A first computation of three-loop master integrals for the production of two off-shell vector bosons with different masses

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based on ongoing work in colaboration with Mattia Pozzoli

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Loop-the-Loop, Online Workshop, 14/11/2024

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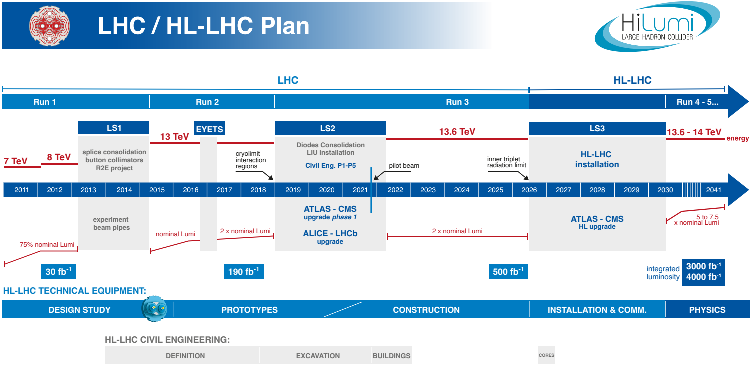


Future 14 TeV HL-LHC runs + Clear signatures left by vector boson productions (leptonic decays)

↓

Observables measured with an accuracy well below one percent

↓

Precision achieved theoretically only if three-loop corrections included!!!

Frontier in Three-Loop Feynman Integral Computations

- Current frontier stands at families with four external particles, where one of them is massive
  - Ladder-box, [S. Di Vita, P. Mastrolia, U. Schubert, V. Yundin, 2014]
  - Planars, [D.C., N. Syrrakos, 2020 & 2021 & 2023]
  - Some Non-Planars, [J.M. Henn, J. Lim, W. Bobadilla, 2023 & 2024]
  - More Non-Planars, [T. Gehrmann, J. Henn, P. Jakubčík, J. Lim, C. Mella, N. Syrrakos, L. Tancredi, W. Bobadilla, 2024] [See Jungwon Lim's talk]
- Recently extended to calculations consisting of two massive particles of the same mass
  - Two Ladder-boxes, [Ming-Ming Long, 2024]
- Amplitudes have been computed in the last years, using some of the results above
  - Planar V+jet production, [T. Gehrmann, P. Jakubčík, C. Mella, N. Syrrakos, L. Tancredi, 2023]
  - Leading-color $N = 4$ form factors for H+jet, [T. Gehrmann, J. Henn, P. Jakubčík, J. Lim, C. Mella, N. Syrrakos, L. Tancredi, W. Bobadilla, 2024] [See Jungwon Lim's talk]

Herein we extend the frontier to families with two external massive particles with unequal masses!!!

- These families studied contribute to planar three-loop QCD amplitudes of processes like

$$q\bar{q}'/gg \to V_1 V_2 \to (l_1 \bar{l}_1')(l_2 \bar{l}_2') \quad \text{with} \quad V_1 V_2 = \gamma^*\gamma^*, W^+W^-, ZZ, W^\pm Z, W^\pm \gamma^*, Z\gamma^*$$

[See talks of Colomba Brancaccio and Simone Zoia for current two-loop frontier]

Notation and Kinematics

- Four-particle scattering + two different external masses → 4 independent invariants $\vec{s}$ (scales)

$$s_{12} = (p_1 + p_2)^2, \quad s_{23} = (p_2 + p_3)^2, \quad m_3^2 = p_3^2, \quad m_4^2 = p_4^2 \quad \text{and} \quad p_1^2 = p_2^2 = 0$$

- Physical region in Mandelstams contains the root $R = \sqrt{m_3^4 + (m_4^2 - s_{12})^2 - 2m_3^2(m_4^2 + s_{12})}$

$$m_3^2, m_4^2 > 0, \quad s_{12} > (m_3 + m_4)^2 \quad \text{and} \quad \frac{m_3^2 + m_4^2 - s_{12} - R}{2} \leq s_{23} \leq \frac{m_3^2 + m_4^2 - s_{12} + R}{2}$$

- R can be rationalized using the parametrization $(R = m_3^2 x(1-y))$ [J. Henn, K. Melnikov, V. Smirnov, 2014]

$$\frac{s_{12}}{m_3^2} = (1+x)(1+xy), \qquad \frac{s_{23}}{m_3^2} = -xz \qquad \text{and} \qquad \frac{m_4^2}{m_3^2} = x^2 y$$

- Physical region in (x, y, z) parametrization takes the form

$$\boxed{x > 0, \qquad 0 < y < 1, \qquad \text{and} \qquad y < z < 1}$$

Integral Families Under Study

- Our three-loop four-point Feynman Integral families are of the form

$$F_{\alpha_1,\ldots,\alpha_{15}} = \int \frac{d^d k_1 d^d k_2 d^d k_3}{(i\pi)^{3d/2}} \frac{e^{3\gamma_E \varepsilon}}{D_1^{\alpha_1} \cdots D_{15}^{\alpha_{15}}} \qquad \text{with} \qquad D_j = \left(\sum_{i=1}^{3} a_{ij} k_i + b_{ij} p_i \right)^2$$

where $a_{ij}, b_{ij} = 0, \pm 1$ and the last 5 propagators are auxiliary ones ($\alpha_i \leq 0$ for $i = 11, \ldots, 15$).

- All the planar FI families can be collected into two propagator super-families: F123 and F132.

- F123 is described by the following set of propagators

$$\begin{aligned}
&D_1 = k_1^2, \quad D_2 = (k_1 + p_1)^2, \quad D_3 = (k_1 + p_{12})^2, \quad D_4 = (k_1 + p_{123})^2, \\
&D_5 = k_2^2, \quad D_6 = (k_2 + p_1)^2, \quad D_7 = (k_2 + p_{12})^2, \quad D_8 = (k_2 + p_{123})^2, \\
&D_9 = k_3^2, \quad D_{10} = (k_3 + p_1)^2, \quad D_{11} = (k_3 + p_{12})^2, \quad D_{12} = (k_3 + p_{123})^2, \\
&D_{13} = (k_1 - k_2)^2, \quad D_{14} = (k_1 - k_3)^2 \quad \text{and} \quad D_{15} = (k_2 - k_3)^2
\end{aligned}$$

- F132 propagators can be obtained from the F123 ones via the transformation $p_2 \leftrightarrow p_3$.

Not-Completely Reducible Families: RL1

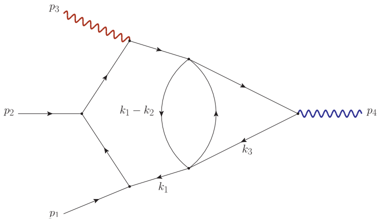


Figure: Top sector of RL1

- Top-Sector Graph: $G[F123, \{1,1,1,1,0,0,0,0,1,0,0,1,1,0,1\}]$
- Number of master integrals (MIs): 27
- Alphabet in the variables $\{x, y, z\}$ consists of the following 12 letters

$$\{x, y, z, 1+x, 1-y, 1-z, 1+xy, 1+xz, z-y, xy+z, 1+y(1+x)-z, z-x(y+z+yz)\}$$

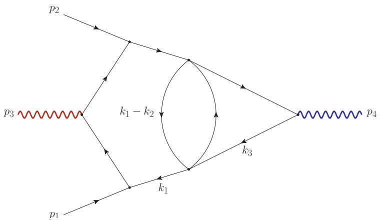


Figure: Top sector of RL2

- Top-Sector Graph: $G[F132, \{1,1,1,1,0,0,0,0,1,0,0,1,1,0,1\}]$
- Number of MIs: 25
- Alphabet in the variables $\{x, y, z\}$ consists of the following 13 letters

$$\{x, y, z, 1+x, 1-y, 1-z, 1+xy, 1+xz, z-y, xy+z, 1+y(1+x)-z, 1+y-z, 1+x(1+y-z)\}$$

Irreducible Families: PL1

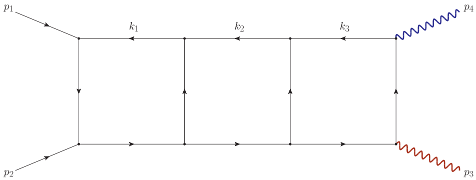


Figure: Top sector of PL1

- Top-Sector Graph: $G[F123, \{1,1,1,0,1,0,1,0,1,0,1,1,1,0,1\}]$
- Number of MIs: 150
- Alphabet in the variables $\{x, y, z\}$ consists of the following 13 letters

$\{x, y, z, 1+x, 1-y, 1-z, 1+xy, 1+xz, z-y, xy+z, 1+y(1+x)-z, 1+y-z, 1+x(1+y-z)\}$

The Irreducible Families: PT4

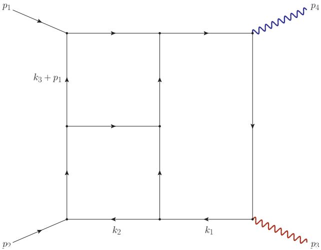
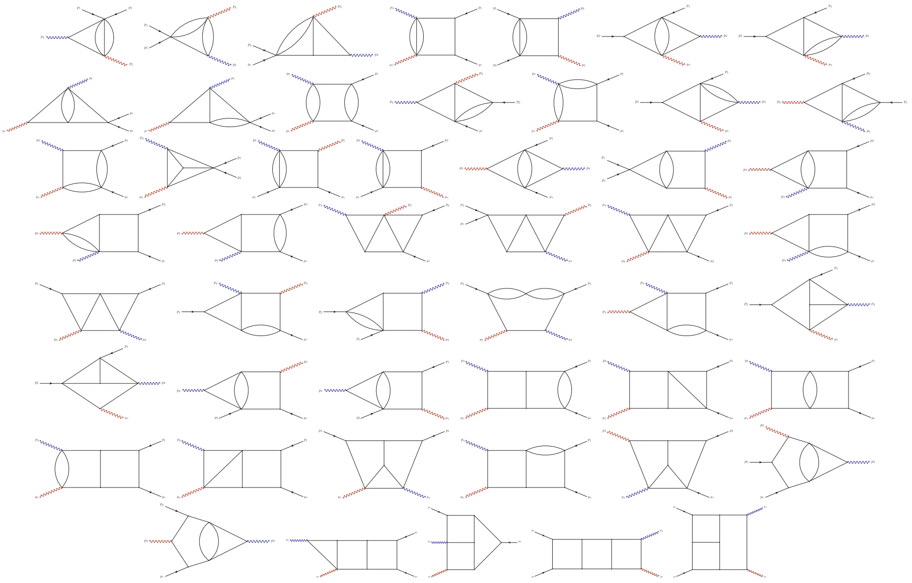


Figure: Top sector of PT4

- Top-Sector Graph: $G[F123, \{1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1\}]$
- Number of MIs: 189
- Alphabet in the variables $\{x, y, z\}$ consists of the following 15 letters

$$\{x,\ y,\ z,\ 1+x,\ 1-y,\ 1-z,\ z-y,\ 1+y-z,\ 1+xy,\ 1+xz,\ z+xy,\ 1-z+y(1+x),\ 1+x(1+y-z),\ z-y(1-z-xz),\ z-x(y-z-yz)\}$$

50 New Sectors with 132 Genuinely New Master Integrals



Differential Equations (DEs)

- Computation of MIs derivatives + Generation of IBPs [K. Chetyrkin, F. Tkachov, 1981] → FFIntRed[1]

$$\int \left(\prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^d} \right) \frac{\partial}{\partial k_b^\mu} \left(u^\mu \frac{\bar{z}_1^{i_1} \cdots \bar{z}_{n_{ir}}^{i_{n_{ir}}}}{\prod_j D_j^{a_j}} \right) = 0 \qquad \text{with} \qquad u^\mu = a^i p_i^\mu + b_i k_i^\mu$$

- IBPs solved over finite fields → Reconstruct DEs [A. Kotikov, 1991] with FiniteFlow [T. Peraro, 2019]

$$\partial_\xi \vec{G} = B_\xi(\vec{s};\epsilon)\ \vec{G} \qquad \text{with} \qquad \xi \in \vec{s},$$

- What we reconstructed is not the above form but the following canonical one [J. Henn, 2013]

$$\partial_\xi \vec{I} = \epsilon\ \tilde{A}_\xi(\vec{s})\ \vec{I} \qquad \to \qquad \boxed{d\vec{I} = \epsilon\ dA(\vec{s})\ \vec{I}} \qquad \text{with} \qquad dA(\vec{s}) = \sum A_i\ d\log w_i$$

- Pure bases constructed employing a bottom-up approach and working sector-by-sector firstly studying the behavior in the maximal cut (MC) → the following methods were used
  - Magnus exponential [M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, 2014] & [T. Gehrmann, A. von Manteuffel, L. Tancredi, E. Weihs, 2014]
  - Building-blocks [Pascal Wasser, 2018]
  - Candidates with integrands of $d\log$ form [J. Henn, B. Mistlberger, V. Smirnov, P. Wasser, 2020]

[1] In-house package written by Tiziano Peraro for generation of Integration-by-Parts identities (IBPs).

Pure Bases Construction: Magnus Exponential

- Choose appropriate candidates (if $N_p \leq 7 \to$ dot, else $\to$ ISPs)[2] that render MC DEs linear on ε

$$\partial_\xi \vec{G}^{\mathrm{MC}} = (H_{0,\xi} + \varepsilon H_{1,\xi})\vec{G}^{\mathrm{MC}}$$

- H_0 can be removed by rescaling MIs a matrix that satisfies the following DEs

$$\partial_\xi \tilde{T}^{\mathrm{MC}} = -\tilde{T}^{\mathrm{MC}} H_{0,\xi}$$

- New candidates defined as $\vec{I}^{\mathrm{MC}} = \tilde{T}^{\mathrm{MC}}\vec{G}^{\mathrm{MC}}$ acquire canonical DEs in MC

$$\partial_\xi \vec{I}^{\mathrm{MC}} = \varepsilon A_\xi^{\mathrm{MC}} \vec{I}^{\mathrm{MC}} \qquad \text{with} \qquad A_\xi^{\mathrm{MC}} = \tilde{T}^{\mathrm{MC}} H_{1,\xi} (\tilde{T}^{\mathrm{MC}})^{-1}$$

- Are $\vec{I}$ pure beyond MC? $\to$ relax cut conditions $\to$ may appear sub-sector entries linear on ε

$$\partial_\xi \vec{I} = \varepsilon A_\xi^{\mathrm{MC}} \vec{I}^{\mathrm{MC}} + (h_{0,\xi} + \varepsilon h_{1,\xi})\vec{I}^{\mathrm{LS}}$$

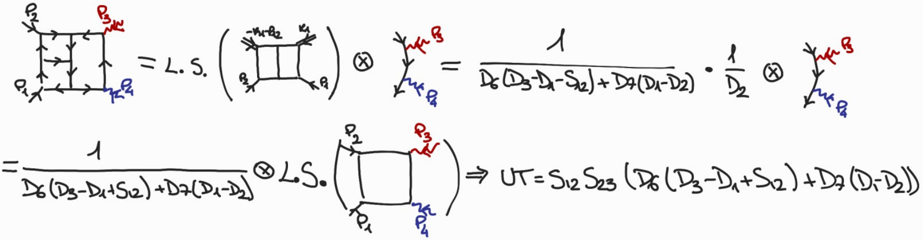
- To set them in canonical form rotate the lower sector ε^0 contributions by integrating out $h_{0,\xi}$

$$\vec{I} = \vec{I}^{\mathrm{MC}} + \tilde{T}^{\mathrm{LS}} \vec{I}^{\mathrm{LS}} \qquad \text{with} \qquad \partial_\xi \tilde{T}^{\mathrm{LS}} = -h_{0,\xi}$$

[2] Where with N_p we denote the number of propagators of the sector at hand

Pure Bases Construction: Building-Blocks + $d\log$*-form*

• Building Blocks: combine leading singularities (LS) of one/two-loop pure candidates as building blocks for creating a three-loop candidate with constant LS, in a graphical/heuristic approach



• DLogBasis: use spinor-helicity formalism (Baikov representation) to bring the 4(d)-dimensional integrand of FI into $d\log$ form and find its LS, e.g. for one-loop massless box

$$\frac{d^4k}{k^2(k-p_1)^2(k-p_{12})^2(k+p_4)^2} \xrightarrow{\texttt{DLogBasis}} \frac{1}{s_{12}s_{23}}\, d\log\frac{k^2}{(k-p_*)^2}\, d\log\frac{(k-p_1)^2}{(k-p_*)^2}\, d\log\frac{(k-p_{12})^2}{(k-p_*)^2}\, d\log\frac{(k+p_4)^2}{(k-p_*)^2}$$

Analytic Solution

- Solve the (x, y, z) canonical DEs in terms of MPLs directly in the physical region

$$\mathcal{G}_{1\ldots n} \equiv \mathcal{G}(a_1, \ldots, a_n; X) = \int_0^X \frac{dt}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t) \quad \text{with} \quad \mathcal{G}(\vec{0}_n; X) = \frac{1}{n!} \log^n(X)$$

- For obtain a minimum number of MPLs → use a different integration path for each superfamily
  - F123: $(0,0,0) \xrightarrow{\gamma_1} (x,0,0) \xrightarrow{\gamma_2} (x,0,z) \xrightarrow{\gamma_3} (x,y,z)$
  - F132: $(0,0,0) \xrightarrow{\gamma_1'} (0,0,z) \xrightarrow{\gamma_2'} (0,y,z) \xrightarrow{\gamma_3'} (x,y,z)$

- The two different paths result to MPLs with same arguments but different indices (backup slides)!

- Solution obtained up to order 6 in the ε expansion → MPLs only up to weight 6 appear

$$\vec{I} = \epsilon^0 \vec{b}_0 + \epsilon \left(\mathcal{G}_i A_i \vec{b}_0 + \vec{b}_1 \right) + \epsilon^2 \left(\mathcal{G}_{ij} A_{ij} \vec{b}_0 + \mathcal{G}_i A_i \vec{b}_1 + \vec{b}_2 \right) + \ldots$$
$$+ \epsilon^6 \left(\mathcal{G}_{ijklmn} A_{ijklmn} \vec{b}_0 + \mathcal{G}_{ijklm} A_{ijklm} \vec{b}_1 + \mathcal{G}_{ijkl} A_{ijkl} \vec{b}_2 + \mathcal{G}_{ijk} A_{ijk} \vec{b}_3 + \mathcal{G}_{ij} A_{ij} \vec{b}_4 + \mathcal{G}_i A_i \vec{b}_5 + \vec{b}_6 \right)$$

where $\vec{b}_i$ the boundary conditions at order ε^i, $A_{i\ldots n} \equiv A_i \ldots A_n$ and $A_i \equiv A_{w_i}$.

Boundary conditions

Regularity Conditions

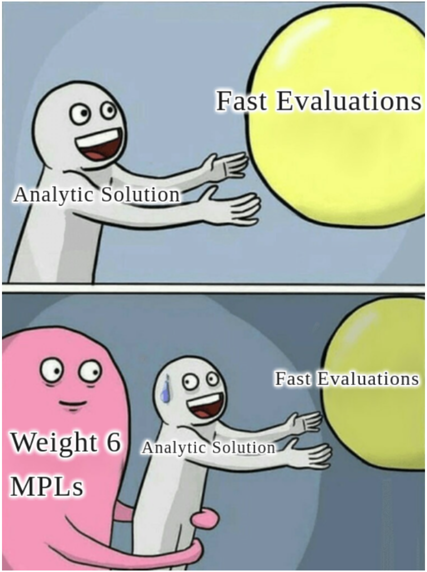
AMFlow + PSLQ

For computing boundaries we followed the procedure of [S. Badger, J. Kryś, R. Moodie, S. Zoia, 2023]:

- Computed MIs in 70 digits precision using AMFlow [X. Liu, Y. Ma, 2022] for the point $(x, y, z) = (3/2, 1/5, 1/2)$.
- Computed the MPLs of the analytic solution in 70 digits precision using `DiffExp` [M. Hidding, 2020] at the same point.
- Known excepted transcendental constants at each order of the ε expansion $\rightarrow$ performed PSLQ order by order.

Expansion	ε^0	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6
Boundaries	1	$i\pi$	π^2	$i\pi^3, \zeta_3$	$\pi^4, i\pi\zeta_3$	$i\pi^5, \pi^2\zeta_3, \zeta_5$	$\pi^6, i\pi^3\zeta_3, \zeta_3^2, i\pi\zeta_5$

Analytic Solution: Evaluation



Partial decomposition into Lyndon words using `PolyLogTools` [C. Duhr, F. Dulat, 2019]

Family	# MPLs	# W6
RL1	3619	1631
PL1	5416	2554
RT4	8290	3709
F123	8360	3739
RL2	10973	5219
ALL	19306	8951

Many MPLs of weight 6 ($\sim 1s$ per MPL)

↓

Much time spent for their evaluation

↓

Analytic solution inefficient in current form!

↓

Generalized Power Series [F. Moriello, 2019]
[See Tommaso Armadillo's talk]

Semi-Analytic Solution using DiffExp

- Solve our DEs using generalized power series → `DiffExp` → results for 7 points (backup slides)[3]

RL1 *(timings in s)*							
$\{s_{12}, s_{23}, m_3^2, m_4^2\}$	159	143	141	145	143	138	144
$\{x, y, z\}$	42	17	36	59	17	42	19
RL2 *(timings in s)*							
$\{s_{12}, s_{23}, m_3^2, m_4^2\}$	160	148	140	148	144	140	142
$\{x, y, z\}$	47	19	42	67	22	47	20
PL1 *(timings in s)*							
$\{s_{12}, s_{23}, m_3^2, m_4^2\}$	2276	2012	1993	2029	2059	1932	1977
$\{x, y, z\}$	618	216	533	914	247	639	230
PT4 *(timings in s)*							
$\{s_{12}, s_{23}, m_3^2, m_4^2\}$	8522	8735	4301	5116	8133	4305	5243
$\{x, y, z\}$	1375	418	987	2603	361	1152	487

- Semi-analytic solution faster than the Analytic one, though not yet efficient!
- DEs on $\{x, y, z\}$ solved 4.5 times faster than them on $\{s_{12}, s_{23}, m_3^2, m_4^2\}$!

[3] Results obtained for 30 digits running one Mathematica Kernel in personal laptop (Apple M1 Pro, 8 cores, 16GB RAM)

Results

- Canonical DEs for a ladder-box, a tennis-court and 2 reducible 3-loop 4-point families with 2 off-shell legs.

- DEs solved analytically (semi-analytically) for these families in terms of MPLs (generalized power series).

What's next

- Investigation of better "massaging" of analytic solution using Lyndon words or Symbol algebra.

- If semi-analytic solution in favor → better sampling of physical phase space → see [S. Abreu et all, 2020].

- Computation of the rest planar and non families.

- Calculation of amplitudes using these families.

Thank you very much for your attention!!!

Backup Slides: MPL indices per path

- Path γ_1 results to MPLs with argument x and indices

$$\{-1, 0\}$$

- Path γ_2 results to MPLs with argument z and indices

$$\left\{0, -\frac{1}{x}, 1, 1+\frac{1}{x}\right\}$$

- Path γ_3 results to MPLs with argument y and indices

$$\left\{0, -\frac{1}{x}, -\frac{z}{x}, z, \frac{z}{x}\frac{1+x}{1-z}, 1, \frac{z-1}{1+x}, z-1-\frac{1}{x}, z-1, \frac{-z}{-1+z+xz}\right\}$$

- Path γ_1' results to MPLs with argument z and indices

$$\{0, 1\}$$

- Path γ_2' results to MPLs with argument y and indices

$$\{0, -1+z, z\}$$

- Path γ_3' results to MPLs with argument x and indices

$$\left\{0, -\frac{1}{z}, -\frac{z}{y}, \frac{1}{z-1-y}, \frac{z-1-y}{y}, -1, -\frac{1}{y}\right\}$$

• Choose an symbolic ordering for the indices of the MPLs, e.g. in our case for the following five indices of RL2 MPLs with arguments x we choose

$$\left\{0 < \frac{1}{-1-y+z} < -\frac{z}{y} < -\frac{1}{y} < -1\right\}$$

• Then all the MPLs whose indices are not sorted according to this ordering can be rewritten in terms of MPLs whose indices do so by using shuffle algebra, e.g.

$$\begin{aligned}\mathcal{G}\left(-\frac{z}{y},-\frac{1}{y},0,\frac{1}{z-1-y},x\right) &= \mathcal{G}\left(0,\frac{1}{z-1-y},x\right)\mathcal{G}\left(-\frac{z}{y},-\frac{1}{y},x\right)-\mathcal{G}\left(-\frac{z}{y},x\right)\mathcal{G}\left(0,-\frac{1}{y},\frac{1}{z-1-y},x\right)\\ &-\mathcal{G}\left(-\frac{z}{y},x\right)\mathcal{G}\left(0,\frac{1}{z-1-y},-\frac{1}{y},x\right)+\mathcal{G}\left(0,-\frac{1}{y},-\frac{z}{y},\frac{1}{z-1-y},x\right)\\ &+\mathcal{G}\left(0,-\frac{1}{y},\frac{1}{z-1-y},-\frac{z}{y},x\right)+\mathcal{G}\left(0,\frac{1}{z-1-y},-\frac{1}{y},-\frac{z}{y},x\right)\end{aligned}$$

• By consistently applying this procedure we can reduce the number of MPLs.

• Decomposition into Lyndon words is implemented in PolyLogTools for up to 5 different indices and up to MPLs with weight 6. For more details on this see [D. Radford, 1979].

- Phase-space points[4] used for the evaluations with `DiffExp` (30 digits precision)

Point	$\{s_{12}, s_{23}, m_3^2, m_4^2\}$
0	$\{\frac{13}{4}, -\frac{3}{4}, 1, \frac{9}{20}\}$
1	$\{\frac{40888134693}{44158}, -\frac{38795165599}{216184}, \frac{24638089237}{897988}, \frac{48727831791}{159737}\}$
2	$\{\frac{92788526010}{118927}, -\frac{39883122197}{344552}, \frac{136801114569}{723634}, \frac{16321101911}{176799}\}$
3	$\{\frac{62287855553}{66469}, -\frac{86538861058}{273175}, \frac{47194257508}{929117}, \frac{183221915556}{715387}\}$
4	$\{\frac{33556102913}{36634}, -\frac{51073960191}{123187}, \frac{13674993650}{1359143}, \frac{41523082884}{125771}\}$
5	$\{\frac{28418180312}{51339}, -\frac{38010266231}{181362}, \frac{27158482990}{126509}, \frac{19338791272}{514135}\}$
6	$\{\frac{27986656140}{61187}, -\frac{19953623257}{109975}, \frac{7866155659}{632225}, \frac{122962065237}{815030}\}$
7	$\{\frac{48080429344}{88767}, -\frac{321222790411}{4410467}, \frac{29824627549}{103071}, \frac{15155449048}{679141}\}$

- The corresponding points for the variables $\{x, y, z\}$ are computed using the relations of slide 5 after taking a numerical value with precision 35 digits and rationalizing it with 30 digits precision.

[4] Point 0 corresponds to the initial point.