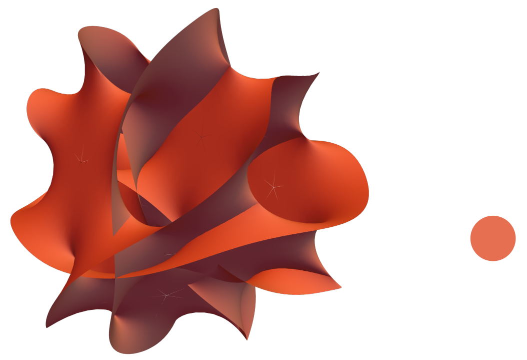
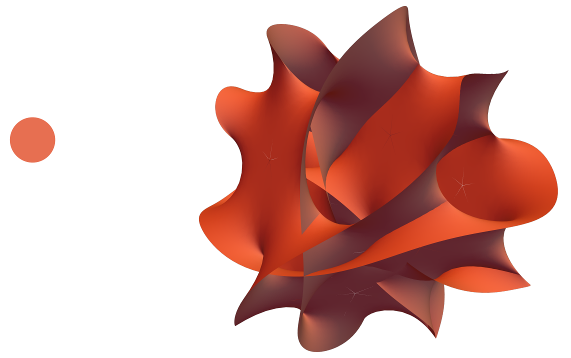
# Tackling Apparent Singularities in Calabi—Yau Feynman Integrals

**Sebastian Pögel, Paul Scherrer Institute**
**Loop-the-loop Conference**
**14th November 2024**



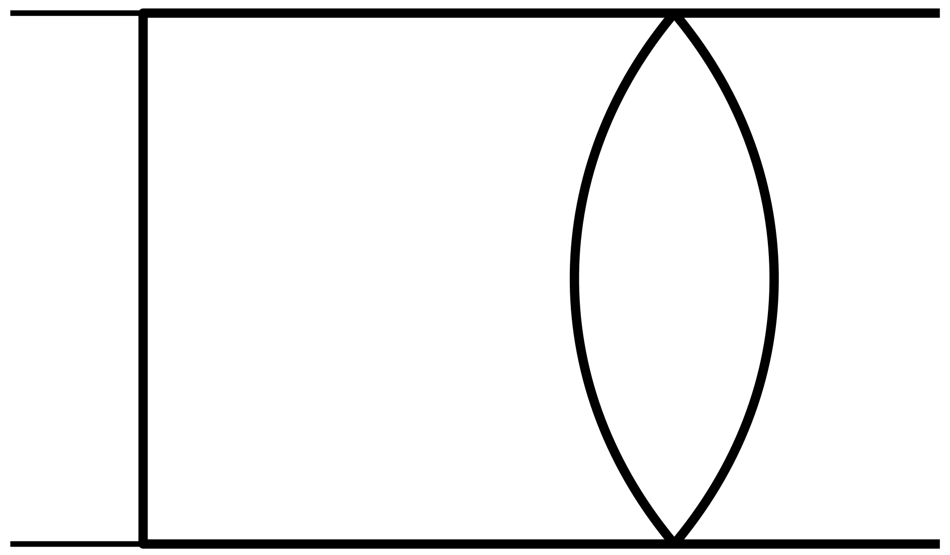
Based on work in collaboration with
Hjalte Frellesvig, Roger Morales, Stefan Weinzierl, Matthias Wilhelm
`24xx.xxxxx`



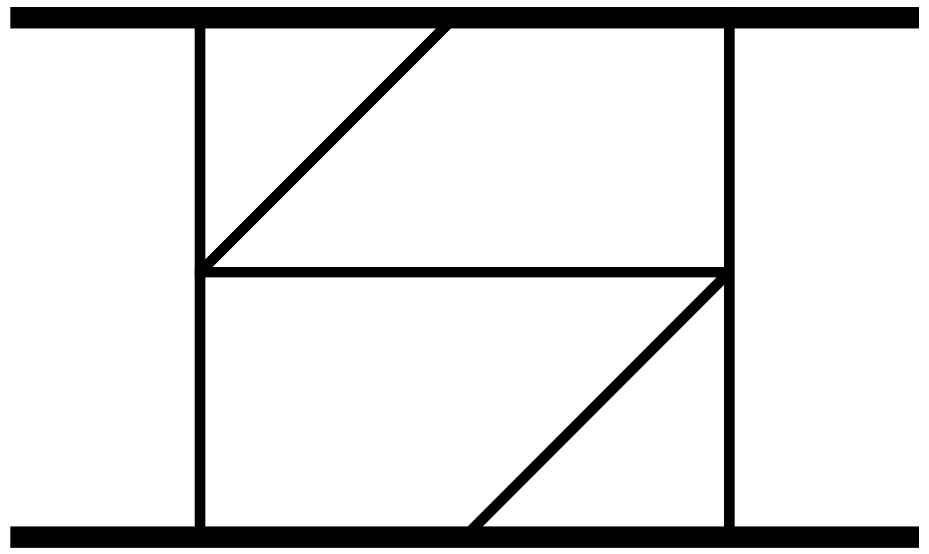
# Feynman Integrals

## A zoo of geometries

QCD



Gravity



Integrals associated to geometries

Determines suitable function space

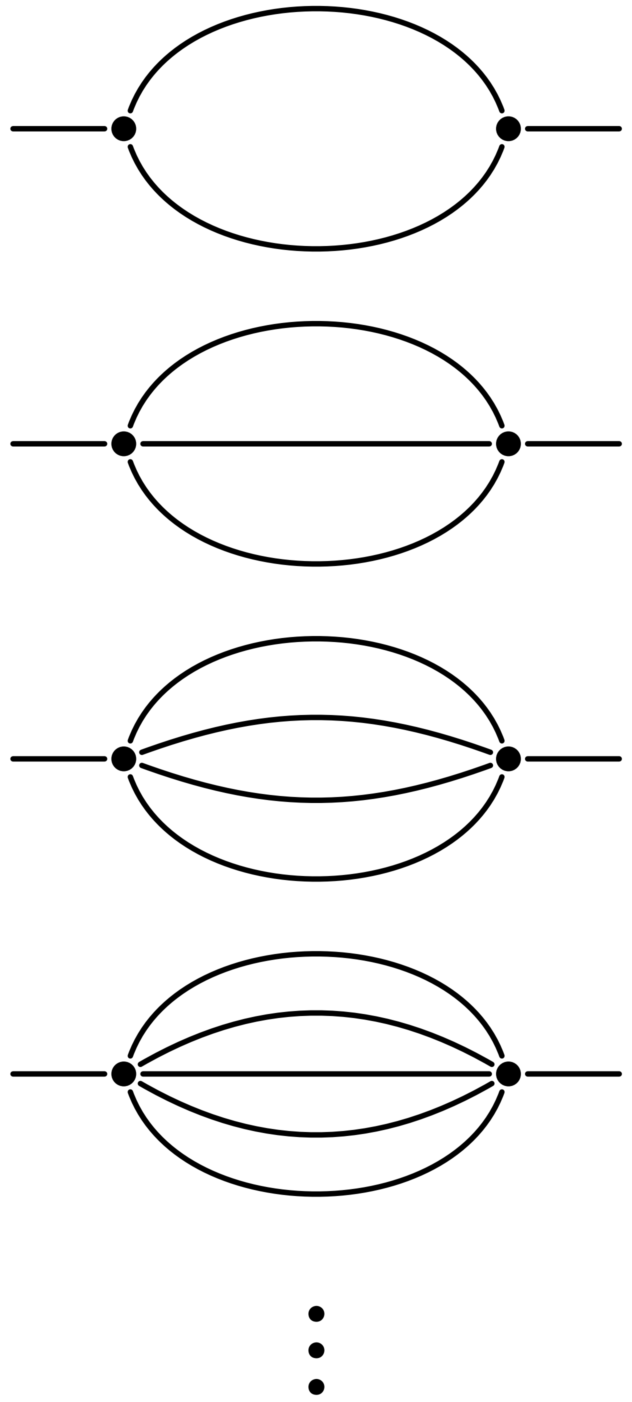
**Beyond elliptics**

| Sphere | Elliptic curves | … | Calabi—Yaus, hyperelliptic curves, … |
|---|---|---|---|
| ↓ | ↓ | | ↓ |
| MPLs | Elliptic Integrals, modular forms, EMPLs | | heMPLs, Siegel modular forms, …? |

[Franziska's talk]

# Bananas: A Calabi–Yau Prototype

Calabi–Yau...



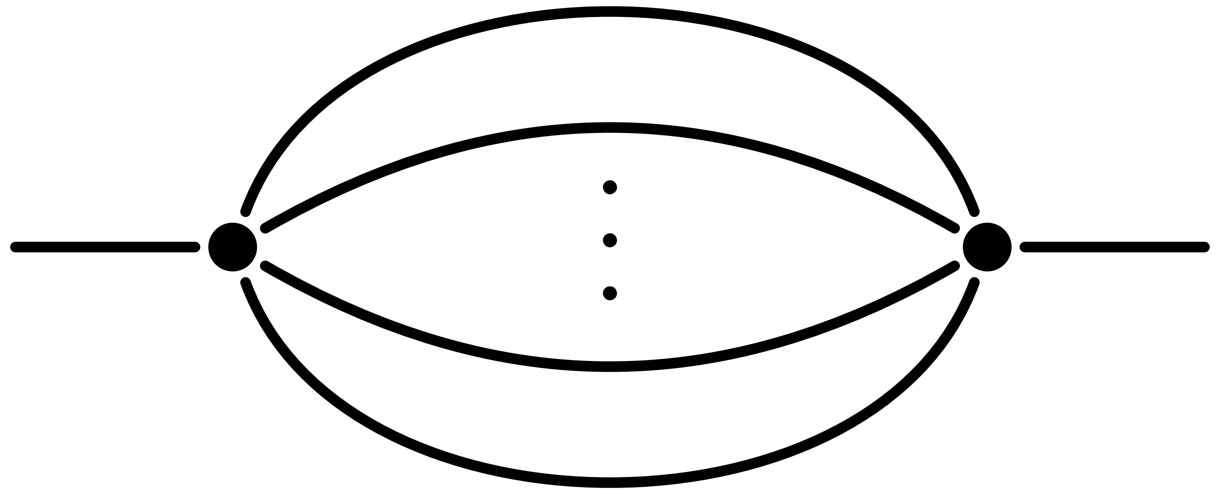
0-fold

1-fold = elliptic curve
"Sunrise"

2-fold = K3 surface

3-fold

⋮



**$\ell$-loop Banana integral**

$\hat{=}$

$(\ell - 1)$ -fold Calabi–Yau manifold

Simplification: Equal-mass → single scale

Kinematic variable $x = \dfrac{p^2}{m^2}$

# $\varepsilon$-factorized Differential Equations for Calabi—Yau Integrals

$$\mathrm{d}I = \varepsilon A I$$

[Also see Sara's talk]

# Just two ingredients

One Seed Integral $I$
via Picard—Fuchs
operator

➕

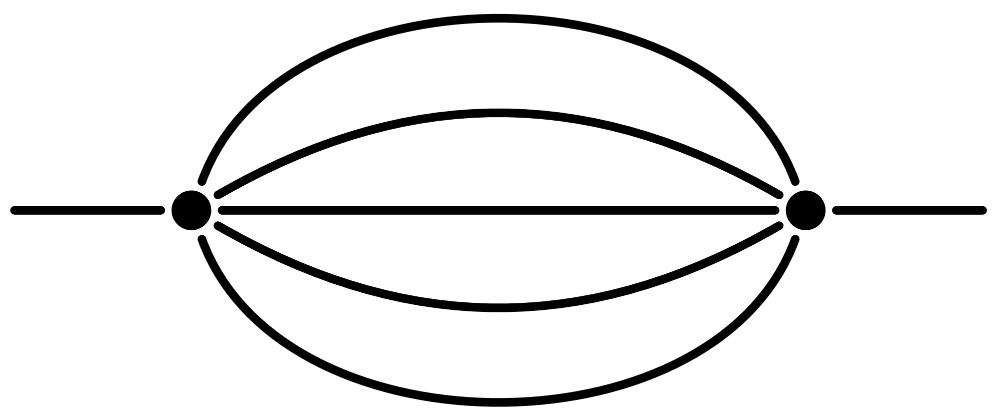
Ansatz for
differential equation $A$

Differential operator
annihilating integral

Idea: Fix Ansatz by eliminating non-$\varepsilon$-factorizing terms

# A CY 3-fold example

## Four-loop banana integral in $d = 2 - 2\varepsilon$



Simplest non-trivial Calabi—Yau integral
Associated to Hulek—Verrill Calabi—Yau 3-fold

**The seed integral:** $I = I_{11111}$

Picard—Fuchs operator
(homogenous ~ maximal cut)

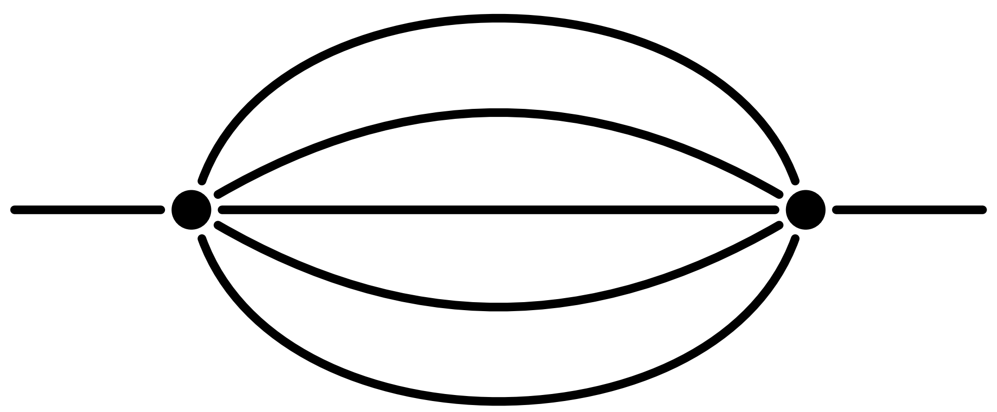
$$I^{(4)}(x)+\frac{2\left(5x^3\epsilon+5x^3-105x^2\epsilon-140x^2+259x\epsilon+777x+225\epsilon-450\right)I'''(x)}{(x-25)(x-9)(x-1)x}+\frac{\left(35x^3\epsilon^2+60x^3\epsilon+25x^3-343x^2\epsilon^2-945x^2\epsilon-518x^2-363x\epsilon^2+1554x\epsilon+1839x-225\epsilon^2+675\epsilon-450\right)I''(x)}{(x-25)(x-9)(x-1)x^2}+\frac{(2\epsilon+1)\left(25x^2\epsilon^2+40x^2\epsilon+15x^2-42x\epsilon^2-322x\epsilon-196x-207\epsilon^2-78\epsilon+285\right)I'(x)}{(x-25)(x-9)(x-1)x^2}+\frac{(2\epsilon+1)(3\epsilon+1)(4\epsilon+1)(x\epsilon+x+3\epsilon-5)}{(x-25)(x-9)(x-1)x^2}$$

**Ansatz:**

$$M_1 = \frac{1}{\varpi} I_{11111}$$
$$M_2 = \frac{1}{\varepsilon} J \frac{\mathrm{d}M_1}{\mathrm{d}x} + F_{11} M_1$$
$$M_3 = \frac{1}{\varepsilon} \frac{J}{K_1} \frac{\mathrm{d}M_2}{\mathrm{d}x} + F_{21} M_1 + F_{22} M_2$$
$$M_4 = \frac{1}{\varepsilon} J \frac{\mathrm{d}M_3}{\mathrm{d}x} + F_{31} M_1 + F_{32} M_2 + F_{33} M_3$$

# A CY 3-fold example

## Four-loop banana integral in $d = 2 - 2\varepsilon$



**Ansatz + Integral:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & 1 & 0 & 0 \\ F_{21} & F_{22} & K_1 & 0 \\ F_{31} & F_{32} & F_{33} & 1 \\ * & * & * & * \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

**Matching:** Requiring $\varepsilon^{<0}$ to vanish

→Constraints on $F_{ij}(x), K_1(x), J(x), \varpi(x)$

**Solving constraints:** Constraints consistently solvable

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \varepsilon \begin{pmatrix} f_2 & 1 & 0 & 0 \\ f_4 & f_2' & K_1 & 0 \\ f_6 & f_4' & f_2' & 1 \\ f_8 & f_6 & f_4 & f_2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix}$$

$\varpi = \omega_1 \quad J(x) = \frac{\mathrm{d}\tau}{\mathrm{d}x} \quad \tau = \frac{\omega_2}{\omega_1}$

$\omega_i$: periods of Calabi—Yau

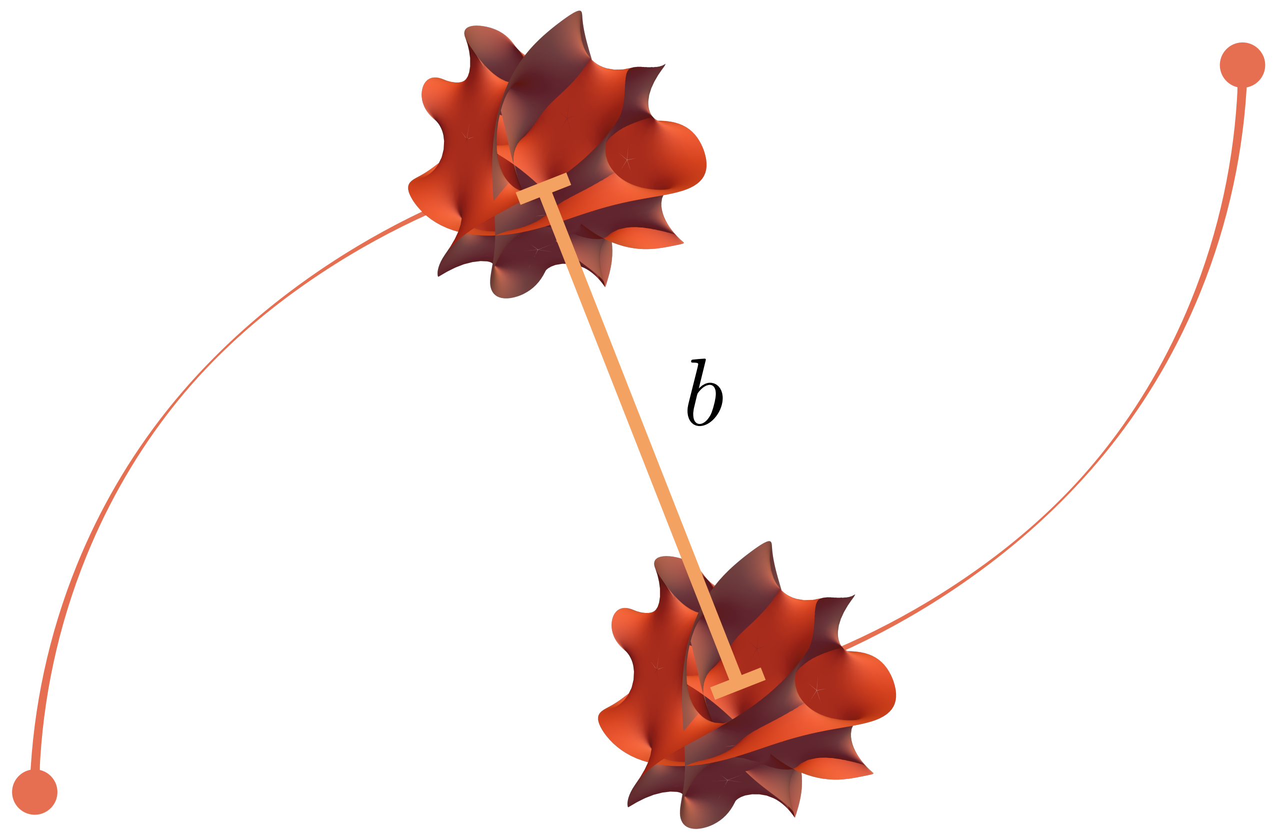
$f_i$: ? (automorphic form, weight $i$?)

# Banana Integrals are very idealized
# Can we find a real-world application?

# Calabi—Yaus in Gravity

## Scattering of black holes [See Benjamin's talk]

Black holes modeled as massive scalars

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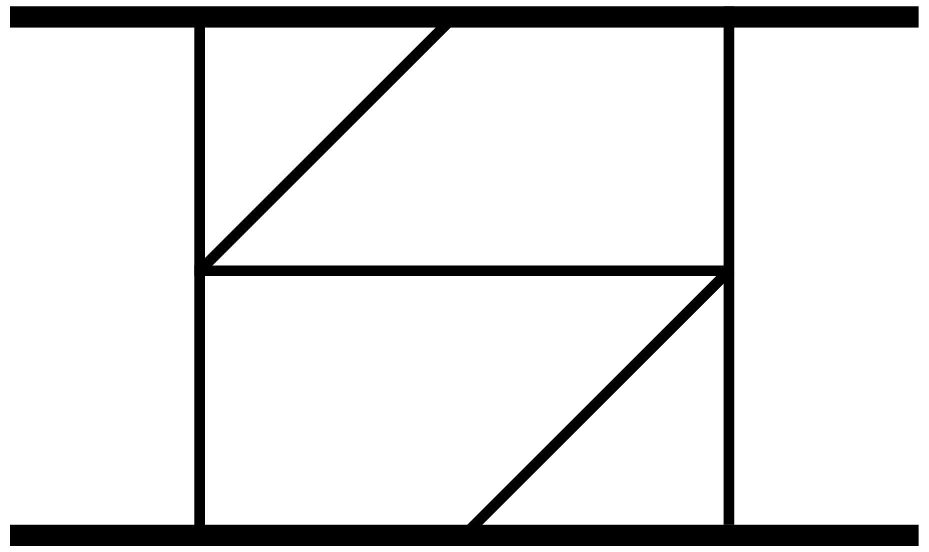


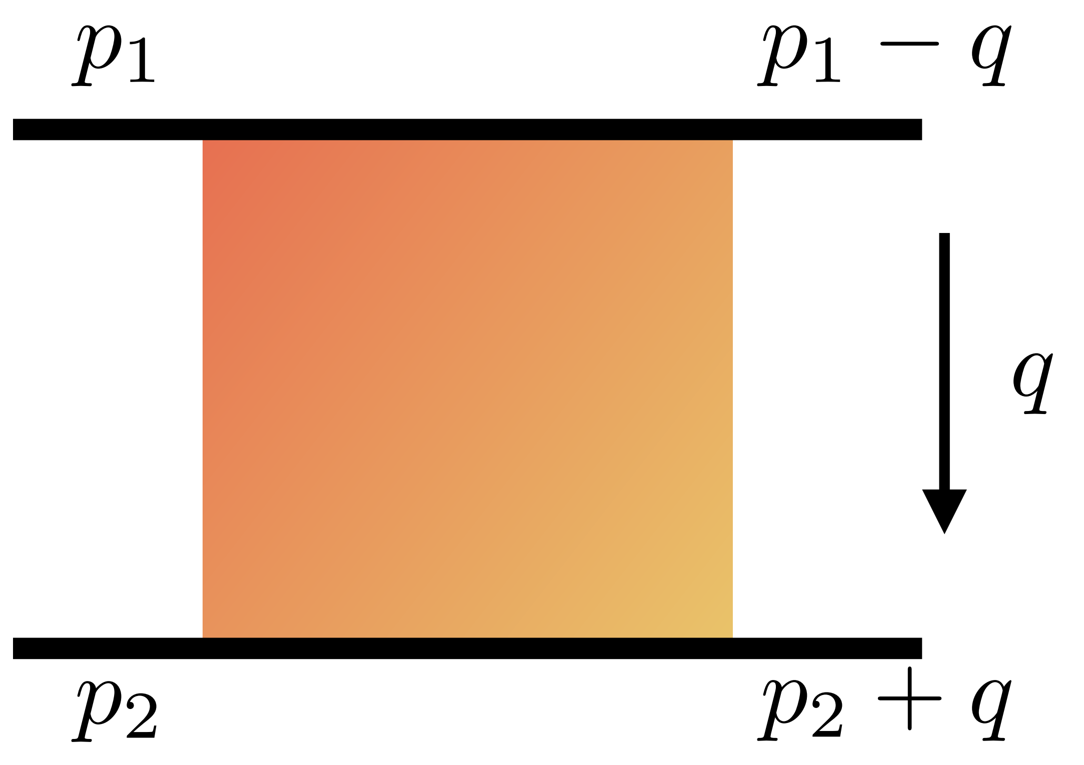
Impact parameter $|b| \sim 1/|q|$

Assume long range interaction $r_s/|b| \ll 1$, thus $Gm|q| \ll 1$ $\longrightarrow$ Compute corrections in Post-Minkowskian expansion in $G^n$

**Extract classical effects from seemingly quantum description**

# Integrals for Black Holes

Classical limit described by soft $|q|$ limit




$$p_1 = \bar{p}_1 - q/2$$
$$p_2 = \bar{p}_2 + q/2$$
$$\bar{p}_i \cdot q = 0$$
$$|q| \ll 1$$

Scalar propagator:

$$\frac{1}{(k+p_i)^2 - m_i^2} \sim \frac{1}{m_i}\frac{1}{2u_i \cdot k} + \mathcal{O}(q^2)$$

$u_i = \frac{\bar{p}_i}{\bar{m}_i} \quad u_i \cdot q = 0 \quad u_i^2 = 1$

$\bar{m}_i^2 = \bar{p}_i^2 = m_i^2 - q^2/4$

At $L$ loops: order $|q|^{L-2}G^{L+1}$

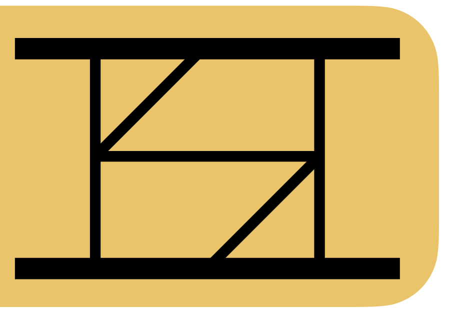
Single kinematic scale: $y = u_1 \cdot u_2$

Great test cases:
- Relevance for gravitational wave physics
- Single kinematic scale
- At 3 and 4 loop: Calabi—Yau 2-folds (K3) and 3-folds appear

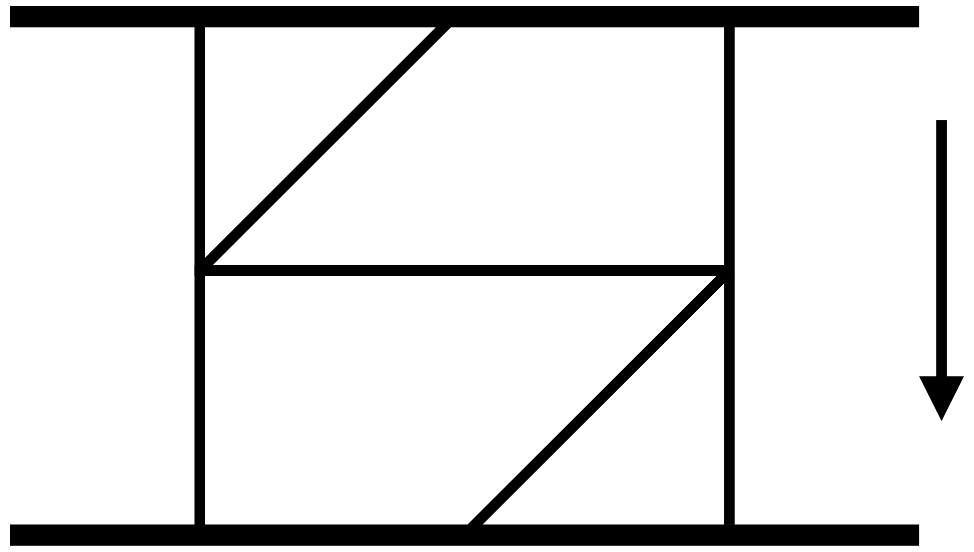
  [Frellesvig, Morales, Wilhelm, '23; Klemm, Nega, Sauer, Plefka, '24]
- Integrals with subtopologies (think DEQ with inhomogeneity)

For most, methods from Banana integrals are sufficient

except for one!



# An Integral for 2-Self-force Correction




$\sim$

$$I_{\nu_1,\nu_2,\ldots,\nu_{11}} = \int \frac{\mathrm{d}^d k_1\, \mathrm{d}^d k_2\, \mathrm{d}^d k_3\, \mathrm{d}^d k_4}{\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_{11}^{\nu_{11}}}$$

In $d = 4$, $I = I_{11111111111}$ is annihilated by

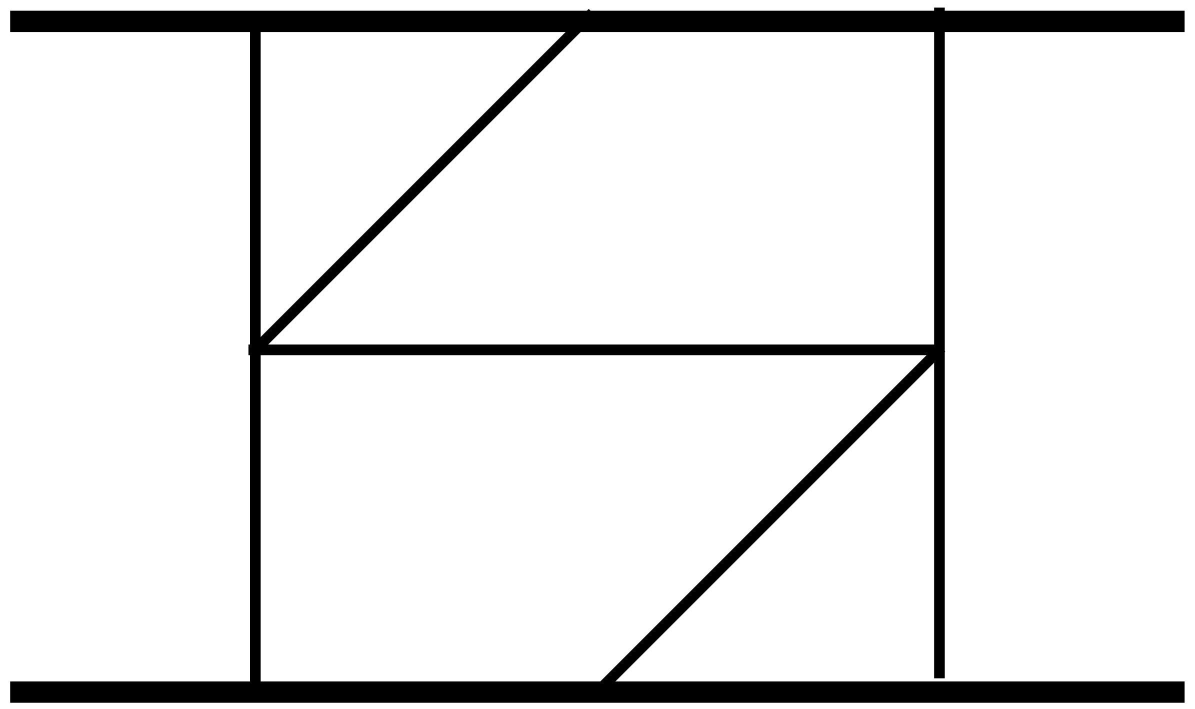
$$I^{(4)}(x) - \frac{\left(10y^2+2\right)}{y-y^3} + \frac{\left(25y^4+y^2+2\right)}{y^2\left(y^2-1\right)^2} I''(x) + \frac{\left(15y^4-6y^2-1\right)}{y\left(y^2-1\right)^3} I'(x) + \frac{4y^4-y^2+4}{4y^2\left(y^2-1\right)^3} I(x) = 0$$

**Calabi—Yau operator (up to normalization of $y$): Associated to Calabi—Yau 3-fold**

**So where is the problem?**

# An Integral for 2-Self-force Correction

In $d = 4 - 2\varepsilon$, $I = I_{1111111111}$ is annihilated by



$$\mathcal{L}^{(5)} I(y) = I^{(5)}(y)$$
$$- I^{(4)}(y) \frac{y\left(y^2\left(16\epsilon^3+60\epsilon^2-532\epsilon-51\right)-800\epsilon^3+3552\epsilon^2+1060\epsilon+70\right)}{\left(y^2-1\right)\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)}$$
$$+ I'''(y) \frac{\left(y^4\left(16\epsilon^4-64\epsilon^3-1184\epsilon^2+2576\epsilon+255\right)-3y^2\left(352\epsilon^4-2936\epsilon^3+6228\epsilon^2+2062\epsilon+141\right)+16\left(800\epsilon^4+220\epsilon^3+262\epsilon^2+75\epsilon\right.\right.}{\left(y^2-1\right)^2\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)}$$
$$+ I''(y) \frac{y\left(4y^4\left(32\epsilon^4+108\epsilon^3-1020\epsilon^2+1009\epsilon+105\right)-y^2\left(128\epsilon^5+9776\epsilon^4-30320\epsilon^3+30120\epsilon^2+12588\epsilon+919\right)+6400\epsilon^5+81104\epsilon\right.}{\left(y^2-1\right)^3\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)}$$
$$+ I'(y) \frac{\left(4y^6\left(208\epsilon^4+1200\epsilon^3-3312\epsilon^2+1604\epsilon+183\right)-y^4\left(64\epsilon^6+2176\epsilon^5+73232\epsilon^4-125184\epsilon^3+35788\epsilon^2+26808\epsilon+2155\right)+y^2\left(8\right.\right.}{4\left(y^2-1\right)^4\left(y^2\left(4\epsilon^2\right.\right.}$$
$$+ I(y) \frac{12y^5(1-2\epsilon)^2\left(4\epsilon^2+32\epsilon+3\right)-y^3\left(64\epsilon^6+1152\epsilon^5+20112\epsilon^4-19136\epsilon^3-980\epsilon^2+1192\epsilon+107\right)+4y\left(800\epsilon^6+400\epsilon^5+12432\epsilon\right.}{4\left(y^2-1\right)^4\left(y^2\left(4\epsilon^2+32\epsilon+3\right)-4\left(50\epsilon^2+15\epsilon+1\right)\right)}$$

Two new features compared to Bananas:

factorization $\mathcal{L}^{(5)} \overset{\varepsilon\to 0}{=} \mathcal{L}^{(1)}\mathcal{L}^{(4)}$
evanescent Master integral in $d = 4$

1. Operator with different dimensions for $\varepsilon \to 0$
2. Operator has unphysical singularity, quadratic in $\varepsilon$

**Apparent Singularity** Singularity of operator at which all solutions are non-singular

# Apparent Singularities in Feynman Integrals

Check empirically: **While undesirable, almost all integrals have them**

When encountering them, you have two options

Option 1: Go back and make a “better" choice for integral ✗ After extensive scan over candidate integrals: no luck

↖ So far always the default solution

However, it can fail:
- All choices you (can) try turn out to be bad
- There might not be a good choice

Option 2: Go back and make a better Ansatz ✓

# Ansatzing $\varepsilon$-factorized DEQs

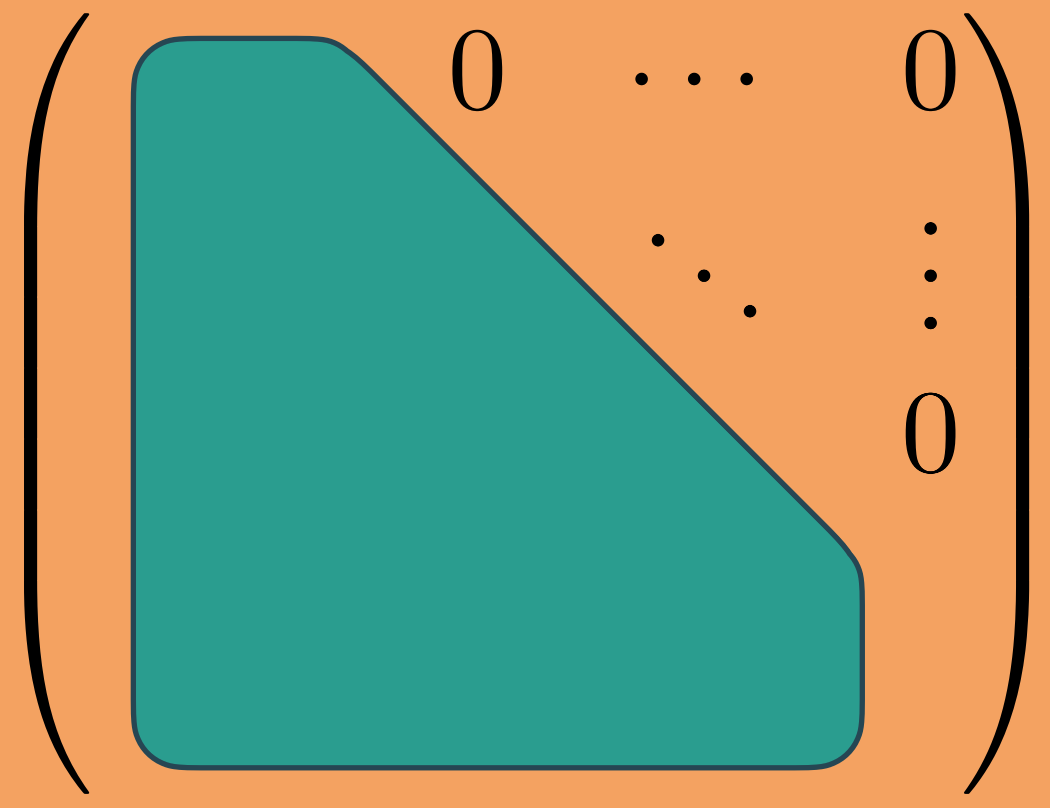
## (Revisited)

For Banana Integrals we made the Ansatz

$$
\begin{aligned}
M_1 &= \frac{1}{\varpi} I_{11111} \\
M_2 &= \frac{1}{\varepsilon} J \frac{\mathrm{d}M_1}{\mathrm{d}x} + F_{11} M_1 \\
M_3 &= \frac{1}{\varepsilon} \frac{J}{K_1} \frac{\mathrm{d}M_2}{\mathrm{d}x} + F_{21} M_1 + F_{22} M_2 \\
M_4 &= \frac{1}{\varepsilon} \frac{J}{K_2} \frac{\mathrm{d}M_2}{\mathrm{d}x} M_3 + F_{31} M_1 + F_{32} M_2 + F_{33} M_3 \\
&\;\;\vdots
\end{aligned}
$$

all $M_{i>1}$ have operators with apparent singularities

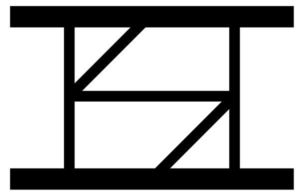
$A \sim \varepsilon$



Let us reverse arrow: don't specify Masters but rather shape of differential equation

Tune shape for operator of $M_1$ to have properties we want

# Our Four-loop Integral

For  with 5 Master Integrals

Assume:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & F_{12} & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} & 0 \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix}$$

Can check: Picard—Fuchs of $M_1$ has apparent singularity, quadratic in $\varepsilon$

**Fits singularity of operator of** $I_{11111111111}$

# Matching Ansatz

(Revisited)

## In case of Bananas

$$M_1 = \frac{I}{\varpi}$$
$$M_2 = \frac{1}{\varepsilon} J(x) \frac{\mathrm{d}M_1}{\mathrm{d}x} + F(x) M_1$$

→

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F(x) & 1 \\ * & * \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

→ **Eliminate non-$\varepsilon$-factorizing pieces**

## For more general Ansatz: Matching of operators

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \varepsilon J(x) \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

→

$$\mathcal{L}_{M_1} = P(x)\mathcal{L}_{I/\varpi}$$

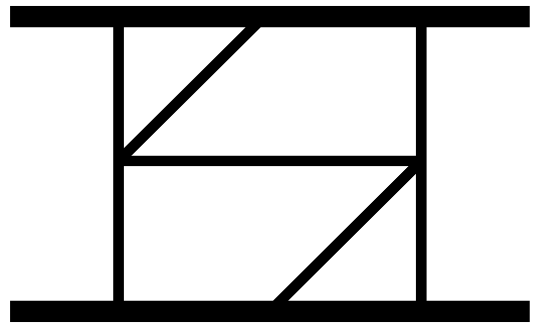
Operator of normalized seed integral

→ **Match coefficients at each order in $\varepsilon$**

**Completely general:**

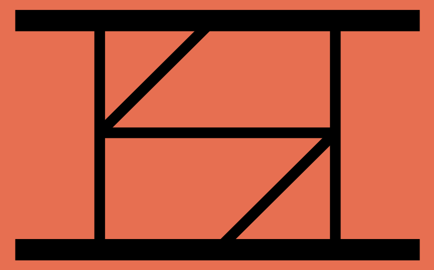
- No a priori knowledge of possible kernels (not limited to Calabi—Yau)
- Accommodates any shape of DEQ

# $\varepsilon$-factorized DEQ

For  performed matching

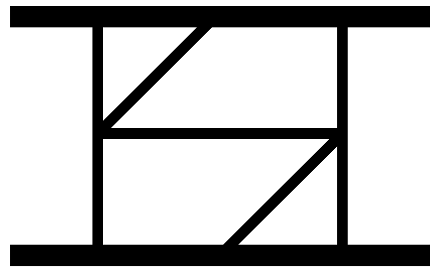
$I = I_{11111111111}$ **+** $\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}M_1\\M_2\\M_3\\M_4\\M_5\end{pmatrix} = \varepsilon J(x)\begin{pmatrix}F_{11} & F_{12} & 0 & 0 & 0\\F_{21} & F_{22} & F_{23} & F_{24} & 0\\F_{31} & F_{32} & F_{33} & F_{34} & 0\\F_{41} & F_{42} & F_{43} & F_{44} & F_{45}\\F_{51} & F_{52} & F_{53} & F_{54} & F_{55}\end{pmatrix}\begin{pmatrix}M_1\\M_2\\M_3\\M_4\\M_5\end{pmatrix}$ **+** $\mathcal{L}_{M_1} = P(x)\mathcal{L}_{I/\varpi}$

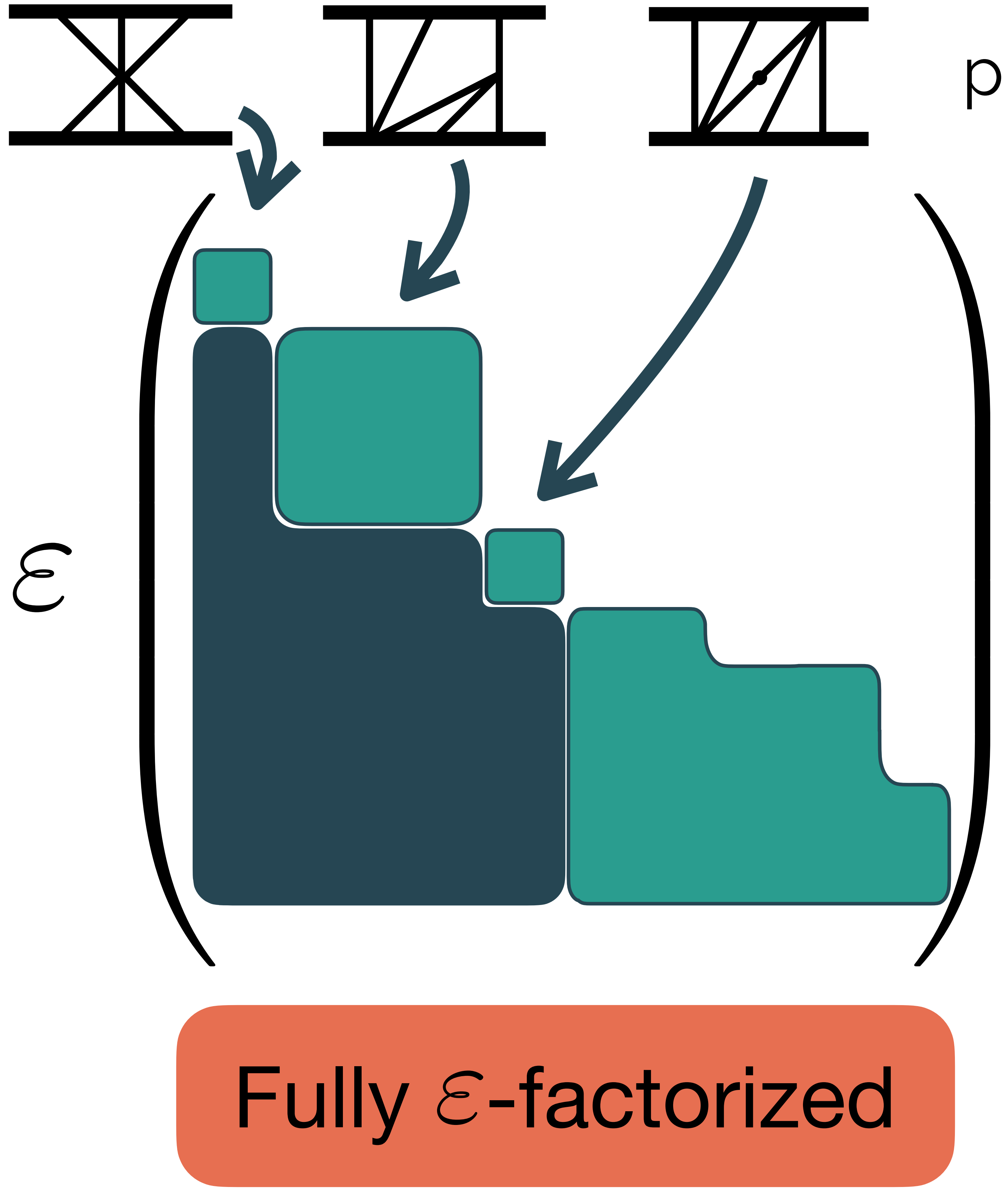
→ Consistently solvable constraints for $F_{ij}(x), J(x), \varpi(x)$

→ $\varepsilon$-factorized DEQ for 

self-dual

# Beyond Top-sector

Full integral  depends on subsectors



# Conclusions

- Feynman Integrals with Calabi—Yau geometries are relevant for both **collider phenomenology and gravitational wave physics**

- Extended existing methods to tackle new features in multi-loop integrals
  - New tools to accommodate **operators with apparent singularities**
  - Allows to **work with sub-optimal seed integral**

- Applied method to derive $\varepsilon$**-factorized differential equation for real-world four-loop integral for 5PM correction**

- Open question: What is the significance of such singularities?
  When can they be avoided?

Thank you!