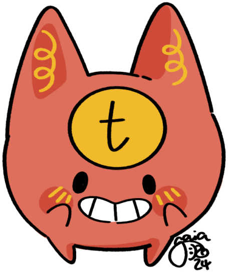
Two-loop QCD corrections to $pp \to t\bar{t}j$

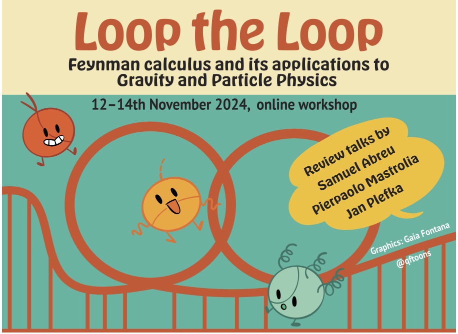
Colomba Brancaccio

University of Turin

In collaboration with: Simon Badger, Matteo Becchetti, Heribertus Bayu Hartanto, Simone Zoia [arXiv:24xx.xxxxx, arXiv:2404.12325, arXiv:2201.12188]

(Thanks also to Gaia Fontana for the wonderful drawings)





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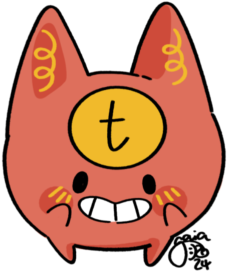
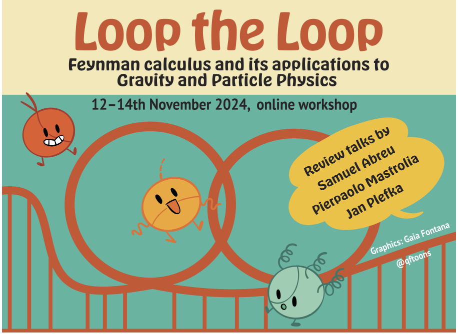




Outline

1. Introduction → What and why?
2. Scattering amplitudes workflow → How?
3. Two-loop $gg \to t\bar{t}g$ amplitudes → What are our results?
4. Conclusions and outlook → What's next?

Precision physics

Monte Carlo generators	***Cross-section predictions***	***Data from colliders***

σ = ∫ PDFs x |A|² x dPS

- ⇒ Current frontier NNLO/N³LO
- ⇒ ***Amplitudes*** are key ingredients for cross-section predictions

Run 1+2+3 + HL-LHC

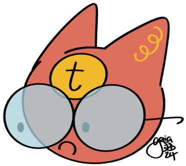
- ⇒ Huge amount of data
- ⇒ Small uncertainties on experimental measurements (***% level accuracy***)
- ⇒ Observe rare processes

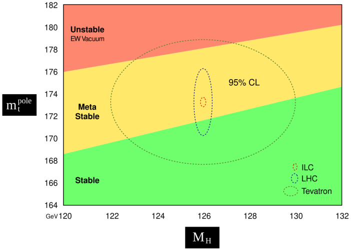
Images generated with PromeAI

Relevance of the top quark

Unique properties of the top quark

- To-date **heaviest** fundamental particle
- Decays before forming hadrons
- Information about its spin state preserved in the decay product distributions



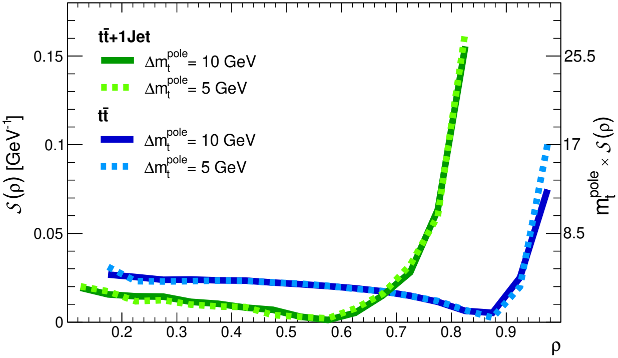


Role in the Standard Model

- Largest coupling to the Higgs boson
- Affects the **EW vacuum stability**

Motivations for $t\bar{t}j$ production

- 50% of $t\bar{t}$ events produced at LHC are associated with a jet
- $t\bar{t}j$ normalised differential cross-section w.r.t. invariant mass of final state particles is **highly sensitive to m_t**

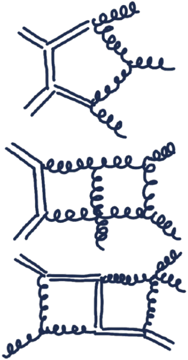


[Alioli, Fernandez, Fuster, Irles, Moch, Uwer '13]

Theory status

What do we know about $t\bar{t}j$?

- NLO QCD corrections [Dittmaier, Uwer, Weinzierl, '07]
- Full off-shell decays and interfaces with parton shower [Melnikov and Schulze '10] [Alioli, Moch, Uwer '12] [Bevilacqua, Czakon, Hartanto, Kraus, Worek '15-'16]
- Mixed QCD and EW corrections [Gütschow, Lindert, Schönherr '18]
- **NNLO QCD corrections needed**
  $\rightarrow$ initial steps toward this challenge
  [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]
  [Badger, Becchetti, Chaubey, Marzucca '23]
  [Badger, Becchetti, Giraudo, Zoia '24]



Current frontier: $2 \to 3$ two-loop scattering amplitudes

Massless external particles:

- $pp \to \gamma\gamma\gamma$
  [Abreu, Page, Pascual, Sotnikov '20]
  [Chawdhry, Czakon, Mitov, Poncelet '21]
  [Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]
- $pp \to \gamma\gamma j$
  [Agarwal, Buccioni, von Manteuffel, Tancredi '21]
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  [Badger, Brönnum-Hansen, Chicherin, [Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21]
- $pp \to \gamma jj$
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- $pp \to jjj$
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One massive external particle: (full colour missing)

- $pp \to Wbb$
  [Badger, Hartanto, Zoia '21]
  [Hartanto, Poncelet, Popescu, Zoia '22]
- $pp \to Wjj$
  [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '22]
- $pp \to Hbb$
  [Badger, Hartanto, Krys, Zoia '21]
- $pp \to W\gamma j$
  [Badger, Hartanto, Krys, Zoia '22]
- $pp \to W/Z + bb$
  [Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22]
  [Mazzitelli, Sotnikov, Wiesemann '24]
- $pp \to W\gamma\gamma^*$
  [Badger, Hartanto, Wu, Zhang, Zoia '24]

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*subleading contribution numerically available

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  [Badger, Hartanto, Krys, Zoia '22]

More masses:

- $pp \to t\bar{t}H$
  [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson, '24]

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*subleading contribution numerically available


INTERNAL MASSES
See also Federico Coro's talk

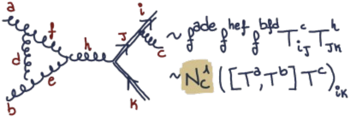
Colour decomposition

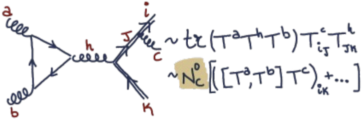
- Consider all diagrams contributing to the process

$$A^{(L)}(\vec{x}, \epsilon) = \sum \Big(\text{Feynman diagrams}\Big)$$

- **Colour expansion** $\to$ take the **leading colour** limit
  $\to$ reduce the complexity of the loop integrals

Example: @1L





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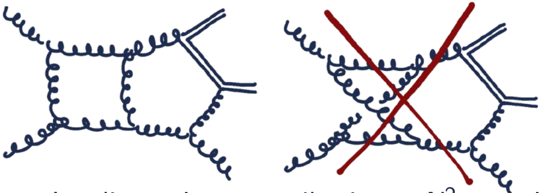
Example: @2L



diagrams	tot	LC
LO	16	6
NLO	384	77
NNLO	11370	1357

Leading colour contribution $\propto N_c^2$ → only planar diagrams

Helicity amplitudes for massive fermions

- **Helicity**: projection of the spin along the direction of momentum

- For massive particles, define the **massless projection**:

$$p^{\flat,\mu} = p^\mu - \frac{m^2}{2p\cdot n} n^\mu$$

with n an arbitrary light-like momentum. The **massive fermion spinor** is:

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^\flat\, n\rangle}, \quad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^\flat\, n]}$$

- Helicity amplitudes encode spin correlation information

$\to$ **inclusion of top-quark decay** in narrow-width approximation

see [Kleiss, Stirling '85] [Arkani-Hamed, Huang, Huang '17] [Badger, Chaubey, Hartanto, Marzucca '21] [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]

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$$u_-(p,m) = \frac{\langle p^\flat n\rangle}{m}\left(u_+(p,m)\big|_{p^\flat \leftrightarrow n}\right),$$

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see [Kleiss, Stirling '85] [Arkani-Hamed, Huang, Huang '17] [Badger, Chaubey, Hartanto, Marzucca '21] [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]

Reduction to MIs

- The amplitude is a linear combination of Feynman integrals:

$$A^{(L)}(\vec{x},\epsilon) = \sum_i c_i(\vec{x},\epsilon) I_i(\vec{x},\epsilon),$$

$$\text{i.e.} \quad I(\vec{x},\epsilon) = \int \frac{d^D k_1 d^D k_2}{k_1^2 (k_1+p_1)^2 (k_1+p_1+p_2)^2 \ldots} \quad \text{and} \quad D = 4-2\epsilon$$

- $I_i(\vec{x},\epsilon)$ written as linear combination of **MIs** using:

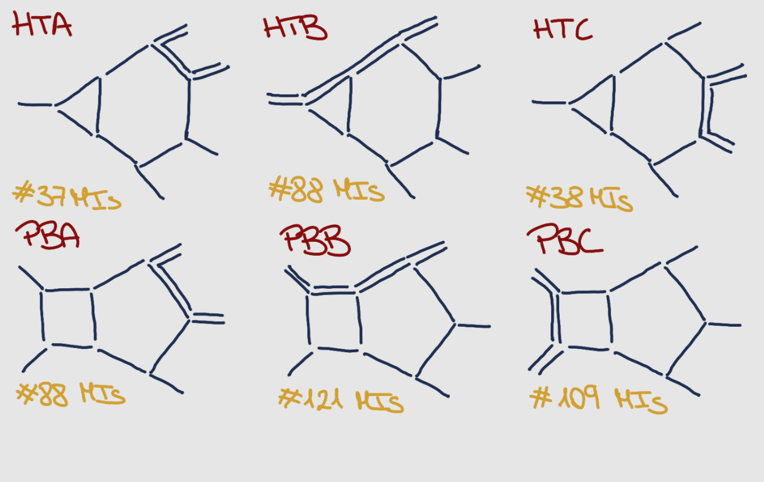
  Integration by Parts Identities (IBPs) [Chetyrkin, Kataev, Tkachov, '80]

$$\text{i.e.} \int d^D k_1 d^D k_2 \frac{\partial}{\partial k_1^\mu} \left(p_1^\mu \frac{1}{k_1^2 (k_1+p_1)^2 (k_1+p_1+p_2)^2 \ldots} \right) = 0$$

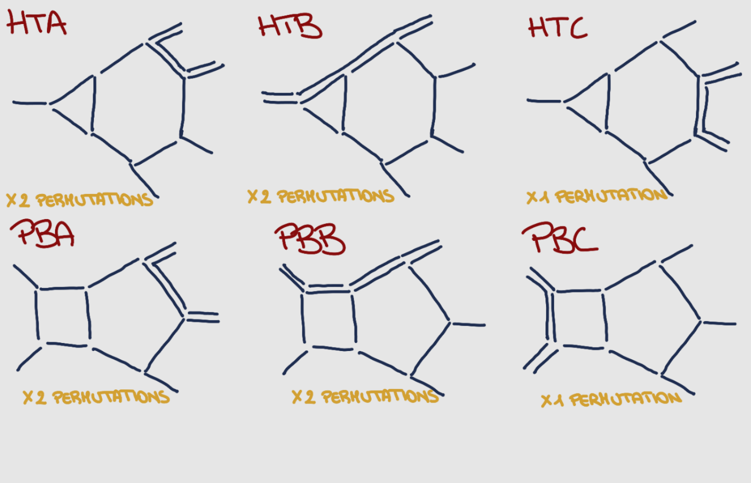
- IBPs generated with `NeatIBP` [Wu, Boehm, Ma, Xu, Zhang '23]

see Rourou Ma's talk

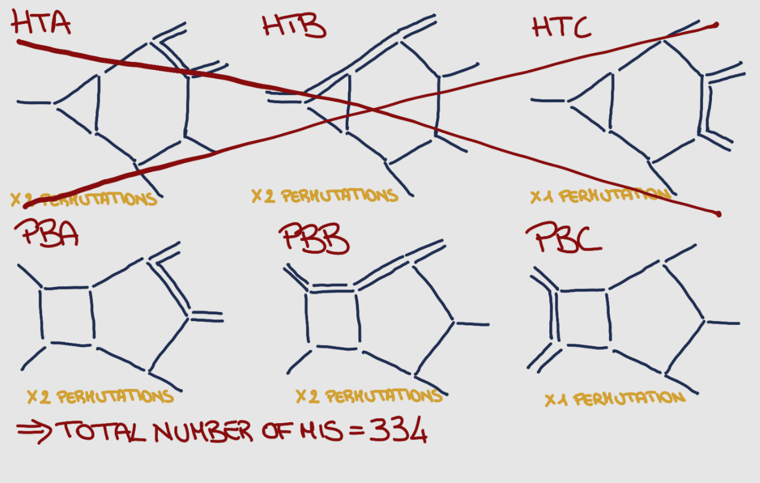
Reduction to MIs



Reduction to MIs

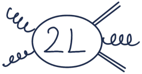


Reduction to MIs

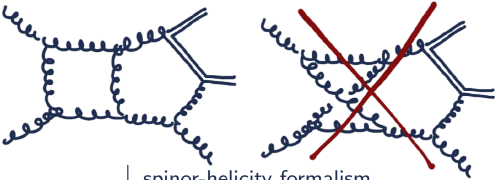


Workflow summary

$$A^{(2)}(\vec{x}, \epsilon) = \sum_i \Big(\; \Big)_i$$



↓ colour decomposition



↓ spinor-helicity formalism

$$A^{hel,(2)}_{\text{LC}}(\vec{x}, \epsilon) = \sum_i c_i(\vec{x}, \epsilon) I_i(\vec{x}, \epsilon)$$

↓ IBP reduction

$$A^{hel,(2)}_{\text{LC}}(\vec{x}, \epsilon) = \sum_i d_i(\vec{x}, \epsilon) \; \text{MI}_i(\vec{x}, \epsilon)$$

⟶ Elliptic sector

[Badger, Becchetti, Giraudo, Zoia '24]

[Badger, Becchetti, Chaubey, Marzucca '23]

$A(hel; n_t, n_{\bar{t}})$ helicity amplitudes encode spin correlations in the narrow width approximation

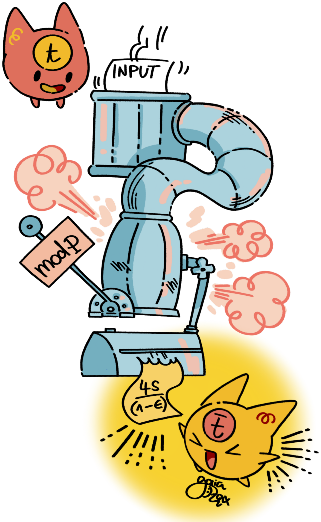
Algebraic complexity

- Intermediate steps in scattering amplitude computations can produce very **large expressions**
- To manage complexity, use numerical methods and then restore analytic dependence
- Replace symbolic operations with numerical evaluations in a **finite field** (integers mod prime P)

[von Manteuffel, Schabinger '14] [Peraro '16]

- Numerical framework: `FiniteFlow`

[Peraro '19]



Analytic complexity: DEs for MIs

MIs satisfy the following **differential equation**:

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)},$$

where $\vec{x}$ are the kinematic invariants $\to$ **6 variables**

For **PBA**[1] and **PBC**[2]:

$$\mathrm{d}A(\vec{x},\epsilon) = \boxed{\epsilon} \sum_j c_j \boxed{\mathrm{dlog}(\alpha_j(\vec{x}))}$$

ϵ-**factorises** ← → **dlog form**

For **PBB**[2]:

$$\mathrm{d}A(\vec{x},\epsilon) = \sum_{k=0}^{2} \boxed{\epsilon^k} \sum_j c_{kj} \boxed{\omega_j(\vec{x})}$$

DEs quadratic in ϵ ← → **One-form**

[1] Badger, Becchetti, Chaubey, Marzucca '23 [2] Badger, Becchetti, Giraudo, Zoia '24

Analytic complexity: DEs for MIs

For **PBB**[2], MIs satisfy the following **differential equation**:

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \sum_{k=0}^{2} \epsilon^k \sum_j c_{kj}\, \omega_j(\vec{x})\, \vec{f}(\vec{x},\epsilon)},$$

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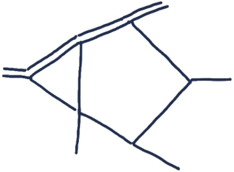
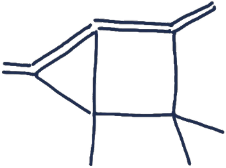


DEs with nested square roots



DEs quadratic in ϵ

$\to$ solution in terms of elliptic functions

[1] Badger, Becchetti, Chaubey, Marzucca '23 [2] Badger, Becchetti, Giraudo, Zoia '24

Analytic complexity: pentagon functions

- **Expand the MIs around $\epsilon = 0$:**

$$f(\vec{x}, \epsilon) = \sum_{k=0}^{4} \epsilon^k f^k(\vec{x})$$

- For **each topology** and **each permutation**

  $\to$ derive the DEs

  $\to$ write the solution in terms of Chen iterated integrals

$$[W_{i_1}, \ldots, W_{i_k}]_{\vec{x_0}}(\vec{x}) = \int_\gamma [W_{i_1}, \ldots, W_{i_{k-1}}]_{\vec{x_0}} \mathrm{dlog}(W_{i_k})$$

  with $\vec{x_0}$ boundaries $\to$ computed with `AMFlow` [Liu, Ma, '22]

- Starting from weight 1 up to weight 4 $\to$ **choose a set of algebraically independet $f^k(\vec{x})$ called $F_i^k(\vec{x})$**

see [Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20] [Chicherin, Sotnikov, Zoia '22] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '24]

Analytic complexity: a basis of special functions for $t\bar{t}j$

- Choose the MIs such that the **"problematic" functions** appearing only at $\mathcal{O}(\epsilon^4)$

  → **only in the finite remainder**

- **Analytic cancellation of the poles**
- Dramatic **simplification of amplitude expressions**
- The amplitude takes now the form

$$A_{\mathrm{LC}}^{hel,2L}(\epsilon, \vec{x}) = \sum_i \sum_{k=-4}^{0} \epsilon^k \, r_{ki}(\vec{x}) \; \mathrm{F}_i(\vec{x})$$

- **$\mathrm{F}_i(\vec{x})$ evaluated using generalised power series** [Moriello '20] method as implemented in `DiffExp` [Hidding '21]

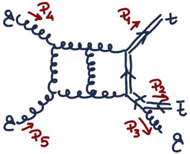
see also Tommaso Armadillo's talk

Notation and kinematics

Evaluated process:

$$g(p_4)g(p_5) \to t(p_1)\bar{t}(p_2)g(p_3),$$

where p_i are external momenta.



Kinematics:

$$p_1^2 = p_2^2 = m_t^2, \quad p_3^2 = p_4^2 = p_5^2 = 0, \quad d_{ij} = p_i \cdot p_j$$

All particles are on-shell and m_t is the top-quark mass

Spin Structure Basis for Helicity States:

$$A_{LC}^{(L)}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5}; n_t, n_{\bar{t}}) = m_t \Phi(3^{h_3}, 4^{h_4}, 5^{h_5}) \sum_{i=1}^{4} \Theta_i(1,2; n_t, n_{\bar{t}})\, A_{LC}^{(L),[i]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$$

see also [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22]

Finite remainder reconstruction

- Mass-renormalised amplitude are gauge invariant

  $\rightarrow$ **Gauge invariance check** ✓

- Two-loop helicity amplitudes for $gg \to t\bar{t}g$ in terms of a basis of special functions

  $\rightarrow$ **simplication of the amplitude**:

helicity	max degrees MIs recon.	max degrees SF recon.
$+++++$	294	131
$+++-+$	384	269
$++++-$	395	264

- UV/IR poles identified analytically and finite remainder computed directly

  $\rightarrow$ **Pole check** ✓

Results

- Example: Numerical evaluations of $A^{(2)}_{\text{LC}}(+++++; n_t n_{\bar{t}})$
- **Decay direction fixed** $(n_t = n_{\bar{t}} = p_3)$
- Finite remainder computed in **'t Hooft-Veltman scheme**

Phase-Space points	$A^{(2)}_{\text{LC}}(+++++; n_t n_{\bar{t}})[\text{GeV}^{-2}]$
$\text{d}_{12} \to 0.1074, \text{d}_{23} \to 0.2719, \text{d}_{34} \to -0.1563,$ $\text{d}_{45} \to 0.5001, \text{d}_{15} \to -0.03196, \text{mt}^2 \to 0.02502$	$19.028262 - 3.1078961\, i$
$\text{d}_{12} \to 0.3915, \text{d}_{23} \to 0.06997, \text{d}_{34} \to -0.06034,$ $\text{d}_{45} \to 0.5002, \text{d}_{15} \to -0.1293, \text{mt}^2 \to 0.02499$	$0.07061470 - 0.00649655\, i$
$\text{d}_{12} \to 0.2167, \text{d}_{23} \to 0.02186, \text{d}_{34} \to -0.01149,$ $\text{d}_{45} \to 0.5007, \text{d}_{15} \to -0.04709, \text{mt}^2 \to 0.02502$	$-29.219122 - 27.542150\, i$
$\text{d}_{12} \to 0.2986, \text{d}_{23} \to 0.1599, \text{d}_{34} \to -0.05978,$ $\text{d}_{45} \to 0.4998, \text{d}_{15} \to -0.2899, \text{mt}^2 \to 0.02500$	$-0.97280521 + 0.86357506\, i$
$\text{d}_{12} \to 0.2882, \text{d}_{23} \to 0.04770, \text{d}_{34} \to -0.1080,$ $\text{d}_{45} \to 0.5000, \text{d}_{15} \to -0.1583, \text{mt}^2 \to 0.02502$	$-0.40407926 - 0.53165671\, i$

Preliminary

with $d_{ij} = p_i \cdot p_j$, normalised here w.r.t. $2\, p_4 \cdot p_5$

TOPline summary

What and why?

- **Two-loop scattering amplitude for $pp \to t\bar{t}j$**
  $\to$ bottleneck for $t\bar{t}j$ **precise theoretical predictions**

What are our results?

- Analytic pole check $\to$ direct determination of the **finite remainder !**
- **Numerical evaluation of the two-loop amplitudes**

4 key questions

How?

- Optimized IBP relations (`NeatIBP`)
- **Finite fields** framework
- **Special function** basis

What's next?

- Deliver **pheno viable** results
- Explore analytical **reconstruction** viability

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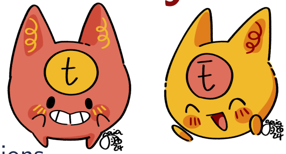

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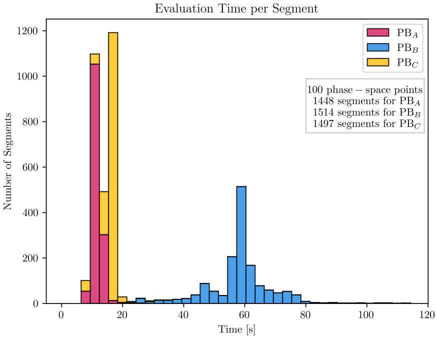
Backup

Numerical evaluation of the MIs



- Comparison of the evaluation performance for the different topologies using `DiffExp`
- Numerical checks of the result against `AMFLow`
- Evaluation strategy still not optimised for phenomenological applications

[Badger, Becchetti, Giraudo, Zoia '24]