Numerical Integration of the N_f Virtual Corrections to Triboson Production at NNLO

in collaboration with **Dario Kermanschah [2407.18051]**

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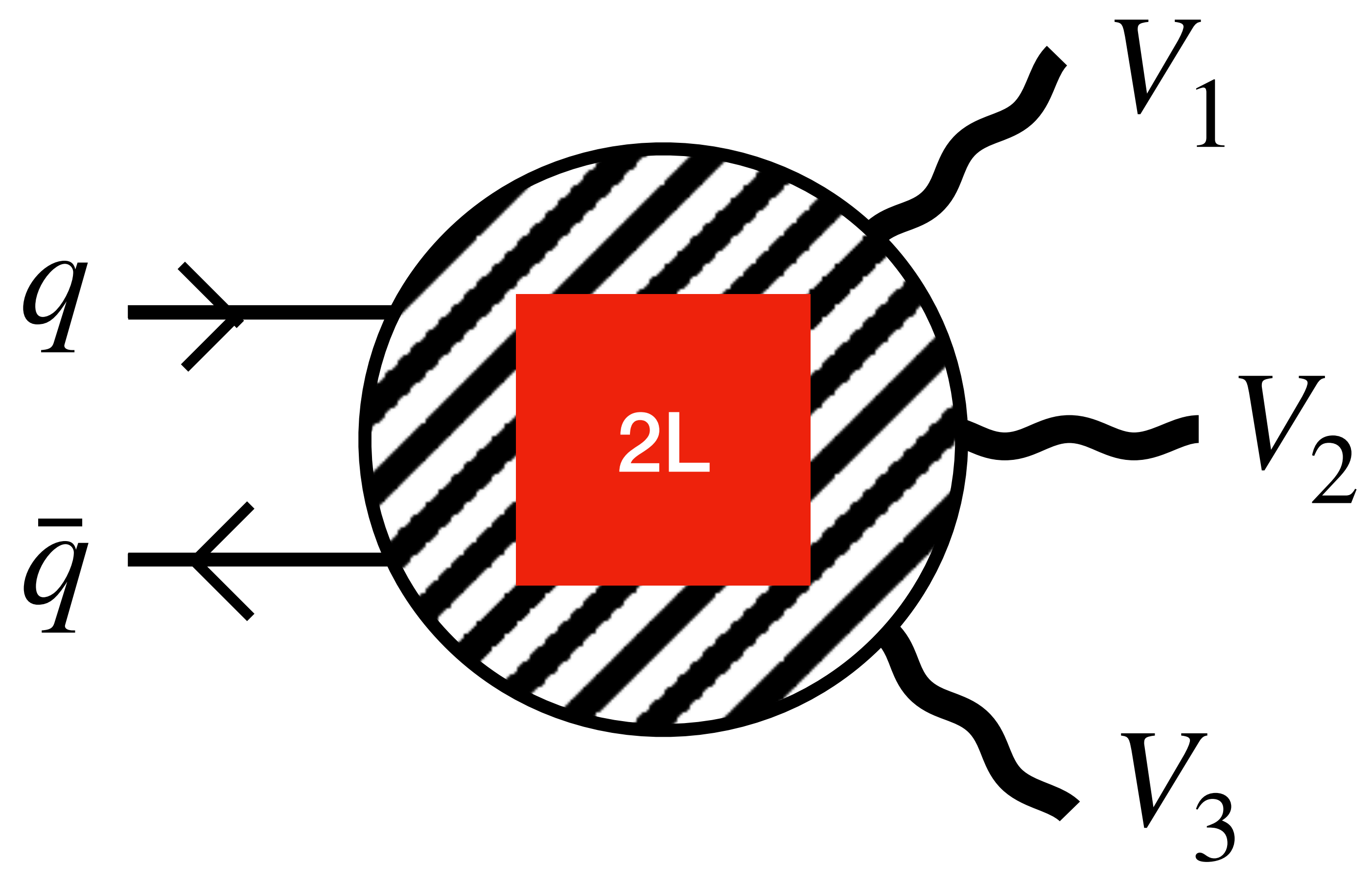
Loop the Loop

14th November 2024

Why numerical methods?

The analytical computation of the two-loop integrals is challenging in $q\bar{q} \to V_1 V_2 V_3$

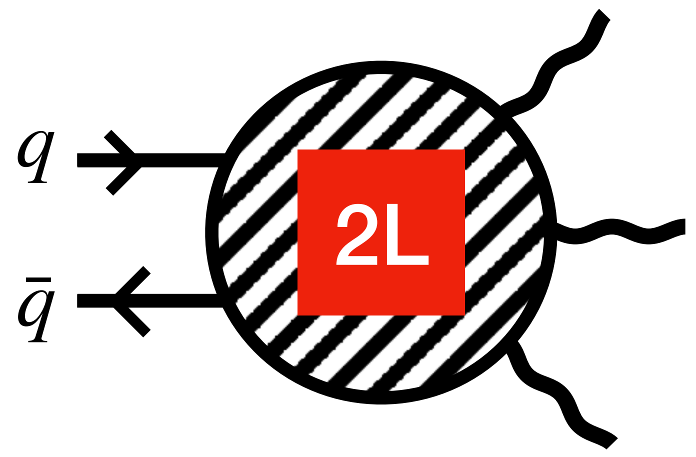


3 external massive vector bosons

High multiplicity in the external legs, many kinematical scales

Analytic integration methods may struggle.

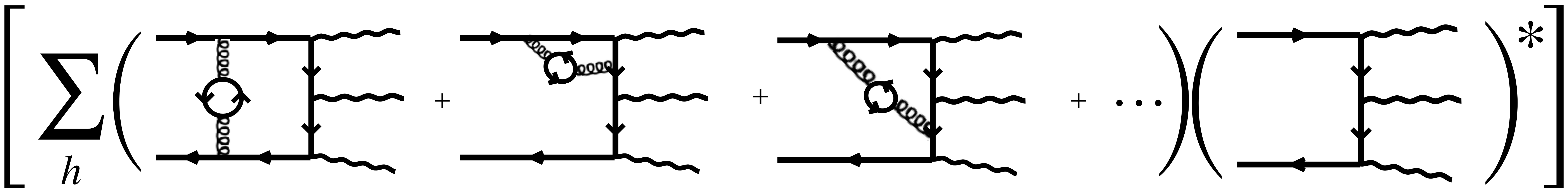
In this talk…



3 external massive vector boson

We tackled the simplest 2L (gauge invariant) contribution at the level of the cross section

$$\sigma_{\mathrm{virt}}^{(2,N_f)} \sim \int \mathbf{d}\Phi_3 \frac{1}{F} 2\,\mathrm{Re}$$



General Framework

To integrate numerically, we needed to remove:

- infrared (IR) singularities
- ultraviolet (UV) singularities
- threshold singularities

General Framework

Subtract local counterterms to remove singularities → difficult but we can use →
- Factorisation of IR singularities — [2008.12293, Anastasiou, Haindl, Sterman, Yang, Zeng]
- R-operation for UV singularities

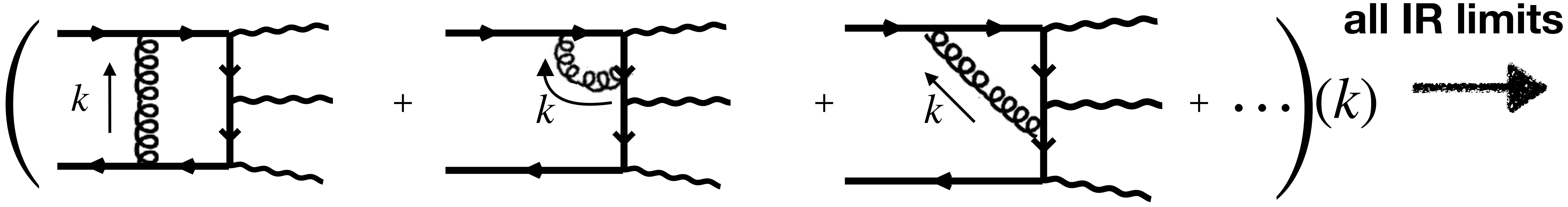
Integrate in $D = 4$ the finite remainder → difficult but let the computer deal with → Monte Carlo integration

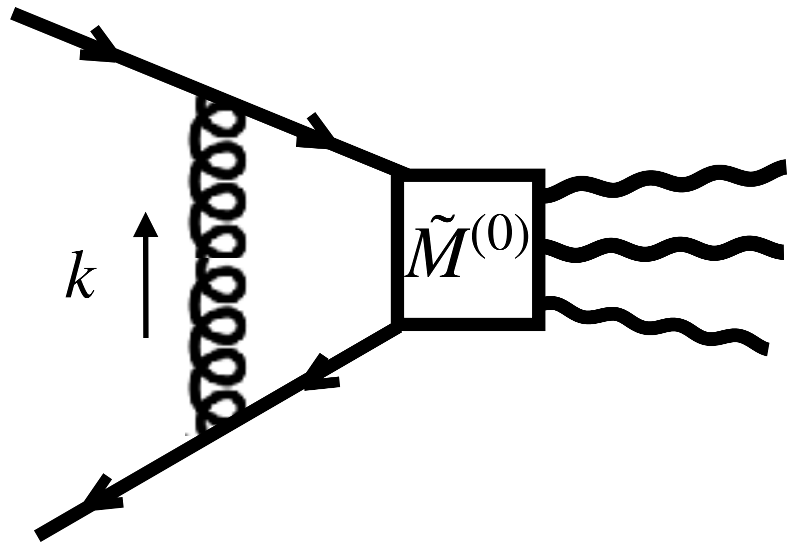
in the physical region → Threshold subtraction — [2110.06869, Kermanschah]

Add back the counterterm integrated analytically → now it is easy by construction! → Easy master integrals

The IR counterterm

[2008.12293, Anastasiou, Haindl, Sterman, Yang, Zeng]



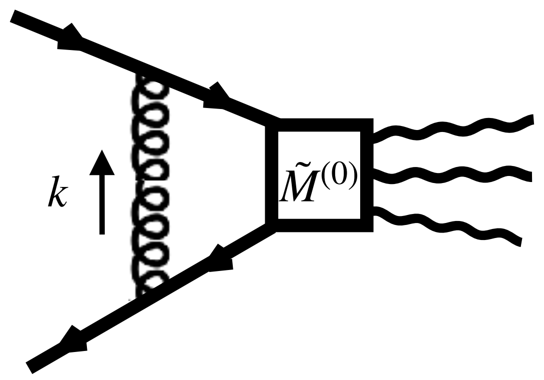


$$= -ig_s^2 C_F \frac{\bar{v}(p_2)\gamma^\mu(\not{k} - \not{p}_2)\left[\widetilde{\mathcal{M}}^{(0)}\right](\not{p}_1 + \not{k})\gamma_\mu u(p_1)}{k^2(k+p_1)^2(k-p_2)^2}$$

The IR form-factor counterterm expresses at the integrand level the integrated Catani-Seymour factorisation of IR divergences:

$$M^{(1)} - I^{(1)}M^{(0)} = \text{finite}$$

To get rid of UV divergences at the local level, we also introduce UV counterterms, one for each UV singular diagram, e.g.:

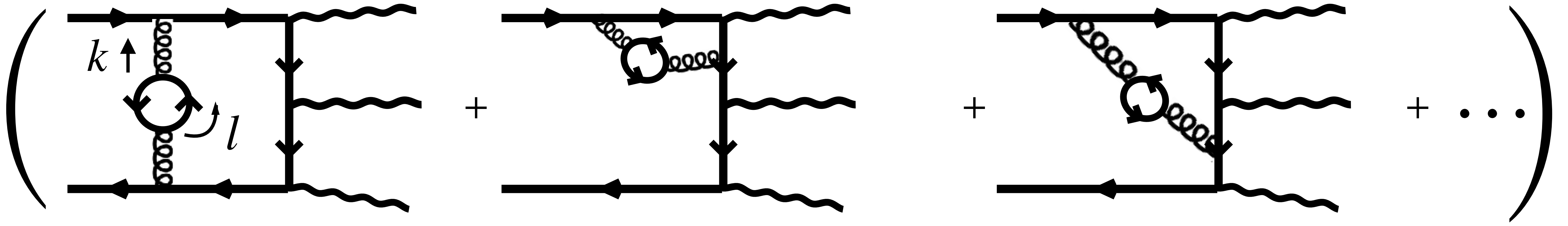
$$\mathcal{R}_{\mathrm{UV}}\Bigg($$


$$\Bigg) = -ig_s^2 C_F \frac{\bar{v}(p_2)\gamma^\mu \not{k}\left[\widetilde{\mathcal{M}}^{(0)}\right]\not{k}\gamma_\mu u(p_1)}{(k^2 - M_{\mathrm{UV}}^2)^3}$$

Alternative integrand for $\mathscr{M}^{(2,N_f)}$

[2008.12293, Anastasiou, Haindl, Sterman, Yang, Zeng]

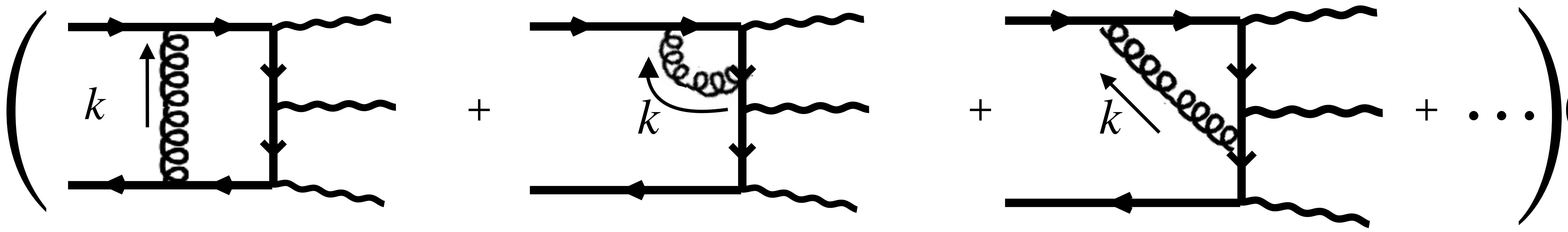
$\mathscr{M}^{(2,N_f)} =$



has power-like IR singularities

First, we subtract the local counterterm represented by the difference of the tensor reduced integrand in l and the original integrand, to get

$\sim$

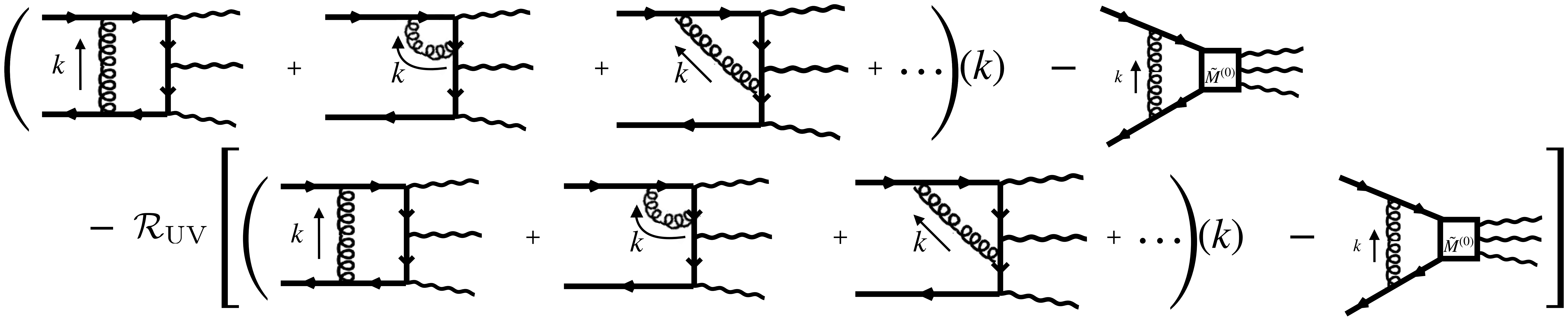


$$(k)\ \frac{1}{l^2(l+k)^2} = \mathscr{M}^{(1)}(k)\frac{1}{l^2(l+k)^2}$$

Now $\mathscr{M}^{(2,N_f)}$ has same IR divergences as $\mathscr{M}^{(1)}$.

Subtracted amplitude in D=4

(1)



$$=: \mathcal{M}^{(1)}_{\text{finite}}(k)$$

$(2,N_f)$

$$\mathcal{M}^{(2,N_f)}_{\text{finite}}(k,l) \sim \left[\frac{1}{l^2(l+k)^2} - \frac{1}{(l^2 - M^2_{UV})^2} \right] \mathcal{M}^{(1)}_{\text{finite}}(k)$$

To integrate $\mathcal{M}_{\text{finite}}$ numerically in D=4 **in the physical region** we need to extract the discontinuities arising from threshold singularities.

Numerical integration in D=4

Several possibilities exist in the literature:

Feynman parameters

- [0004013, Binoth, Heinrich]
- [0703282, Anastasiou, Beerli, Daleo]
- [0807.4129, Smirnov,Tentyukov]
- [0703273, Lazopoulos, Melnikov, Petriello]
- [1011.5493, Carter, Heinrich]
- [1703.09692, Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk,..]
- [2302.08955, Borinsky, Munch, Tellander]
- …

Loop momentum space

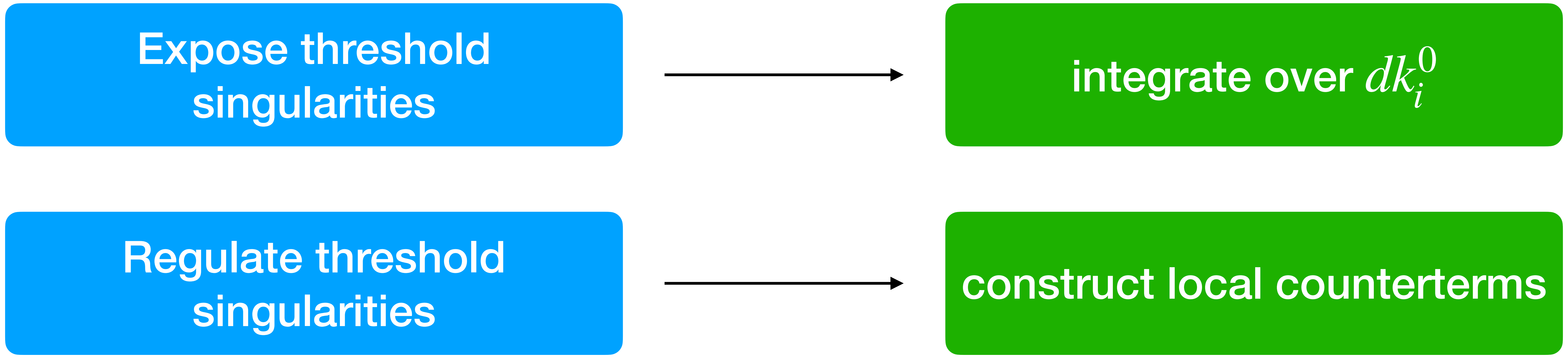
- [9804454, Soper]
- [0812.3686, Gong, Nagy, Soper]
- [1010.4187, Becker, Reuschle, Weinzierl]
- [1111.1733, Becker, Goetz, Reuschle, Schwan, Weinzierl]
- [1211.0509, Becker, Weinzierl]
- [1510.00187, Buchta, Chachamis, Draggiotis, Rodrigo]
- [0912.3495, Kilian, Kleinschmidt]
- [1912.09291, Capatti, Hirschi, Kermanschah, Pelloni, Ruijl]
- [2110.06869, Kermanschah]
- …

Numerical integration in D=4

Threshold subtraction

[0912.3495, Kilian, Kleinschmidt]
[2110.06869, Kermanschah]
[2407.18051, Kermanschah, MV]

To remove threshold singularities in momentum space:

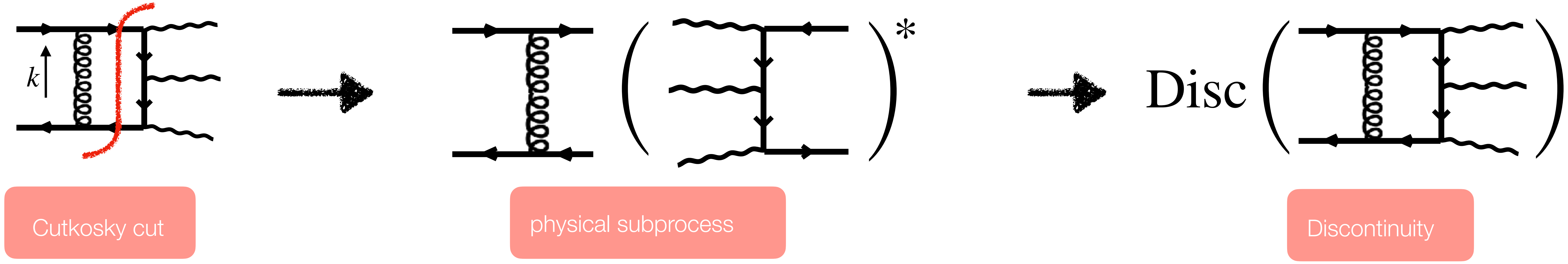


Local threshold subtraction has several advantages with respect to numerical contour deformation including

No need for extra parameters that need to be fine-tuned

Flatter integrand, better numerical convergence

Threshold singularities in loop momentum space



To expose all the threshold singularities in loop momentum space:

$$\int dk^4 \mathcal{M}_{\text{finite}}(k) = \int d^3\vec{k}\, f^{3\text{d}}\left(\mathcal{M}_{\text{finite}}(k)\right) \sim \int d^3\vec{k} \left\{ \frac{1}{E_1 + E_3 - p_1^0 - p_2^0} \frac{1}{E_0 + E_1 - p_1^0} \cdots + \cdots \right\}$$

E_i : on-shell energy of propagator i

$\int dk^0$ **via residue theorem**

f^{3d} : 3d representation of the integrand

Cross-free-family representation (CFF)

[2211.09653, Capatti]

Loop tree duality (LTD)

[0804.3170, Catani, Gleisberg, Krauss, Rodrigo, Winter]

[1904.08389, Aguilera-Verdugo, Driencourt-Mangin, Plenter, Ramírez-Uribe, Rodrigo, Sborlini et al.,]

[1906.06138, Capatti, Hirschi, Kermanschah, Ruijl]

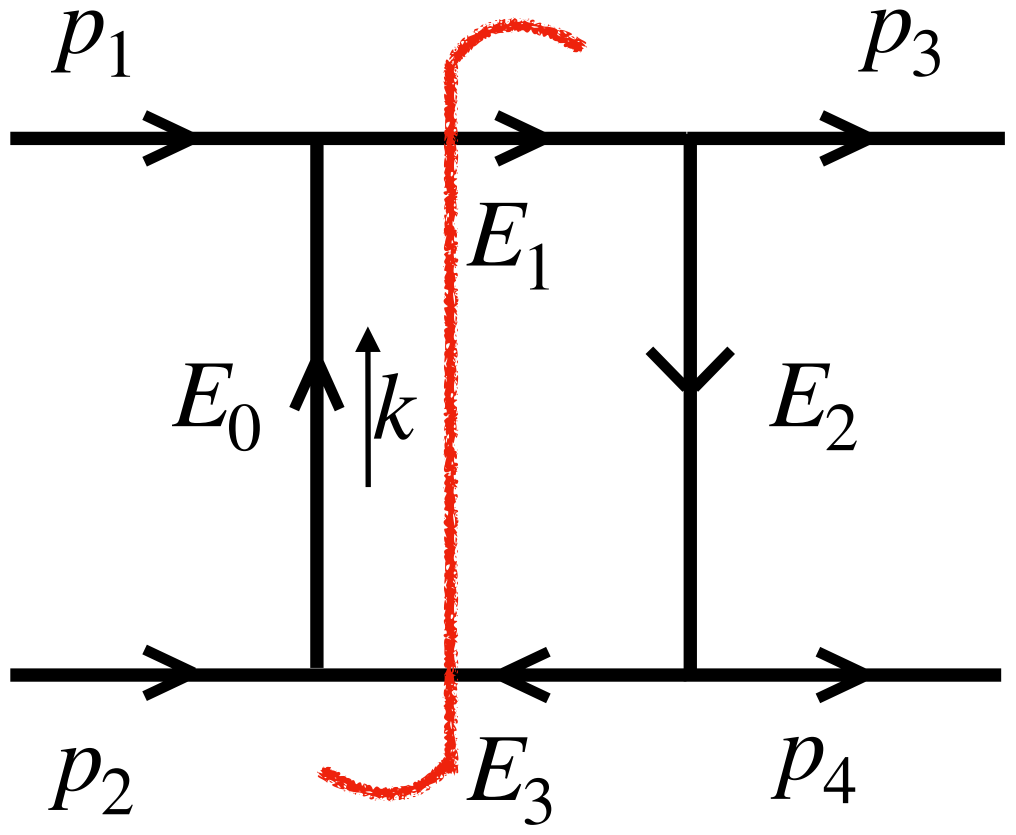
[2009.05509, Capatti, Hirschi, Kermanschah, Pelloni, Ruijl]

…

Threshold singularities in loop momentum space

$$\int d^3\vec{k}\, f^{3\mathrm{d}}\left(\mathscr{M}(k)\right) \sim \int d^3\vec{k} \left\{ \frac{1}{E_1 + E_3 - p_1^0 - p_2^0} \frac{1}{E_0 + E_1 - p_1^0} \cdots + \cdots \right\}$$

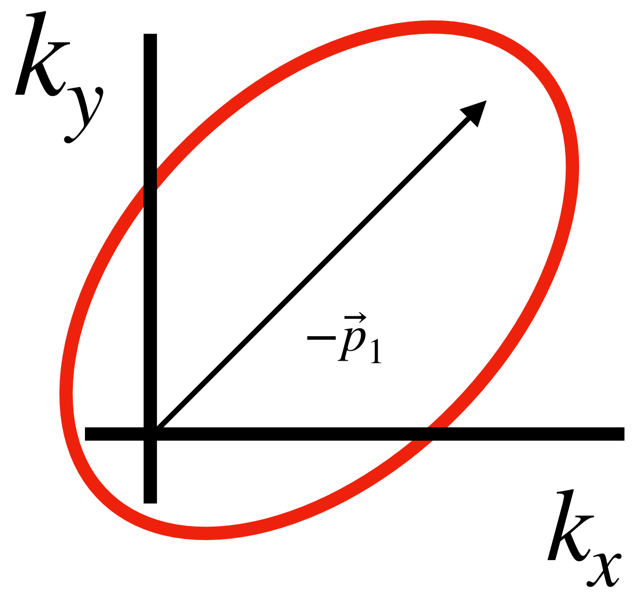
Setting each denominator = 0 identifies a bounded region in $\vec{k}$ space, e.g.

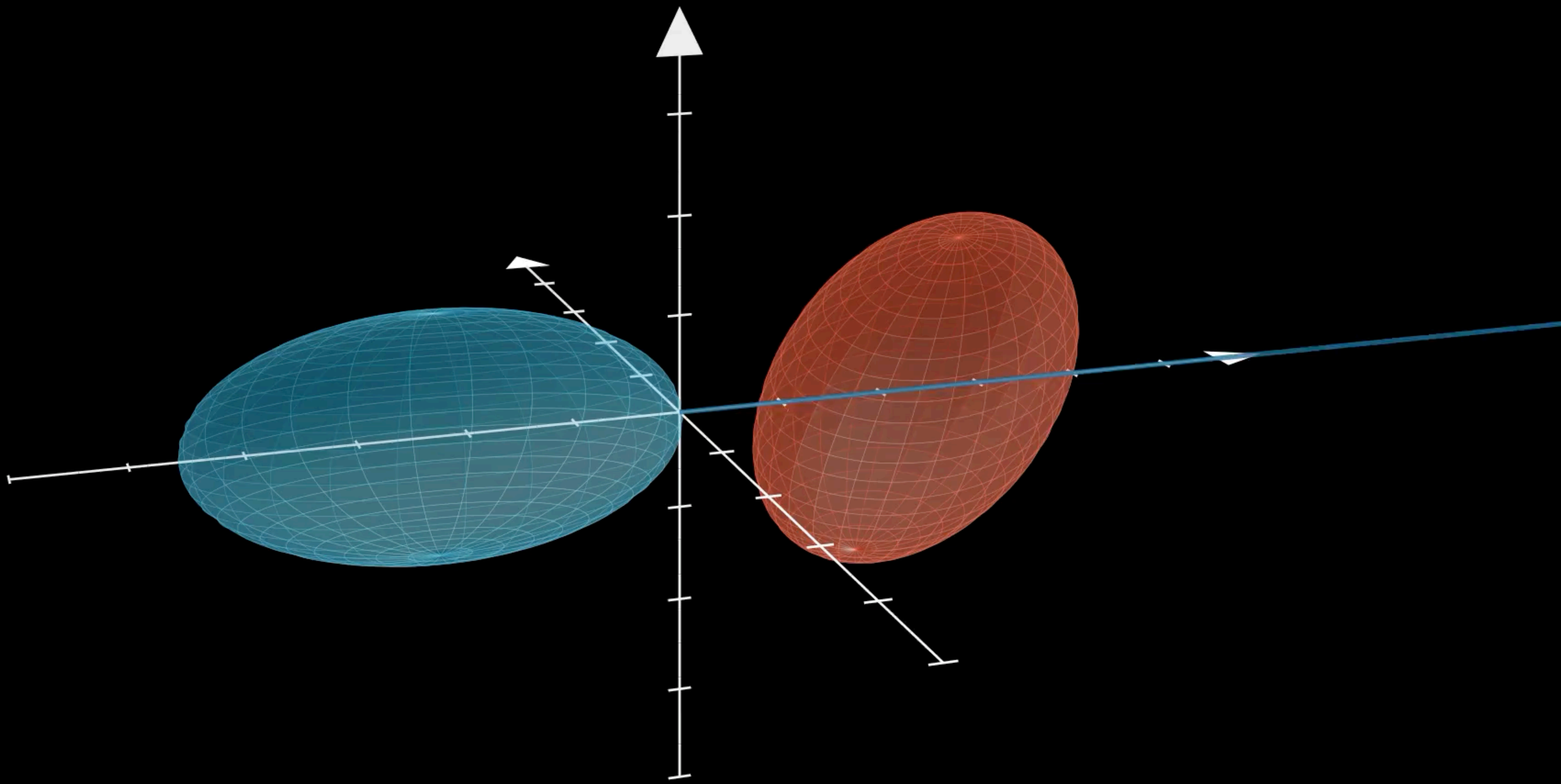


$$E_1 + E_3 - p_1^0 - p_2^0 = 0$$

for some $\vec{k} = \vec{k}^*$

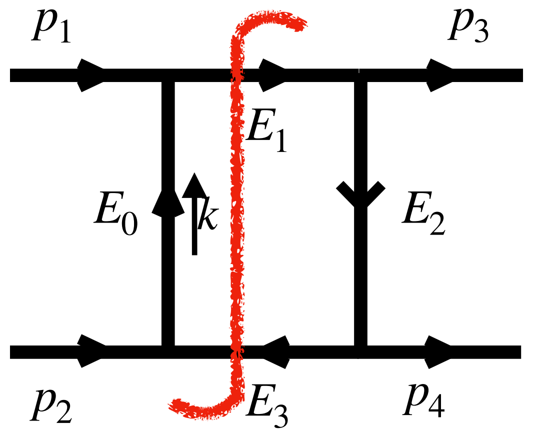
if $(p_1 + p_2)^2 > 0\,, (p_1^0 + p_2^0) > 0$





animation generated with Manim

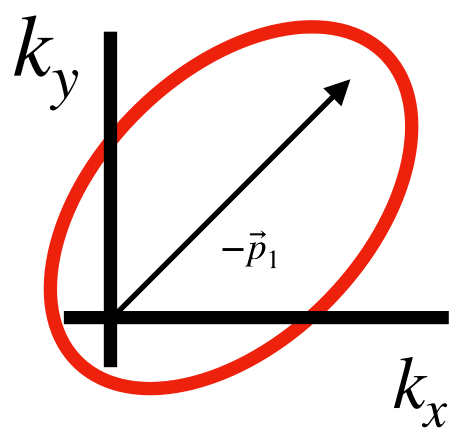
Numerical threshold subtraction



$$E_1 + E_3 - p_1^0 - p_2^0 = 0$$

for some $\vec{k} = \vec{k}^*$

if $(p_1 + p_2)^2 > 0\,, (p_1^0 + p_2^0) > 0$

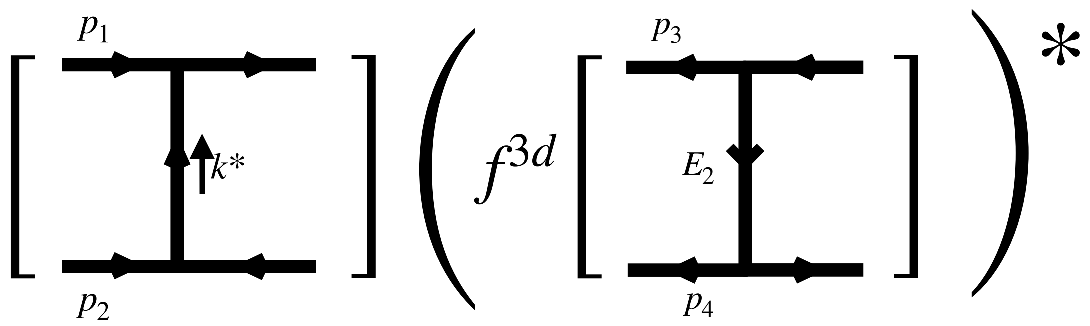


[2110.06869, Kermanschah]

- around the threshold singularity at $\vec{k} = \vec{k}^*$ the integrand behaves as:

$$\frac{\textbf{Res}_{\vec{k}=\vec{k}^*}[f^{3d}(\mathscr{M})]}{|\vec{k}| - k^* \pm i\varepsilon}\,,$$

$\text{Res}_{\vec{k}=\vec{k}^*}[f^{3d}(\mathscr{M})] \sim f^{3d}\,[\;\cdot\;]\left(f^{3d}\,[\;\cdot\;]\right)^*$



- build a local threshold counterterm CT_*

$$\int d^3\vec{k}\, f^{3d}\left(\mathscr{M}(k)\right) = \int d^3\vec{k}\, \left\{ f^{3d}\left(\mathscr{M}(k)\right) - \frac{\textbf{Res}_{\vec{k}=\vec{k}^*}[f^{3d}(\mathscr{M})]}{|\vec{k}| - k^* \pm i\varepsilon}\chi(\vec{k},\vec{k}^*) \right\} + \int CT_*\,,$$

χ : suppression function when $|\vec{k}| \to \infty$

- to find $\int CT_*$, use Sokhotski–Plemelj theorem

$$\lim_{\epsilon\to 0} \frac{1}{x - a \pm i\epsilon} = PV\frac{1}{x-a} \mp i\pi\delta(x-a) \quad \Longrightarrow \quad \int CT_* = \mp i\pi \int d^2\hat{k}\ \text{Res}_*(\hat{k})$$

for smart choice of χ

Overlapping thresholds

[2110.06869, Kermanschah]
[2407.18051, Kermanschah, MV]

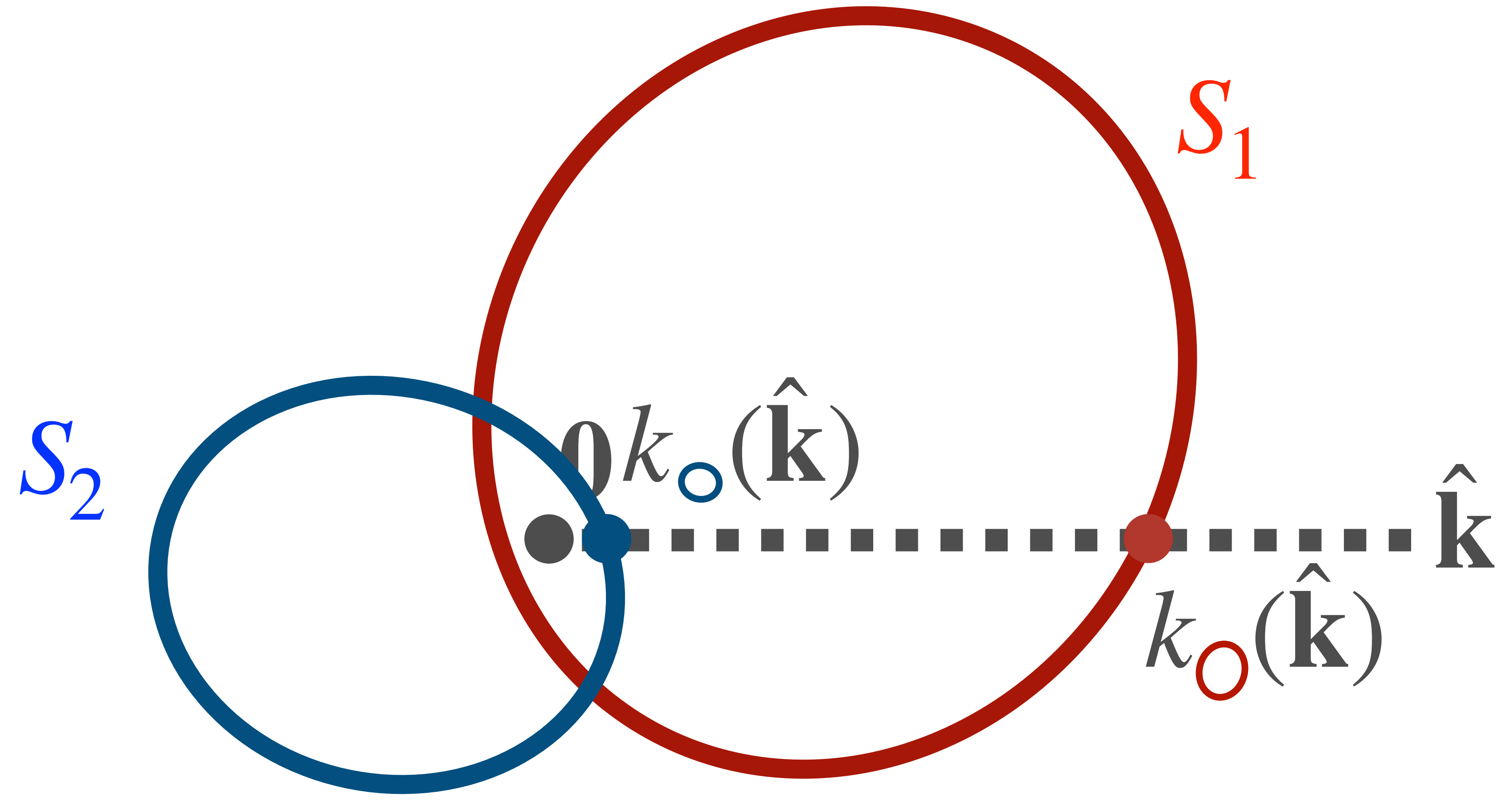
$$\mathscr{I} = \int d^3\vec{k} \left(\frac{1}{{\color{red}S_1}{\color{blue}S_2}\cdots} + \cdots \right)$$

$$\mathrm{Res}_{\color{red}\bigcirc}\mathscr{I} \sim (\ldots)\frac{1}{{\color{blue}S_2}\cdots} \qquad \mathrm{Res}_{\color{blue}\circ}\mathscr{I} \sim (\ldots)\frac{1}{{\color{red}S_1}\cdots}$$

If ${\color{red}S_1} \cap {\color{blue}S_2} \neq 0$:

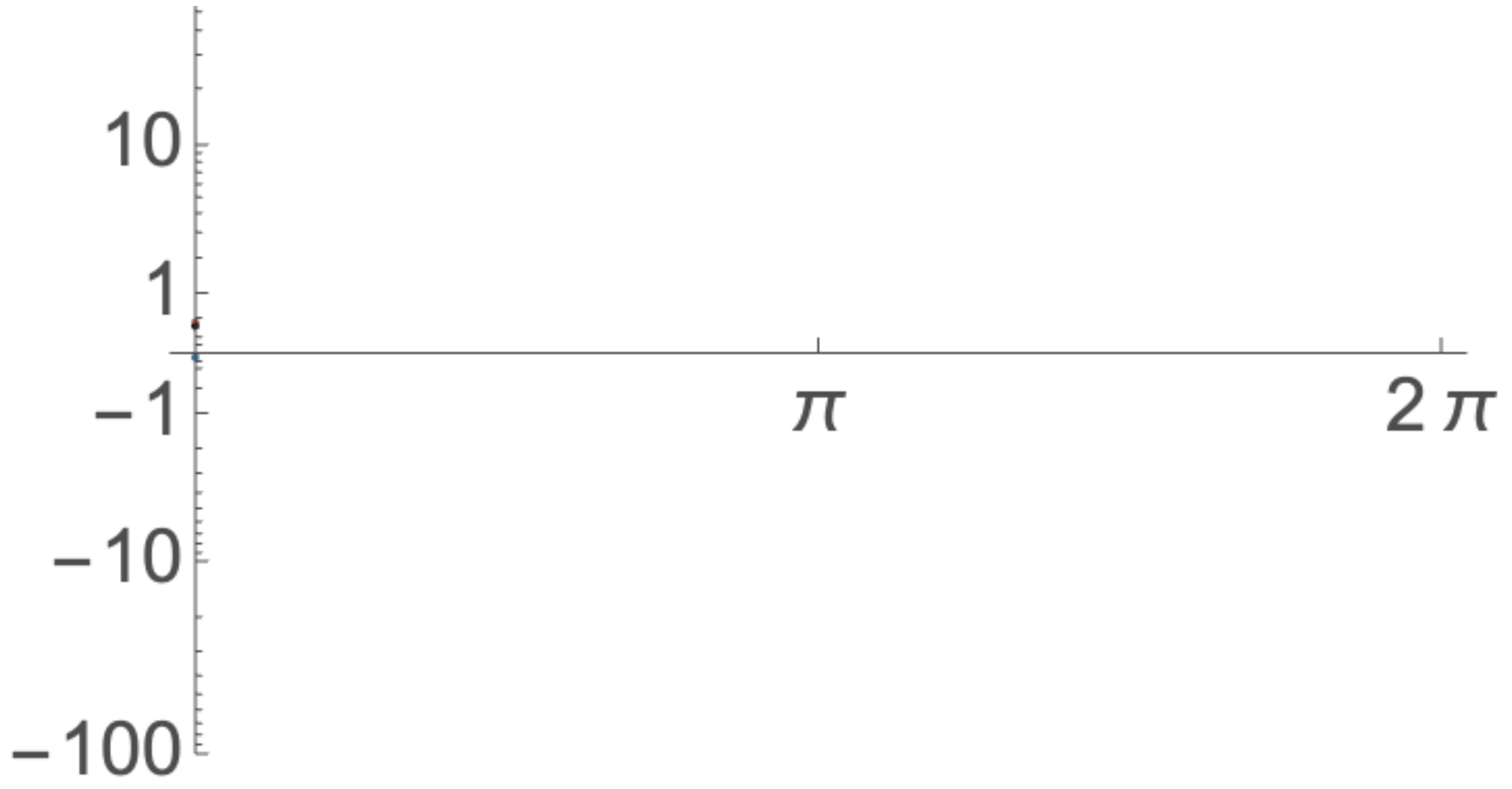
For higher order poles the same $i\epsilon$ prescription needs to appear in the counterterms, so that the residues are summed correctly!

The origin of the coordinate system in loop momentum space needs to lie inside ${\color{red}S_1}$, ${\color{blue}S_2}$ simultaneously



parameterisation aligns singularities

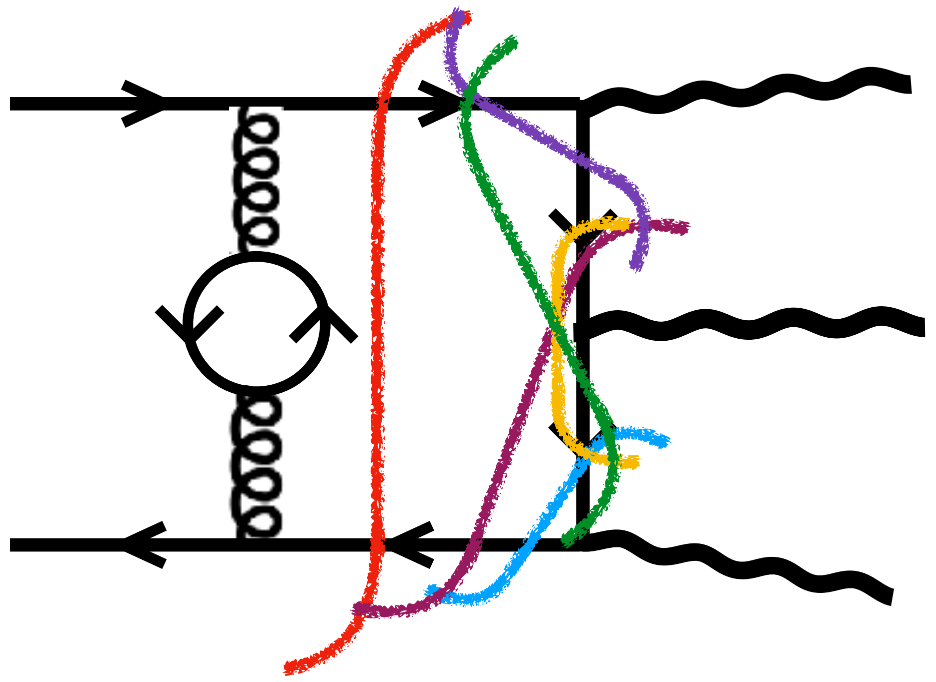
$$\mathrm{Res}_{\color{red}\bigcirc}\mathscr{I} + \mathrm{Res}_{\color{blue}\circ}\mathscr{I}$$



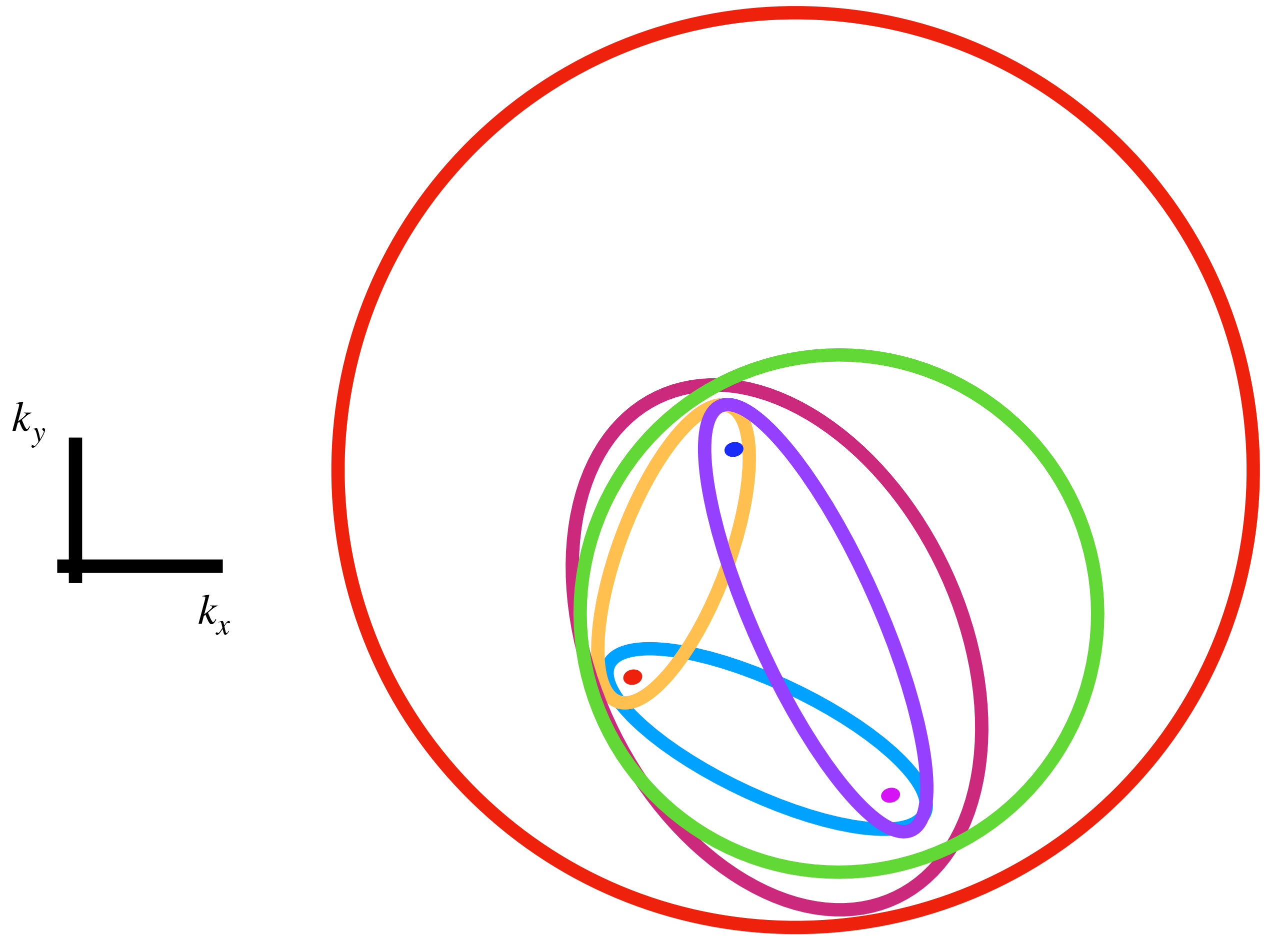
animation by Dario Kermanschah

Overlapping thresholds

Back to our example:



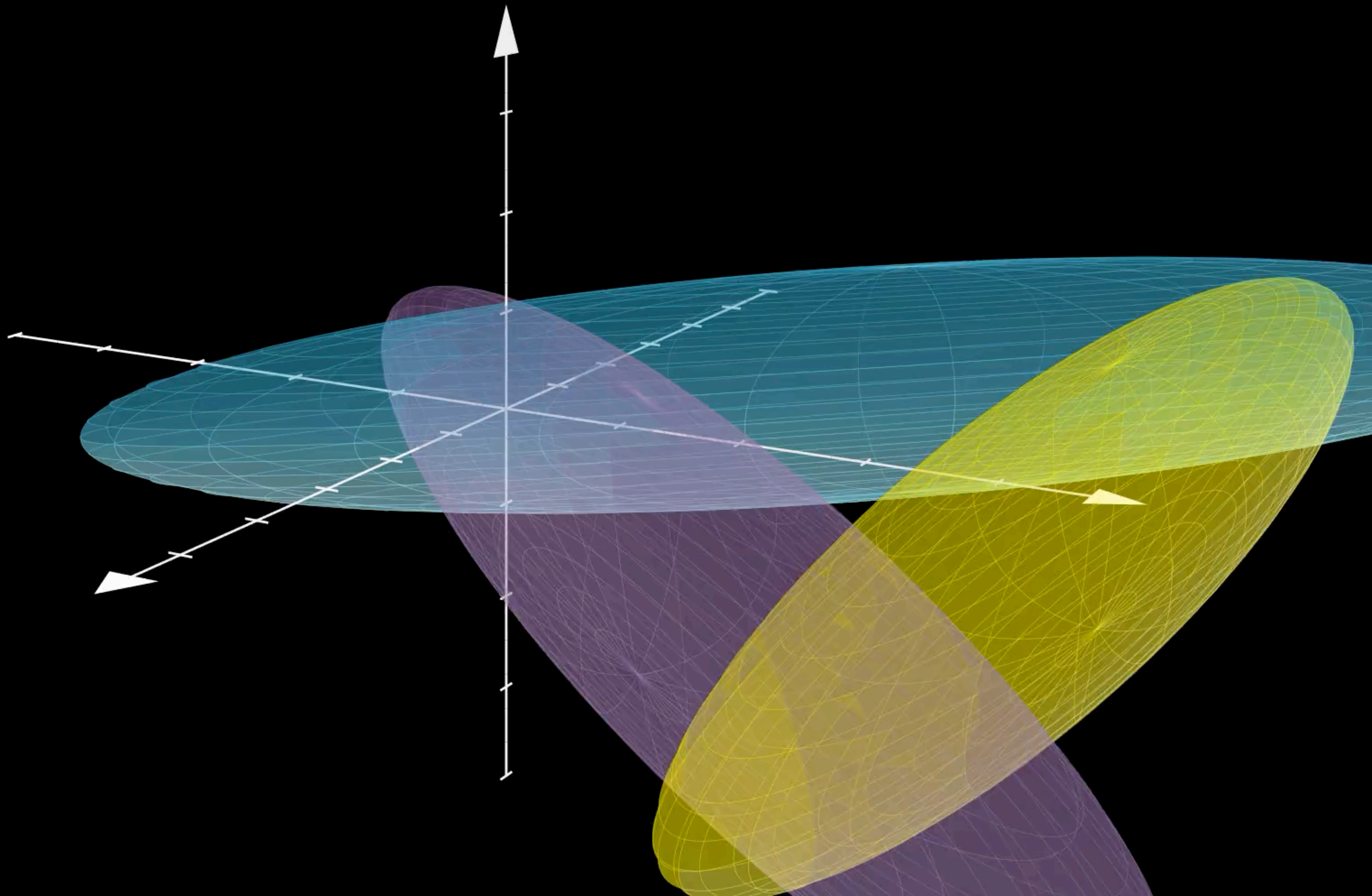
- 6 threshold surfaces
- some intersections can lead to higher-order poles!



Separate the relevant intersecting regions via multi-channelling:

$$1 = \frac{S_1^2 + S_2^2 + S_3^2}{S_1^2 + S_2^2 + S_3^2}$$

$$\mathcal{I} = \mathcal{I} \cdot 1 = \frac{S_3^2}{S_1^2 + S_2^2 + S_3^2}\mathcal{I} + \frac{S_2^2}{S_1^2 + S_2^2 + S_3^2}\mathcal{I} + \frac{S_1^2}{S_1^2 + S_2^2 + S_3^2}\mathcal{I}$$



animation generated with Manim

Does threshold structure change with phase space points?

$$\int d^3\vec{k}\, f^{3\mathrm{d}}\left(\mathscr{M}(k)\right) = \int d^3\vec{k}\, \left\{ f^{3\mathrm{d}}\left(\mathscr{M}(k)\right) - \sum_{*} \frac{\mathbf{Res}_{\vec{k}=\vec{k}^*}[f^{3d}(\mathscr{M})]}{|\vec{k}| - k^* \pm i\varepsilon} \chi(\vec{k}, \vec{k}^*) \right\} + \sum_{*} \int CT_*$$

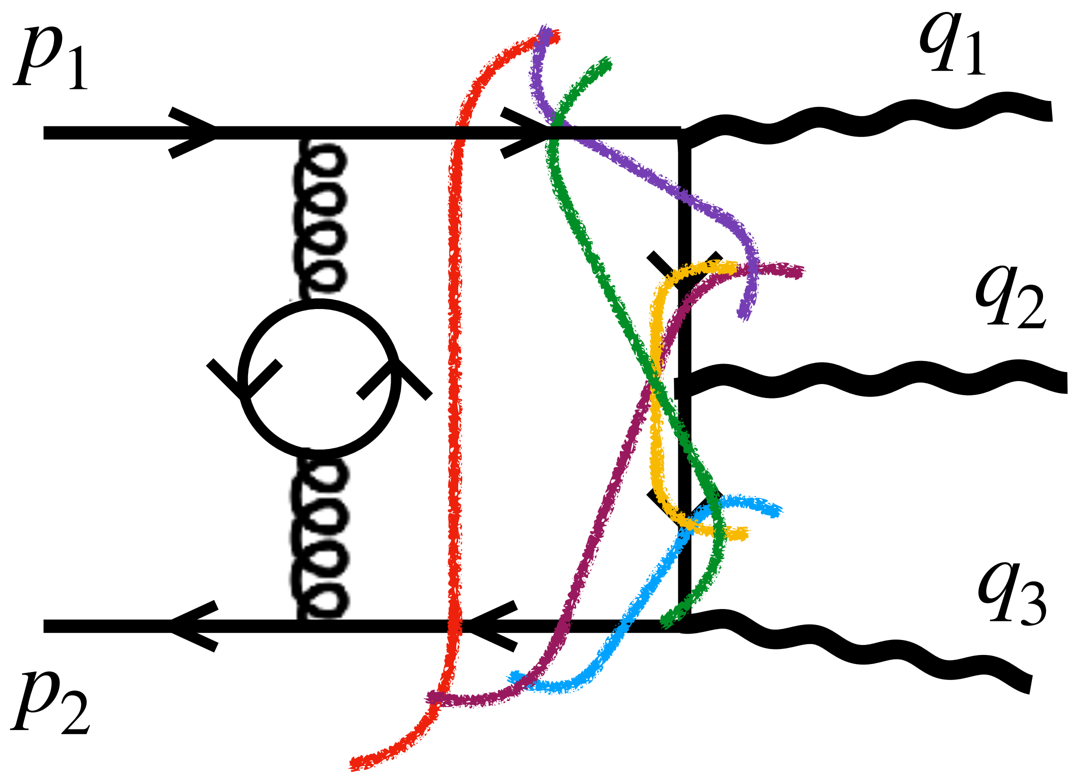
$Re\mathscr{M}$ $Im\mathscr{M}$

$Re\mathscr{M}$, $Im\mathscr{M}$ are separately locally finite for each set of external momenta in the physical region for the correct coordinate system in the sense specified before.

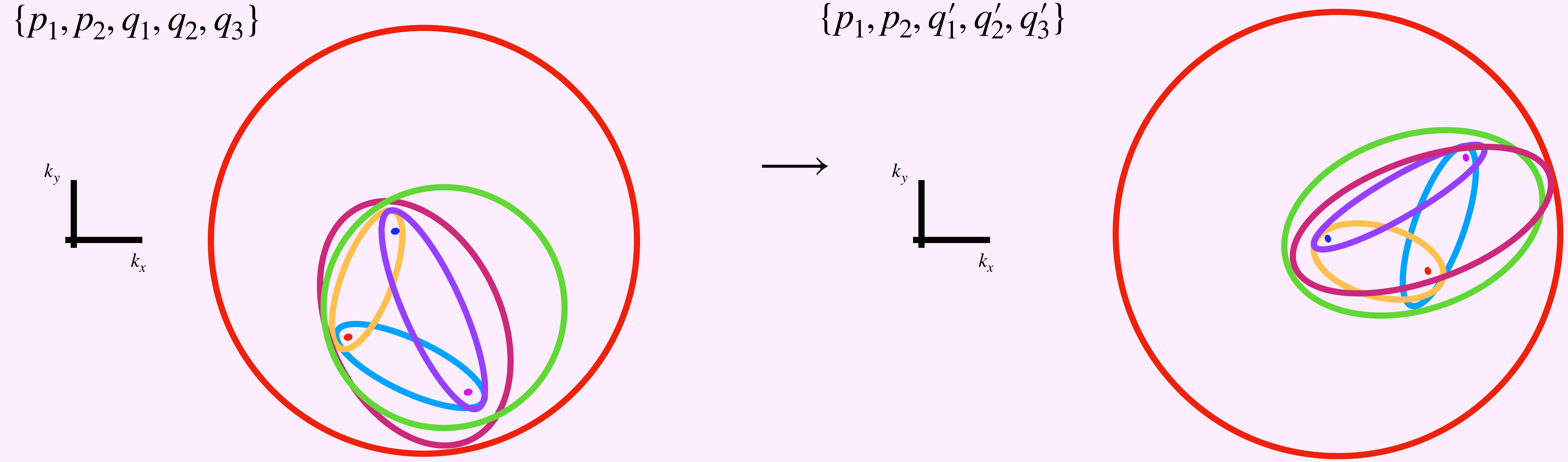
If we keep the Lorentz frame constant, does the intersection of threshold surfaces change?

Integration over phase space

For this specific example, threshold structure varies with q_1, q_2, q_3 sampled from the phase space generation in this way:



Plot the threshold surfaces in the COM frame for two different phase space points:

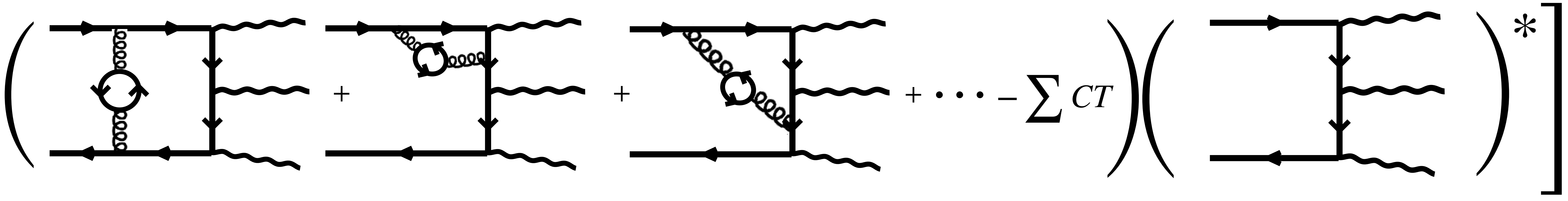


The structure of the relevant intersections stays constant

This allows to…

Perform simultaneous Monte-Carlo integration $\mathrm{d}\Phi_3 d^3\vec{k} d^3\vec{l}$ in:

$$\int \mathbf{d}\Phi_3 \frac{1}{F} 2\,\mathrm{Re}\left[\sum_h \left(\;\cdots\;\right)\left(\;\cdots\;\right)^*\right]$$



Gauge-invariant finite corrections to the virtual cross section!

Save computing time by sampling simultaneously phase space and loop measure.

Integrate IR/UV counterterms analytically

Just simple master integrals:

Results

same pipeline & same computer with 24 cores

DK, Matilde Vicini [2407.18051]
numerical integration over loop & phase space
summed over helicities and convoluted with PDFs

NLO and NNLO-Nf virtual cross sections

	Order	Result [pb]	Δ[%]	total time
$pp \to \gamma\gamma$	NLO	5.2851 ± 0.0164 e-01	0.3	10 min
	NNLO-Nf	-6.1475 ± 0.0349 e-02	0.6	1 h 30 min
$pp \to \gamma^*\gamma^*$	NLO	4.3172 ± 0.0089 e-01	0.2	2 min
	NNLO-Nf	-3.6943 ± 0.0322 e-02	0.9	40 min
$p_d p_d \to ZZ$	NLO	7.0067 ± 0.0159 e-01	0.2	4 min
	NNLO-Nf	-5.9363 ± 0.0520 e-02	0.9	1 h 30 min
$pp \to \gamma\gamma\gamma$	NLO	1.4874 ± 0.0140 e-04	0.9	2 h 30 min
	NNLO-Nf	-2.5460 ± 0.0237 e-05	0.9	1 day
$pp \to \gamma^*\gamma^*\gamma^*$	NLO	1.4692 ± 0.0144 e-04	1.0	2h 45 min
	NNLO-Nf	-1.4301 ± 0.0137 e-05	1.0	4 days
$p_d p_d \to Z\gamma_1^*\gamma_2^*$	NLO	2.4600 ± 0.0210 e-04	0.9	1 day 12 h
	NNLO-Nf	-2.5301 ± 0.0229 e-05	0.9	1 month

potential for optimization!

Catani finite remainder
NLO in BLHA
NNLO-Nf in $\overline{\text{MS}}$

NLO cross checked interferences with OpenLoops

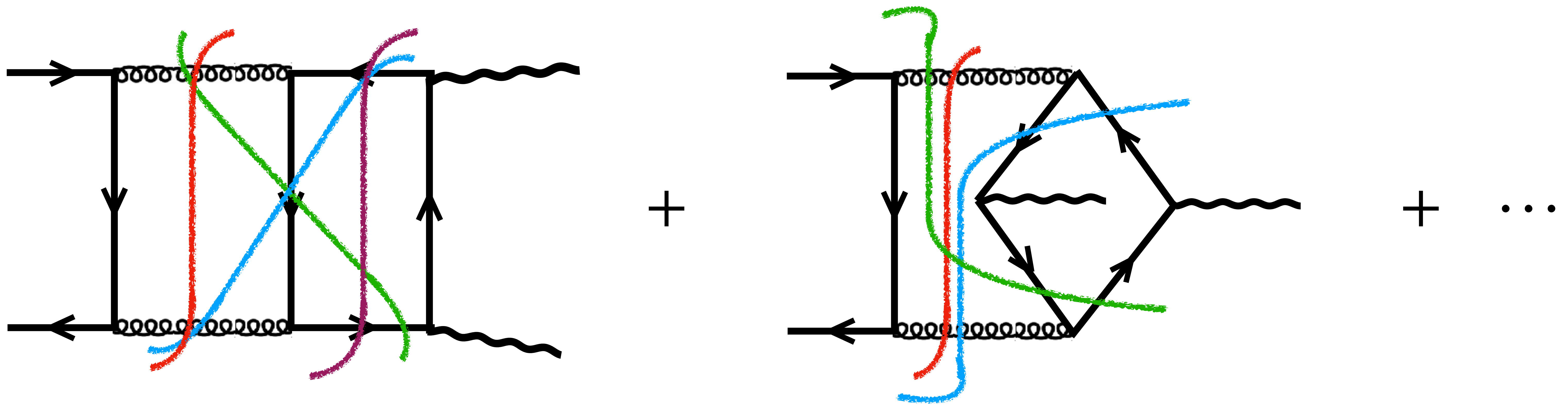
in agreement with
`FivePoint Amplitudes-cpp`
Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov [2305.17056]

$\times 3!$

new!

Towards the full NNLO result

Other fermion loop contributions



The IR (and UV) counterterms are already available.

[2403.13712, Anastasiou, Karlen, Sterman, Venkata]
[2008.12293, Anastasiou, Haindl, Sterman, Yang, Zeng]

Two-loop type threshold conditions can be solved numerically.

More challenging: longer evaluation time!

Summary and Outlook

Showed new result for NNLO virtual cross section for 3 massive vector boson production

Flexible and robust framework suited for automation, based on

- local infrared (IR) counterterms built in order to exploit universal IR factorisation
- analytic loop energy integration (CFF/LTD)
- threshold subtraction