NESTED SOFT-COLLINEAR INFRARED SUBTRACTION

Federica Devoto

Loop-the-Loop, 11/14/2024

SLAC NATIONAL ACCELERATOR LABORATORY

LONG STORY SHORT

K. Asteriadis, A. Behring, F. Buccioni, F. Caola, M. Delto, FD, K. Melnikov, R. Roenstch, C. Signorile-Signorile, D.M. Tagliabue, …

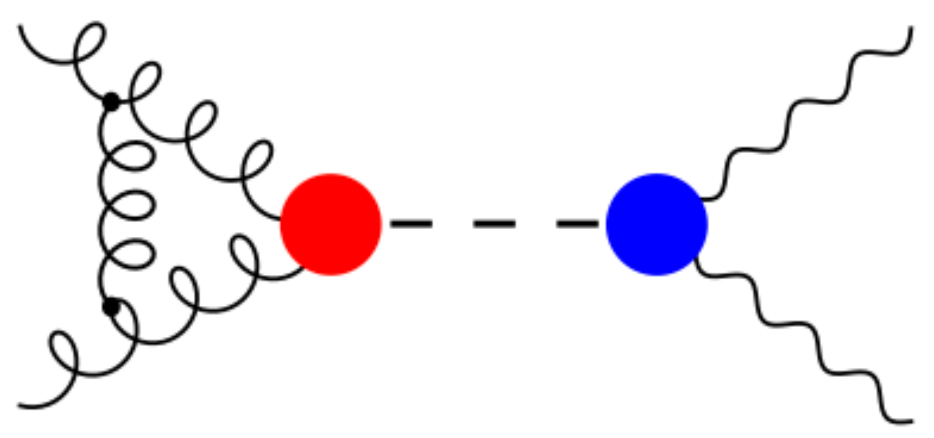
$$d\sigma = \sum_{i_1,i_2} \int dx_1 dx_2 f_{i_1}(x_1) f_{i_2}(x_2) d\sigma_{i_1 i_2}(x_1, x_2) F_J \left(1 + \mathcal{O}(\Lambda^n_{\rm QCD}/Q^n)\right), \quad n \geq 1.$$

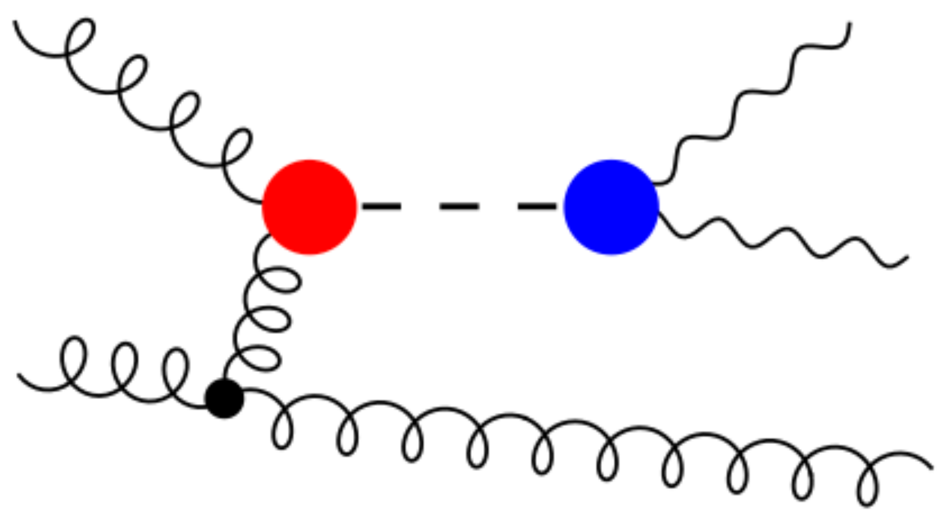
Parton distribution functions

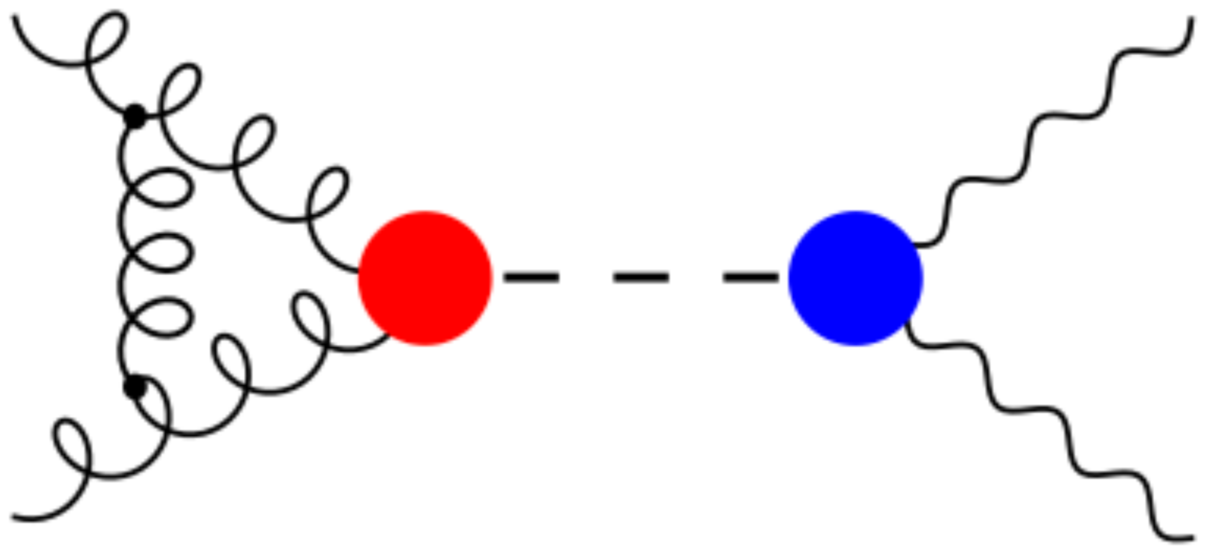
Partonic cross section

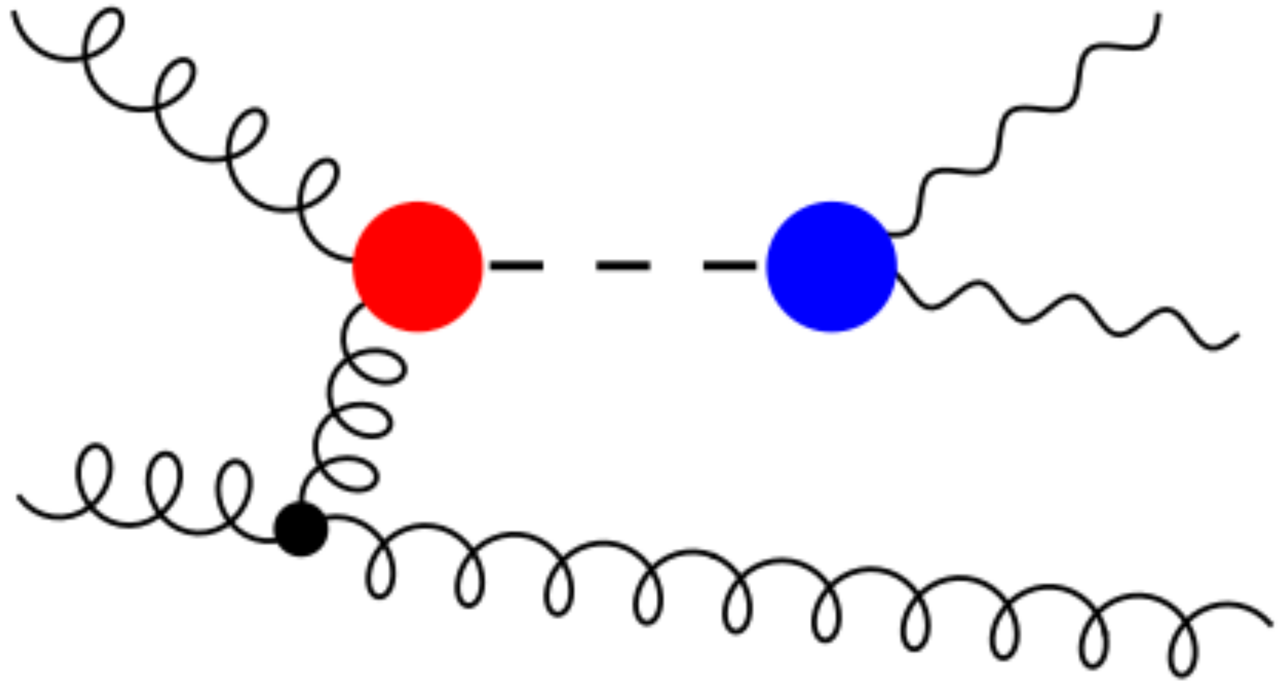
Encodes non-perturbative effects

$$d\sigma_{ij} = d\sigma_{ij,\rm LO}\left(1 + \alpha_s \Delta^{QCD}_{ij,\rm NLO} + \alpha_{ew} \Delta^{EW}_{ij,\rm NLO} + \alpha_s^2 \Delta^{QCD}_{ij,\rm NNLO} + \alpha_s \alpha_{ew} \Delta^{QCD\otimes EW}_{ij,\rm NNLO} + \dots\right)$$









In general, UV & IR divergent

Understood

Needs to be combined with legs

Divergences are **explicit**

IR divergent when integrated over radiation PS

Divergences are **implicit,** i.e. appear after integrating

How do we deal with these divergences?

= finite!

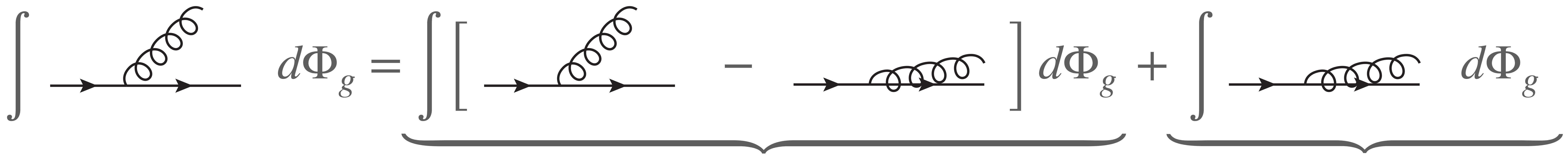
Inclusive: not a problem, KLN ensures cancellation. Relatively interesting

Differential: more interesting, but need proper subtraction of divergences

Structure of divergences get more and more involved at higher orders, important to understand and organise the cancellation

“Subtraction scheme”

Main idea of subtraction: add and subtract the divergent configurations



Finite in d=4

Divergent "counterterm"

The choice of the counterterm defines a given subtraction scheme

This talk: Nested soft-collinear subtraction scheme

For a review of all subtraction schemes on the market see talk by Gloria Bertolotti

Identikit of suitable counterterm:

- Approximate full matrix element in all singular limits
- Easy to integrate
- Other optional (?) features: locality, Lorentz invariance, limit number of spurious singularities etc.

Other approaches: slicing

$$\int |\mathcal{M}|^2 F_J \, d\phi_d = \int_0^{\delta} \left[|\mathcal{M}|^2 F_J \, d\phi_d \right]_{\text{approx}} + \int_{\delta}^{\infty} |\mathcal{M}|^2 F_J \, d\phi_4 \; + \mathcal{O}(\delta)$$

Match to resummation,…

Slicing parameter

Transverse momentum of colorless/ colored (massive) system

N-jettiness $\tau_N = \frac{2}{Q^2} \sum_k \min\left\{ q_a p_k, q_b p_k, \ldots, q_N p_K \right\}$

- qT slicing [Catani, Grazzini]
- Jettiness slicing [R. Boughezal et al., J. Gaunt et al.]

$$\lim_{\tau \to 0} d\sigma_{pp \to X} \approx B \otimes B \otimes S \otimes H \otimes J \otimes d\sigma^{\text{LO}}_{pp \to X}$$

Non local ✗

Easier generalization to higher orders ✓

NSC AT NLO (FKS)

Simple (yet instructive) example: Z+j production in proton collision

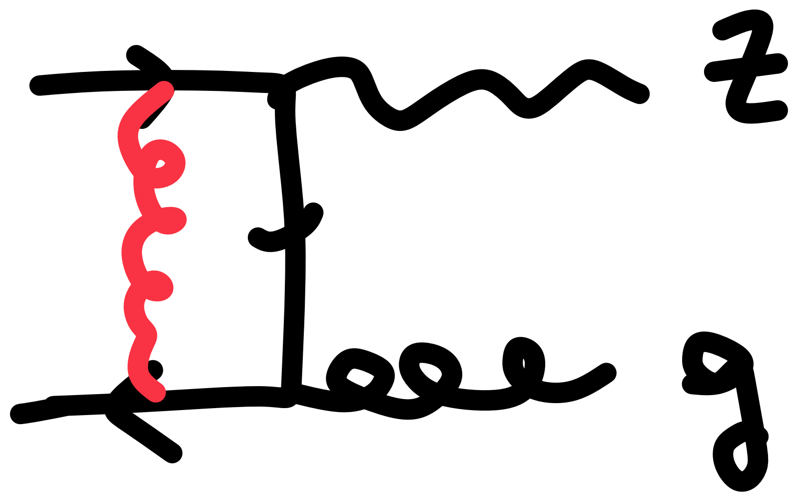
$$pp \to Z + j \sim \alpha_s^1$$

At **NLO QCD**: include contributions $\sim \alpha_s^2$

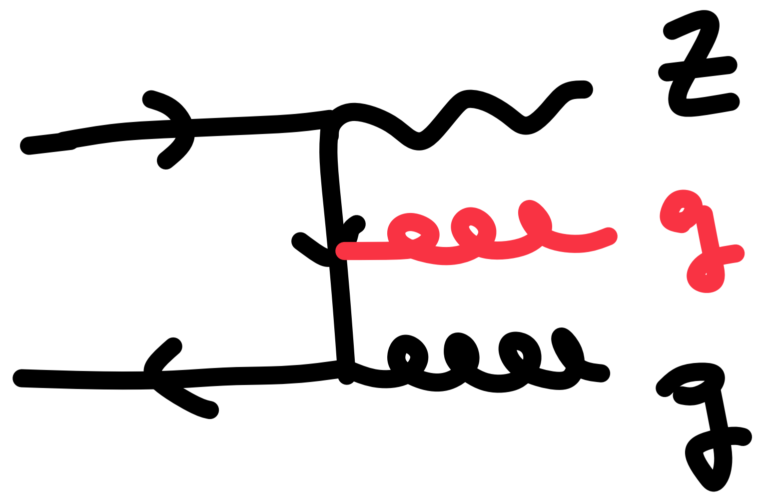
Let's procede step-by-step:

- Identify singular configurations
- Make singularities explicit
- Combine them to get finite result

Loop



Real



Simple (yet instructive) example: Z+j production in proton collision

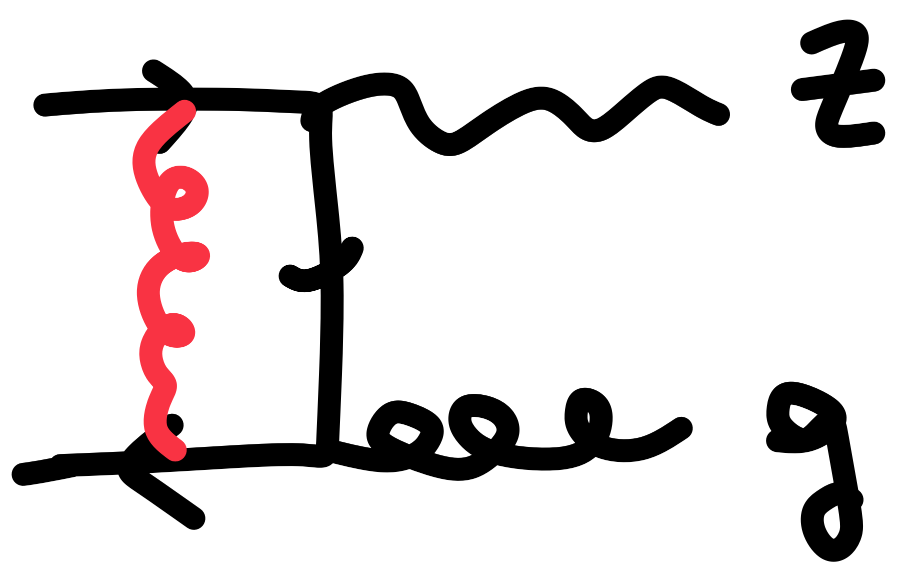
$pp \to Z + j \sim \alpha_s^1$

At **NLO QCD**: include contributions $\sim \alpha_s^2$

Let's procede step-by-step:

- **Identify singular configurations**
- Make singularities explicit
- Combine them to get finite result

Loop corrections



Divergent when loop momentum becomes soft/collinear

Real emissions

Simple (yet instructive) example: Z+j production in proton collision

$$pp \to Z + j \sim \alpha_s^1$$

At **NLO QCD**: include contributions $\sim \alpha_s^2$

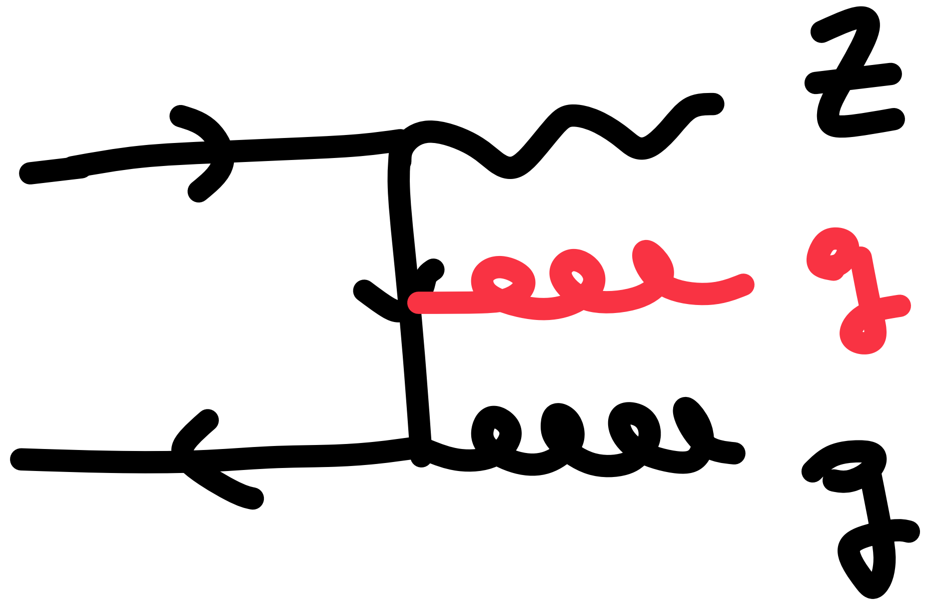
Let's procede step-by-step:

- **Identify singular configurations**

Loop corrections

Real emissions

Integration over gluon
phase space divergent
in soft/collinear regions



Simple (yet instructive) example: Z+j production in proton collision

$$pp \to Z + j \sim \alpha_s^1$$

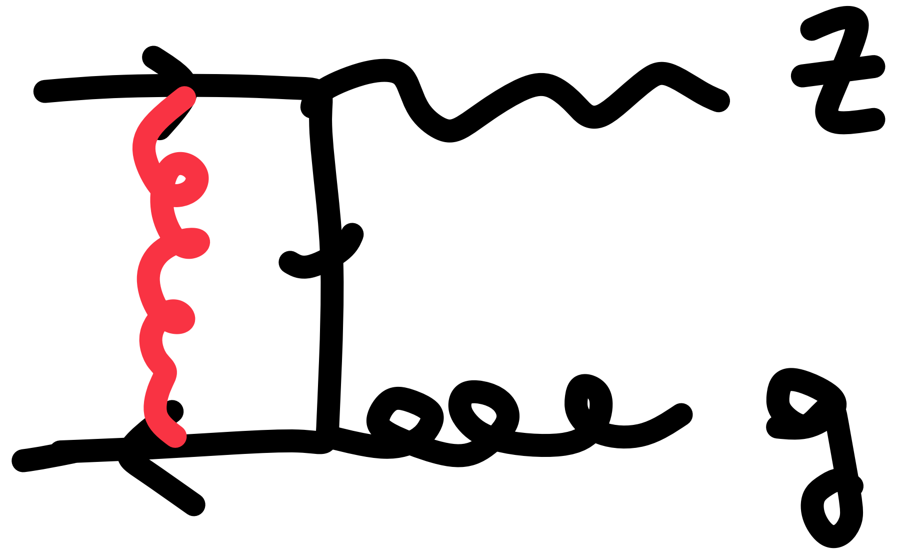
At **NLO QCD**: include contributions $\sim \alpha_s^2$

Let's procede step-by-step:

- Identify singular configurations
- **Make singularities explicit**
- Combine them to get finite result

Loop corrections

Divergent when loop momentum becomes soft/collinear



$$\langle F_{\rm LV}(1\ldots n)\rangle = \frac{\alpha_s}{2\pi}\Big\langle 2\Re\big(\mathcal{I}_1(\epsilon)\big)\,F_{\rm LM}\Big\rangle$$

$$\mathcal{I}_1(\epsilon) = \frac{1}{2}\frac{e^{\epsilon\,\gamma_E}}{\Gamma(1-\epsilon)}\sum_i \frac{1}{\mathbf{T}_i^2}\Big(\mathbf{T}_i^2\frac{1}{\epsilon^2}+\gamma_i\frac{1}{\epsilon}\Big)\sum_{j\neq i}\mathbf{T}_i\cdot\mathbf{T}_j\Big(\frac{\mu^2}{2p_i\cdot p_j}\Big)^{\epsilon}e^{i\lambda_{ij}\pi\epsilon}$$

Catani, 1998

Color correlations

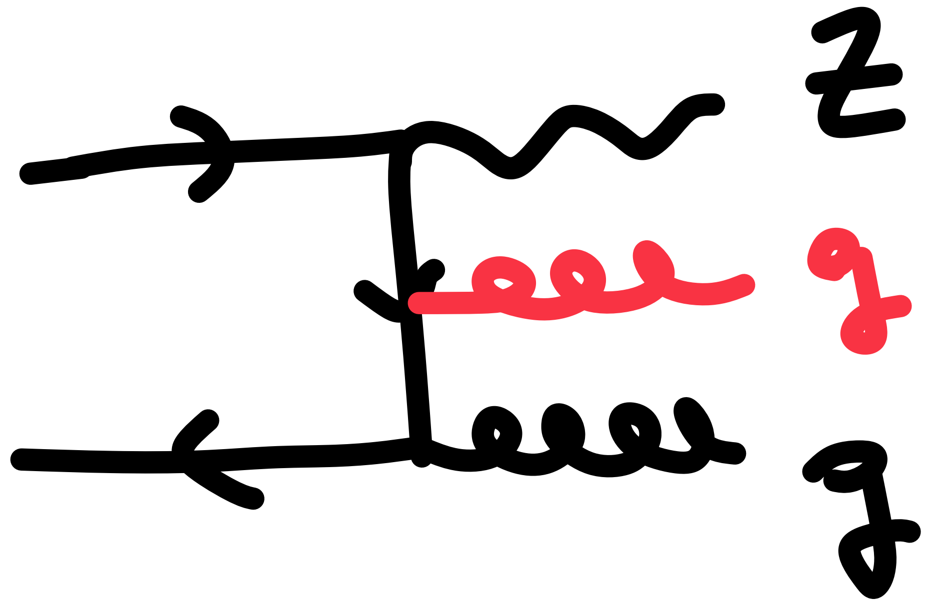
$$\langle F_{\rm LV}(1_q,2_{\bar q};3_g)\rangle = [\alpha_s]\Big\{(C_A-2C_F)\left[\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right](s_{12})^{-\epsilon}\cos(\pi\epsilon) - \left[\frac{C_A}{\epsilon^2}+\frac{3C_A+2\beta_0}{4\epsilon}\right]\left((s_{13})^{-\epsilon}+(s_{23})^{-\epsilon}\right)\Big\}\langle F_{\rm LM}(1_q,2_{\bar q};3_g)\rangle + \langle F_{\rm LV}^{\rm fin}(1_q,2_{\bar q};3_g)\rangle$$

Simple (yet instructive) example: Z+j production in proton collision

$$pp \to Z + j \sim \alpha_s^1$$

At **NLO QCD**: include contributions $\sim \alpha_s^2$

Real emissions



Let's procede step-by-step:

- Identify singular configurations

Integration over gluon phase space divergent in soft/collinear regions

- **Make singularities explicit**

NSC philosophy: subtract singularities in a nested way, i.e. regulate soft first, then collinear

$$< F_{LM}(1,2,3,4) > = < (I - S_4) F_{LM} \Delta^{(4)}(1,2,3,4) > + < S_4 \Delta^{(4)} F_{LM}(1,2,3,4) >$$

$$= < (I - S_4)(I - C_{4i}) \Delta^{(4)} F_{LM}(1,2,3,4) > + \sum_i < (I - S_4) C_{4i} \Delta^{(4)} F_{LM}(1,2,3,4) > + < S_4 \Delta^{(4)} F_{LM}(1,2,3,4) >$$

Fully-regulated — Hard-collinear counterterm — Soft counterterm

- Combine them to get finite result

```
sigmaNLO =
 - braαs/ϵ^2 (4 Emax^2/mu2)^-ϵ (CA (eta[1, 2]^-ϵ KK[1, 2] - eta[1, 3]^-ϵ KK[1, 3] - eta[2, 3]^-ϵ KK[2, 3]) - 2 CF eta[1, 2]^-ϵ KK[1, 2] )
   FLM[p1_q, p2_q̄, p3_g] + braαs/ϵ CA (4 E3^2/mu2)^-ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] PggNLOFS[L3] × FLM[p1_q, p2_q̄, p3_g] +
   braαs/ϵ TR Nf (4 E3^2/mu2)^-ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] γzgqq22 2 FLM[p1_q, p2_q̄, p3_g] +
   braαs ((CA - 2 CF) (1/ϵ^2 + 3/(2 ϵ)) (s12/mu2)^-ϵ Cos[π ϵ] - (CA/ϵ^2 + (3 CA + 2 β0)/(4 ϵ)) ((s13/mu2)^-ϵ + (s23/mu2)^-ϵ)) FLM[p1_q, p2_q̄, p3_g] +
   asontwopi FLVfin[p1_q, p2_q̄, p3_g] +
   braαs CF 1/ϵ Gamma[1 - ϵ]^2/Gamma[1 - 2 ϵ] (((4 E1^2/mu2)^-ϵ (3/2 + 1/ϵ (1 - (Emax^2/E1^2)^-ϵ)) + (4 E2^2/mu2)^-ϵ (3/2 + 1/ϵ (1 - (Emax^2/E2^2)^-ϵ)))) FLM[p1_q, p2_q̄, p3_g] +
   asontwopi ONLO[2 Δ34 FLM[p1_q, p2_q̄, p3_g, p4_g]] +
   asontwopi ONLO[Δ34 (FLM[p1_q, p2_q̄, p3_qp, p4_q̄p] + FLM[p1_q, p2_q̄, p3_q̄p, p4_qp])] +
   braαs CF (Pqqfin[z, E1] × FLM[z p1_q, p2_q̄, p3_g, z ] + Pqqfin[z, E2] × FLM[p1_q, z p2_q̄, p3_g, z]);
```

Use factorization of matrix element in the singular limits to evaluate each subtraction term

Check analytic pole cancellation

```
Normal[Series[sigmaNLO //. softfunctions //. splittings /. replaceSij /. L3 → Log[Emax/E3] /. β0 → 11/6 CA - 2/3 Nf TR,
    {ϵ, 0, -1}]] // PowerExpand // Simplify

0
```

∘ Combine them to get finite result

```
Coefficient[%, ϵ, 0]

asontwopi FLVfin[p1_q, p2_q̄, p3_g] +
 braαs CA FLM[p1_q, p2_q̄, p3_g] (67/9 - 2 π^2/3 - 2 (Log[E3] - Log[Emax])^2 + (11/6 - 2 Log[E3] + 2 Log[Emax]) (-2 Log[E3] + Log[mu2/4])) +
 1/9 braαs Nf TR FLM[p1_q, p2_q̄, p3_g] (-23 + Log[4096] + 12 Log[E3] - 6 Log[mu2]) +
 braαs CF FLM[p1_q, p2_q̄, p3_g] (-6 Log[2] + 2 Log[E1]^2 + 2 Log[E2]^2 - 8 Log[2] Log[Emax] - 4 Log[Emax]^2 +
    Log[E1] (-3 + Log[16] - 2 Log[mu2]) + Log[E2] (-3 + Log[16] - 2 Log[mu2]) + 3 Log[mu2] + 4 Log[Emax] Log[mu2]) +
 braαs CF (FLM[z p1_q, p2_q̄, p3_g, z] (1 - z + 4 D1[z] - 1/2 (2 + 2 z - 4 D0[z] - 3 delta[1 - z]) (Log[4] + 2 Log[E1] - Log[mu2]) -
       2 (1 + z) Log[1 - z]) + FLM[p1_q, z p2_q̄, p3_g, z]
     (1 - z + 4 D1[z] - 1/2 (2 + 2 z - 4 D0[z] - 3 delta[1 - z]) (Log[4] + 2 Log[E2] - Log[mu2]) - 2 (1 + z) Log[1 - z])) +
 1/6 braαs FLM[p1_q, p2_q̄, p3_g] (-3 (CA - 2 CF) (π^2 - 4 Log[2]^2 + Log[64] + 3 (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]]) -
       4 Log[2] (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]]) - (Log[E1] + Log[E2] - Log[mu2] + Log[eta[1, 2]])^2) +
    2 (5 CA - Nf TR) (Log[E1] + Log[E2] + 2 Log[E3] - 2 Log[mu2] + Log[16 eta[1, 3]] + Log[eta[2, 3]]) - 3 CA (8 Log[2]^2 +
       4 Log[2] (Log[E1] + Log[E3] - Log[mu2] + Log[eta[1, 3]]) + (Log[E1] + Log[E3] - Log[mu2] + Log[eta[1, 3]])^2 +
       4 Log[2] (Log[E2] + Log[E3] - Log[mu2] + Log[eta[2, 3]]) + (Log[E2] + Log[E3] - Log[mu2] + Log[eta[2, 3]])^2)) +
 asontwopi ONLO[2 Δ34 FLM[p1_q, p2_q̄, p3_g, p4_g]] + asontwopi ONLO[Δ34 (FLM[p1_q, p2_q̄, p3_qp, p4_q̄p] + FLM[p1_q, p2_q̄, p3_q̄p, p4_qp])] +
 braαs FLM[p1_q, p2_q̄, p3_g]
  (-CF π^2/3 + 1/2 (CA + 2 CF) (Log[4] + 2 Log[Emax] - Log[mu2])^2 + CF Log[eta[1, 2]]^2 -
    (-2 Log[Emax] + Log[mu2/4]) (-((CA - 2 CF) Log[eta[1, 2]]) + CA (Log[eta[1, 3]] + Log[eta[2, 3]])) +
```

And then try to find recurring structures that make final result nice and cute

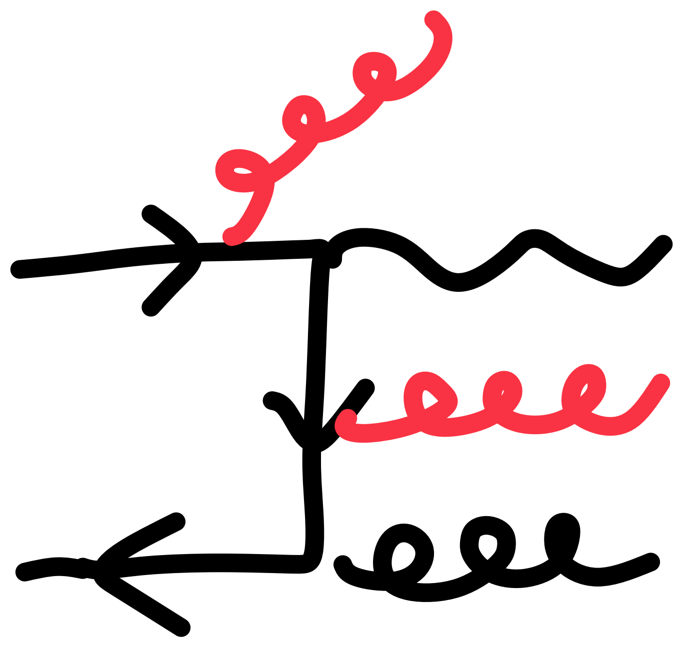
$$
\begin{aligned}
\mathrm{d}\sigma^{qq}_{\mathrm{NLO}} = {}& n_f \left\langle \mathcal{O}^{(4)}_{\mathrm{nlo}} F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_q, 4_{\bar{q}}) \right\rangle + \left\langle \mathcal{O}^{(4)}_{\mathrm{nlo}}\, 2\,\Delta^{(3)}_{\perp,34}\, F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_g, 4_g) \right\rangle \\
& + \left\langle F^{\mathrm{fin}}_{\mathrm{LV}}(1_q, 2_{\bar{q}}; 3_q) \right\rangle \\
& + [\alpha_s]\, C_F \sum_{i=1}^{2} \int_0^1 dz \left\langle \tilde{P}^{\mathrm{NLO}}_{qq}(z, E_c)\, F^{(i)}_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_g | z) \right\rangle \\
& + [\alpha_s] \left\langle F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_q) \right\rangle \Bigg[T_R\, n_f \left(-\frac{23}{9} + \frac{2}{3} \log\left(\frac{E_3}{E_c}\right) - \frac{1}{3} \log\left(\eta_{13}\, \eta_{23}\right) \right) \\
& \qquad + C_F \left(\frac{2\pi^2}{3} + 6 \log\left(\frac{2E_c}{\mu}\right) \right) + C_A \left(\frac{67}{9} - \frac{4\pi^2}{3} + \frac{1}{3} \log\left(\frac{E_c}{E_3}\right) + \log^2\left(\frac{E_c}{E_3}\right) \right. \\
& \qquad \left. \frac{1}{3}\left(5 + 3 \log\left(\frac{E_c}{E_3}\right) \right) \log\left(\eta_{13}\, \eta_{23}\right) \right) + \mathrm{Li}_2(1 - \eta_{13}) + \mathrm{Li}_2(1 - \eta_{23}) \Bigg] ,
\end{aligned}
$$

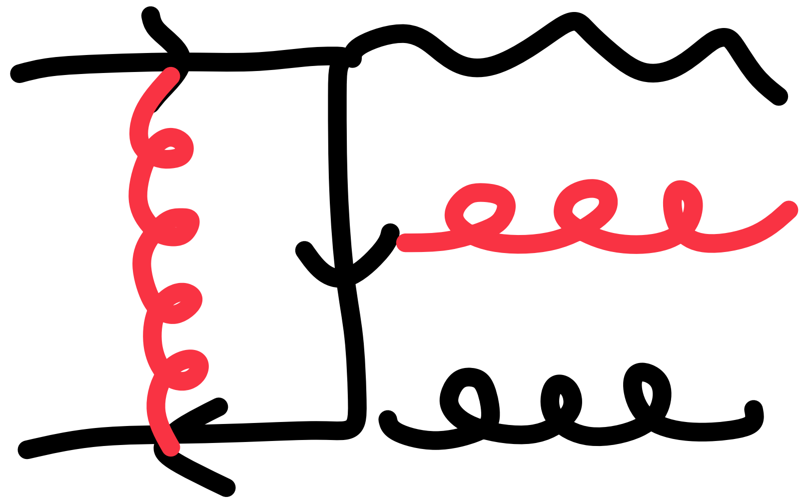
Can I get directly the nice and cute result?

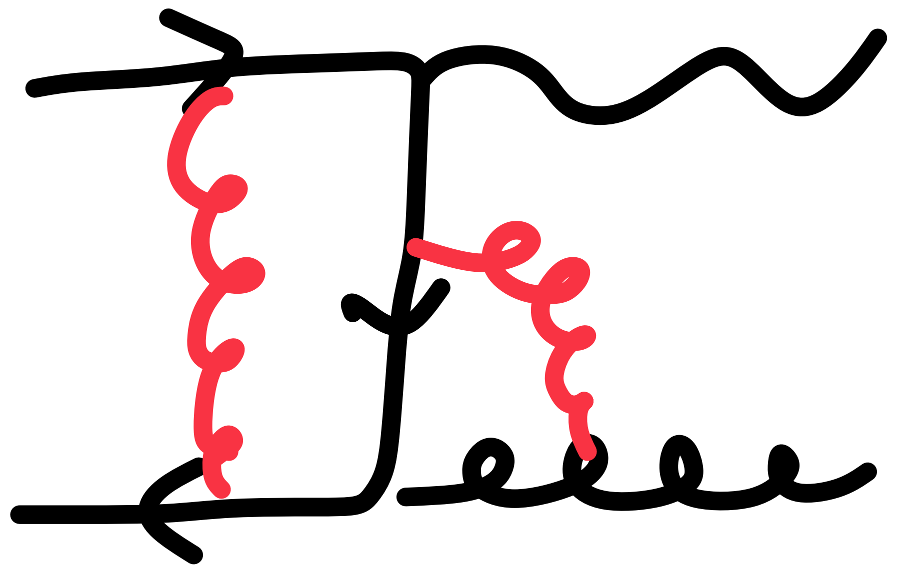
Will come back to this later

NNLO COMPLEXITY

$$d\sigma_{NNLO} = d\sigma_{RR} + d\sigma_{RV} + d\sigma_{VV} + d\sigma_{PDF}$$







Catani, 1998

IR singularities at 2-loops encoded in Catani's 2-loop operator

Many singular configurations…

VV is "simple", RV and RR much more intricate

- Overlapping singularities
- Interplay soft/collinear limits

} ?!

$$\begin{aligned} \boldsymbol{I}^{(2)}_{\rm RS}(\epsilon,\mu^2;\{p\}) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) + 4\pi\beta_0\frac{1}{\epsilon}\right) \\ &+ \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(2\pi\beta_0\frac{1}{\epsilon}+K\right)\boldsymbol{I}^{(1)}(2\epsilon,\mu^2;\{p\}) \\ &+ \boldsymbol{H}^{(2)}_{\rm RS}(\epsilon,\mu^2;\{p\}) \ , \end{aligned}$$

NSC AT NNLO

Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

→ No interplay between soft and collinear → subtract soft limits first, then collinear

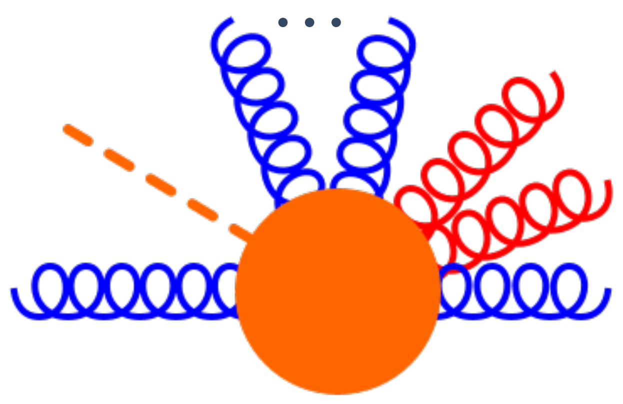
"Nested approach"

Three steps:

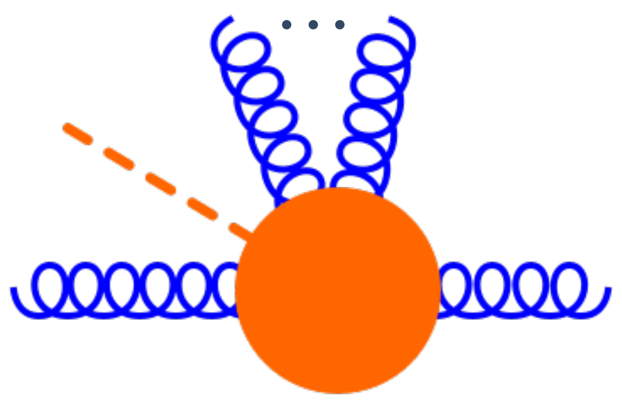
- Globally remove double soft singularity

Universal, contains all explicit double soft (DS) singularities

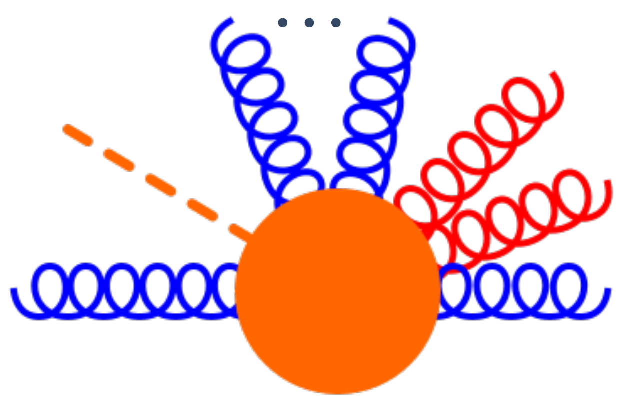
Free from DS singularities



$$I = \mathbb{S} + (I - \mathbb{S}) \longrightarrow$$



$$\times \langle \mathbb{S} \rangle \quad + (I - \mathbb{S})$$



BACK TO NNLO

Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

→ No interplay between soft and collinear → subtract soft limits first, then collinear

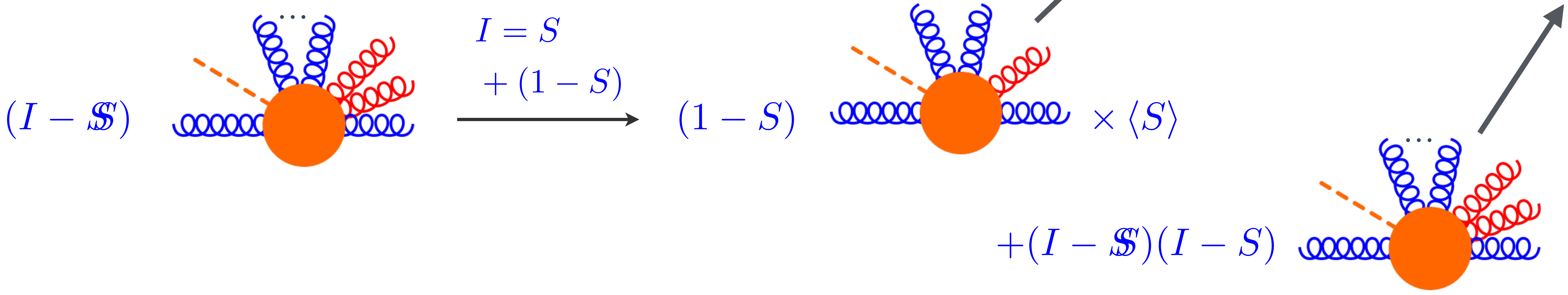
"Nested approach"

Three steps:

- Globally remove double soft singularity
- Globally remove single soft singularity

~NLO

Only contains collinear singularities



BACK TO NNLO

Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

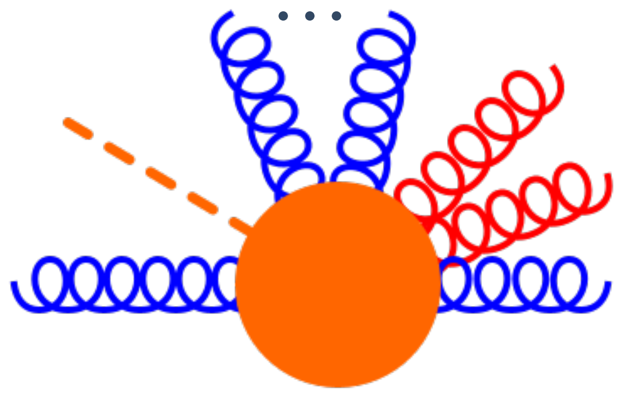
No interplay between soft and collinear → subtract soft limits first, then collinear

Three steps:

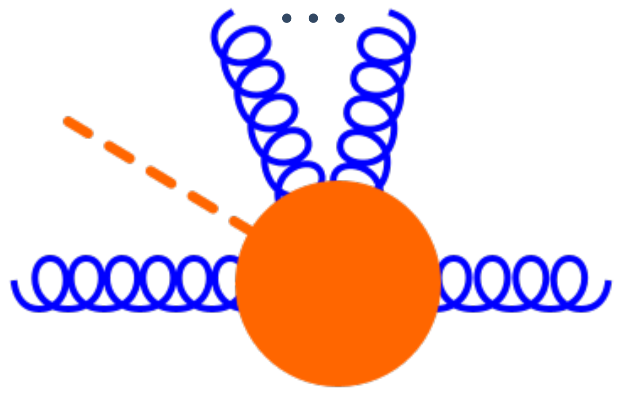
"Nested approach"

- Globally remove double soft singularity
- Globally remove single soft singularity
- FKS partition and treat one collinear singularity at a time

$(I - \mathbb{S})(I - S)$

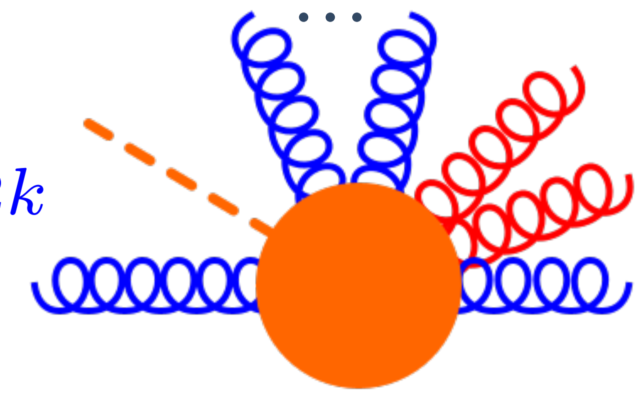


$\xrightarrow{I = \mathbb{C} + (I - \mathbb{C})} \sum_k$



$\langle P_{ijk} \rangle$

$+ \sum_k (I - \mathbb{S})(I - S)(I - \mathbb{C}_{ijk})\Delta^{1k,2k}$



BACK TO NNLO

Use of color coherence is what distinguishes original **sector decomposition (Czakon, 2010)** from the **nested soft-collinear subtraction scheme (Caola, Melnikov, Roentsch 2017)**

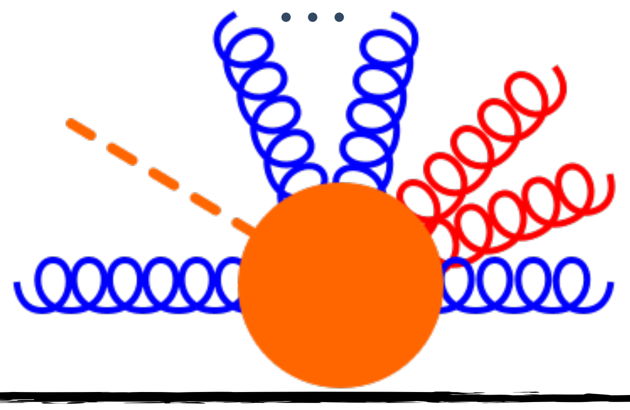
No interplay between soft and collinear → subtract soft limits first, then collinear

"Nested approach"

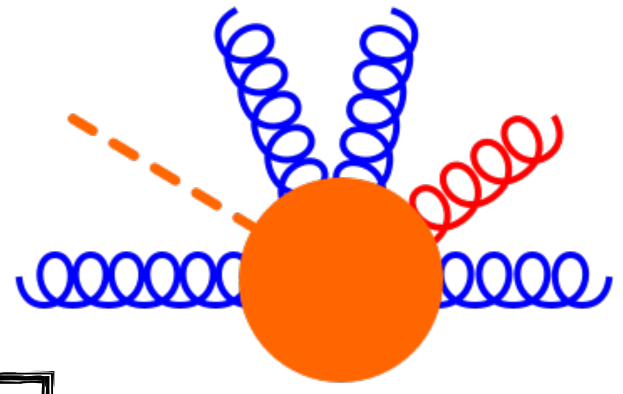
Three steps:

- Globally remove double soft singularity
- Globally remove single soft singularity
- FKS partition and treat one collinear singularity at a time

$$+\sum_k (I-\mathbb{S})(I-S)(I-\mathbb{C}_{ijk})\Delta^{1k,2k}$$



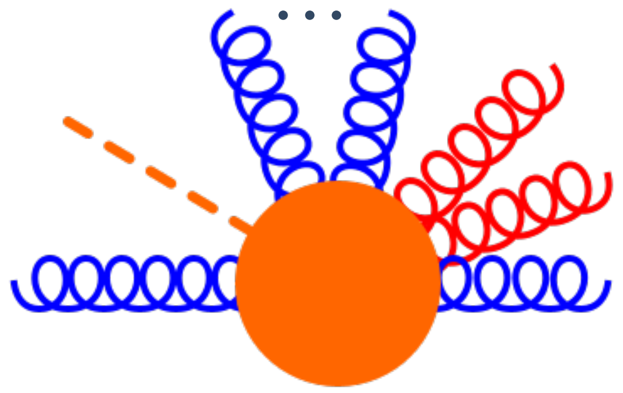
$$\xrightarrow{I = C + (1-C)} \quad (I-S)(I-C)$$



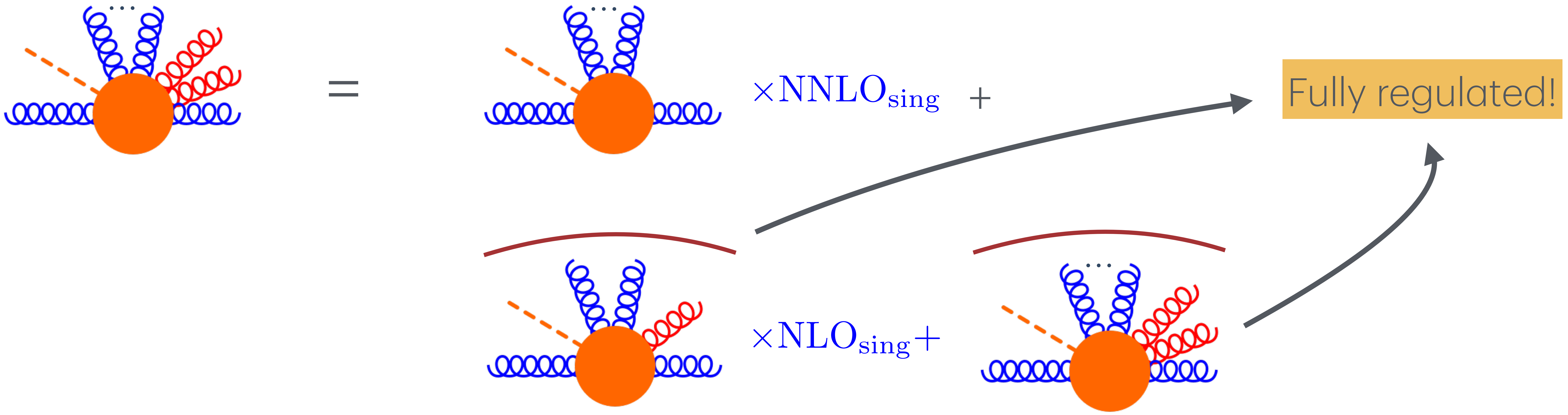
$$\times \langle P \rangle$$

Fully regulated!

$$+\sum_{k,a} (I-\mathbb{S})(I-S)(I-\mathbb{C}_{ijk})\theta^{(a)}(I-C_a)$$



FINAL RESULT



- Singularities are explicitly extracted in the counterterms, constructed from universal structures
- Counterterms are integrated analytically *"once and for all"*
- Minimal approach: only *inequivalent* physical limits are subtracted

Reverse unitarity method to compute the PS integrals

Local in phase space & Analytic

[Caola, Delto, Frellesvig, Melnikov, 2018+19]

TOWARDS COLORFUL FINAL STATES

In principle, everything is known to deal with a completely generic process

In practice, several issues encountered:

- Bookkeeping increases dramatically
- Color correlations become crucial, SU(Nc) algebra does not close for n>=4

Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen "by eye", even for a generic process?

A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N-gluon final states in $q\bar{q}$ annihilation

Federica Devoto,[a] Kirill Melnikov,[b] Raoul Röntsch,[c] Chiara Signorile-Signorile[b,d,e] Davide Maria Tagliabue[c]

JHEP 02 (2024) 016, arXiv:2310.17598

Main idea: look at the pole structure of the virtuals to infer similar structures for the reals

With this new approach, we are able to analytically prove the cancellation of IR singularities for generic processes at NNLO (only gluons for now)

PHASE SPACE INTEGRATION OF ONE LOOP SOFT GLUON CURRENT

Consider the soft limit of the real-virtual contribution

$$
\begin{aligned}
&S_{\mathfrak{m}} F_{\mathrm{RV}}(\mathfrak{m}) \\
&= -g_{s,b}^2 \sum_{(ij)}^{N_p} \Bigg\{ 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LV}} - \frac{\alpha_s(\mu)}{2\pi} \frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
&\quad - 2 \frac{[\alpha_s]}{\epsilon^2} C_A\, A_K(\epsilon) \left(S_{ij}(p_{\mathfrak{m}}) \right)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
&\quad - [\alpha_s] \frac{4\pi\, \Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1 \\ k \neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \left(S_{ij}(p_{\mathfrak{m}}) \right)^{\epsilon} f_{abc}\, T_k^a T_i^b T_j^c\, F_{\mathrm{LM}} \Bigg\}
\end{aligned}
$$

Triple color correlated term

Only contributes for processes with at least 4 partons at Born level

Last missing integral for fully generic processes! Computed in [2310.17598] via **Mellin-Barnes** representation

Largely used also for loop integrals!

$$
S_{ij}(p_m) = \frac{p_i \cdot p_j}{2(p_i \cdot p_m)(p_j \cdot p_m)}
$$

Energy integration is trivial, angular integrand in MB representation

[G. Somogyi, *J.Math.Phys.* 52 (2011) 083501]

$$
G^{kij} = \int \frac{\mathrm{d}\Omega_{\mathfrak{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathfrak{m}}\rho_{i\mathfrak{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\rho_{j\mathfrak{m}}} \right)^{\epsilon} \qquad \rho_{ij} = 1 - \cos\theta_{ij}
$$

PHASE SPACE INTEGRATION OF ONE LOOP SOFT GLUON CURRENT

Energy integration is trivial, angular integrand in MB representation

[G. Somogyi, *J.Math.Phys.* 52 (2011) 083501]

$$G^{kij} = \int \frac{\mathrm{d}\Omega_{\mathrm{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathrm{m}}\rho_{i\mathrm{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathrm{m}}\rho_{j\mathrm{m}}}\right)^{\epsilon}$$

$$\rho_{ij} = 1 - \cos\theta_{ij}$$

$$\eta_{ij} = \left(1 - \cos\theta_{ij}\right)/2$$

$$v_{kl} \equiv \begin{cases} \frac{p_k \cdot p_l}{2} & ; \ k \neq l \\ \frac{p_k^2}{4} & ; \ k = l \end{cases}$$

Three massless denominators:

$$\begin{aligned} G^{kij} &= \int \frac{\mathrm{d}\Omega_{\mathrm{m}}^{(d-1)}}{2(2\pi)^{d-1}} \frac{\rho_{ki}}{\rho_{k\mathrm{m}}\rho_{i\mathrm{m}}} \left(\frac{\rho_{ij}}{\rho_{i\mathrm{m}}\rho_{j\mathrm{m}}}\right)^{\epsilon} \\ &= \rho_{ki}\,\rho_{ij}^{\epsilon} \int_{-i\infty}^{+i\infty} \frac{dz_{ij}\,dz_{jk}\,dz_{ki}}{(2\pi i)^3} \frac{\pi^{-2+\epsilon}}{2^{4+2\epsilon}} \Gamma(-z_{ij})\Gamma(-z_{ki})\Gamma(-z_{jk}) \\ &\times \Gamma(1+\epsilon+z_{ij}+z_{ki})\Gamma(-1-3\epsilon-z_{ij}-z_{ki}-z_{jk})\Gamma(\epsilon+z_{ij}+z_{jk}) \\ &\times \Gamma(1+z_{ki}+z_{jk}) \frac{1}{\Gamma(-4\epsilon)\Gamma(\epsilon)\Gamma(1+\epsilon)} \eta_{ij}^{z_{ij}}\,\eta_{ki}^{z_{ki}}\,\eta_{jk}^{z_{jk}}\,. \end{aligned}$$

Angular integration traded for 3-fold Mellin-Barnes integration

Integration contour to be chosen in such a way that poles of $\Gamma(\ldots + x)$ are separated from poles of $\Gamma(\ldots - x)$

Involved singularity structure due to various Gamma functions involving three variables

`MBresolve` [M. Czakon, Comput. Phys. Commun. 175 (2006) 559–571]

`MB` [A.V. Smirnov, V.A. Smirnov, Eur. Phys. J. C 62 (2009) 445–449]

PHASE SPACE INTEGRATION OF ONE LOOP SOFT GLUON CURRENT

At this point singularities are resolved and integration contours are straight vertical lines in the complex plane, can integrate via residue theorem

Final result written in terms of **classical + generalized polylogs up to weight 3**

$$\begin{aligned}
\overline{G}^{kij}_{r,\text{fin}} = {} & \text{Li}_2(\eta_{ij}) \log\left(\frac{\eta_{ik}}{\eta_{jk}}\right) - \text{Li}_2(\eta_{ik}) \log\left(\frac{\eta_{jk}}{\eta_{ij}\eta_{ik}}\right) + \text{Li}_2(\eta_{jk}) \log\left(\frac{\eta_{ik}}{\eta_{ij}\eta_{jk}}\right) \\
& + \log(\eta_{ik})\text{Li}_2\left(-\frac{\eta_{ik}-\eta_{jk}}{1-\eta_{ik}}\right) + \log(\eta_{ik})\text{Li}_2\left(-\frac{\eta_{ik}-\eta_{jk}}{\eta_{jk}}\right) + 3\text{Li}_3(1-\eta_{ik}) \\
& - 3\text{Li}_3(1-\eta_{jk}) + \text{Li}_2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right) \log(\eta_{ik}\eta_{jk}) + \text{Li}_2\left(\frac{\eta_{ik}}{\eta_{jk}}\right) \log(\eta_{ik}\,\eta_{jk}) \\
& - \log(\eta_{jk})\text{Li}_2\left(-\frac{\eta_{jk}-\eta_{ik}}{\eta_{ik}}\right) - \log(\eta_{jk})\text{Li}_2\left(-\frac{\eta_{jk}-\eta_{ik}}{1-\eta_{jk}}\right) + \text{Li}_3(\eta_{ik}) \\
& - \text{Li}_3(\eta_{jk}) + \log^2(\eta_{ik})\left[\frac{1}{2}\log\left(\frac{1-\eta_{jk}}{\eta_{ij}}\right) + \log\left(\frac{\eta_{jk}-\eta_{ik}}{\eta_{jk}}\right)\right] \qquad + \ldots
\end{aligned}$$

$$
\begin{aligned}
&+\log(\eta_{ik})\Big[-\frac{1}{2}\log^2(\eta_{ij})+\log(1-\eta_{ij})\log(\eta_{ij})+\frac{1}{2}\log^2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\\
&+\frac{1}{2}\log^2\left(\frac{\eta_{ik}}{\eta_{jk}}\right)+\log(1-\eta_{jk})\log(\eta_{jk}(\eta_{jk}-\eta_{ik}))+\log^2(\eta_{jk})-\frac{13\pi^2}{6}\Big]\\
&+\log(1-\eta_{jk})\Big[-\log(\eta_{jk})\log\left(\frac{\eta_{ij}}{\eta_{jk}-\eta_{ik}}\right)-\log^2(\eta_{jk})-\frac{\pi^2}{6}\Big]\\
&+\log(1-\eta_{ik})\Big[\log(\eta_{ik})\Big[\log\left(\frac{\eta_{ij}}{\eta_{jk}-\eta_{ik}}\right)-\log(1-\eta_{jk})\Big]+\log^2(\eta_{ik})\\
&-\log(\eta_{jk})\log(\eta_{jk}-\eta_{ik})-\frac{\log^2(\eta_{jk})}{2}+\frac{\pi^2}{6}\Big]+\frac{1}{2}\log(\eta_{ij})\log^2(\eta_{jk})\\
&+\frac{1}{2}\log^2(\eta_{ij})\log(\eta_{jk})-\log(1-\eta_{ij})\log(\eta_{ij})\log(\eta_{jk})-\frac{1}{3}\log^3\left(\frac{\eta_{ik}}{\eta_{jk}}\right)\\
&-\frac{1}{2}\log\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\log^2\left(\frac{\eta_{ik}}{\eta_{jk}}\right)-\frac{1}{2}\log^2\left(\frac{1-\eta_{jk}}{1-\eta_{ik}}\right)\log\left(\frac{\eta_{ik}}{\eta_{jk}}\right)\\
&-\log^2(\eta_{jk})\log(\eta_{jk}-\eta_{ik})+\frac{2}{3}\pi^2\log\left(\frac{\eta_{ik}}{\eta_{jk}}\right)+\log\left(\frac{\eta_{ik}}{1-\eta_{ik}}\right)\\
&\times\Big[\frac{\pi^2}{6}-\log(1-\eta_{jk})\log(\eta_{jk})\Big]-\frac{1}{2}\log^3(\eta_{ik})+\log^2(1-\eta_{ik})\log(\eta_{ik})\\
&+\frac{\log^3(\eta_{jk})}{2}-\log^2(1-\eta_{jk})\log(\eta_{jk})+\frac{3}{2}\pi^2\log(\eta_{jk})-\frac{1}{6}\pi^2\log\left(\frac{\eta_{jk}}{1-\eta_{jk}}\right)\\
&+\log(\eta_{ij})\Big[G(\tilde\eta_{ik},w^+,1)-G(\tilde\eta_{jk},w^+,1)+G(\tilde\eta_{ik},w^-,1)-G(\tilde\eta_{jk},w^-,1)\Big]\\
&-G(\tilde\eta_{ik},w^+,\tilde\eta_{jk},1)+G(\tilde\eta_{jk},w^+,\tilde\eta_{ik},1)+G(\tilde\eta_{ik},w^+,1,1)-G(\tilde\eta_{ik},w^+,\tilde\eta_{ik},1)\\
&-G(\tilde\eta_{jk},w^+,1,1)+G(\tilde\eta_{jk},w^+,\tilde\eta_{jk},1)-G(\tilde\eta_{ik},w^-,\tilde\eta_{jk},1)+G(\tilde\eta_{jk},w^-,\tilde\eta_{ik},1)\\
&+G(\tilde\eta_{ik},w^-,1,1)-G(\tilde\eta_{ik},w^-,\tilde\eta_{ik},1)-G(\tilde\eta_{jk},w^-,1,1)+G(\tilde\eta_{jk},w^-,\tilde\eta_{jk},1)\ ,
\end{aligned}
$$

$$\tilde\eta_{ab}=1/(1-\eta_{ab}).$$

$$w^\pm=\frac{2-\eta_{ij}-\eta_{ik}-\eta_{jk}\pm\sqrt{(\eta_{ij}-\eta_{ik}-\eta_{jk})^2-4\eta_{ik}\eta_{jk}(1-\eta_{ij})}}{2\,(\eta_{ik}\eta_{jk}-\eta_{ik}-\eta_{jk}+1)}$$

CONCLUSIONS

- Higher order calculations are important for LHC precision physics program; they involve infrared singularities that need to be regulated

- There is a freedom in how to regulate such divergences -> subtraction scheme. I presented generalities and mentioned advances in the context of the nested soft-collinear subtraction scheme, goal is to deal with **high-multiplicity final states** (beyond 2->2)

- With the new approach, we are able to **analytically prove** the cancellation of IR singularities for a generic NNLO process (only gluons for now)

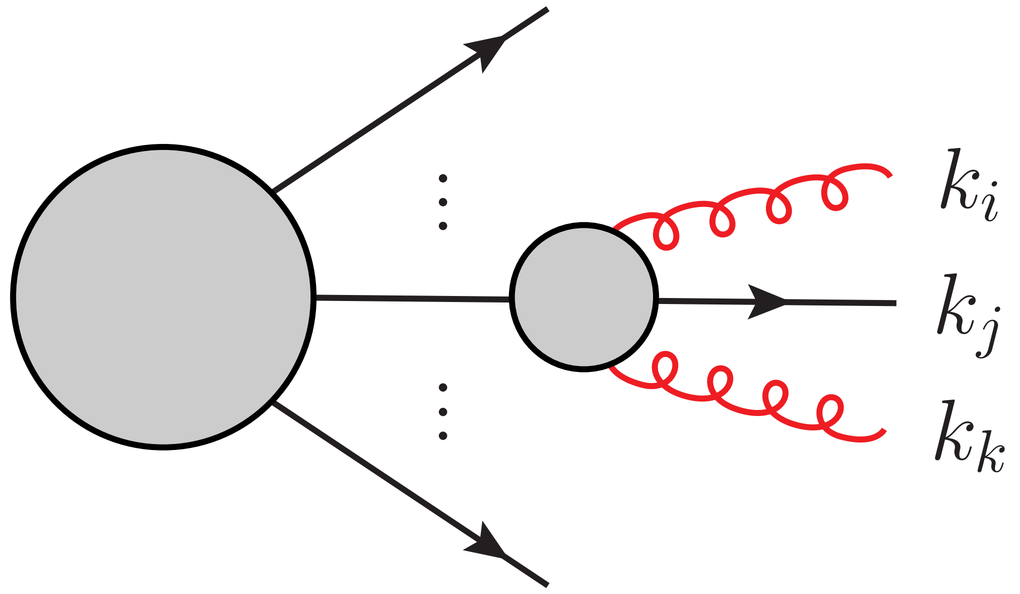
- What's next: generalization of the NSC "new" approach to off-diagonal partonic channels, include quarks in the final state, etc., numerical studies, phenomenological applications

Thank you for your attention!

BACKUP

Under IR limits, RR factorises into **universal kernels** x **lower multiplicity matrix elements**

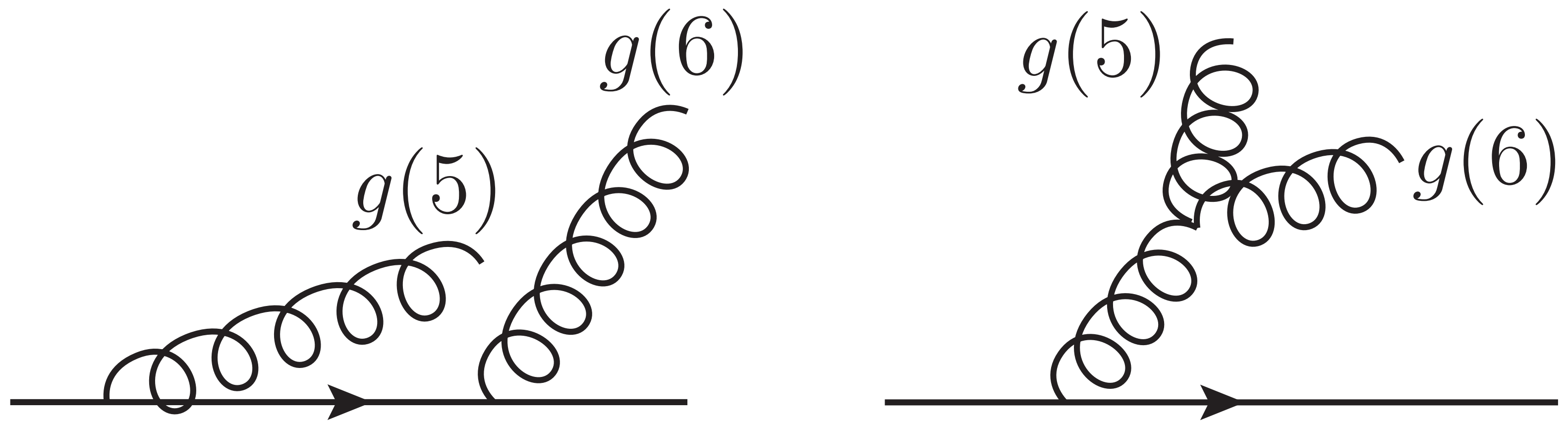
- Double soft *[Catani, Grazzini 9908523]*
- Triple collinear limit *[Catani, Grazzini 9810389]*



$$|\mathcal{M}_{a_1,a_2,a_3,\ldots}(p_1,p_2,p_3,\ldots)|^2 \simeq \frac{4}{s_{123}^2}(4\pi\mu^{2\epsilon}\alpha_S)^2\, \mathcal{T}^{ss'}_{a,\ldots}(p,\ldots)\, \hat{P}^{ss'}_{a_1a_2a_3}$$

$$\mathcal{T}^{s_1s'_1}_{a_1,\ldots}(p_1,\ldots) \equiv \sum_{\text{spins}\neq s_1,s'_1}\sum_{\text{colours}} \mathcal{M}^{c_1,c_2,\ldots;s_1,s_2,\ldots}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\left[\mathcal{M}^{c_1,c_2,\ldots;s'_1,s_2,\ldots}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\right]^\dagger$$

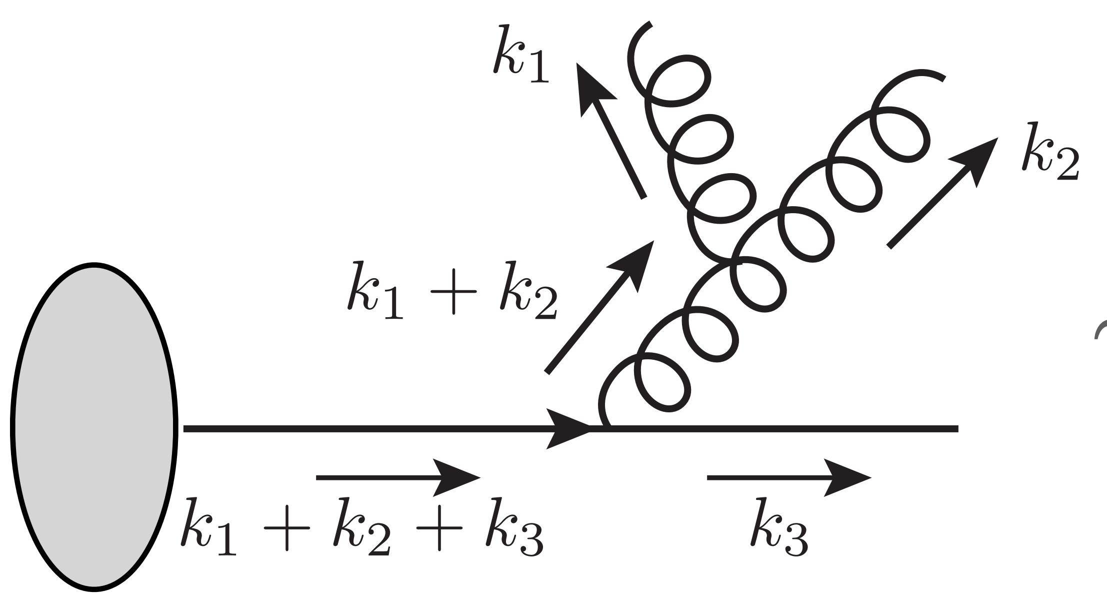
Different triple collinear topologies to disentangle



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$

OVERLAPPING SINGULARITIES – SOFT/COLLINEAR INTERPLAY

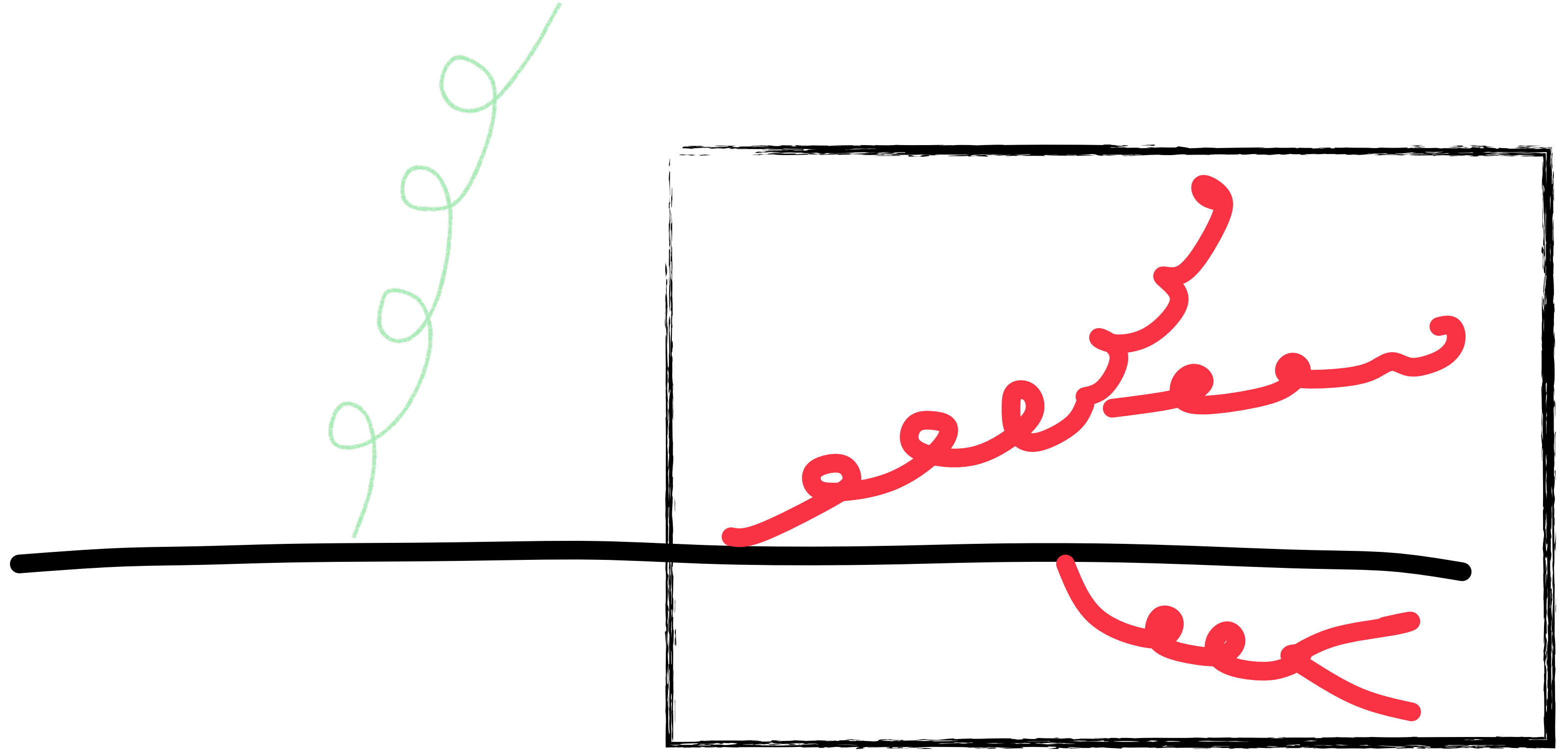


$$\sim \frac{1}{(k_1+k_2)^2}\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Overlapping energy/angle (e.g. soft/collinear) singularity!

Turns out to be an artifact of individual Feynman diagrams. **On-shell scattering amplitudes are free from entangled singularities**

Color coherence



Soft gluon only sensitive to color charge of collinear subsystem, no S/C interplay!

TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO

$$
\begin{aligned}
\frac{1}{3!}\langle F_{\mathrm{LM}}(1_q, 2_{\bar{q}}; 3_g, 4_g, 5_g)\rangle = {} & \langle S_{45}\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\rangle + \langle (I-S_4)S_5\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\rangle \\
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\Big[\Theta^{(a)}C_{45,i}(I-C_{5i}) + \Theta^{(b)}C_{45,i}(I-C_{45}) \\
& \quad + \Theta^{(c)}C_{45,i}(I-C_{4i}) + \Theta^{(d)}C_{45,i}(I-C_{45})\Big]\omega_{4i5i}\Big\}\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\Big\rangle \\
& - \Big\langle (I-S_{45})(I-S_5)\sum_{(ij)\in\mathrm{DC}} C_{4i}C_{5j}\,\omega_{4i5j}\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\Big\rangle \\
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\big[\Theta^{(a)}C_{5i} + \Theta^{(b)}C_{45} + \Theta^{(c)}C_{4i} + \Theta^{(d)}C_{45}\big]\,\omega_{4i5i} \\
& \quad + \sum_{(ij)\in\mathrm{DC}}\big[C_{4i} + C_{5j}\big]\,\omega_{4i5j}\Big\}\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\Big\rangle \\
& + \Big\langle (I-S_{45})(I-S_5)\Big\{\sum_{i\in\mathrm{TC}}\Big[\Theta^{(a)}(I-C_{45,i})(I-C_{5i}) + \Theta^{(b)}(I-C_{45,i})(I-C_{45}) \\
& \quad + \Theta^{(c)}(I-C_{45,i})(I-C_{4i}) + \Theta^{(d)}(I-C_{45,i})(I-C_{45})\Big]\omega_{4i5i} \\
& \quad + \sum_{(ij)\in\mathrm{DC}}(I-C_{4i})(I-C_{5j})\,\omega_{4i5j}\Big\}\,\Delta^{(45)} F_{\mathrm{LM}}^{4>5}\Big\rangle
\end{aligned}
$$

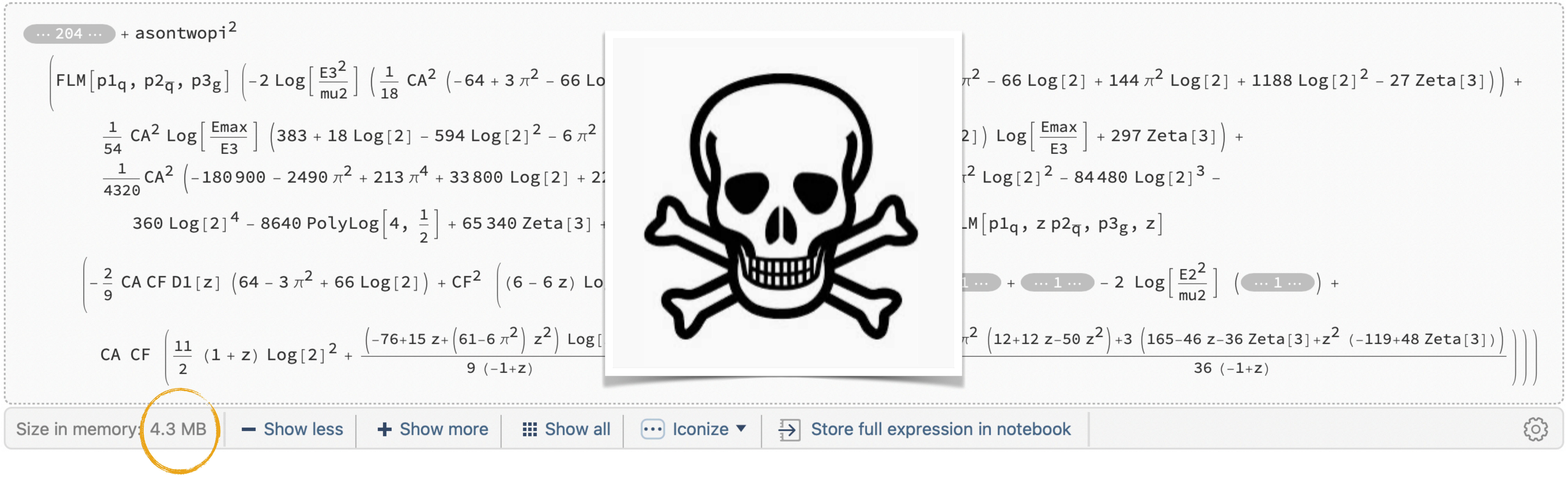
In principle, everything is known to deal with a completely generic process

In practice, several issues encountered:

- Bookkeeping increases dramatically
- Color correlations become crucial, SU(Nc) algebra does not close for n>=4

TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO

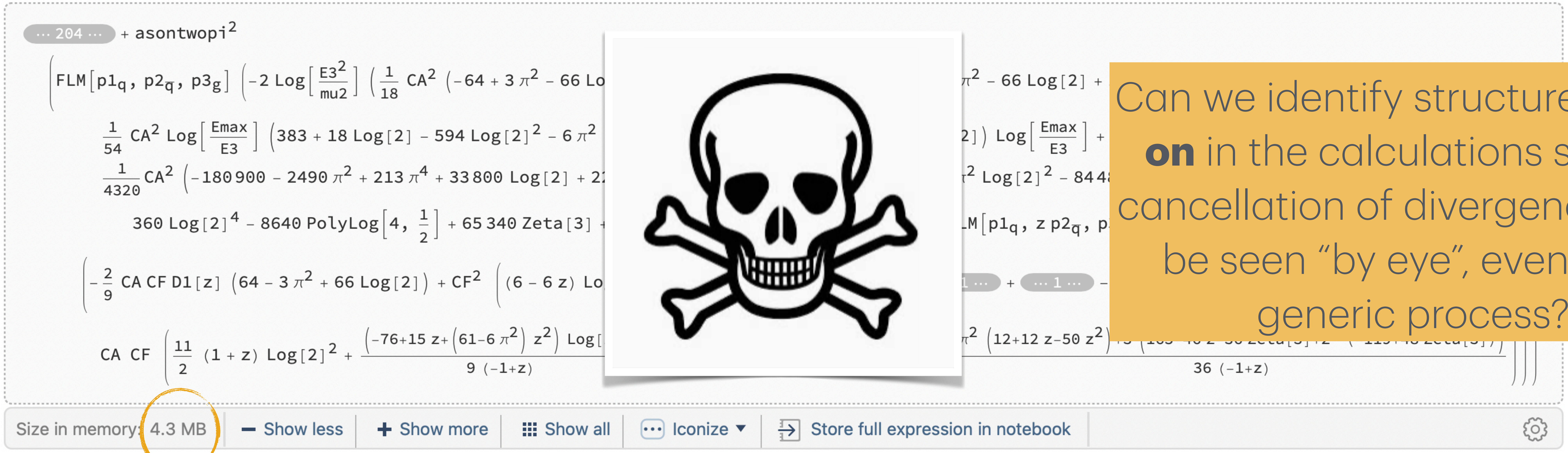


Evaluating each subtraction term explicitly hides structures & simplifications

“Asymmetry”: VV very simple pole structure, RR structure obscured by energy ordering, partitioning, etc..

TOWARDS COLORFUL FINAL STATES

Back to our original example: Z+j @NNLO



Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen “by eye”, even for a generic process?

Evaluating each subtraction term explicitly hides structures & simplifications

“Asymmetry”: VV very simple pole structure, RR structure obscured by energy ordering, partitioning, etc..

Main idea: look at the pole structure of the virtuals to infer similar operators for the reals

Warm-up @ NLO

- Virtuals: $\boldsymbol{I_{\mathrm{V}}(\epsilon)} = \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon)$

$$\bar{I}_1(\epsilon) = \frac{1}{2}\sum_{i\neq j}^{Np} \frac{\mathcal{V}_i^{\mathrm{sing}}(\epsilon)}{\boldsymbol{T}_i^2}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\left(\frac{\mu^2}{2p_i\cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\mathrm{sing}}(\epsilon) = \frac{\boldsymbol{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

$$N_p = N+2$$

- Reals: $\boldsymbol{I_{\mathrm{S}}(\epsilon)} = -\frac{(2E_{\mathrm{max}}/\mu)^{-2\epsilon}}{\epsilon^2}\sum_{i\neq j}^{N_p}\eta_{ij}^{-\epsilon}K_{ij}(\boldsymbol{T}_i\cdot\boldsymbol{T}_j)$

$$\boldsymbol{I_{\mathrm{C}}(\epsilon)} = \sum_{i=1}^{N_p}\frac{\boxed{\Gamma_{i,f_i}}}{\epsilon}$$

$\boldsymbol{I_{\mathrm{V}}(\epsilon)} + \boldsymbol{I_{\mathrm{S}}(\epsilon)}$: Highest pole trivially cancel; Color correlations cancel } Remnant single pole canceled by $I_{\mathrm{C}}(\epsilon)$

Γ_{i,f_i}: "Generalised anomalous dimensions"

$$\boldsymbol{I_{\mathrm{T}}(\epsilon)} = I_{\mathrm{V}}(\epsilon) + I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon)$$ Finite!

$$\mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = [\alpha_s]\langle I_{\mathrm{T}}(\epsilon)\cdot F_{\mathrm{LM}}\rangle + [\alpha_s]\Big[\langle P_{aa}^{\mathrm{NLO}}\otimes F_{\mathrm{LM}}\rangle + \langle F_{\mathrm{LM}}\otimes P_{aa}^{\mathrm{NLO}}\rangle\Big] + \langle F_{\mathrm{LV}}^{\mathrm{fin}}\rangle + \langle \mathcal{O}_{\mathrm{NLO}}\,\Delta^{(\mathfrak{m})}F_{\mathrm{LM}}(\mathfrak{m})\rangle$$

NEW APPROACH AT NNLO

Think about **structures** arising in VV and look for their friends in RV and RR

Ideally the result will be ~NLO^2 as much as possible

$$\begin{aligned} \boldsymbol{I}^{(2)}_{\rm R.S.}(\epsilon,\mu^2;\{p\}) &= -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\left(\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) + 4\pi\beta_0\frac{1}{\epsilon}\right) \\ &+ \frac{e^{+\epsilon\psi(1)}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(2\pi\beta_0\frac{1}{\epsilon}+K\right)\boldsymbol{I}^{(1)}(2\epsilon,\mu^2;\{p\}) \\ &+ \boldsymbol{H}^{(2)}_{\rm R.S.}(\epsilon,\mu^2;\{p\})\ , \end{aligned}$$

Catani, 1998

$$\begin{aligned} \langle F_{\rm VV}\rangle = [\alpha_s]^2 &\left\langle \left[\frac{1}{2}I_{\rm V}^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_{\rm E}}}\left(\frac{\beta_0}{\epsilon}I_{\rm V}(\epsilon) - \left(\frac{\beta_0}{\epsilon}+K\right)I_{\rm V}(2\epsilon)\right)\right]\cdot F_{\rm LM}\right\rangle \\ + [\alpha_s]^2 &\left\langle \left[-\frac{1}{2}\left[\overline{I}_1(\epsilon),\overline{I}_1^\dagger(\epsilon)\right] + \mathcal{H}_{2,\rm tc} + \mathcal{H}_{2,\rm tc}^\dagger + \mathcal{H}_{2,\rm cd} + \mathcal{H}_{2,\rm cd}^\dagger\right]\cdot F_{\rm LM}\right\rangle \\ + [\alpha_s] &\left\langle I_{\rm V}(\epsilon)\cdot F_{\rm LV}^{\rm fin}\right\rangle + \left\langle F_{\rm LV^2}^{\rm fin}\right\rangle + \left\langle F_{\rm VV}^{\rm fin}\right\rangle . \end{aligned}$$

Color correlated contributions:

$$\begin{cases} \sim T_i\cdot T_j \\ \sim T_i\cdot T_j\cdot T_k \\ \sim (T_i\cdot T_j)\cdot(T_k\cdot T_l) \end{cases}$$

Different patterns of cancellations!

COLOR CORRELATIONS AND WHERE TO FIND THEM

We know they can only arise from soft real emissions and loop amplitudes

Double soft → $\sim T_i \cdot T_j$

$$\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}$$
$$= g_{s,b}^4 \sum_{i<j}^{N_p} \int [dp_{\mathfrak{m}}][dp_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}})\langle \widetilde{S}_{ij}(p_{\mathfrak{m}},p_{\mathfrak{n}})\ (\boldsymbol{T}_i\cdot\boldsymbol{T}_j)\cdot F_{\rm LM}\rangle$$
$$= [\alpha_s]^2\left[\frac{C_A}{\epsilon^2}c_1(\epsilon)+\frac{\beta_0}{\epsilon}c_2(\epsilon)+\beta_0\,c_3(\epsilon)\right]\langle \widetilde{I}_{\rm S}(2\epsilon)\cdot F_{\rm LM}\rangle + \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\rm fin}$$

Pole content identical to $I_S(2\epsilon)$!

Double soft → $\sim (T_i \cdot T_j)\cdot(T_k \cdot T_l)$

"Factorised contribution"

$$\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^4} = 2g_{s,b}^4 \sum_{(ij),(kl)}^{N_p}\left\langle \int [dp_{\mathfrak{m}}][dp_{\mathfrak{n}}]\Theta(E_{\mathfrak{m}}-E_{\mathfrak{n}})S_{ij}(p_{\mathfrak{m}})S_{kl}(p_{\mathfrak{n}}) \times \{\boldsymbol{T}_i\cdot\boldsymbol{T}_j,\boldsymbol{T}_k\cdot\boldsymbol{T}_l\}\cdot F_{\rm LM}\right\rangle$$
$$= [\alpha_s]^2\frac{1}{2}\langle I_{\rm S}^2(\epsilon)\cdot F_{\rm LM}\rangle\ .$$

$$\langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle$$
$$= [\alpha_s]^2\left\langle\left[\frac{1}{2}I_{\rm S}^2(\epsilon)+\left(\frac{C_A}{\epsilon^2}c_1(\epsilon)+\frac{\beta_0}{\epsilon}c_2(\epsilon)+\beta_0\,c_3(\epsilon)\right)\widetilde{I}_{\rm S}(2\epsilon)\right]\cdot F_{\rm LM}\right\rangle + \langle S_{\mathfrak{mn}}\Theta_{\mathfrak{mn}}F_{\rm LM}(\mathfrak{m},\mathfrak{n})\rangle_{T^2}^{\rm fin}\ .$$

$I_S^2(\epsilon) + I_V^2(\epsilon)$ takes care of "quartic" color-correlated poles

COLOR CORRELATIONS AND WHERE TO FIND THEM

We know they can only arise from soft real emissions and loop amplitudes

Soft real-virtual

$$
\begin{aligned}
& S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \\
&= -\, g_{s,b}^2 \sum_{(ij)}^{N_p} \Big\{ 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LV}} - \frac{\alpha_s(\mu)}{2\pi} \frac{\beta_0}{\epsilon}\, 2\, S_{ij}(p_{\mathfrak{m}})\ (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
&\quad - 2 \frac{[\alpha_s]}{\epsilon^2} C_A\, A_K(\epsilon) \Big(S_{ij}(p_{\mathfrak{m}}) \Big)^{1+\epsilon} (\boldsymbol{T}_i \cdot \boldsymbol{T}_j) \cdot F_{\mathrm{LM}} \\
&\quad - [\alpha_s] \frac{4\pi\, \Gamma(1+\epsilon)\Gamma^3(1-\epsilon)}{\epsilon\, \Gamma(1-2\epsilon)} \sum_{\substack{k=1 \\ k \neq i,j}}^{N_p} \kappa_{ij}\, S_{ki}(p_{\mathfrak{m}}) \Big(S_{ij}(p_{\mathfrak{m}}) \Big)^{\epsilon} f_{abc}\, T_k^a\, T_i^b\, T_j^c\, F_{\mathrm{LM}} \Big\}
\end{aligned}
$$

Triple color correlators

Only contributes in processes with 2 colored particles in the initial state and for processes with Np >=4

Non-trivial phase space integral

$$
\begin{aligned}
\langle S_{\mathfrak{m}}\, F_{\mathrm{RV}}(\mathfrak{m}) \rangle &= [\alpha_s]^2 \Big\langle \frac{1}{2} \Big[I_{\mathrm{S}}(\epsilon) \cdot I_{\mathrm{V}}(\epsilon) + I_{\mathrm{V}}(\epsilon) \cdot I_{\mathrm{S}}(\epsilon) \Big] \cdot F_{\mathrm{LM}} \Big\rangle \\
&\quad + [\alpha_s] \Big\langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LV}}^{\mathrm{fin}} \Big\rangle - [\alpha_s]^2 \frac{\Gamma(1-\epsilon)}{e^{\epsilon \gamma_E}} \frac{\beta_0}{\epsilon} \Big\langle I_{\mathrm{S}}(\epsilon) F_{\mathrm{LM}} \Big\rangle \\
&\quad - \frac{[\alpha_s]^2}{\epsilon^2} C_A\, A_K(\epsilon) \Big\langle \widetilde{I}_{\mathrm{S}}(2\epsilon) \cdot F_{\mathrm{LM}} \Big\rangle \\
&\quad + [\alpha_s]^2 \Big\langle \Big(\frac{1}{2} \Big[I_{\mathrm{S}}(\epsilon)\, ,\, \overline{I}_1(\epsilon) - \overline{I}_1^{\dagger}(\epsilon) \Big] + I_{\mathrm{tri}}^{\mathrm{RV}}(\epsilon) \Big) \cdot F_{\mathrm{LM}} \Big\rangle
\end{aligned}
$$

The subtraction term can be almost fully written in terms of our NLO Catani-like operators

And so on…….(hard-collinear RV, single soft RR etc.)

CANCELLATION OF DOUBLE COLOR-CORRELATED POLES

Recall: $I_T = I_V + I_S + I_C$ = finite! $= I_T^2 - I_C^2$ ➡ No color correlated poles!

$$\Sigma_N^{(\mathrm{V+S}),\mathrm{el}} = [\alpha_s]^2 \frac{1}{2}\left\langle \left[I_\mathrm{V}^2 + I_\mathrm{V} I_\mathrm{S} + I_\mathrm{S} I_\mathrm{V} + I_\mathrm{S}^2 + 2I_\mathrm{C} I_\mathrm{V} + 2 I_\mathrm{C} I_\mathrm{S}\right] \cdot F_\mathrm{LM}\right\rangle$$
$$+ [\alpha_s]^2 \frac{\beta_0}{\epsilon} \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}} \left\langle \left[-\left[I_\mathrm{S}(\epsilon) + I_\mathrm{V}(\epsilon)\right] + I_\mathrm{V}(2\epsilon) + \tilde{c}(\epsilon)\, \widetilde{I}_\mathrm{S}(2\epsilon)\right] \cdot F_\mathrm{LM}\right\rangle$$
$$+ [\alpha_s]^2 \left\langle \left[K \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_\mathrm{E}}} I_\mathrm{V}(2\epsilon) + C_A \left(\frac{c_1(\epsilon)}{\epsilon^2} - \frac{A_K(\epsilon)}{\epsilon^2} - 2^{2+2\epsilon} \delta_g^{C_A}(\epsilon)\right)\right.\right.$$
$$\left.\left. \times\, \widetilde{I}_\mathrm{S}(2\epsilon)\right] \cdot F_\mathrm{LM}\right\rangle + [\alpha_s]\left\langle \left[I_\mathrm{V}(\epsilon) + I_\mathrm{S}(\epsilon)\right] \cdot F_\mathrm{LV}^\mathrm{fin}\right\rangle ,$$

No color correlated poles!

$$\sim -I_{V+S}(\epsilon) + I_{V+S}(2\epsilon) + (\tilde{c}(\epsilon) - 1)\tilde{I}_S(2\epsilon) + \tilde{I}_S(2\epsilon) - I_S(2\epsilon)$$

$\mathcal{O}(\epsilon)$ $\mathcal{O}(\epsilon^2)$ $\mathcal{O}(\epsilon)$

With similar arguments one can show that all terms are free of color correlated poles

CANCELLATION OF TRIPLE COLOR–CORRELATED POLES

Double origin: **explicit** or **commutators** of I operators

From VV ($\mathscr{H}_2$) and soft RV (I_{tri}^{RV})

Again present in VV and soft RV

$$\Sigma_N^{\text{tri}} = [\alpha_s]^2 \left\langle \left(\frac{1}{2} \left[I_{\text{S}}(\epsilon) \,,\, \overline{I}_1(\epsilon) - \overline{I}_1^\dagger(\epsilon) \right] + I_{\text{tri}}^{\text{RV}}(\epsilon) \right) \cdot F_{\text{LM}} \right\rangle ,$$

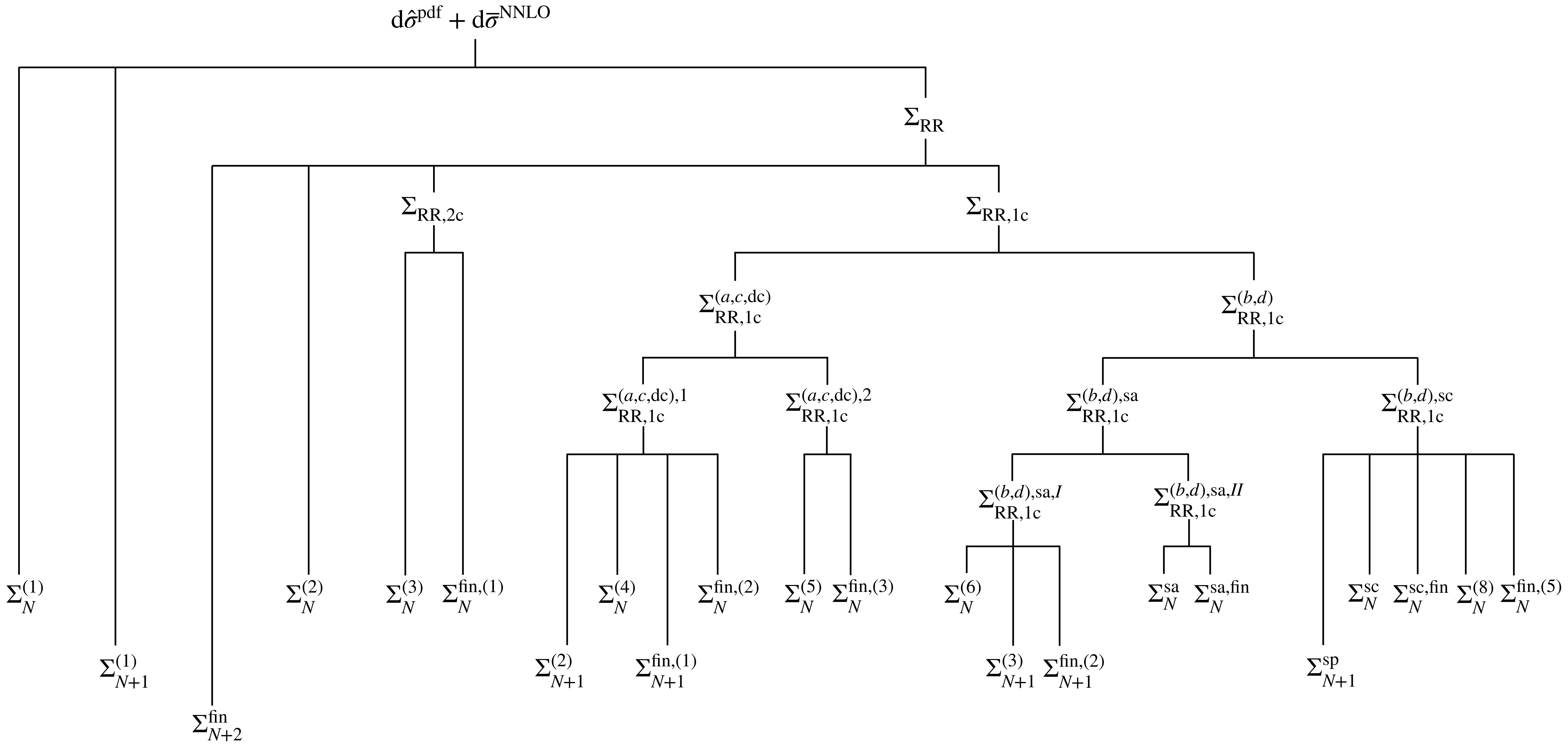
$$+ [\alpha_s]^2 \left\langle \left(-\frac{1}{2} \left[\overline{I}_1(\epsilon), \overline{I}_1^\dagger(\epsilon) \right] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger \right) \cdot F_{\text{LM}} \right\rangle$$

Computed explicitly up to $\mathcal{O}(\epsilon^0)$

$$I_{\text{tri}}^{(\text{cc})} = -\left[I_+, I_- \right] + \left[2I_+ + I_{\text{S}}, I_- \right] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger$$

By computing these commutators one can see that the poles exactly cancel!

$$\mathcal{H}_{2,\text{tc}} = \frac{1}{2\epsilon} \left[\Gamma, C \right] \qquad \overline{I}_1^{(\text{cc})} = \frac{\Gamma}{\epsilon} + C + \mathcal{O}(\epsilon)$$



$d\hat{\sigma}^{\rm pdf} + d\overline{\sigma}^{\rm NNLO}$
$\Sigma_{\rm RR}$
$\Sigma_{\rm RR,2c}$
$\Sigma_{\rm RR,1c}$
$\Sigma_{\rm RR,1c}^{(a,c,{\rm dc})}$
$\Sigma_{\rm RR,1c}^{(b,d)}$
$\Sigma_{\rm RR,1c}^{(a,c,{\rm dc}),1}$
$\Sigma_{\rm RR,1c}^{(a,c,{\rm dc}),2}$
$\Sigma_{\rm RR,1c}^{(b,d),{\rm sa}}$
$\Sigma_{\rm RR,1c}^{(b,d),{\rm sc}}$
$\Sigma_{\rm RR,1c}^{(b,d),{\rm sa},I}$
$\Sigma_{\rm RR,1c}^{(b,d),{\rm sa},II}$
$\Sigma_N^{(1)}$
$\Sigma_{N+1}^{(1)}$
$\Sigma_{N+2}^{\rm fin}$
$\Sigma_N^{(2)}$
$\Sigma_N^{(3)}$
$\Sigma_N^{{\rm fin},(1)}$
$\Sigma_{N+1}^{(2)}$
$\Sigma_N^{(4)}$
$\Sigma_{N+1}^{{\rm fin},(1)}$
$\Sigma_N^{{\rm fin},(2)}$
$\Sigma_N^{(5)}$
$\Sigma_N^{{\rm fin},(3)}$
$\Sigma_N^{(6)}$
$\Sigma_{N+1}^{(3)}$
$\Sigma_{N+1}^{{\rm fin},(2)}$
$\Sigma_N^{\rm sa}$
$\Sigma_N^{\rm sa,fin}$
$\Sigma_{N+1}^{\rm sp}$
$\Sigma_N^{\rm sc}$
$\Sigma_N^{\rm sc,fin}$
$\Sigma_N^{(8)}$
$\Sigma_N^{{\rm fin},(5)}$

FINAL RESULT

Finite remainders for the generic process $q\bar{q} \to X + Ng$

$$2s\, \mathrm{d}\hat{\sigma}^{\mathrm{NNLO}}_{\mathrm{db}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \left\langle \mathcal{P}^{\mathrm{NLO}}_{qq} \otimes F_{\mathrm{LM}} \otimes \mathcal{P}^{\mathrm{NLO}}_{qq} \right\rangle$$

$$\begin{aligned}2s\, \mathrm{d}\hat{\sigma}^{\mathrm{NNLO}}_{\mathrm{sb}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Big\{ & \left\langle \mathcal{P}^{\mathrm{NLO}}_{qq} \otimes \left[I^{(0)}_{\mathrm{T}} \cdot F_{\mathrm{LM}} \right] \right\rangle + \left\langle \left[I^{(0)}_{\mathrm{T}} \cdot F_{\mathrm{LM}} \right] \otimes \mathcal{P}^{\mathrm{NLO}}_{qq} \right\rangle \\ & + \left\langle \mathcal{P}^{\mathcal{W}}_{qq} \otimes \left[\mathcal{W}^{1\|\mathfrak{n},\mathrm{fin}}_{1} \cdot F_{\mathrm{LM}} \right] \right\rangle + \left\langle \left[\mathcal{W}^{2\|\mathfrak{n},\mathrm{fin}}_{2} \cdot F_{\mathrm{LM}} \right] \otimes \mathcal{P}^{\mathcal{W}}_{qq} \right\rangle \\ & + \left\langle \mathcal{P}^{\mathrm{NNLO}}_{qq} \otimes F_{\mathrm{LM}} \right\rangle + \left\langle F_{\mathrm{LM}} \otimes \mathcal{P}^{\mathrm{NNLO}}_{qq} \right\rangle \\ & + \left\langle \mathcal{P}^{\mathrm{NLO}}_{qq} \otimes F^{\mathrm{fin}}_{\mathrm{LV}} \right\rangle + \left\langle F^{\mathrm{fin}}_{\mathrm{LV}} \otimes \mathcal{P}^{\mathrm{NLO}}_{qq} \right\rangle \Big\} ,\end{aligned}$$

$$\begin{aligned}2s\, \mathrm{d}\hat{\sigma}^{\mathrm{NNLO}}_{\mathrm{el}} = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \Big\{ & \left\langle \left[I^{\mathrm{fin}}_{\mathrm{cc}} + I^{\mathrm{fin}}_{\mathrm{tri}} + I^{\mathrm{fin}}_{\mathrm{unc}} \right] \cdot F_{\mathrm{LM}} \right\rangle \\ & + \sum_{i=1}^{N_p} \left\langle \left[\gamma^{\mathcal{W}}(L_i)\, \theta_{i2}\, \mathcal{W}^{i\|\mathfrak{n},\mathrm{fin}}_{i} + \delta^{(0)}_g\, \mathcal{W}^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}}_{i} + \delta^{\perp}_g \mathcal{W}^{(i)}_{r} \right] \cdot F_{\mathrm{LM}} \right\rangle \Big\} \\ & + \left[\frac{\alpha_s(\mu)}{2\pi}\right] \left\langle I^{(0)}_{\mathrm{T}} \cdot F^{\mathrm{fin}}_{\mathrm{LV}} \right\rangle + \left\langle S_{\mathfrak{mn}} \Theta_{\mathfrak{mn}} F_{\mathrm{LM}}(\mathfrak{m},\mathfrak{n}) \right\rangle^{\mathrm{fin}}_{T^2} + \left\langle F^{\mathrm{fin}}_{\mathrm{LV}^2} \right\rangle + \left\langle F^{\mathrm{fin}}_{\mathrm{VV}} \right\rangle\end{aligned}$$

- Ready to be implemented in a numerical code
- Trivial dependence on number of partons
- Analytic proof of pole cancellation for generic process at NNLO!!