A Physical Basis for Cosmological Correlators from Cuts

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Based on 2411.xxxxx with A. Pokraka

Loop-the-Loop Workshop, MITP

Cosmological Correlators as Twisted Integrals

→ Object of interest: Wavefunction for cosmological fluctuations

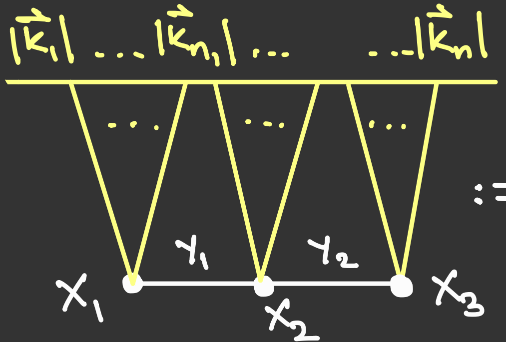
Flat-space wavefunction $(\hat{\Omega}_n^{(l)})$ $\xrightarrow{\text{seeds}}$ Wavefunction in power-law FRW cosmology $(\Psi_n^{(l)})$

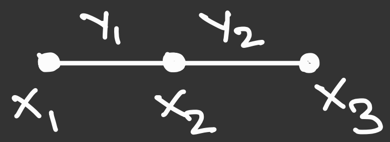
Flat-space Wavefunction:

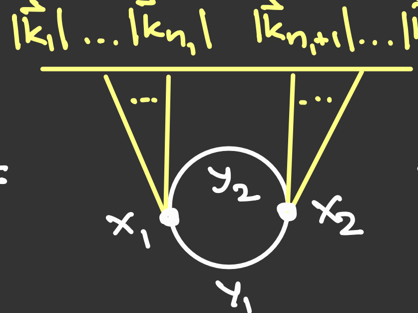
(Arkani-Hamed, Benincasa, Postnikov '17; Benincasa '22)

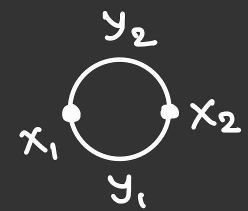
→ Setup: $$S = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \sum_{p>2} \frac{\lambda_p}{p!} \phi^p \right]$$

→ $\hat{\Omega}_n^{(l)}$: computed using Feynman diagrammatics

$\hat{\Omega}_3^{(0)} =$ 

$:=$ 

; $\hat{\Omega}_2^{(1)} =$ 

$:=$ 

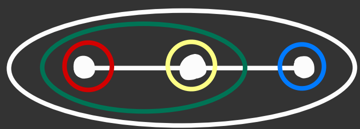
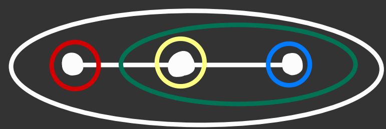
Combinatorial def^{n} of $\hat{\Omega}_n^{(0)}$: (Arkani-Hamed, Benincasa, Postnikov '17)

→ Form complete tubings – maximal set of non-overlapping graph tubings τ_i and sum over all possible complete tubings.

→ To each τ_i, the associated hyperplane polynomial S_i:

$$\boxed{\tau_i \longrightarrow S_i = \sum_{v \in \mathcal{V}_i} x_v + \sum_{e \in \mathcal{E}_i} y_e} \, .$$

→ For example, the 3-site graph:

$\hat{\Omega}_3^{(0)} =$  $+$ 

$$= \frac{1}{(x_1+x_2+x_3)(x_1+y_1)(x_2+y_1+y_2)(x_3+y_3)} \left(\frac{1}{x_1+x_2+y_2} + \frac{1}{x_2+x_3+y_1} \right)$$

$$= \frac{1}{S_1 S_2 S_3 S_4} \left(\frac{1}{S_5} + \frac{1}{S_6} \right).$$

Cosmological Wavefunction:

→ Setup: • Theory of conformally coupled scalars with (non-conformal) polynomial interactions in a power-law FRW background:

$$S = \int d^3x \, d\eta \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12} R\phi^2 - \sum_{p>2} \frac{\lambda_p}{p!} \phi^p \right]$$

where the metric is $ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right]$.

• We assume that the scale factor takes the form of a power law:

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{-(1+\varepsilon)} \begin{cases} \varepsilon = 0 & \text{(de Sitter)} \\ \varepsilon \approx 0 & \text{(inflation)} \\ \varepsilon = -1 & \text{(Minkowski)} \\ \varepsilon = -2 & \text{(Radiation domination)} \\ \varepsilon = -3 & \text{(Matter domination)} \end{cases}$$

From flat-space to FRW spacetimes:

→ FRW Wavefunction coefficients $\psi_n^{(l)}$ ⟶ Integrate shifted flat-space wavef$^{n\text{'s}}$ against a "twist" factor

$$\boxed{\psi_n^{(l)} = \int_0^\infty u\, \Psi_n^{(l)}} \qquad \text{(twisted FRW integrals)}$$

where $\Psi_n^{(l)} = \hat{\Omega}_n^{(l)}(x_v + X_v, y_e)\, d^n x_v$ (FRW form),

$$u = \prod_{i=1}^{n} (x_v)^{\varepsilon} = \prod_{i=1}^{n} T_v^{\varepsilon} \qquad \text{(twist)}.$$

→ Denote shifted linear factors in $\Psi_n^{(l)}$ by:

$$\boxed{B_i(x; X, y) = S_i(x + X, y)}, \text{ s.t., } \Psi_n^{(l)} = d^n x \sum_{\tau \in T} \prod_{t \in \tau} \frac{1}{B_t}.$$

Example: the 3-site tree-level FRW wavefunction coefficient $\psi_3^{(0)}$

$$\psi_3^{(0)} = \int_0^\infty (x_1 x_2 x_3)^\varepsilon \, \Psi_3^{(0)}$$

where $\Psi_3^{(0)} = \dfrac{d^3x}{B_1 B_2 B_3 B_4} \left(\dfrac{1}{B_5} + \dfrac{1}{B_6} \right)$

with the linear factors B_i defined by:

$$B_1 = X_1 + x_1 + y_1 \,, \qquad B_4 = X_1 + X_2 + X_3 + x_1 + x_2 + x_3$$

$$B_2 = X_2 + x_2 + y_1 + y_2 \,, \qquad B_5 = X_1 + X_2 + x_1 + x_2 + y_2$$

$$B_3 = X_3 + x_3 + y_3 \,, \qquad B_6 = X_2 + X_3 + x_2 + x_3 + y_1 \,.$$

$\boxed{\text{Each twisted integral is associated to a \underline{hyperplane arrangement}}}$

Recent progress

→ Twisted integrals enormously rich in structure – mathematics governed by the theory of twisted cohomology.

→ The twisted integrals $\psi_n^{(l)}$ form a finite-dimensional vector space, spanned by a basis of master integrals.

→ $\psi_n^{(l)}$ are shown to satisfy first-order differential equations

$$\boxed{d_{kin} \int u \varphi_j = \varepsilon A_{ij} \int u \varphi_j}$$

(Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel '23; SD, Pokraka '23; He, Jiang, Liu, Yang, Zhang '24, ...)

→ Twisted cohomology predicts the size of the vector space:

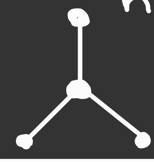
# of independent master integrals = # of bounded regions defined by divisors of the integrand (χ)

(Aomoto '75, Mastrolia, Mizera '19)

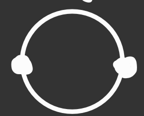
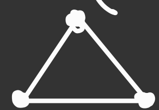
→ Surprisingly, universal rules of kinematic flow pick out a distinguished subset of master integrals $\ll \chi$!

(Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel '23)

At tree-level:

	$n=2$ 	$n=3$ 	$n=4$ (chain) 	$n=4$ (star) 	...
χ	4	25	213	312	
4^{n-1}	4	16	64	64	

At 1-loop:

	$n=2$ (bubble) 	$n=3$ (triangle) 	...
χ	10	99	
$4^n - 2(2^n-1)$	10	50	

(He, Jiang, Liu, Yang, Zhang '24, Baumann, Goodhew, Lee '24, Benincasa, Brunello, Mandal, Mastrolia, Vazão, '24)

QUESTION: Are there physical and geometrical arguments that govern this characterization of the physical subspace of integrals onto which the DEQ closes?

ANSWER: Yes, based on unitarity cuts and the geometry of the associated hyperplane arrangement!

(SD, A. Pokraka; 2411.xxxxx)

Relative twisted cohomology and cuts

→ The twisted cohomology of $\Psi_n^{(l)}$:

$$H^n(M\setminus\mathbb{B};\nabla) := \frac{\text{covariantly closed } n\text{-forms on } M\setminus\mathbb{B}}{\text{covariantly exact } n\text{-forms on } M\setminus\mathbb{B}} \quad \text{— } (*)$$

where $M := \mathbb{C}^n \setminus \{\cup_{i=1}^{n} x_i\}$, $\mathbb{B} := \cup_i \mathbb{B}_i$, $\mathbb{B}_i := \{B_i = 0\}$

classifies all non-trivial differential n-forms on $M\setminus\mathbb{B}$

→ The dual cohomology to $(*)$ is the relative twisted cohomology :

$$\check{H}^n := H^n(M,\mathbb{B},\check{\nabla}) \subseteq \bigoplus_{p=0}^{n} \bigoplus_{|J|=p} \delta_J\left(H^{n-p}\left(M_J ; \check{\nabla}|_J\right)\right)$$

multi-index denoting a cut

$= M\cap\mathbb{B}_J$ is the space associated to the cut $\left(\mathbb{B}_J = \cap_{j\in J}\{B_j=0\}\right)$.

$$\boxed{\text{Dual cohomology} = \text{direct sum of twisted cohomology of each cut.}}$$

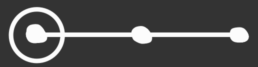
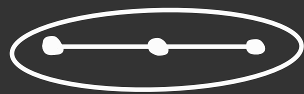
Result: Physical subspace is spanned by all FRW forms that have non-overlapping residues with $\Psi_n^{(l)}$:

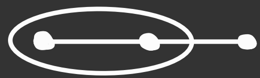
$$\mathcal{H}^n_{\text{phys}} \subset \text{Span}\left\{ d\log_J \wedge \tilde{\Omega}_J \right\}_J \quad \text{s.t.} \quad \text{Res}_J\left[\Psi_n^{(l)}\right] \neq 0$$

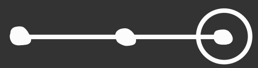
Task ahead: Given a graph contributing to $\Psi_n^{(l)}$, organize a basis of forms that have compatible sequential residues with the physical FRW form $\Psi_n^{(l)}$.

A pedagogical example: the 3-site chain

→ Hyperplane polynomials B_i:

$B_1 =$  $= x_1 + X_1 + y_1$, $B_{T=4} =$  $= x_1 + x_2 + x_3 + X_1 + X_2 + X_3$

$B_2 =$  $= x_2 + X_2 + y_1 + y_2$, $B_5 =$  $= x_1 + x_2 + X_1 + X_2 + y_2$

$B_3 =$  $= x_3 + X_3 + y_2$, $B_6 =$  $= x_2 + x_3 + X_2 + X_3 + y_1$.

→ FRW form:

$$\boxed{\Psi_3^{(0)} = \frac{d^3x}{B_1 B_2 B_3 B_4}\left(\frac{1}{B_5} + \frac{1}{B_6}\right)}$$

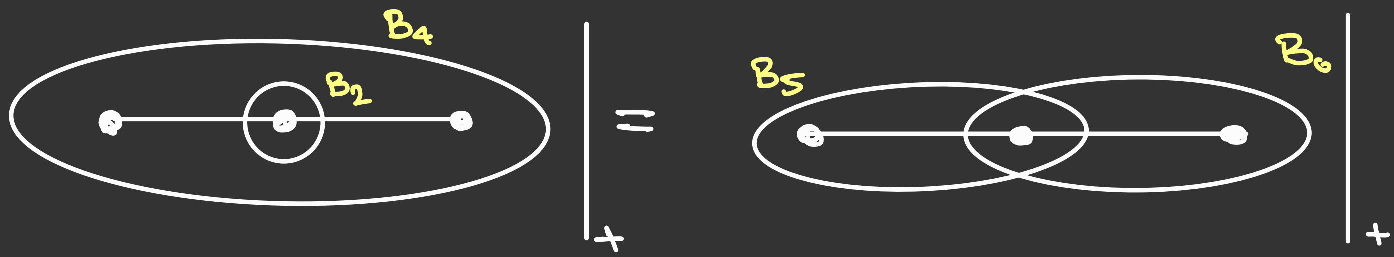
differential form on $M \setminus \mathbb{B}$ w/ $M = \mathbb{C}^3 \setminus \{x_1 x_2 x_3\} = 0$.

→ Now, interested in M after taking a sequential residue Res_J:

$$\boxed{M_J := M \cap \mathbb{B}_J \quad \text{where } \mathbb{B}_J = \cap_{j \in J} \mathbb{B}_j}$$

→ Linear relation: $\boxed{B_2 + B_4 = B_5 + B_6}$

$$\Updownarrow$$



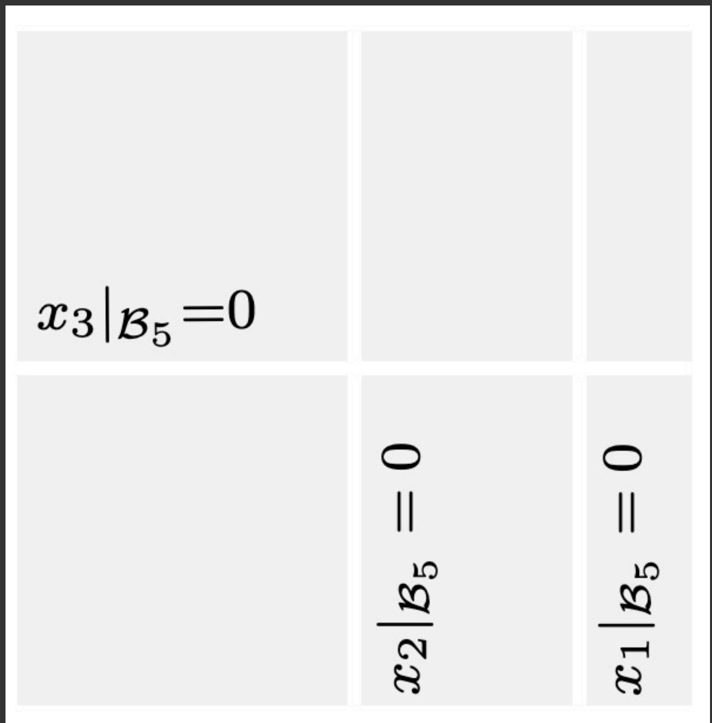
STEP 1: Classify all non-trivial cohomologies corresponding to each cut.

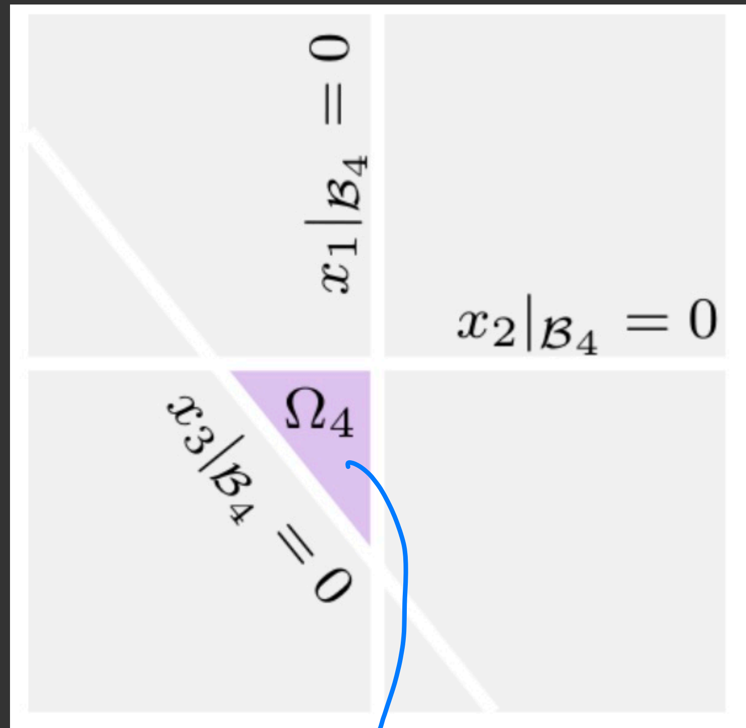
→ 1-cuts / Single residues – 6 possible 1-cuts corresponding to 6 B_i's:

$$\{M_1, M_2, M_3, M_4, M_5, M_6\}.$$

(These are 2-dim$^{n'}$ topological spaces).

Observation: Of all the 1-cuts, only M_4 has a bounded chamber $\Rightarrow$ only 1-cut with non-trivial cohomology.

$M_5 =$ 

, $M_4 =$ 

No bounded chambers $\Rightarrow$ trivial cohomology (M_1, M_2, M_3, M_5, M_6)

1-bounded chamber $\Rightarrow$ one-dimensional cohomology (M_4)

→ <u>2-cuts / Double residues:</u> $\binom{6}{2} = 15$ possible 2-cuts:

$$\Big\{ M_{12}, M_{13}, M_{51}, M_{23}, M_{62}, M_{63}, \boxed{M_{41}, M_{61}, M_{42}, M_{43}, M_{53}, M_{45}, M_{46}, M_{56}} \Big\}$$

$= 8$

only these have non-trivial cohomologies

→ 3-cuts / Triple residues: $\binom{6}{3} = 20 - 4 = 16$ possible 3-cuts:

$$\Big\{ \boxed{M_{123}, M_{412}, M_{612}, M_{413}, M_{513}, M_{613}, M_{451}, M_{561}, M_{423}, M_{523}, M_{463}, M_{563}, M_{452}, M_{462}, M_{562}, M_{456}} \Big\}.$$

$= 16$

$\therefore$ $1 + 8 + 16 = 25$ bounded chambers → 25 FRW forms/ master integrals.

STEP 2: Classify all non-trivial cut cohomologies into physical, unphysical and degenerate/mixed cuts.

- Physical cuts have non-trivial sequential residues on $\Psi_3^{(0)}$:

$$\boxed{\mathrm{Res}_J\left[\Psi_3^{(0)}\right] \neq 0}$$

where $J \in \{\{1\}, \{4,1\}, \{6,1\}, \{4,3\}, \{5,3\}, \{4,5\}, \{4,6\}, \{1,2,3\}, \{6,1,2\}, \{4,1,3\}, \{5,1,3\}, \{6,1,3\}, \{4,5,1\}, \{5,2,3\}, \{4,6,3\}\}$.

- Unphysical cuts have trivial sequential residues on $\Psi_3^{(0)}$:

$$\boxed{\mathrm{Res}_J\left[\Psi_3^{(0)}\right] = 0}$$

(Steinmann-relations) (Benincasa, Mcleod, Vergu '20; Benincasa, Bobadilla '22)

where $J \in \{\{4,2\}, \{5,6\}, \{4,1,2\}, \{5,6,1\}, \{4,2,3\}, \{5,6,3\}\}$.

NB: J corresponds to a sequence that always saturates either side of the linear relation:

$\Longleftrightarrow \; B_2 + B_4 = B_5 + B_6$

- Degenerate / mixed cuts may or may not annihilate $\Psi_3^{(0)}$:

$$\underbrace{\mathrm{Res}_{4,5,2}\left[\Psi_3^{(0)}\right] = -\mathrm{Res}_{4,6,2}\left[\Psi_3^{(0)}\right] = \mathrm{Res}_{4,5,6}\left[\Psi_3^{(0)}\right] = \frac{1}{4y_1y_2},}$$

contributes only one physical form.

while $\mathrm{Res}_{5,6,2}\left[\Psi_3^{(0)}\right] = 0. \longrightarrow$ becomes blue

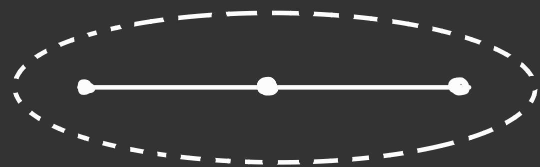
$$\boxed{\text{Left with 16 physical forms for DEQ system.}}$$

→ This two-step classification lands on the physical subspace of FRW forms for any n-site l-loop graph $\Psi_n^{(l)}$!

- needs knowledge of $\boxed{\begin{array}{c}\text{non-trivial cut cohomologies} \\ + \\ \text{system of linear relations}\end{array}}$

→ Universal graphical rules that systematize this algorithm via residue or cut tubings:

$$\mathrm{Res}_{B_4} \Psi_3^{(0)} :=$$

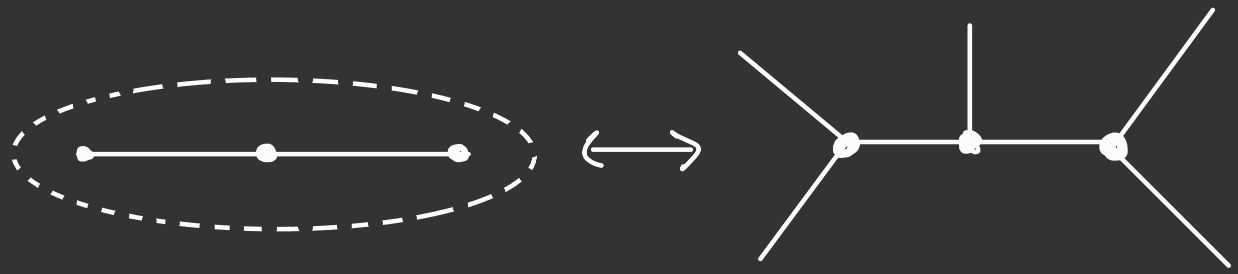


- Correspond to all ways to factorize $\Psi_n^{(l)}$ into product(s) of flat amplitudes $A_n^{(l)}$ only.
- Correspond to all l-dim^{n}' cut cohomologies, counting degen. cut once.

Cut tubings – the 3-site tree example

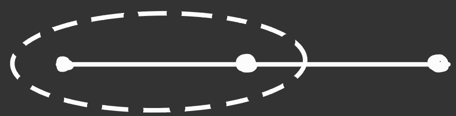
1-cut tubing

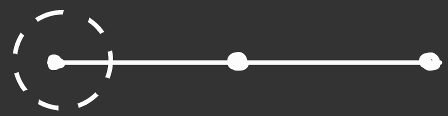
1) The scattering facet :



The uni-dimensional cohomology condition(s):

i) Cut-tubing(s) must enclose all sites of a graph $\Psi_n^{(1)}$



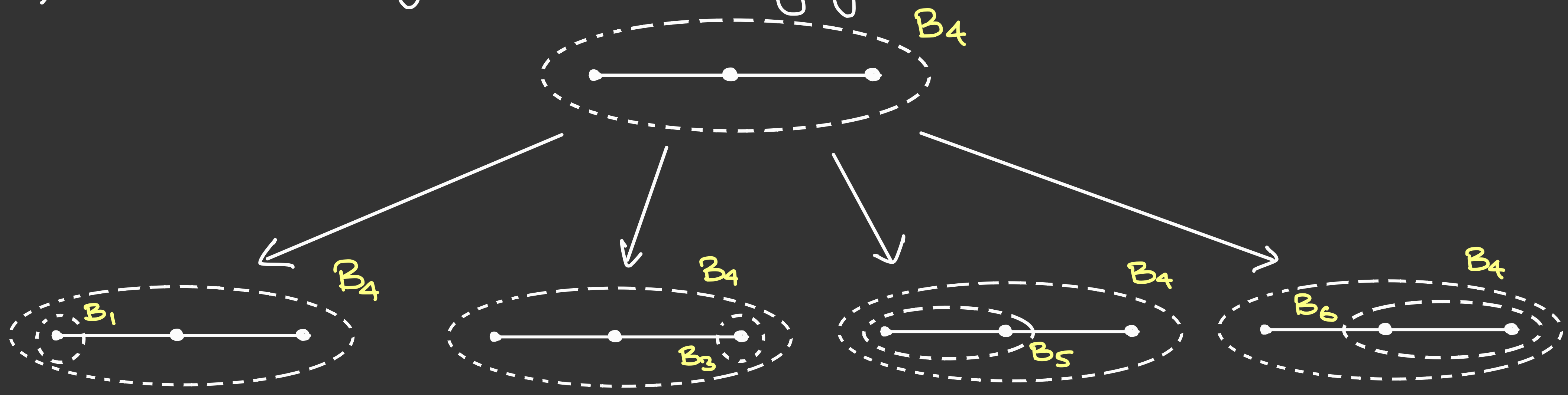


(Not allowed)

ii) Inside each cut-tubing, leave a vertex unenclosed.

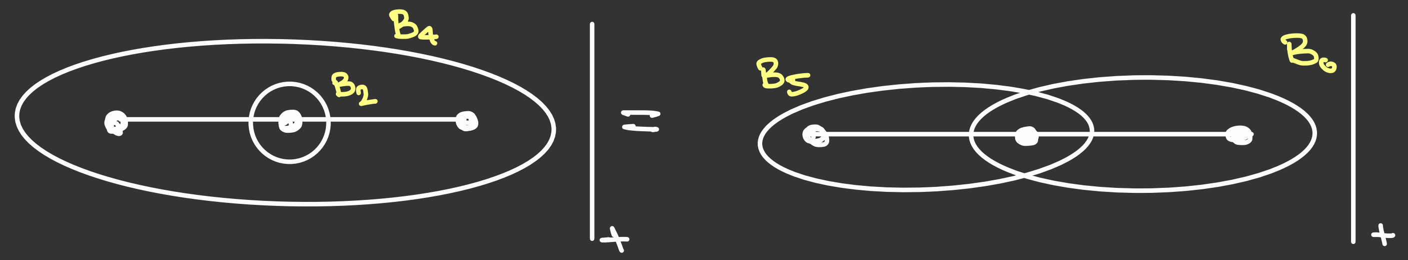
2-cut tubings

2a) Evolution of the scattering facet:

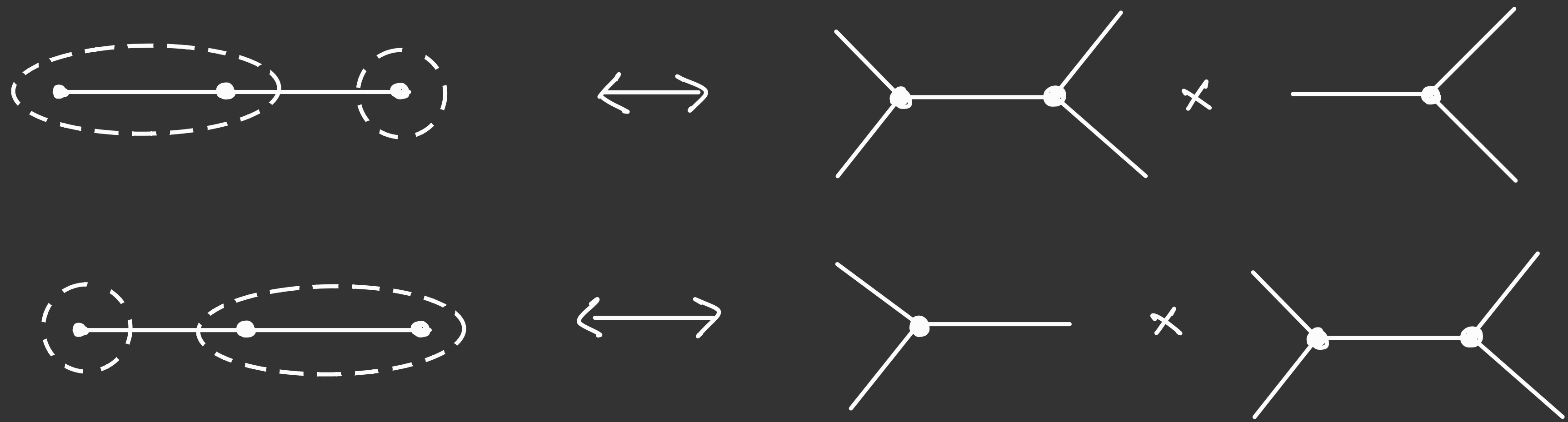


The good cut condition: Given a cut-tubing, identify all linear relations I_α (in which it is the largest tubing) and pair tubings that do not saturate either side of I_α.

Recall:



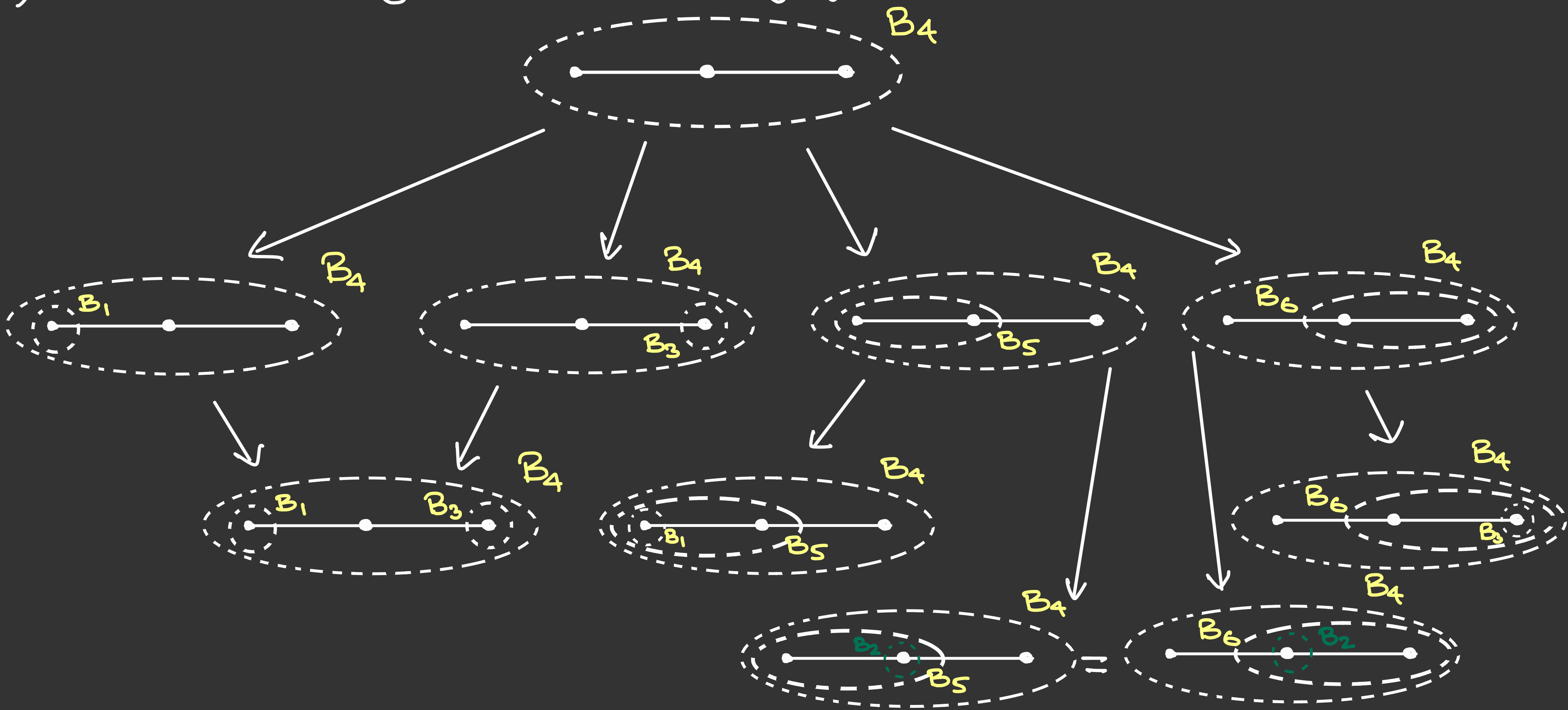
2b) We also have the following 2-cuts:



consistent with the uni-dimensional cohomology condition.

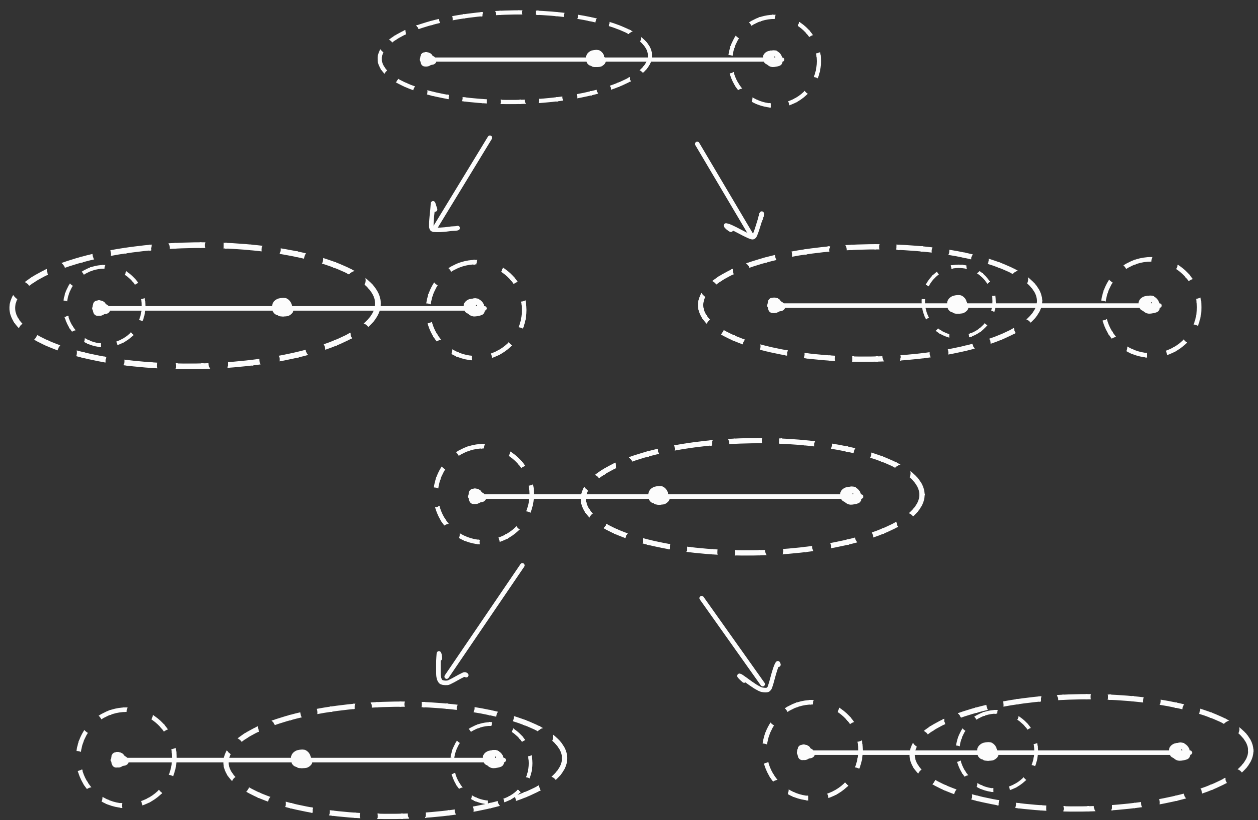
3-cut tubings

3a) Keep evolving the scattering facet:

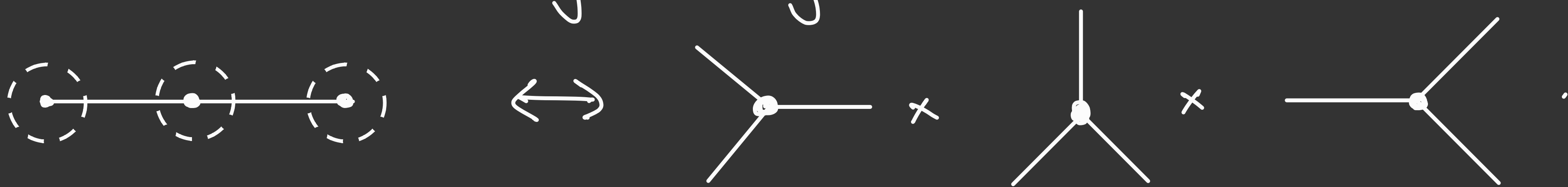


The degenerate cut condition: Identify cut-tubings that set all but one B_i in I_α to zero.

3b) Evolve 2-cuts obtained in (2b):



3c) We also have the following 3-cut:



Cut Stratification - All Trees and Loops

→ Simple universal rules for cut tubings enumerate the physical subspace of FRW forms for any n-site, l-loop graph $\Psi_n^{(l)}$!

→ At tree-level, explains the counting $(=4^{n-1})$ predicted by kinematic flow, with b-cuts organized in a Pascal's triangle:

	1-cuts	2-cuts	3-cuts	4-cuts	5-cuts	6-cuts ...
	3^0	3^1	3^2	3^3	3^4	3^5 ...
n=2	1	1				
n=3	1	2	1			
n=4	1	3	3	1		
n=5	1	4	6	4	1	
n=6	1	5	10	10	5	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

(He, Jiang, Liu, Yang, Zhang '24)

→ Matches the couting $\left(=4^n-2(2^n-1)\right)$ for 1-loop n-gons:

	1-cuts	2-cuts	3-cuts	4-cuts	...
n=2 (bubble)	3	7			
n=3 (triangle)	4	21	25		
n=4 (box)	5	42	100	79	
⋮	⋮	⋮	⋮	⋮	

→ New prediction: For 2-site l-loop coefficients $\psi_2^{(l)}$:

$$\boxed{\#\text{ physical forms} = \underbrace{\left(2^{l+1}-1\right)}_{\#\text{ 1-cuts}} + \underbrace{\left(2^{l+2}-1\right)}_{\#\text{ 2-cuts}} = 2\left(3.2^{l}-1\right)}$$

Matches counting for sunset ($l=2$) graph predicted by loop-level kinematic flow! (Baumann, Goodhew, Lee '24; Hang '24)

Outlook & Discussions

→ Cut/residue stratification of relative twisted cohomology + physics, geometry of hyperplane arrangements explains the origin of the physical DEQ subspace predicted recently.

→ Algorithm in terms of cut-tubings whose construction dictated by simple universal rules; holds for any $\Psi_n^{(L)}$.

Future Directions

→ Why does the characterization of the physical subspace favour only cuts corresponding to products of flat-space amplitudes $A_n^{(1)}$ only?

→ Does the flow of cut-tubings encode the connection matrix A_{ij} in the DEQ system – could it explain the physio-geometric origins of kinematic flow?

→ Does this analysis extend to recent progress (Chen, Feng, Tao '24; Gasparotto, Mazloumi, Xu '24) beyond our toy model theory of conformally-coupled scalars?