$$

Fresan, Sabbah, Yu (2020)

- String-Theory Amplitudes: **KLT relations = TRPR** Mizera (2016/17)

$$\mathcal{A}^{\mathrm{GR}} = \sum_{\beta,\gamma} \mathcal{A}^{\mathrm{YM}}(\beta)\, m^{-1}(\beta|\gamma)\, \mathcal{A}^{\mathrm{YM}}(\gamma)$$

$$\mathcal{A}^{\mathrm{closed}} = \sum_{\beta,\gamma} \mathcal{A}^{\mathrm{open}}(\beta)\, m_{\alpha'}^{-1}(\beta|\gamma)\, \mathcal{A}^{\mathrm{open}}(\gamma)$$