MAX-PLANCK-INSTITUT
FÜR PHYSIK

On one-loop corrections to the Bunch-Davies wavefunction of the universe

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Based on: P. Benincasa, G. Brunello, M. K. Mandal, P. Mastrolia and F.V. [2408.16386]

Loop-the-Loop: Feynman Calculus and its Applications to Gravity and Particle Physics

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Outline

- Motivation to study loop corrections to cosmological observables
- Background
- Cosmological integrals as twisted period integrals
- Differential equations for cosmological integrals
- Bubble Integral: differential equations and full integration
- Triangle Integral: “zero-sector” and elliptic sector
- Outlook

Motivation

- Infrared effects in de Sitter: for theories with light scalar fields in a FLRW spacetime, at late times the accumulation of long wavelength modes leads to the breakdown of perturbation theory.

[Tsamis, Woodard], [Polyakov], [Baumgart, Sundrum], [Senatore, Gorbenko], [Green, Cohen], [Céspedes, David, Wang], [Benincasa, F.V.], …

- Phenomenology: the inflaton couples to fermions at loop level.

[Chen, Wang, Xianyu]

Cosmological Wavefunction

$$\Psi[\Phi] = \mathcal{N} \int_{\phi(-\infty)=0}^{\phi(0)=\Phi} \mathcal{D}\phi \, e^{iS^{(\varepsilon)}[\phi]} \longrightarrow \langle f \rangle = \int \mathcal{D}\Phi |\Psi[\Phi]|^2 f[\Phi]$$

The action for a general scalar in power-law FLRW cosmologies, after a conformal transformation:

$$S[\phi] = -\int_{-\infty}^{0} d\eta \int d^d x \left[\frac{1}{2} (\partial\phi)^2 - \frac{1}{2} \mu^2(\eta) \, \phi^2 - \sum_{k \geq 3} \lambda_k(\eta) \phi^k \right]$$

With couplings:

$$\mu^2(\eta) = m^2 a^2(\eta) + 2d \left(\xi - \frac{d-1}{4d} \right) \left[\partial_\eta \left(\frac{\dot{a}}{a} \right) + \frac{d-1}{2} \left(\frac{\dot{a}}{a} \right)^2 \right]$$

$$\lambda_k(\eta) = \lambda_k \, [a(\eta)]^{2 + \frac{(d-1)(2-k)}{2}}$$

$\xi = \frac{d-1}{4d} \longrightarrow$ Conformal Coupling

$\xi = 0 \longrightarrow$ Minimal Coupling

Wavefunction Expansion

Expanding the fields around the classical solution, the wavefunction acquires the form:

$$\Psi[\Phi] = e^{iS_2^{(\varepsilon)}[\Phi]}\mathcal{N}\int_{\varphi(-\infty)=0}^{\varphi(0)=0}\mathcal{D}\varphi\, e^{iS_2^{(\varepsilon)}[\varphi]+iS_{\text{int}}^{(\varepsilon)}[\Phi,\varphi]}$$

$$= e^{iS_2^{(\varepsilon)}[\Phi]}\left\{1+\sum_{n\geq 1}\int\prod_{j=1}^{n}\left[\frac{d^d p_j}{(2\pi)^d}\Phi[\vec{p}_j]\right]\sum_{L\geq 0}\tilde{\psi}_n^{(L)}(\vec{p}_1,\ldots,\vec{p}_n;\,\varepsilon)\right\}$$

Sum of connected Feynman graphs:

$$\tilde{\psi}_n^{(L)}\Big|_{\text{connected}} = \sum_{\mathcal{G}\subset\mathcal{G}_n^{(L)}}\tilde{\psi}_{\mathcal{G}} \longrightarrow \tilde{\psi}_{\mathcal{G}} = \int_{-\infty}^{0}\prod_{s\in\mathcal{V}}\left[d\eta_s\, V_s\, \phi_\circ^{(s)}\right]\prod_{e\in\mathcal{E}}G_e(y_e;\,\eta_{s_e},\eta_{s'_e})$$

Momentum flowing in the internal edge (→ y_e)

These can be seen as Feynman integrals for scalars in FRW cosmologies.

Cosmological Integrals

[Arkani-Hamed, Benincasa, Postnikov]

[Benincasa]

More concretely, for a general scalar in FRW cosmologies the Feynman integrals become:

$$\tilde{\psi}_{\mathcal{G}}^{\{l_e\}} = \prod_{j=1}^{n} \left[\frac{1}{E_j^l} \hat{\mathcal{O}}_l(E_j) \right] \prod_{e\in\mathcal{E}} \left[\frac{1}{y_e^{2l_e}} \right] \int_{-\infty}^{0} \prod_{s\in\mathcal{V}} \left[d\eta_s \, \frac{\lambda_k(\eta_s)}{(-\eta_s)^{\rho_s}} \, e^{iX_s\eta_s} \prod_{e\in\mathcal{E}} G(y_e; \eta_{s_e}, \eta_{s'_e}) \right]$$

Boundary energies for external states are:

$$X_s := \sum_{j=1}^{k_s} E_j$$

Here l_j labels the external states and l_e the internal:

$$\rho_s := \sum_{j=1}^{k_s} l_j + \sum_{e\in\mathcal{E}_s} l_e$$

$l = 0$ Conformally coupled

$l = 1$ Minimally coupled

The explicit form of the couplings is:

$$\lambda_k(\eta_s) = \lambda_k \left[-\ell_\gamma/\eta_s \right]^{\gamma\left[2-\frac{(k-2)(d-1)}{2}\right]}$$

And after the Fourier Transform:

$$\frac{\lambda_k(\eta_s)}{(-\eta_s)^{\rho_s}} = i^{\beta_{k_s,l}} (i\lambda_k) \ell_\gamma^{\gamma\left[2-\frac{(k-2)(d-1)}{2}\right]} \int_{-\infty}^{+\infty} d\epsilon_s \, e^{i\epsilon_s\eta_s} \, \epsilon_s^{\beta_{k_s,l}-1} \vartheta(\epsilon_s)$$

One can get the momentum representation of these cosmological Feynman integrals in terms of $x_s = \epsilon_s + X_s$ and y_e

Cosmological Integrals

[Benincasa, FV]

$\alpha := \rho_s + \gamma[2 - (k-2)(d-1)/2]$

$d = 3$

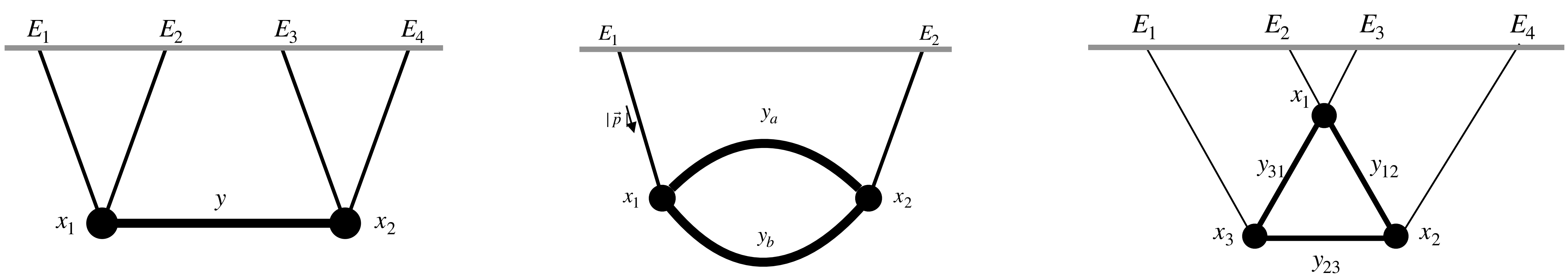
Cosmology | Loop Integration | Rational Function

$$\mathcal{I}_{\mathcal{G}}[\alpha,\,\beta;\,\mathcal{X}] \;=\; \mathfrak{c}_{d,n_e^{(L)},L}\int_{\mathbb{R}_+^{n_s}}\prod_{s\in\mathcal{V}}\left[\frac{dx_s}{x_s}\,x_s^{\alpha_s}\right]\int_{\Gamma}\prod_{e\in\mathcal{E}^{(L)}}\left[\frac{dy_e}{y_e}\,y_e^{\beta_e}\right]\left[\frac{\mathrm{Vol}^2(y_e^2,P^2_{i\ldots j})}{\mathrm{Vol}^2(P^2_{i\,.\,.\,.\,j})}\right]^{\frac{d-n_s-L}{2}}\frac{\mathfrak{n}_\delta(x_s,y_e,\,X_s)}{\prod_{\mathfrak{g}\subseteq\mathcal{G}}[q_{\mathfrak{g}}(x_s,y_e,\,X_s)]^{\tau_{\mathfrak{g}}}}$$

Polynomial of degree δ

Linear polynomials

Examples of graphs:



Cosmological Loop Integrals as Twisted Period Integrals

$$\mathcal{I}^{(\mathcal{G})}_{\{\tau_{\mathfrak{g}}\}} := \int_{\Gamma} \prod_{e\in\mathcal{E}^{(L)}} \left[\frac{dy_e}{y_e}\, y_e^{\beta_e}\right] \mu_d\left(y_e;\mathcal{X}\right)\, \frac{\mathfrak{n}_{\delta}(x,y;\mathcal{X})}{\prod_{\mathfrak{g}\subseteq\mathcal{G}} \left[q_{\mathfrak{g}}(x,y;\mathcal{X})\right]^{\tau_{\mathfrak{g}}}}$$

Twisted period integral structure:

$$\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} := \int_{\Gamma} u\,\varphi \qquad\qquad \varphi := \frac{\prod_{e\in\mathcal{E}^{(L)}} dy_e}{\prod_{\mathfrak{g}\in\mathfrak{G}^{(j)}\cup\{e\}} \left[q_{\mathfrak{g}}(y)\right]^{\tau_{\mathfrak{g}}}}$$

With:

$$u := \mu_d = \kappa_0\,\kappa^{\chi} \qquad\qquad \kappa := \mathrm{Vol}^2\left\{\Sigma_{n_e^{(L)}}\left(y^2, P^2_{i\ldots j}\right)\right\}$$

$$\chi := \frac{d-n_s-L}{2} \qquad\qquad \kappa_0 := c_{d,n_e^{(L)},L}\left[\mathrm{Vol}^2\left\{\Sigma_{n_e^{(L)}-L}\left(P^2_{i\ldots j}\right)\right\}\right]^{-\chi}$$

The integrals are dimensionally regulated.

The twist vanishes in the boundary of the integration contour:

$$\kappa\big|_{\partial\Gamma} = 0$$

Differential Equations for Loop Cosmological Integrals

Triangulations of Cosmological Polytope help decomposing the universal integrand into a sum of rational functions with trivial numerator. Bubble example:

[Benincasa, Torres Bobadilla]

$$\frac{2(\tilde{x}_1+\tilde{x}_2+y_{12}+y_{21})}{(\tilde{x}_1+\tilde{x}_2)(\tilde{x}_1+y_{12}+y_{21})(\tilde{x}_2+y_{12}+y_{21})(\tilde{x}_1+\tilde{x}_2+2y_{12})(\tilde{x}_1+\tilde{x}_2+2y_{21})} =$$

$$= \frac{1}{(\tilde{x}_1+\tilde{x}_2)(\tilde{x}_1+y_{12}+y_{21})(\tilde{x}_2+y_{12}+y_{21})(\tilde{x}_1+\tilde{x}_2+2y_{12})} + \frac{1}{(\tilde{x}_1+\tilde{x}_2)(\tilde{x}_1+y_{12}+y_{21})(\tilde{x}_2+y_{12}+y_{21})(\tilde{x}_1+\tilde{x}_2+2y_{21})}$$

Integral relations:

Partial fraction identities: $q_{\mathfrak{g}'}(y) = c_{\mathfrak{g}'0} + \sum_{\tilde{\mathfrak{g}}\in\mathfrak{G}_{\mathrm{B}}} c_{\mathfrak{g}'\tilde{\mathfrak{g}}}\, q_{\tilde{\mathfrak{g}}}(y) \longrightarrow \mathcal{I}^{(j)}_{\tau_{\mathfrak{g}'}-1} = c_{\mathfrak{g}'0}\mathcal{I}^{(j)}_{\{\tau_{\mathfrak{g}}\}} + \sum_{\tilde{\mathfrak{g}}\in\mathfrak{G}_{\mathrm{B}}} c_{\mathfrak{g}'\tilde{\mathfrak{g}}}\mathcal{I}^{(j)}_{\tau_{\tilde{\mathfrak{g}}}-1}$

Integration by parts identities:

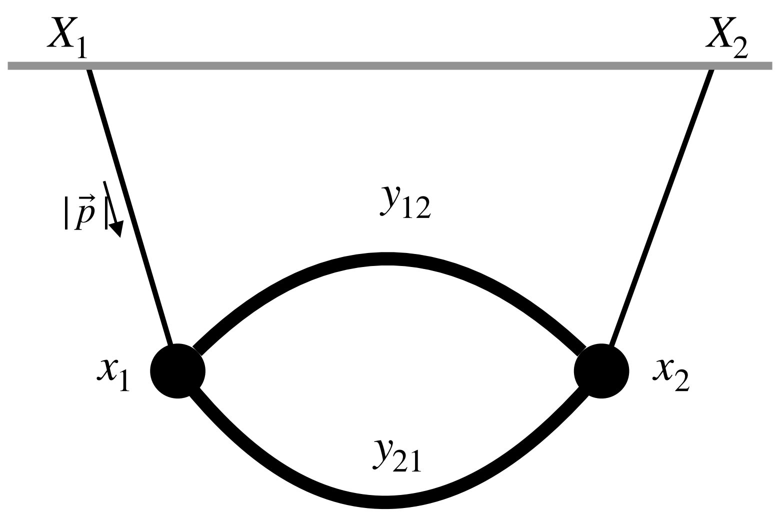
$$\sum_{e'\in\mathcal{E}^{(L)}} \int_{\Gamma} \left[\prod_{e\in\mathcal{E}^{(L)}} dy_e\right] \frac{\partial}{\partial y_{e'}} \left[\kappa^{\chi} \frac{n_e(x,y,\mathcal{X})}{\prod_{g\subseteq\mathcal{G}} [q_{\mathfrak{g}}]^{\tau_{\mathfrak{g}}}}\right] = 0 \longleftarrow \sum_{e\in\mathcal{E}^{(L)}} (\partial_{y_e}\kappa)\, n_e = n_0\,\kappa$$

[Singular] Solution of *Syzygy* equations

Differential Equations:

$$\partial_x \mathcal{I}_i = \sum_j (\mathbb{A}_x)_{ij}\,\mathcal{I}_j \longleftarrow \partial_x\kappa = \sum_{i=1}^{n_e^{(1)}} w_i\,\partial_{y_i}\kappa + w_0\,\kappa$$

Bubble Integral



$$\mathcal{I}^{(2,1)}_{\{1\}} = \kappa_0 \int_\Gamma \prod_{e\in\mathcal{E}^{(1)}} [dy_e\, y_e]\, \frac{\kappa^\chi}{q_{\mathcal{G}}\, q_{\mathfrak{g}_1}\, q_{\mathfrak{g}_2}} \left(\frac{1}{q_{\mathcal{G}_{12}}} + \frac{1}{q_{\mathcal{G}_{21}}}\right)$$

$$\kappa = -\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & y_{12}^2 & y_{21}^2 \\ 1 & y_{12}^2 & 0 & P^2 \\ 1 & y_{21}^2 & P^2 & 0 \end{vmatrix} = \left[(y_{12}+P)^2 - y_{21}^2\right]\left[y_{21}^2 - (y_{12}-P)^2\right]$$

$$\begin{aligned} q_{\mathcal{G}} &= \tilde{x}_1 + \tilde{x}_2 \\ q_{\mathfrak{g}_1} &= \tilde{x}_1 + y_{12} + y_{21} \\ q_{\mathfrak{g}_2} &= \tilde{x}_2 + y_{12} + y_{21} \\ q_{\mathcal{G}_{12}} &= \tilde{x}_1 + \tilde{x}_2 + 2y_{12} \\ q_{\mathcal{G}_{21}} &= \tilde{x}_1 + \tilde{x}_2 + 2y_{21} \end{aligned}$$

Consider the twisted period integral:

$$\mathcal{I}^{(2,1)}_{\tau_{\mathfrak{g}_1}\tau_{\mathcal{G}_{12}}} := \int_\Gamma \kappa^\chi\, \varphi_{\tau_{\mathfrak{g}_1}\tau_{\mathcal{G}_{12}}} \qquad \varphi_{\tau_{\mathfrak{g}_1}\tau_{\mathcal{G}_{12}}} := \frac{dy_{12}dy_{21}}{q_{\mathfrak{g}_1}^{\tau_{\mathfrak{g}_1}}\, q_{\mathcal{G}_{12}}^{\tau_{\mathcal{G}_{12}}}}$$

Basis of master integrals of dimension 6:

$$\{\boxed{\mathcal{I}_{00}}, \boxed{\mathcal{I}_{10}}, \boxed{\mathcal{I}_{01}, \mathcal{I}_{02}, \mathcal{I}_{-11}}, \boxed{\mathcal{I}_{11}}\}$$

“Zero-Sector” is a Feynman integral with half-integer powers in the propagators:

$$\mathcal{I}_{00} := \int \frac{d\vec{\ell}}{|\vec{\ell}\,|\,|\vec{\ell}+\vec{P}\,|}$$

Bubble Integral

After rotating to get an ϵ-factorised form:

[Lee], [CANONICA], [Initial]

$$\partial_x \mathcal{I}_i = \epsilon \sum_j (\mathbb{A}_x)_{ij} \mathcal{I}_j \qquad d\mathbb{A} = \hat{\mathbb{A}}_{\tilde{x}_1} d\tilde{x}_1 + \hat{\mathbb{A}}_{\tilde{x}_2} d\tilde{x}_2 + \hat{\mathbb{A}}_P \, dP = \sum_{i=1}^{8} \mathbb{M}_i \, d\log(w_i)$$

With symbol letters:

$$w_i \in \{P, \tilde{x}_1 + \tilde{x}_2, \tilde{x}_1 + P, \tilde{x}_2 + P, \tilde{x}_1 + \tilde{x}_2 + 2P, \tilde{x}_1 - P, \tilde{x}_2 - P, \tilde{x}_1 + \tilde{x}_2 - 2P\}$$

One can then integrate the differential equations and fix boundary conditions by:

- Computing the zero-sector integral directly.
- Demanding a finite solution at the singularities: $\{\tilde{x}_1 - P, \tilde{x}_2 - P, \tilde{x}_1 + \tilde{x}_2 - 2P\}$.
- Computing one integral directly in the line $\tilde{x}_1 + \tilde{x}_2 \to 0$.

$$\begin{aligned} \mathcal{I}^{(2,1)}_{\{1\}} &= -\frac{1}{\epsilon(\tilde{x}_1 + \tilde{x}_2)} + \frac{(-2\log(P) - \gamma_E + 2 - \log(4\pi))}{\tilde{x}_1 + \tilde{x}_2} + \frac{2}{\tilde{x}_1^2 - \tilde{x}_2^2}\left[\tilde{x}_2 \log\left(\frac{P + \tilde{x}_1}{P}\right) - \tilde{x}_1 \log\left(\frac{P + \tilde{x}_2}{P}\right)\right] \\ &- \frac{1}{P}\left[\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{P - \tilde{x}_2}{P + \tilde{x}_1}\right) + \mathrm{Li}_2\left(\frac{P - \tilde{x}_1}{P + \tilde{x}_2}\right) + \frac{1}{2}\log^2\left(\frac{P + \tilde{x}_1}{P + \tilde{x}_2}\right)\right] \end{aligned}$$

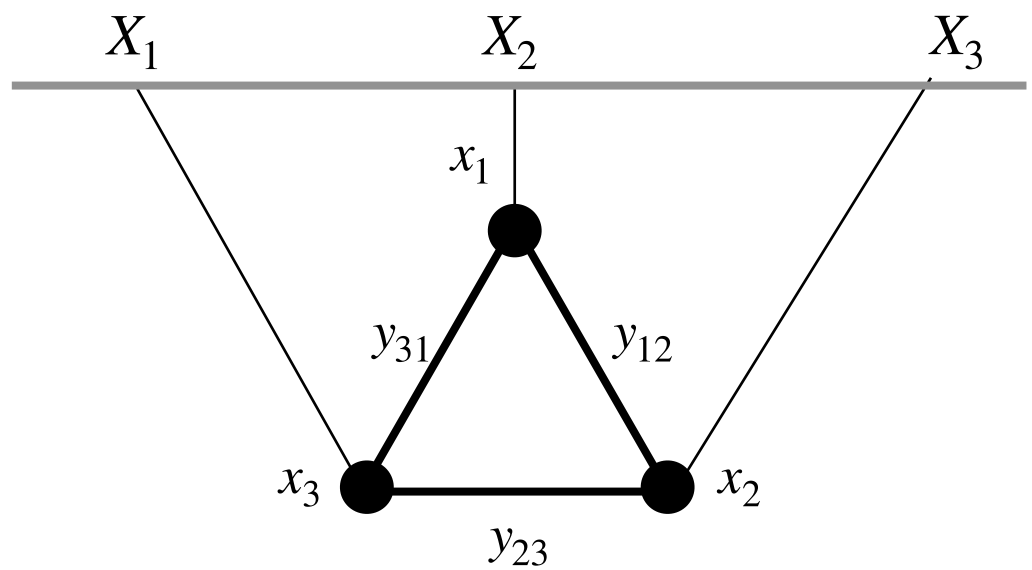
Bubble Integral: Site Integration

One can integrate the flat-space wavefunction over the site-weights via Method of Brackets:

$$
\begin{aligned}
I_2^{(1)} = & \frac{2^{-3-2\alpha}\pi^{3/2}\left(X_1+X_2\right)^{1+2\alpha}\csc(\pi\alpha)^2\Gamma\left(-\frac{1}{2}-\alpha\right)}{\Gamma[-\alpha]}\left(2-\frac{1}{\epsilon}-\log\left(4\pi e^{\gamma_E}P^2\right)\right) \\
& +\frac{\pi^{3/2}\csc^2(\pi\alpha)}{8(\alpha+1)^2P}\left[-4\sqrt{\pi}\left(\left(P+X_1\right)^{\alpha+1}-2\left(X_1-P\right)^{\alpha+1}\right)\left(P+X_2\right)^{\alpha+1}-\frac{4^{-\alpha}\Gamma\left(-\alpha-\frac{1}{2}\right)\left(X_1+X_2\right)^{2\alpha+2}}{\Gamma(-\alpha)}{}_2F_1\left(1,-2(\alpha+1);-\alpha;\frac{P+X_1}{X_1+X_2}\right)\right] \\
& +\frac{\pi^2\csc(\pi\alpha)\csc(2\pi\alpha)\left(P+X_1\right)^{\alpha}}{4\alpha+2}\left[-2\left(P+X_1\right)\left(\left(P-X_2\right)^{\alpha}+(-1)^{\alpha}\left(P+X_2\right)^{\alpha}\right)+(-1)^{\alpha}\left(X_1-X_2\right)\left(P+X_1\right)^{\alpha}{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1-X_2}{P+X_1}\right)\right. \\
& \left.+\left(X_1+X_2\right)\left(P+X_1\right)^{\alpha}{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1+X_2}{P+X_1}\right)\right] \\
& -\frac{\pi^{5/2}4^{-\alpha-1}\csc(\pi\alpha)\csc(2\pi\alpha)}{\Gamma(-\alpha)\Gamma\left(\alpha+\frac{3}{2}\right)\left(P+X_1\right)}\left[(-1)^{\alpha}\left(X_1-X_2\right)^{2\alpha+2}{}_3F_2\left(1,1,\alpha+2;2,2\alpha+3;\frac{X_1-X_2}{P+X_1}\right)+\left(X_1+X_2\right)^{2\alpha+2}{}_3F_2\left(1,1,\alpha+2;2,2\alpha+3;\frac{X_1+X_2}{P+X_1}\right)\right] \\
& +\frac{\pi^{5/2}2^{-2\alpha-1}\csc(\pi\alpha)\csc(2\pi\alpha)\left((-1)^{\alpha}\left(X_1-X_2\right)^{2\alpha+1}+\left(X_1+X_2\right)^{2\alpha+1}\right)}{\Gamma(-\alpha)\Gamma\left(\alpha+\frac{3}{2}\right)}\log\left(\frac{P+X_1}{P}\right) \\
& +\left(X_1\leftrightarrow X_2\right)
\end{aligned}
$$

In the limit $\alpha \to 1$, only Log, Log^2 and Li_2 appear.

Triangle Integral



$$\mathcal{I}^{(3,\,1)}_{\{1\}} = \kappa_0 \int_\Gamma \prod_{e\in\mathcal{E}^{(1)}} [dy_e\, y_e]\, \frac{\kappa^\chi}{q_{\mathcal{G}} \prod_{j=1}^{3} q_{\mathfrak{g}_j}} \left[\frac{1}{q_{\mathcal{G}_{12}}}\left(\frac{1}{q_{\mathfrak{g}_{23}}}+\frac{1}{q_{\mathfrak{g}_{31}}}\right) + +\frac{1}{q_{\mathcal{G}_{23}}}\left(\frac{1}{q_{\mathfrak{g}_{31}}}+\frac{1}{q_{\mathfrak{g}_{12}}}\right) + \frac{1}{q_{\mathcal{G}_{31}}}\left(\frac{1}{q_{\mathfrak{g}_{12}}}+\frac{1}{q_{\mathfrak{g}_{23}}}\right)\right]$$

$$\kappa = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & y_{12}^2 & y_{23}^2 & y_{31}^2 \\ 1 & y_{12}^2 & 0 & P_2^2 & P_1^2 \\ 1 & y_{23}^2 & P_2^2 & 0 & P_3^2 \\ 1 & y_{31}^2 & P_1^2 & P_3^2 & 0 \end{vmatrix}$$

$$q_{\mathcal{G}} = \sum_{i=1}^{3} X_i$$

$$q_{\mathfrak{g}_j} = y_{j-1,j} + X_j + y_{j,j+1}$$

$$q_{\mathcal{G}_{j,j+1}} = \sum_{s=1}^{3} X_s + y_{j,j+1}$$

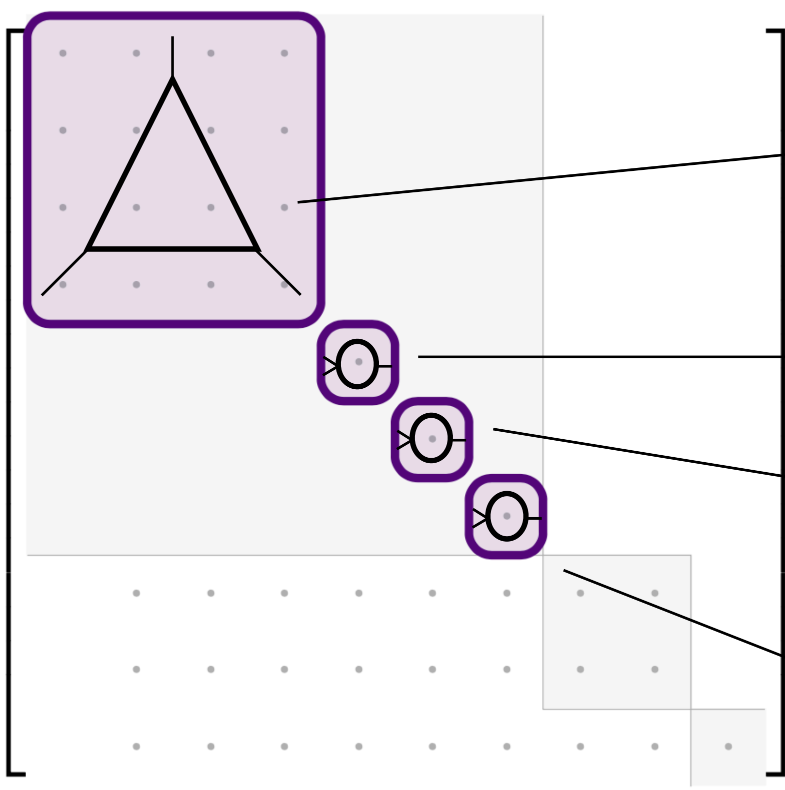
We will consider the cases where there is only one external leg per site:

$$P_1 \to X_1, P_2 \to X_2, P_3 \to X_3$$

Triangle Integral

$$\mathcal{I}_{\tau_{\mathfrak{g}_1}\tau_{\mathfrak{g}_2}\tau_{\mathfrak{g}_3}\tau_{\mathcal{G}_{12}}\tau_{\mathfrak{g}_{23}}} = \int_\Gamma \mu_d\, \varphi_{\tau_{\mathfrak{g}_1}\tau_{\mathfrak{g}_2}\tau_{\mathfrak{g}_3}\tau_{\mathcal{G}_{12}}\tau_{\mathfrak{g}_{23}}} \qquad \varphi_{\tau_{\mathfrak{g}_1}\tau_{\mathfrak{g}_2}\tau_{\mathfrak{g}_3}\tau_{\mathcal{G}_{12}}\tau_{\mathfrak{g}_{23}}} = \frac{\prod_{e\in\mathcal{E}^{(1)}} dy_e}{q_{\mathfrak{g}_1}^{\tau_{\mathfrak{g}_1}} q_{\mathfrak{g}_2}^{\tau_{\mathfrak{g}_2}} q_{\mathfrak{g}_3}^{\tau_{\mathfrak{g}_3}} q_{\mathcal{G}_{12}}^{\tau_{\mathcal{G}_{12}}} q_{\mathfrak{g}_{23}}^{\tau_{\mathfrak{g}_{23}}}}$$

The “zero-sector” of the Triangle can be decomposed into homogeneous blocks:



$$\mathcal{I}_{00000} = \int_{\mathbb{R}^3} d\vec{l}\, \frac{1}{|\vec{l}||\vec{l}+\vec{P}_1||\vec{l}+\vec{P}_1+\vec{P}_2|}$$

$$y_{12}\varphi_{00000} = \int_{\mathbb{R}^3} d\vec{l}\, \frac{1}{|\vec{l}+\vec{P}_1||\vec{l}+\vec{P}_1+\vec{P}_2|}$$

$$y_{23}\varphi_{00000} = \int_{\mathbb{R}^3} d\vec{l}\, \frac{1}{|\vec{l}||\vec{l}+\vec{P}_1+\vec{P}_2|}$$

$$y_{31}\varphi_{00000} = \int_{\mathbb{R}^3} d\vec{l}\, \frac{1}{|\vec{l}||\vec{l}+\vec{P}_1|}$$

This sector has a basis of dimension seven:

$$\{\varphi_{0000},\ y_{12}^2\varphi_{0000},\ y_{23}^2\varphi_{0000},\ y_{31}^2\varphi_{0000},\ y_{12}\varphi_{0000},\ y_{23}\varphi_{0000},\ y_{31}\varphi_{0000}\}$$

Zero Sector for General one loop graphs

$$\mathcal{I}_{\{1\}}^{(n_s\,,\,1;\,0)} = \frac{\kappa_0}{2^n}\int \frac{\prod\limits_{e\in\mathcal{E}^{(1)}} dy_e^2}{\prod\limits_{e\in\mathcal{E}^{(1)}} \left[y_e^2\right]^{1/2}} \left[\kappa(y_e^2)\right]^{\epsilon} \longrightarrow \mathcal{I}_{\{1\}}^{(n_s\,,\,1;\,0)} = \frac{1}{2^n}\int_{\mathbb{R}^n} d\vec{l}\, \frac{1}{\left|\vec{l}\right| \left|\vec{l}+\vec{P}_1\right| \cdots \left|\vec{l}+\sum\limits_{j=1}^{n_s-1}\vec{P}_j\right|}$$

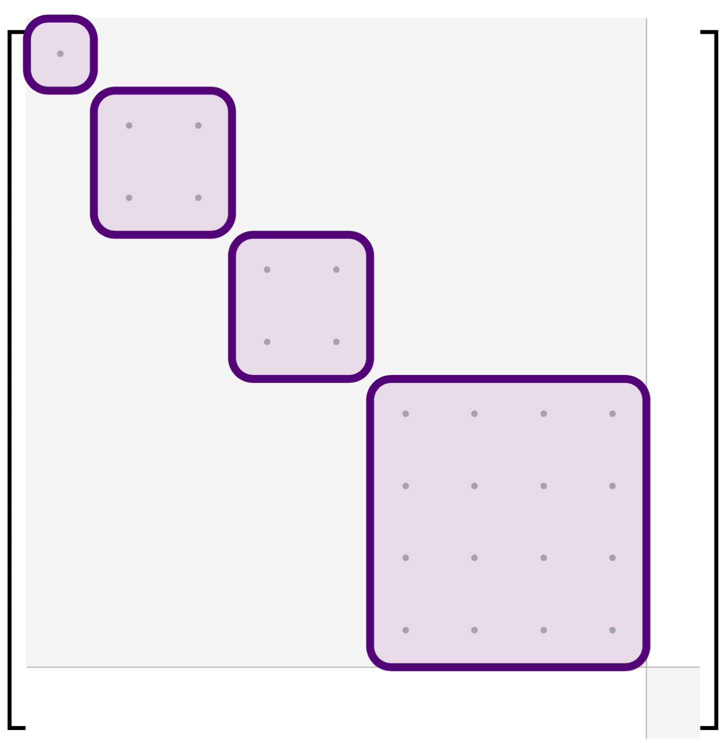
- Integration by parts identities relate integrals shifted by integer powers
- Each sub-sector will be one combination of integer and half-integer powers
- The “zero-sector” admits the following dimension for a one-loop n_s-sites graph:

$$\nu_{n_s}^{(\mathrm{CI})} = 3^{n_s} - 2^{n_s-1}(2+n_s)$$

Triangle: Elliptic Sector

$$\mathcal{I}_{\tau_{\mathcal{G}_{12}}} = \int_\Gamma \mu_d \, \varphi_{\tau_{\mathcal{G}_{12}}} \qquad \varphi_{\tau_{\mathcal{G}_{21}}} = \frac{\prod_{e \in \mathcal{E}^{(1)}} dy_e}{q_{\mathcal{G}_{12}}^{\tau_{\mathcal{G}_{12}}}} \qquad q_{\mathcal{G}_{12}} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + 2y_{12}$$

The Picard-Fuchs operator of the 4x4 homogenous block factorises: $\mathcal{L}_4 = \mathcal{L}_1 \mathcal{L}_1 \mathcal{L}_2$



$y_{23}y_{31}\varphi_{001}$
$y_{23}\varphi_{001}$
$y_{23}\varphi_{002}$
$y_{31}\varphi_{001}$
$y_{31}\varphi_{002}$
φ_{002}
φ_{001}
$y_{23}^2\varphi_{001}$
$y_{31}^2\varphi_{001}$

$x_i \to a_i \lambda$

$$\mathcal{L}_2 = \frac{d^2}{d\lambda^2} + \frac{5\left(a^2-1\right)^2\lambda^4 - 6\left(a^2+1\right)\lambda^2 + 1}{\left(a^2-1\right)^2\lambda^5 - 2\left(a^2+1\right)\lambda^3 + \lambda}\frac{d}{d\lambda} + \frac{3\left(a^2-1\right)^2\lambda^2 - 2\left(a^2+1\right)}{\left(a^2-1\right)^2\lambda^4 - 2\left(a^2+1\right)\lambda^2 + 1}$$

Solution: $\psi_{1,2}(K^2)\,, \quad K^2 = \dfrac{(a-1)^2\lambda^2 - 1}{(a+1)^2\lambda^2 - 1}$

In the total energy pole $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \to 0$, it factorises in two linear operators:

$$\mathcal{L}_1 = \frac{d}{d\lambda} + \frac{2\lambda^4 + 5\lambda^2 - 1}{(\lambda-1)\lambda(\lambda+1)\left(\lambda^2+1\right)} \qquad \mathcal{L}_1 = \frac{d}{d\lambda} + \frac{\lambda}{(\lambda-1)(\lambda+1)}$$

Outlook

- Find canonical form for the differential equations of the Triangle graph and integrate.
- Classification of the geometries appearing in higher point one loop graphs.
- Differential equations for site- and loop-integration.

[Baumann, Goodhew, Lee]