UNIVERSITY OF SUSSEX

Subtraction of NNLO singularities with a universal analytic algorithm in massless QCD

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University of Sussex

Loop-the-Loop Workshop, 14/11/24

In collaboration with L. Magnea, G. Pelliccioli, A. Ratti, C. Signorile-Signorile, P. Torrielli, S. Uccirati

[JHEP12(2022)042, JHEP07(2023)140]

Introduction

- Vast increase in collider data accuracy as well as in the complexity of observables being probed, as the LHC moves into the high-luminosity phase

$$d\sigma_{AB} = \sum_{ab} \int_0^1 dx_a \int_0^1 dx_b\, f_{a|A}(x_a)\, f_{b|B}(x_b)\, d\hat{\sigma}_{ab}(x_a, x_b) \Big(1 + \mathcal{O}(\Delta^n_{QCD}/Q^n)\Big), \;\; n \geq 1$$

PDFs

Non-perturbative effects

Hard scattering in perturbation theory

$$d\hat{\sigma}_{ab} = d\hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right) d\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2 d\hat{\sigma}^{(2)}_{ab} + \ldots$$

LO *NLO* *NNLO*

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PDFs

Non-perturbative effects

Hard scattering in perturbation theory

- Enormous progress is rapidly setting **NNLO in the strong coupling** as the **standard**; state-of-the-art predictions for $2 \to 3$ collider processes

$$d\hat{\sigma}_{ab} = d\hat{\sigma}^{(0)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right) d\hat{\sigma}^{(1)}_{ab} + \left(\frac{\alpha_s}{2\pi}\right)^2 d\hat{\sigma}^{(2)}_{ab} + \ldots$$

LO *NLO* *NNLO*

$pp \to Wb\bar{b}$ (massless) *[H. B. Hartanto, et al. 2022]*

$pp \to t\bar{t}H$ *[S. Catani, et al. 2022]*

$pp \to Wb\bar{b}$ (massive) *[L. Buonocore, et al. 2022]*

$pp \to \gamma jj$ *[S. Badger, et al. 2023]*

$pp \to t\bar{t}W$ *[L. Buonocore, et al. 2023]*

$pp \to b\bar{b}Z$ *[J. Mazzitelli, et al. 2021]*

→ Multi-loop scattering amplitudes

→ **Cancellation of infrared singularities**

From the literature

- Automated subtractions at **NLO**

Solved problem

* **Frixione-Kunszt-Signer (FKS)** subtraction *[9512328]*
* **Catani-Seymour (CS)** dipole subtraction *[9602277]*
* Nagy-Soper subtraction *[0308127]*

- Proposed methods at **NNLO**

* Antenna subtraction *[Gerhmann, Glover, et al.]*
* ColorfulNNLO subtraction *[Del Duca, Trocsanyi, et al.]*
* Sector-improved residue subtraction *[Czakon, Mitov, et al.]*
* Nested soft-collinear subtraction *[Caola, Melnikov, et al.]*
* Projection to Born *[Cacciari, Salam, et al.]*
* qT subtraction *[Catani, Grazzini, et al.]*
* N-jettiness *[Boughezal, Petriello, et al.]*
* Local Unitarity *[Hirschi, et al.]*
* …

See talk by F. Devoto

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Why a new scheme at NNLO?

- Room for improvement in universality and efficiency, to overcome high computational complexity

- Proposed methods at **NNLO**
  - Antenna subtraction *[Gerhmann, Glover, et al.]*
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  - …

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Our proposal:

Local Analytic Sector Subtraction

[JHEP12(2022)042 , JHEP07(2023)140]

→ **Ambitious goal**: automation of QCD predictions in fixed NNLO event generator

Exploring the framework...

Local Analytic Sector Subtraction

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \frac{d\sigma_{\rm NLO}}{dX} + \frac{d\sigma_{\rm NNLO}}{dX} + \dots$$

σ = partonic cross section
X = generic IRC-safe observable

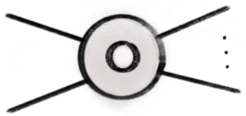
- Introduction to subtraction strategy at NLO FSR
- Subtraction formula at NNLO FSR
- Status & Perspective

Generalities at NLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

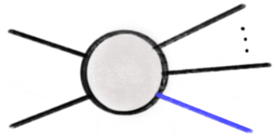
$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \, V \, \delta_{X_n}$$

Explicit ϵ poles



$$+ \int d\Phi_{n+1} \, R \, \delta_{X_{n+1}}$$

Singular in PS



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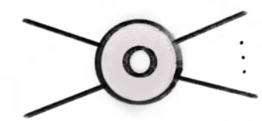

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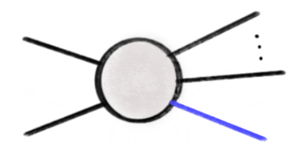

- ***Subtraction algorithm:*** introduce **local counterterm** K and **phase-space factorisation**

$$\int d\Phi_{n+1} \, K \, \delta_{X_n} = \int d\Phi_n \, I \, \delta_{X_n} \qquad d\Phi_{n+1} = d\Phi_n \, d\Phi_{\text{rad}}$$

- ***Result*****:** subtracted NLO cross section **numerically integrable** in $d = 4$ dimensions

$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \left(V + I \right) \delta_{X_n}$$

Finite in ϵ

$$+ \int d\Phi_{n+1} \left(R \, \delta_{X_{n+1}} - K \, \delta_{X_n} \right)$$

Integrable in PS

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

$$R = \sum_{i,j\neq i} R\mathcal{W}_{ij} \qquad \text{with} \qquad \sum_{i,j\neq i} \mathcal{W}_{ij} = 1$$

Minimal approach to disentangle overlapping singularities

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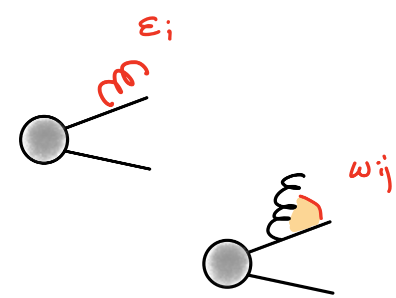
Minimal approach to disentangle overlapping singularities

- Single-unresolved configurations

$\mathbf{S}_i$ **soft** parton *i* $\qquad \mathcal{E}_i = \dfrac{s_{qi}}{s} \to 0$

$\mathbf{C}_{ij}$ **collinear** pair *ij* $\qquad w_{ij} = \dfrac{s\, s_{ij}}{s_{qi} s_{qj}} \to 0$



- Example of NLO ***sector function*** $\quad (s_{qi} = 2\, q_{\rm cm}\cdot k_i \,,\, s_{ij} = 2\, k_i \cdot k_j \,,\, s = q_{\rm cm}^2)$

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{a,b\neq a} \sigma_{ab}} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, \omega_{ij}}$$

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- ***Sum rules:*** limits of sector functions still form a unitary partition

$$\mathbf{S}_i \sum_{l\neq i} \mathcal{W}_{il} = 1$$

$$\mathbf{C}_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1$$

Key for integration

- Example of NLO *sector function* $\quad (s_{qi} = 2\, q_{\rm cm} \cdot k_i\,, s_{ij} = 2\, k_i \cdot k_j\,, s = q_{\rm cm}^2)$

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{a,b\neq a} \sigma_{ab}} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, \omega_{ij}}$$

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ [Frixione, Kunszt, Signer 9512328]

- Collect the relevant **IRC limits** for a given sector

$$R\mathcal{W}_{ij} - \Big[\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\mathbf{C}_{ij}\Big] R\mathcal{W}_{ij} \to \text{integrable}$$

Soft + Collinear - Overlap

Notation

$$\mathbf{L}\, R\,\mathcal{W}_{ij} = (\mathbf{L}\, R)\,(\mathbf{L}\,\mathcal{W}_{ij})$$

for $\mathbf{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$

Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*
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$$RW_{ij} - \Big[\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\mathbf{C}_{ij}\Big] RW_{ij} \to \text{integrable}$$

Soft + Collinear - Overlap

Notation

$$\mathbf{L}\, R\,\mathcal{W}_{ij} = (\mathbf{L}\, R)(\mathbf{L}\,\mathcal{W}_{ij})$$

for $\mathbf{L} = \mathbf{S}_i, \mathbf{C}_{ij}, \ldots$

✱ Products of known **splitting kernels** x **Born-level MEs**

Not yet parametrised

$$\mathbf{S}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_{\not i})$$

$$\mathbf{C}_{ij}\, R = \mathcal{N}_1 \frac{P_{ij}^{\mu\nu}}{s_{ij}} B_{\mu\nu}(\{k\}_{\not i \not j}, k_i + k_j)$$

missing proper
n-body on-shell kinematics!

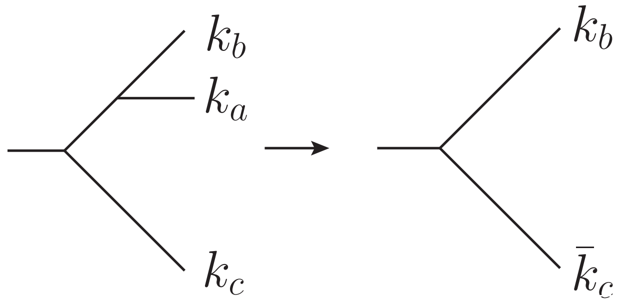
Strategy of the algorithm

- ***Unitary partition*** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$ *[Frixione, Kunszt, Signer 9512328]*

- Collect the relevant **IRC limits** for a given sector

- ***Catani-Seymour*** final-state **dipole mapping** *[Catani, Seymour 9605323]*

$$\{k_1, \ldots, k_{n+1}\} \to \{\bar{k}_1, \ldots, \bar{k}_n\}^{(abc)}$$



$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{y}{1-y} k_c$$
$$\bar{k}_c^{(abc)} = \frac{1}{1-y} k_c$$

$$y = \frac{s_{ab}}{s_{ab} + s_{ac} + s_{bc}}, \quad z = \frac{s_{ac}}{s_{ab} + s_{bc}}$$

* Phase-space factorisation and parametrisation

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)} = d\Phi_n(\{\bar{k}\}^{(abc)})\, d\Phi_{\rm rad}(\bar{s}_{bc}^{(abc)}; y, z, \phi)$$

$$\int d\Phi_{\rm rad}^{(abc)} \propto (\bar{s}_{bc}^{(abc)})^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \Big[y(1-y^2)z(1-z)\Big]^{-\epsilon}(1-y)$$

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- Promotion to **counterterms**: ***Improved limits***

Adapt momenta mapping to each kernel,
while tuning action on sector functions when necessary

$$K \;=\; \sum_{i,j\neq i} \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \right] R \mathcal{W}_{ij}$$

Notation

$$\overline{\mathbf{L}}\, R\, \mathcal{W}_{ij} \;=\; (\overline{\mathbf{L}}\, R)(\overline{\mathbf{L}}\, \mathcal{W}_{ij})$$

$$\overline{\mathbf{S}}_i\, R \;=\; \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik} s_{il}} \bar{B}_{kl}^{(ikl)}$$

$$\overline{\mathbf{C}}_{ij}\, R \;=\; \mathcal{N}_1 \frac{P_{ij}^{\mu\nu}}{s_{ij}} \bar{B}_{\mu\nu}^{(ijr)}$$

$$\overline{\mathbf{S}}_i\, \mathcal{W}_{ij} \;\equiv\; \mathbf{S}_i\, \mathcal{W}_{ij} \;=\; \frac{\frac{1}{w_{ij}}}{\sum_{l\neq i} \frac{1}{w_{il}}}$$

$$\overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} \;\equiv\; \frac{e_j w_{ir}}{e_i w_{ir} + e_j w_{jr}}$$

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- **Locality** of the cancellation ensured by ***consistency relations***

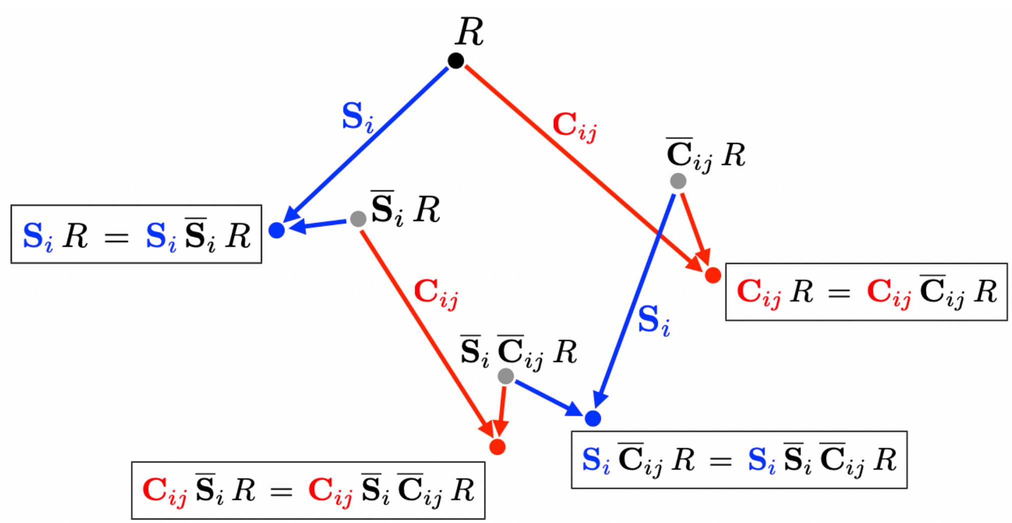
$$\mathbf{S}_i\, R = \mathbf{S}_i\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right)R$$

$$\mathbf{C}_{ij}\, R = \mathbf{C}_{ij}\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right)R$$

as well as

$$\mathbf{S}_i\, \mathcal{W}_{ij} = \mathbf{S}_i\, \overline{\mathbf{S}}_i\, \mathcal{W}_{ij}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ij} = \mathbf{C}_{ij}\, \overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij}$$



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$$\frac{d\sigma_{NLO}}{dX} = \int d\Phi_n \Big(V + I\Big)\delta_{X_n} + \int d\Phi_{n+1}\Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n}\Big)$$

☑ **Finite** in phase space (integrable in $d = 4$)

$$\mathbf{S}_i\, R = \mathbf{S}_i\Big(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\Big)R$$

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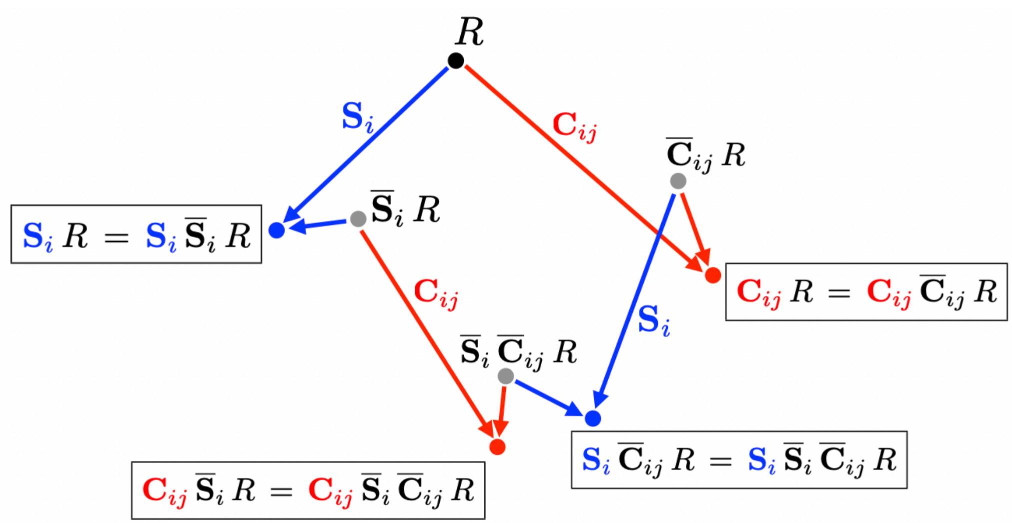

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- $\mathcal{W}_{ij}$ sum rules + mapping adaptation = **simple analytic** counterterm integration

$$K = \sum_{i,j\neq i}\left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right] R\mathcal{W}_{ij} = \sum_i\left[\overline{\mathbf{S}}_i\, R + \sum_{j>i}\overline{\mathbf{C}}_{ij}(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j)\, R\right]$$

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☑ **Free** from ϵ poles

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$$I^S \propto \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{1}{\bar{s}_{kl}^{(ikl)}}\int d\Phi_{\rm rad}(\bar{s}_{kl}^{(ikl)}; y, z, \phi)\frac{1-z}{yz}$$

$$= \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{kl}^{(ikl)})^{\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

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- [x] **Finite** in phase space (integrable in $d = 4$)
- [x] **Free** from ϵ poles

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$$I^S \propto \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{1}{\bar{s}_{kl}^{(ikl)}}\int d\Phi_{\rm rad}(\bar{s}_{kl}^{(ikl)}; y, z, \phi)\frac{1-z}{yz}$$
$$= \sum_{k,l}\bar{B}_{kl}^{(ikl)}\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{kl}^{(ikl)})^{\epsilon}}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

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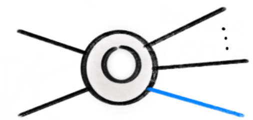
$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$

Up to $1/\epsilon^4$ poles

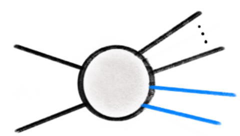
$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$



Up to $1/\epsilon^2$ poles

Singular in PS

$$+ \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$



Singular in PS

Generalities at NNLO

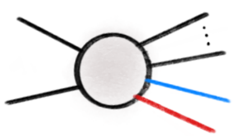
- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$
\begin{aligned}
\frac{d\sigma_{NNLO}}{dX} = & \int d\Phi_n \Big(VV \Big) \delta_{X_n} \\
& + \int d\Phi_{n+1} \Big[\Big(RV \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} \Big) \delta_{X_n} \Big] \\
& + \int d\Phi_{n+2} \Big[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big]
\end{aligned}
$$

- Introduce **local counterterms** and proper **phase-space factorisations**

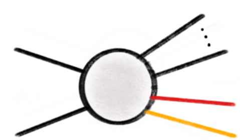
single-unresolved

$K^{(1)}$



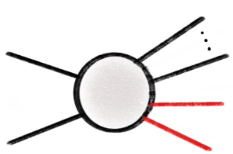
strongly-ordered double-unresolved

$K^{(12)}$



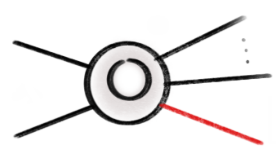
double-unresolved

$K^{(2)}$



single-unresolved

$K^{(\mathrm{RV})}$



Generalities at NNLO

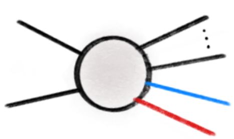
- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$
\begin{aligned}
\frac{d\sigma_{NNLO}}{dX} = & \int d\Phi_n \left(VV + I^{(2)} + I^{(RV)} \right) \delta_{X_n} \\
& + \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right] \\
& + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]
\end{aligned}
$$

- Introduce **local counterterms** and proper **phase-space factorisations**

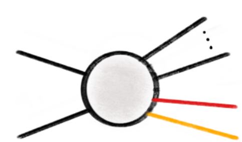
single-unresolved

$K^{(1)}$



strongly-ordered double-unresolved

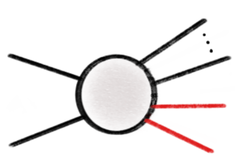
$K^{(12)}$



$$d\Phi_{n+2} = d\Phi_{n+1}\, d\Phi_{\mathrm{rad},1}$$

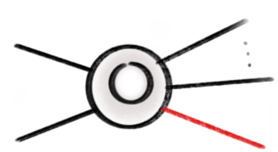
double-unresolved

$K^{(2)}$



single-unresolved

$K^{(\mathrm{RV})}$



$$d\Phi_{n+2} = d\Phi_n\, d\Phi_{\mathrm{rad},2}$$

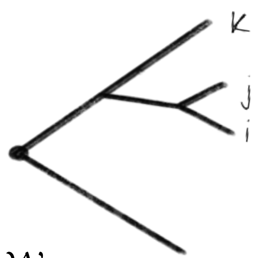
$$d\Phi_{n+1} = d\Phi_n\, d\Phi_{\mathrm{rad},1}$$

Subtracting RR singularities

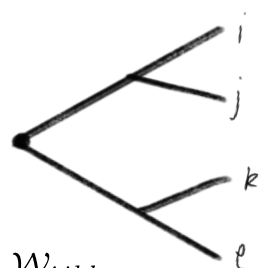
- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl} \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

- 3 ***topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

Subtracting RR singularities

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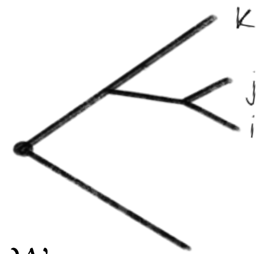
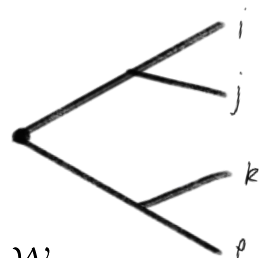


$\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

		single-unresolved limits		double-unresolved limits			
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

single-unresolved limits

double-unresolved limits

$\mathbf{S}_{ij}$ **double-soft** partons *i* and *j*

$\mathbf{C}_{ijk}$ **triple-collinear** partons *(i,j,k)*

$\mathbf{C}_{ijkl}$ **double-collinear** partons *(i,j)* and *(k,l)*

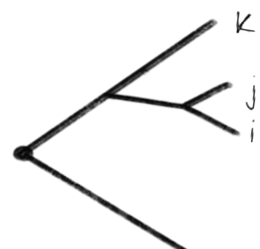
$\mathbf{SC}_{ijk}$ **soft** parton *i* and **collinear** partons *(j,k)*

Subtracting RR singularities

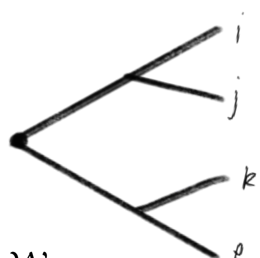
- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\, \mathcal{W}_{ijkl} \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

* 3 ***topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}, \mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

		single-unresolved limits		double-unresolved limits			
$\mathcal{W}_{ijjk}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ij}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	
$\mathcal{W}_{ijkj}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijk}$	$\mathbf{SC}_{ijk}$	$\mathbf{SC}_{kij}$
$\mathcal{W}_{ijkl}$	:	$\mathbf{S}_i$	$\mathbf{C}_{ij}$	$\mathbf{S}_{ik}$	$\mathbf{C}_{ijkl}$	$\mathbf{SC}_{ikl}$	$\mathbf{SC}_{kij}$

single-unresolved limits

double-unresolved limits

* ***Sum rules:*** limits of sector functions still form a unitary partition

Key for integration

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right) = 1$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\pi(ijk)}\left(\mathcal{W}_{abbc} + \mathcal{W}_{abcb}\right) = 1$$

...

$\mathbf{S}_{ij}$ **double-soft** partons i and j

$\mathbf{C}_{ijk}$ **triple-collinear** partons (i,j,k)

$\mathbf{C}_{ijkl}$ **double-collinear** partons (i,j) and (k,l)

$\mathbf{SC}_{ijk}$ **soft** parton i and **collinear** partons (j,k)

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

$$RR\,\mathcal{W}_\tau - \left[\mathbf{L}_{ij}^{(1)} + \mathbf{L}_\tau^{(2)} - \mathbf{L}_{ij}^{(1)}\mathbf{L}_\tau^{(2)}\right] RR\,\mathcal{W}_\tau \;\rightarrow\; \text{integrable}$$

single-unresolved

$$\mathbf{L}_{ij}^{(1)} = \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

double-unresolved $(\tau = ijjk, ijkj, ijkl)$

$$\mathbf{L}_{ijjk}^{(2)} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}_{ijkj}^{(2)} = \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$

$$\mathbf{L}_{ijkl}^{(2)} = \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

Notation

$$\mathbf{L}\,RR\,\mathcal{W}_{ijkl} = (\mathbf{L}\,RR)\,(\mathbf{L}\,\mathcal{W}_{ijkl})$$

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

$$RR\,\mathcal{W}_\tau - \left[\mathbf{L}_{ij}^{(1)} + \mathbf{L}_\tau^{(2)} - \mathbf{L}_{ij}^{(1)}\mathbf{L}_\tau^{(2)}\right] RR\,\mathcal{W}_\tau \;\to\; \text{integrable}$$

single-unresolved

$$\mathbf{L}_{ij}^{(1)} = \mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)$$

double-unresolved $(\tau = ijjk, ijkj, ijkl)$

$$\mathbf{L}_{ijjk}^{(2)} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}_{ijkj}^{(2)} = \mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})$$
$$\mathbf{L}_{ijkl}^{(2)} = \mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})$$

Notation

$$\mathbf{L}\,RR\,\mathcal{W}_{ijkl} = (\mathbf{L}\,RR)(\mathbf{L}\,\mathcal{W}_{ijkl})$$

* Products of known **splitting kernels** x **lower-multiplicities MEs**

[Catani, Grazzini 9810389, 9908523]

$$\mathbf{S}_{ik}\,RR \supset \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,k\\ d\neq i,k,c}}\left\{\frac{s_{cd}}{s_{ic}s_{id}}\left[\sum_{\substack{e\neq i,k,c,d\\ f\neq i,k,c,d,e}}\frac{s_{ef}}{s_{ke}s_{kf}}B_{cdef}(\{k\}_{\not i\not k}) + 2\,\frac{s_{cd}}{s_{kc}s_{kd}}B_{cdcd}(\{k\}_{\not i\not k})\right]\right\}$$

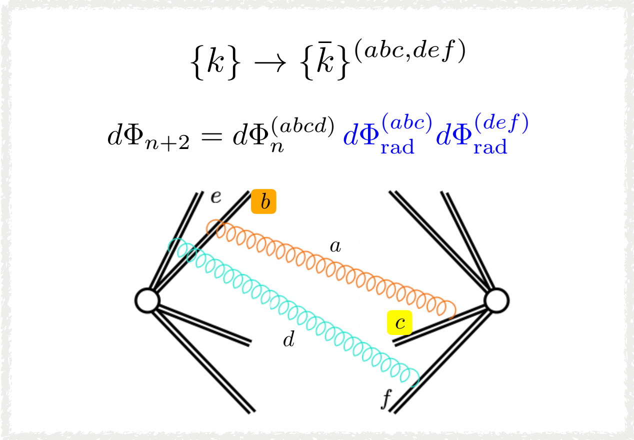
Missing momentum conservation out of the double singular region!

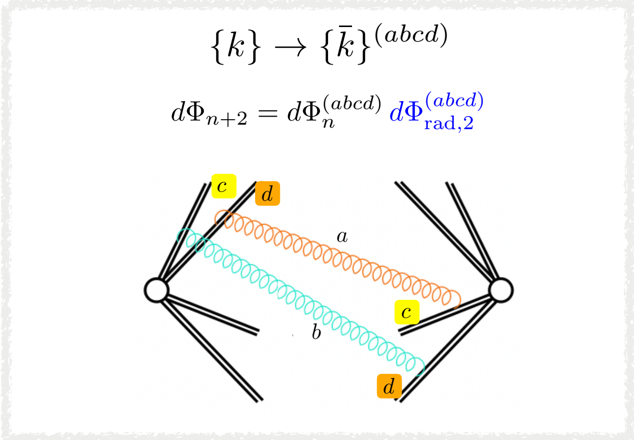
Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$
- Collect the limited ***relevant*** **IRC limits** reproducing all singularities
- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics

  * Minimal set of involved momenta
  * Still clear factorisation of radiative d.o.f.
  * Smart and adaptive parametrisation simplifies kernel expressions

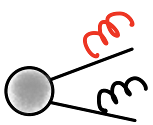
$$\overline{\mathbf{S}}_{ik}\, RR \supset \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c\neq i,k \\ d\neq i,k,c}} \left\{ \frac{s_{cd}}{s_{ic} s_{id}} \left[\sum_{\substack{e\neq i,k,c,d \\ f\neq i,k,c,d,e}} \frac{s_{ef}}{s_{ke} s_{kf}} \bar{B}^{(icd,kef)}_{cdef} + 2 \frac{s_{cd}}{s_{kc} s_{kd}} \bar{B}^{(icd,kcd)}_{cdcd} \right] \right\}$$





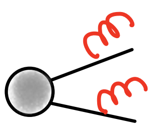
Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$
- Collect the limited ***relevant*** **IRC limits** reproducing all singularities
- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics
- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel, while tuning action on sector functions when necessary



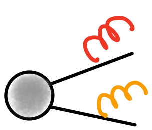
$$K^{(\mathbf{1})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ij}^{(\mathbf{1})} \, RR \, \mathcal{W}_{ijkl}$$

single-unresolved limits



$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}_{ijkl}^{(\mathbf{2})} \, RR \, \mathcal{W}_{ijkl}$$

uniform double-unresolved limits



$$K^{(\mathbf{12})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq ,k}} \overline{\mathrm{L}}_{ij}^{(\mathbf{1})} \, \overline{\mathrm{L}}_{ijkl}^{(\mathbf{2})} \, RR \, \mathcal{W}_{ijkl}$$

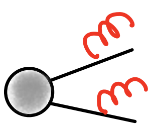
strongly-ordered double-unresolved limits

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$
- Collect the limited ***relevant*** **IRC limits** reproducing all singularities
- **Nested *Catani-Seymour* mappings** from $(n+2) \to n$ kinematics
- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel, while tuning action on sector functions when necessary

$$K^{(1)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij} \, RR \, \mathcal{W}_{ijkl}$$

single-unresolved limits



$$K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl} \, RR \, \mathcal{W}_{ijkl}$$

uniform double-unresolved limits

$$\begin{aligned}
&= \Big\{ \sum_{i,k>i} \overline{\mathbf{S}}_{ik} + \sum_{i,j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \Big(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\Big) \\
&\quad + \sum_{i,j>i} \sum_{\substack{k\neq j \\ k>i}} \sum_{\substack{l\neq j \\ l>k}} \overline{\mathbf{C}}_{ijkl} \Big[1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl} \\
&\qquad - \overline{\mathbf{SC}}_{ikl}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il}) - \overline{\mathbf{SC}}_{jkl}(1 - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl}) \\
&\qquad - \overline{\mathbf{SC}}_{kij}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) - \overline{\mathbf{SC}}_{lij}(1 - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}) \Big] \\
&\quad + \sum_{i,j>i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk}(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik})(1 - \overline{\mathbf{C}}_{ijk}) \Big\} RR
\end{aligned}$$

Collection of universal kernels!

Subtracting RR singularities

- Smooth **unitary partition** of double-unresolved phase space Φ_{n+2} into ***sectors*** $\mathcal{W}_{ijkl}$

- Collect the limited ***relevant*** **IRC limits** reproducing all singularities

- **Nested** ***Catani-Seymour*** **mappings** from $(n+2) \to n$ kinematics

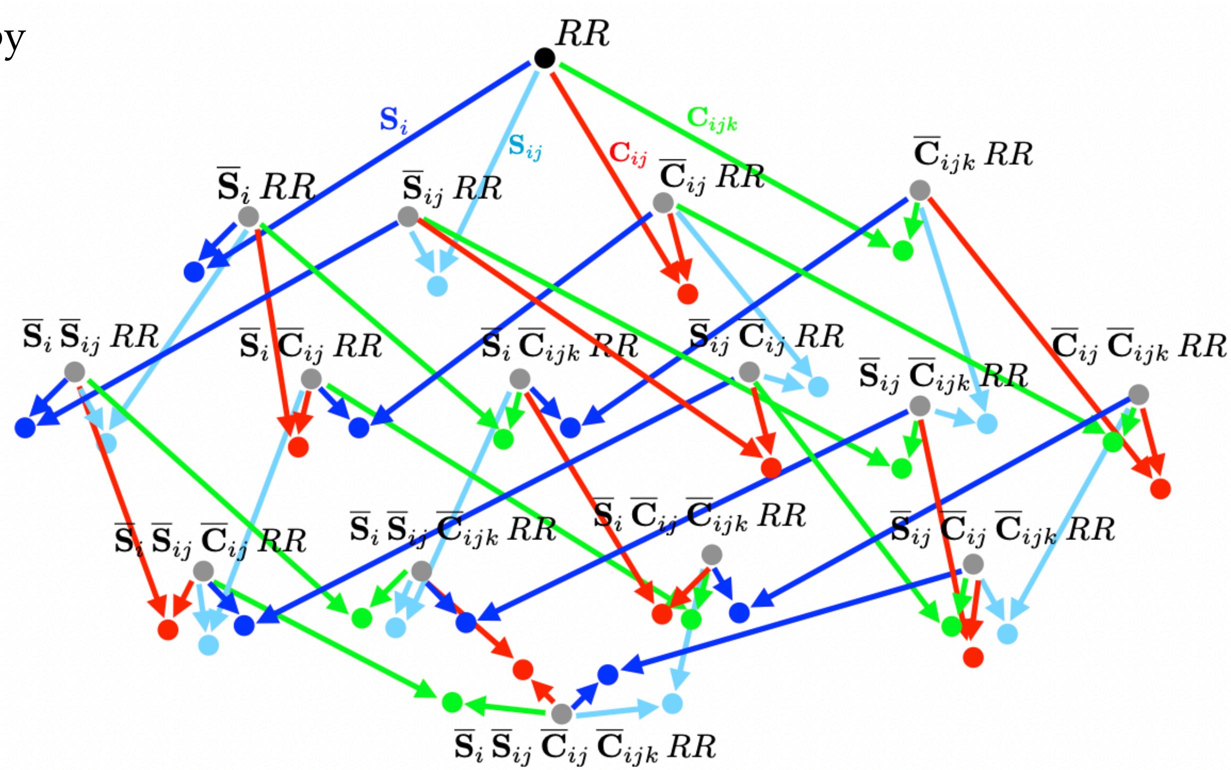
- Promotion to **counterterms**: ***Improved limits*** adapting momenta mapping to each kernel,
  while tuning action on sector functions when necessary

- **Locality** of the cancellation ensured by
  consistency relations

verified sector by sector

Selection of displayed limits

$\mathbf{S}_i,\ \mathbf{C}_{ij},\ \mathbf{S}_{ij},\ \mathbf{C}_{ijk}$



Subtracting RR singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n} + \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}} + \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]$$

☑ **Finiteness** of double-real correction (integrable in $d = 4$)

Subtracting RR singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \Big) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} + I^{(12)} \Big) \delta_{X_n} \Big]$$

$$+ \int d\Phi_{n+2} \Big[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n} \Big]$$

☑ **Finiteness** of double-real correction (integrable in $d = 4$)

- $\mathcal{W}_{ijkl}$ sum rules + mapping adaptation → **analytic integrations** by means of ***standard techniques***

[Magnea, et al 2010.14493]

$$I^{(\mathbf{1})} = \int d\Phi_{\mathrm{rad}} \, K^{(\mathbf{1})} \qquad I^{(\mathbf{12})} = \int d\Phi_{\mathrm{rad}} \, K^{(\mathbf{12})} \qquad I^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},2} \, K^{(\mathbf{2})}$$

NNLO complexity

- Logarithmic (trivial) dependence on Mandelstam invariants
- **Note**: no approximations in local terms!

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right]$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right] ✓$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$RV + I^{(1)} \to$ finite in ϵ — Still singular in PS ✗

$I^{(1)} - I^{(12)} \to$ integrable — Contains poles in ϵ ✗

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad\qquad \right)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\Big[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\Big]$$
$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\Big]$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$RV + I^{(1)} \to$ finite in ϵ

$I^{(1)} - I^{(12)} \to$ integrable

Analytically checked finiteness of RV contribution!

$RV - K^{(\mathrm{RV})} \to$ integrable

$K^{(\mathrm{RV})} + I^{(12)} \to$ finite in ϵ

- Apply NLO strategy to define the **real-virtual local term** *[Bern, at al 9903516]*

$$K^{(\mathrm{RV})} = \sum_{i,j\neq i}\Big[\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i)\right) RV\,\mathcal{W}_{ij} + \Delta_{ij}\Big]$$

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(RV)} \right) \delta_{X_n}$$
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(RV)} + I^{(12)} \right) \delta_{X_n} \right] ✓$$
$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right] ✓$$

- More intricate cancellation pattern involving both ***poles*** and ***phase-space singularities***

$$RV + I^{(1)} \to \text{finite in } \epsilon$$
$$I^{(1)} - I^{(12)} \to \text{integrable}$$

☑ **Analytically** checked finiteness of RV contribution!

$$RV - K^{(RV)} \to \text{integrable}$$
$$K^{(RV)} + I^{(12)} \to \text{finite in } \epsilon$$

- Apply NLO strategy to define the **real-virtual local term** *[Bern, at al 9903516]*

- $K^{(RV)} = \sum_{i,j\neq i} \left[\left(\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1 - \bar{\mathbf{S}}_i) \right) RV\, \mathcal{W}_{ij} + \Delta_{ij} \right]$
- $I^{(RV)} = \int d\Phi_{\rm rad}\, K^{(RV)}$

Subtracting RV singularities

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} + I^{(\mathrm{RV})}\Big)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\Big[\Big(RV + I^{(1)}\Big)\delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} + I^{(12)}\Big)\delta_{X_n}\Big]$$
$$+ \int d\Phi_{n+2}\Big[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)}\Big)\delta_{X_n}\Big]$$

Removing VV poles

- Extract the ϵ poles of the ***double-virtual*** correction and sum counterterm integrations

☑ **Analytically** verified for an **arbitrary** number of final-state partons

$$VV + I^{(2)} + I^{(\mathrm{RV})} \rightarrow \text{free from } \epsilon \text{ poles}$$

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$
$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$
$$+ \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(1)}\delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)}\right)\delta_{X_n}\right]$$

- Verified for an **arbitrary number** of final-state coloured particles (as well as of arbitrary massive/massless colourless ones)
- **No approximations** introduced in local and integrated terms
- **Analytic finite remainder** retaining mostly ***simple logarithmic dependence*** on kinematic invariants

NNLO subtraction formula

Massless QCD final-state radiation

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$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

Analytic and compact!

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \left[I^{(0)} + \sum_j I^{(1)}_j \mathbf{L}_{jr} + \sum_j I^{(2)}_j \mathbf{L}^2_{jr} + \frac{1}{2}\sum_{j,l\neq j} \gamma^{\mathrm{hc}}_j \gamma^{\mathrm{hc}}_l \mathbf{L}_{jr'}\mathbf{L}_{lr'} \right] \mathbf{B} \\
& + \sum_j \left[I^{(0)}_{jr} + I^{(1)}_{jr}\mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma^{\mathrm{hc}}_j \left(2 - \mathbf{L}_{cr}\right) \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \left[I^{(0)}_{cd} + I^{(1)}_{cd}\mathbf{L}_{cd} + \frac{\beta_0}{12}\mathbf{L}^2_{cd} + (4 - \mathbf{L}_{cd}) \sum_j \gamma^{\mathrm{hc}}_j \mathbf{L}_{jr} \right] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3)\,\mathbf{L}_{cd} \right] \mathbf{B}_{cdcd} \\
& + (1-\zeta_2) \sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\mathbf{L}_{ed}\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}} \mathbf{L}_{cd}\mathbf{L}_{ef}\left[1 - \frac{1}{2}\mathbf{L}_{cd}\left(1 - \frac{1}{8}\mathbf{L}_{ef}\right)\right] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \left[\ln\frac{s_{ce}}{s_{de}}\mathbf{L}^2_{cd} + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}}\right) \right] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \left[\Sigma_\phi - \sum_j \gamma^{\mathrm{hc}}_j \mathbf{L}_{jr} \right] \mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd}\left(2 - \frac{1}{2}\mathbf{L}_{cd}\right) \mathbf{V}^{\mathrm{fin}}_{cd} \Bigg\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$
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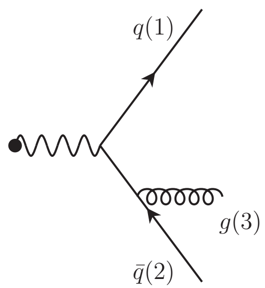
- Verified for an **arbitrary number** of final-state coloured particles (as well as of arbitrary massive/massless colourless ones)
- **No approximations** introduced in local and integrated terms
- **Analytic finite remainder** retaining mostly ***simple logarithmic dependence*** on kinematic invariants
- Ready to be implemented in a numerical framework equipped with the relevant matrix elements

Symmetrised sector functions

$\mathcal{Z}_{ij} \equiv \mathcal{W}_{ij} + \mathcal{W}_{ji}$

$e^+e^- \to jj$ at NLO

Real configuration

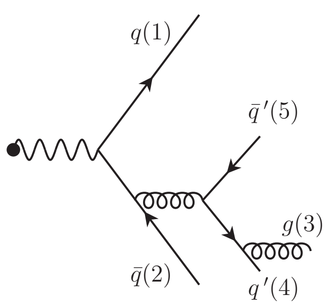


	# limits per sector	# sectors	# limits
$\mathcal{W}_{13}, \mathcal{W}_{23}$	1	4	8
$\mathcal{W}_{31}, \mathcal{W}_{32}$	3		
$\mathcal{Z}_{13}, \mathcal{Z}_{23}$	2	2	4

$e^+e^- \to jjj$ at NNLO

Double-real configuration for selected channel

$e^+e^- \to q\bar{q}q'\bar{q}'g$



	# limits per sector	# sectors	# limits
$\mathcal{W}_{3445}, \mathcal{W}_{3554}, \mathcal{W}_{3454}, \mathcal{W}_{3545}, \mathcal{W}_{4535}, \mathcal{W}_{5434}$	11	12	88
$\mathcal{W}_{4335}, \mathcal{W}_{4553}, \mathcal{W}_{5334}, \mathcal{W}_{5443}$	3		
$\mathcal{W}_{4353}, \mathcal{W}_{5343}$	5		
$\mathcal{Z}_{345}$	15	1	15

$$\mathcal{Z}_{ijk} = \mathcal{W}_{ijjk} + \mathcal{W}_{ikkj} + \mathcal{W}_{jiik} + \mathcal{W}_{jkki} + \mathcal{W}_{kiij} + \mathcal{W}_{kjji} + \mathcal{W}_{ijkj} + \mathcal{W}_{ikjk} + \mathcal{W}_{jiki} + \mathcal{W}_{jkik} + \mathcal{W}_{kiji} + \mathcal{W}_{kjij}$$

Exploring the framework...

Local Analytic Sector Subtraction

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\mathrm{LO}}}{dX} + \frac{d\sigma_{\mathrm{NLO}}}{dX} + \frac{d\sigma_{\mathrm{NNLO}}}{dX} + \dots$$

σ = partonic cross section
X = generic IRC-safe observable

- Introduction to subtraction strategy at NLO FSR
- Subtraction formula at NNLO FSR
- Status & Perspective

Status & Perspective

☑ General analytic **subtraction formula** for **massless FSR and ISR at NLO**

☑ **Numerical validation** of the NLO subtraction formula in MadNkLO

[GB, Torrielli, Uccirati, Zaro 2209.09123]

☑ General analytic **subtraction formula** for **massless FSR at NNLO**

Next steps…

- **Numerical implementation** of the NNLO FSR formula
  - Improved-MadNkLO *[GB, Limatola, Torrielli, Uccirati to appear soon]*
  - e+ e- > 3 jets *[Kardos, Bevilacqua, Chargeishvili, Moch, Trocsanyi 2407.02194, 2407.02195]*
- Extension to **initial-state** coloured particles for LHC applications
  (expected integrals of complexity similar to massless FSR)
- Treatment of the massive case: less singular limits, but more involved integrals

[GB, Limatola, Torrielli, Uccirati "Massive @ NLO" to appear soon]

Status & Perspective

- ☑ General analytic **subtraction formula** for **massless FSR and ISR at NLO**
- ☑ **Numerical validation** of the NLO subtraction formula in MadNkLO
  [GB, Torrielli, Uccirati, Zaro 2209.09123]
- ☑ General analytic **subtraction formula** for **massless FSR at NNLO**

Next steps…

Thanks for your attention!

- **Numerical implementation** of the NNLO FSR formula
  - Improved-MadNkLO *[GB, Limatola, Torrielli, Uccirati to appear soon]*
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Backup slides

NNLO subtraction formula

Massless QCD final-state radiation

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Analytic and compact!

$$VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2\Bigg\{\left[I^{(0)} + \sum_j I_j^{(1)}\mathbf{L}_{jr} + \sum_j I_j^{(2)}\mathbf{L}_{jr}^2 + \ldots\right.$$

$$+ \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)}\mathbf{L}_{jr}\right]\mathbf{B}_{jr} - 2\,(1-\zeta_2 \ldots$$

$$+ \sum_{c,d\neq c}\mathbf{L}_{cd}\left[I_{cd}^{(0)} + I_{cd}^{(1)}\mathbf{L}_{cd} + \frac{\beta_0}{12}\mathbf{L}_{cd}^2 + \ldots\right.$$

$$+ \sum_{c,d\neq c}\left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1 \ldots\right.$$

$$+ (1-\zeta_2)\sum_{\substack{c,d\neq c\\ e\neq d}}\mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}}\mathbf{L} \ldots$$

$$+ \pi\sum_{\substack{c,d\neq c\\ e\neq c,d}}\left[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}_{cd}^2 + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathbf{L} \ldots\right.$$

$$+ \left(\frac{\alpha_s}{2\pi}\right)\Bigg\{\left[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\right]\mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c}\mathbf{L}_{cd} \ldots$$

$$I^{(0)} = N_q^2 C_F^2\left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4\right] + N_g N_q C_F\left[C_A\left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4\right) + \beta_0\left(\frac{77}{12} - \frac{53}{12}\zeta_2\right)\right]$$
$$+ N_g^2\left[C_A^2\left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4\right) + \beta_0^2\left(\frac{73}{72} - \frac{1}{8}\zeta_2\right) + C_A\beta_0\left(-\frac{1}{9} - \frac{11}{3}\zeta_2\right)\right]$$
$$+ N_q C_F\left[C_F\left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4\right) + C_A\left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4\right)\right.$$
$$\left. + \beta_0\left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$
$$+ N_g\left[C_F C_A\left(-\frac{737}{48} + 11\zeta_3\right) + C_F\beta_0\left(\frac{67}{16} - 3\zeta_3\right) + \beta_0^2\left(\frac{73}{72} - \frac{3}{8}\zeta_2\right)\right.$$
$$\left. + C_A^2\left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4\right) + C_A\beta_0\left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3\right)\right]$$

$$I_j^{(1)} = \delta_{f_a\{q,\bar{q}\}}C_F\left[N_q C_F\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A\left(\frac{1}{3} - \frac{7}{4}\zeta_2\right) + \frac{2}{3}N_g\beta_0\right.$$
$$\left. + C_F\left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3\right) + C_A\left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3\right) + \beta_0\left(\frac{1}{24} + \zeta_2\right)\right]$$
$$+ \delta_{f_a g}\left[N_q C_F C_A\,(10 - 7\zeta_2) - N_q C_F\beta_0\left(\frac{5}{2} - \frac{7}{4}\zeta_2\right) + N_g C_A^2\left(\frac{4}{3} - 7\zeta_2\right) + N_g C_A\beta_0\left(\frac{7}{3} + \frac{7}{4}\zeta_2\right)\right.$$
$$\left. - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4}C_F C_A - \frac{3}{4}C_F\beta_0 + C_A^2\left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3\right) - C_A\beta_0\left(\frac{2}{3} - \frac{5}{2}\zeta_2\right)\right]$$

$$I_j^{(2)} = \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15)C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\,C_{f_j}^2$$

$$I_{jr}^{(0)} = (-1 + 3\zeta_2 - 2\zeta_3)C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3)C_{f_j} + (5 + 2\zeta_3)\gamma_j$$

$$I_{jr}^{(1)} = (1 - \zeta_2)C_A + \frac{1}{2}(4 + 7\zeta_2)C_{f_j} - (2 + \zeta_2)\gamma_j$$

$$I_{cd}^{(0)} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3\right)C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\,C_{f_d}$$

$$I_{cd}^{(1)} = -\left(\frac{1}{3} - \frac{1}{2}\zeta_2\right)C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi$$

Consistency relations

Primary

Limits selected by the *full* sector functions

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sum\limits_{a,b\neq a} \sum\limits_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd}}$$

$$RR\,\mathcal{W}_{ijjk} : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}$$
$$RR\,\mathcal{W}_{ijkj} : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{SC}_{kij}$$
$$RR\,\mathcal{W}_{ijkl} : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{SC}_{kij}$$

$$\sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^{\alpha}} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}$$

$[\alpha > 0]$

Limits damped by the ***full*** sector functions, but not by *improved limits* of sector functions

Secondary

Generated by the *specific definition* chosen for sector functions

$$\mathbf{L}_{\rm S} \left[RR\,\mathcal{W}_{\tau} - \left(\overline{\mathbf{L}}_{ij}^{(\mathbf{1})} + \overline{\mathbf{L}}_{\tau}^{(\mathbf{2})} - \overline{\mathbf{L}}_{\tau}^{(\mathbf{12})} \right) RR\,\mathcal{W}_{\tau} \right] \quad \to \quad \text{integrable}$$

Non singular — *Still potentially singular in PS*

$(\tau = ijjk, ijkj, ijkl)$

Auxiliary

Generated by *spurious singularities* in collinear kernels

$$P^{\mu\nu}_{ij(r)}(z_i) \supset \# \frac{z_j}{z_i} \sim \# \frac{s_{jr}}{s_{ir}}$$

$$\mathbf{C}_{ir} \left[\overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i\right) \right] R\mathcal{W}_{ij}$$

Singular in PS

$$\overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} = \frac{e_j}{e_i + e_j}$$

Integrable *in PS*

$$\overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} \equiv \frac{e_j w_{ir}}{e_i w_{ir} + e_j w_{jr}}$$