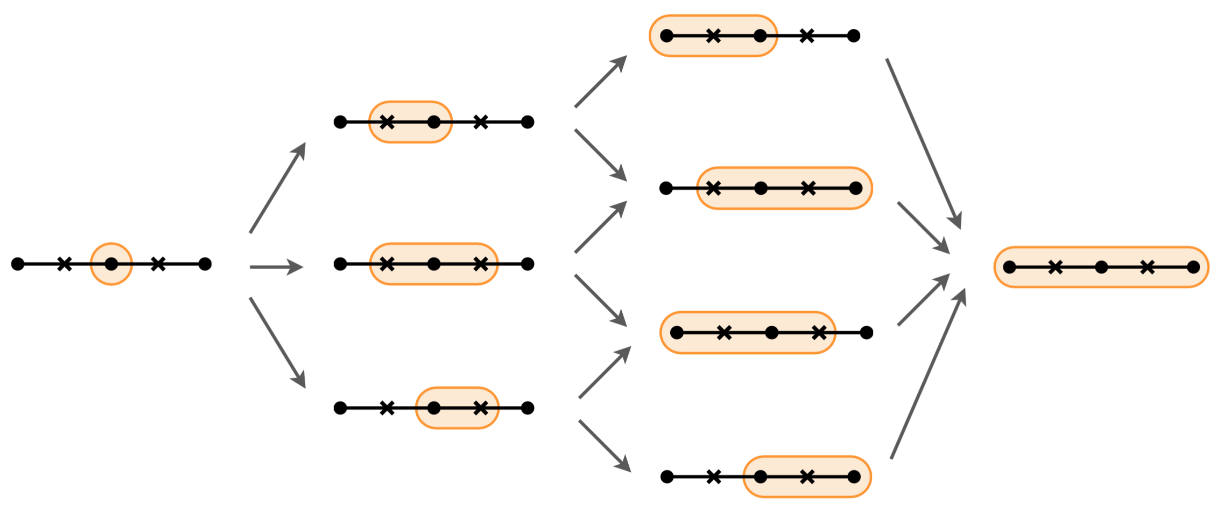
Differential Equations for Cosmological Correlators



Hayden Lee

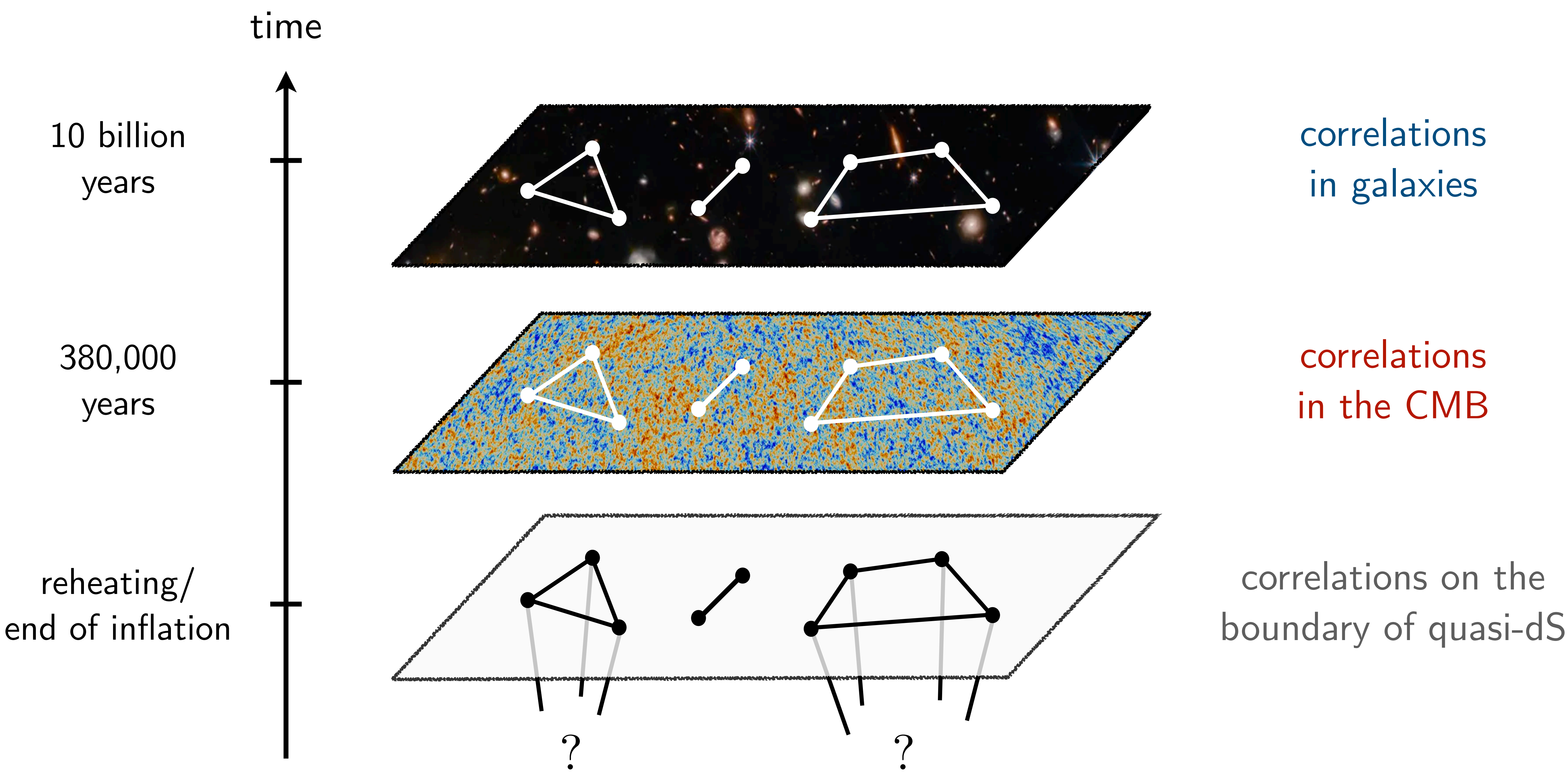
University of Pennsylvania

w/ N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, G. Pimentel [2312.05300, 2312.05303]

w/ D. Baumann, H. Goodhew [2410.17994]

[Loop-the-Loop @ MITP, November 2024]

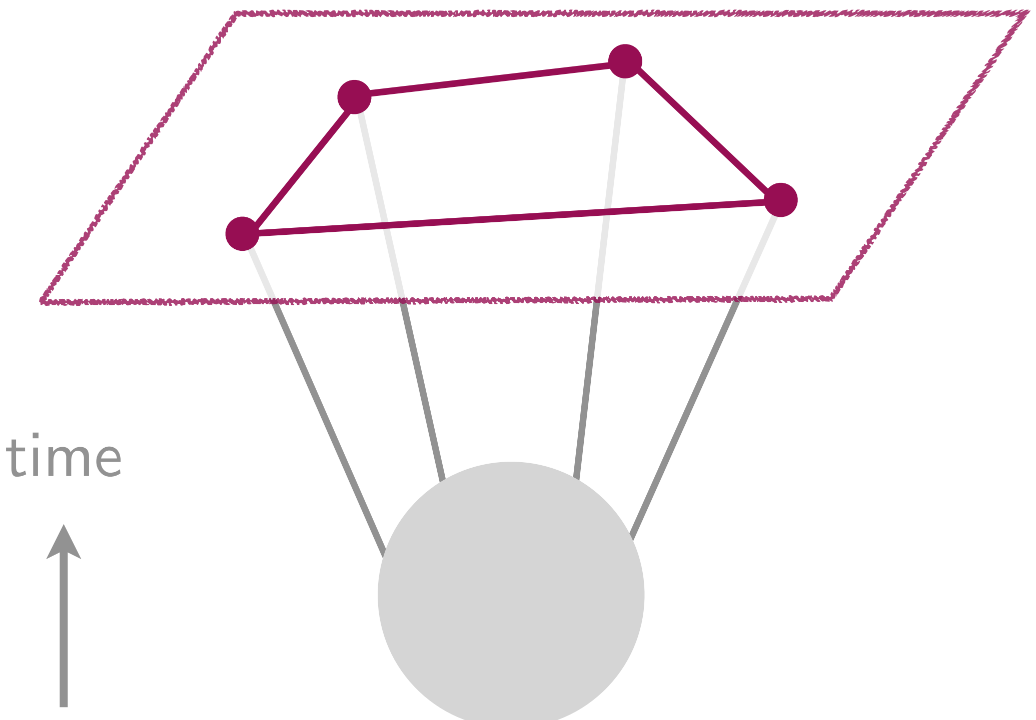
Cosmological Correlators



Cosmological correlators encode the physics of the primordial universe.

Cosmological Bootstrap

Cosmological correlators live on the future boundary of quasi-dS.



bootstrap boundary correlators in **kinematic space**

Forget about time evolution in the bulk

$$\int \prod_v d\eta_v \prod_n \mathcal{K}_n \prod_e \mathcal{G}_e$$

practical advantage: simplify calculations

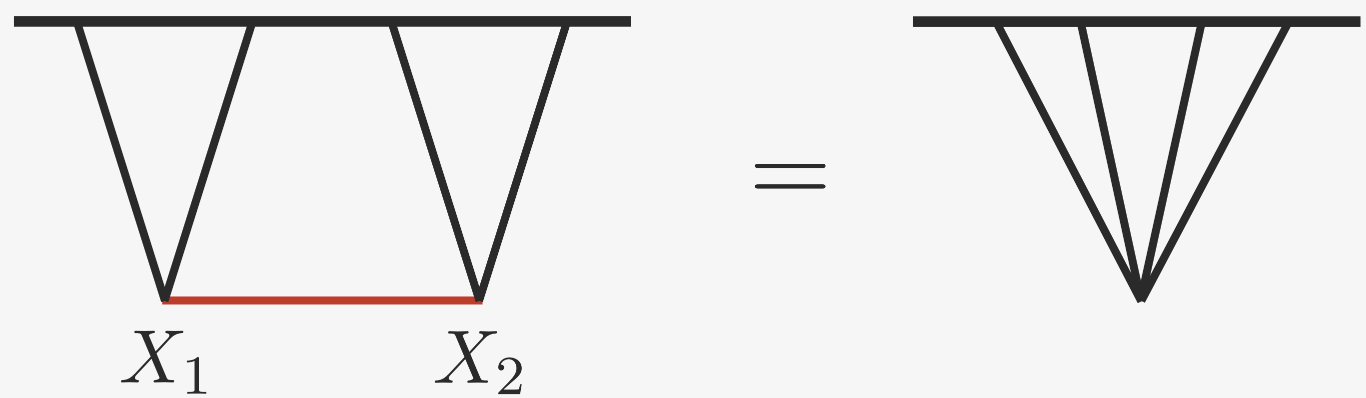
conceptual advantage: reveals hidden structures

[See e.g. Snowmass 2203.08121]

Differential Equations in dS

In de Sitter, boundary correlators satisfy conformal Ward identities.

$$(D_{X_1} - m^2)$$



conformal Casimir

$$(X_1^2 - 1)\partial_{X_1}^2 - 2X_1\partial_{X_1}$$

cf. $(\Box - m^2)G(\eta, \eta') = i\delta(\eta - \eta')$

Arkani-Hamed, Maldacena [2015]
Arkani-Hamed, Baumann, HL, Pimentel [2018]

Chen, Feng, Tao [2024]
Gasparotto, Mazloumi, Xu [2024]
[See also J. Chen's talk]

Do similar differential equations exist **beyond dS**?

Toy Model in FRW

Consider a **conformally coupled scalar** with **polynomial interactions**:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3\right]$$

conformal mass

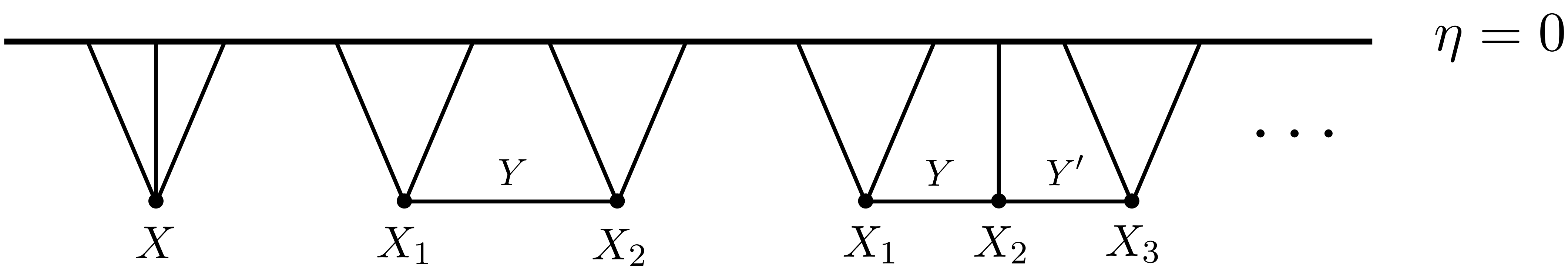
non-conformal interaction

in an FRW spacetime expanding as a **power law**:

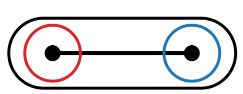
$$\mathrm{d}s^2 = a^2(\eta)[-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2] \qquad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}} \quad \begin{cases} \varepsilon = 0: & \text{dS} \\ \varepsilon = -1: & \text{flat} \\ \varepsilon = -2: & \text{radiation} \\ \varepsilon = -3: & \text{matter} \end{cases}$$

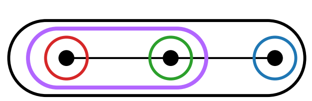
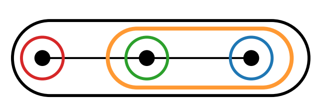
Wavefunction in Flat Space

Consider the **tree-level wavefunction/correlators** in this theory.



In **flat space**, the WF is given by **rational functions** with simple poles.

$\psi^{(2)}_{\text{flat}}$ =  $= \dfrac{1}{(X_1+X_2)(X_1+Y)(X_2+Y)}$

$\psi^{(3)}_{\text{flat}}$ =  + 

$$= \frac{1}{(X_1+X_2+X_3)(X_1+Y)(X_2+Y+Y')(X_3+Y')}\left(\frac{1}{X_1+X_2+Y'}+\frac{1}{X_2+X_3+Y}\right)$$

Arkani-Hamed, Benincasa, Postnikov [2017]

Wavefunction in FRW

In **FRW**, the wavefunction can be represented as **twisted integrals**:

$$u = \prod_v x_v^{\varepsilon} \quad \text{twist (multi-valued)}$$

$$\psi_{\text{FRW}}(\mathbf{X}, \mathbf{Y}) = \int_0^{\infty} u\, \Omega_\psi$$

rational form (single-valued)

$$\Omega_\psi = \psi_{\text{flat}}(\mathbf{X} + \mathbf{x}, \mathbf{Y}) \bigwedge_v \mathrm{d}x_v$$

Modern amplitude approaches to compute integrals of this type include:

- twisted cohomology
- method of differential equations

[See also S. De's talk]

Twisted Cohomology

Two integrands differing by a total differential give the same integral.

$$0 = \int \mathrm{d}(u\,\Omega) = \int u \underbrace{(\mathrm{d} + \mathrm{d}\log u \wedge)}_{\equiv \nabla_\omega} \Omega \qquad \Rightarrow \qquad \Omega \sim \Omega + \nabla_\omega \xi$$

The set of equivalence classes of integrands = **twisted cohomology**

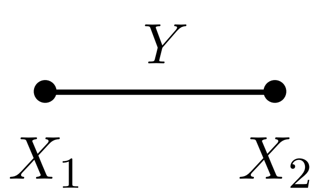
Basis size = # **bounded regions** formed by the singular hyperplanes.

$\Rightarrow$ satisfies a closed system of **differential equations**.

[See yesterday's talks]

Two-Site Chain

The integral for the **two-site chain** takes the form



$$= \int_0^\infty (x_1 x_2)^\varepsilon \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 + X_1 + Y)(x_2 + X_2 + Y)(x_1 + x_2 + X_1 + X_2)}$$

We consider a family of integrals with the same singularities:

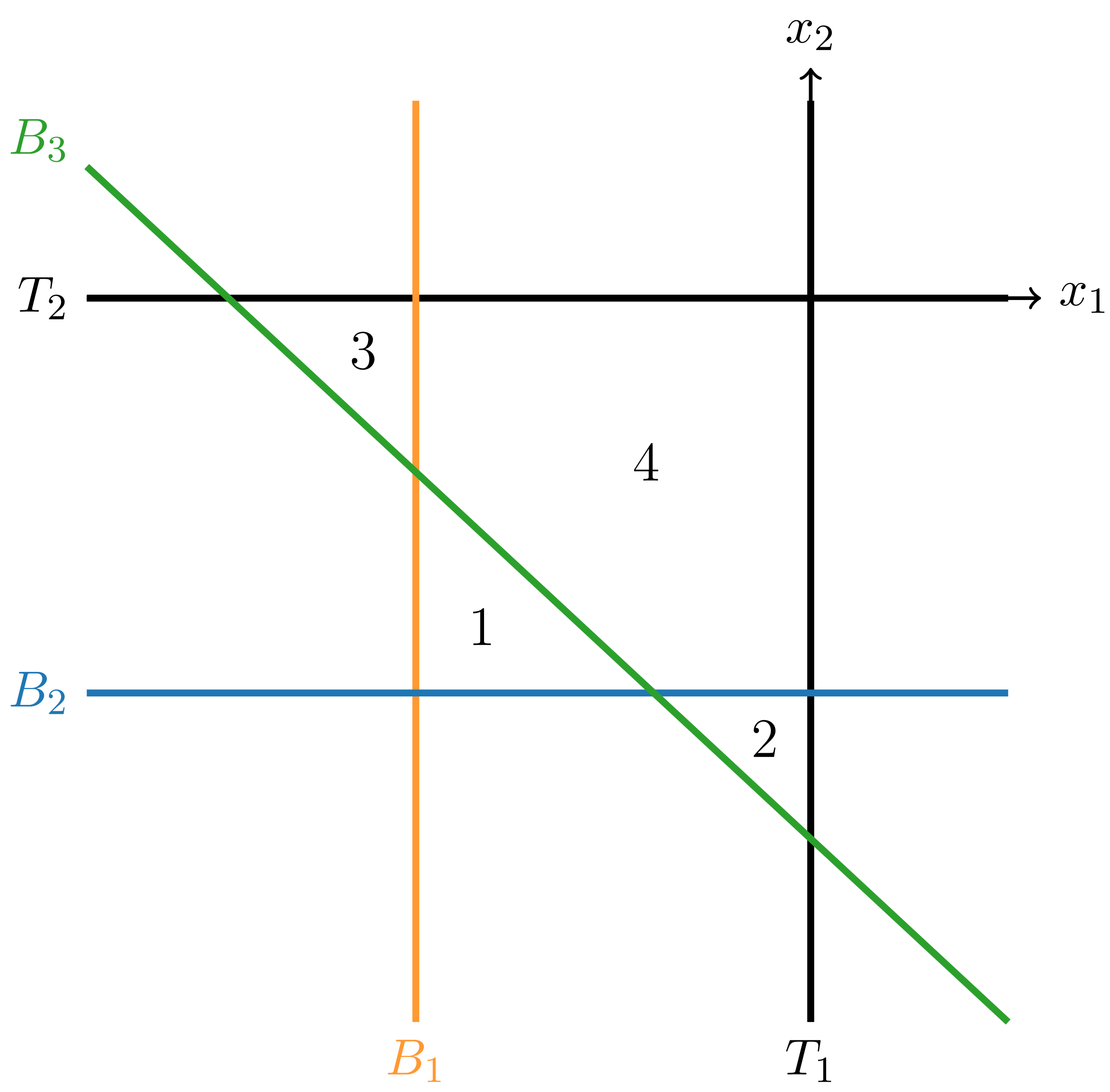
$$\int_0^\infty (x_1 x_2)^\varepsilon \, \Omega_{\mathbf{n}} \,, \qquad \Omega_{\mathbf{n}} = \frac{\mathrm{d}x_1 \mathrm{d}x_2}{T_1^{n_1} T_2^{n_2} B_1^{n_3} B_2^{n_4} B_3^{n_5}} \qquad (n_i \in \mathbb{Z})$$

$$T_1 = x_1 \,, \qquad B_1 = x_1 + X_1 + Y \,,$$

$$T_2 = x_2 \,, \qquad B_2 = x_2 + X_2 + Y \,, \qquad B_3 = x_1 + x_2 + X_1 + X_2$$

Master Integrals

master integrals = # bounded regions formed by the singular lines.

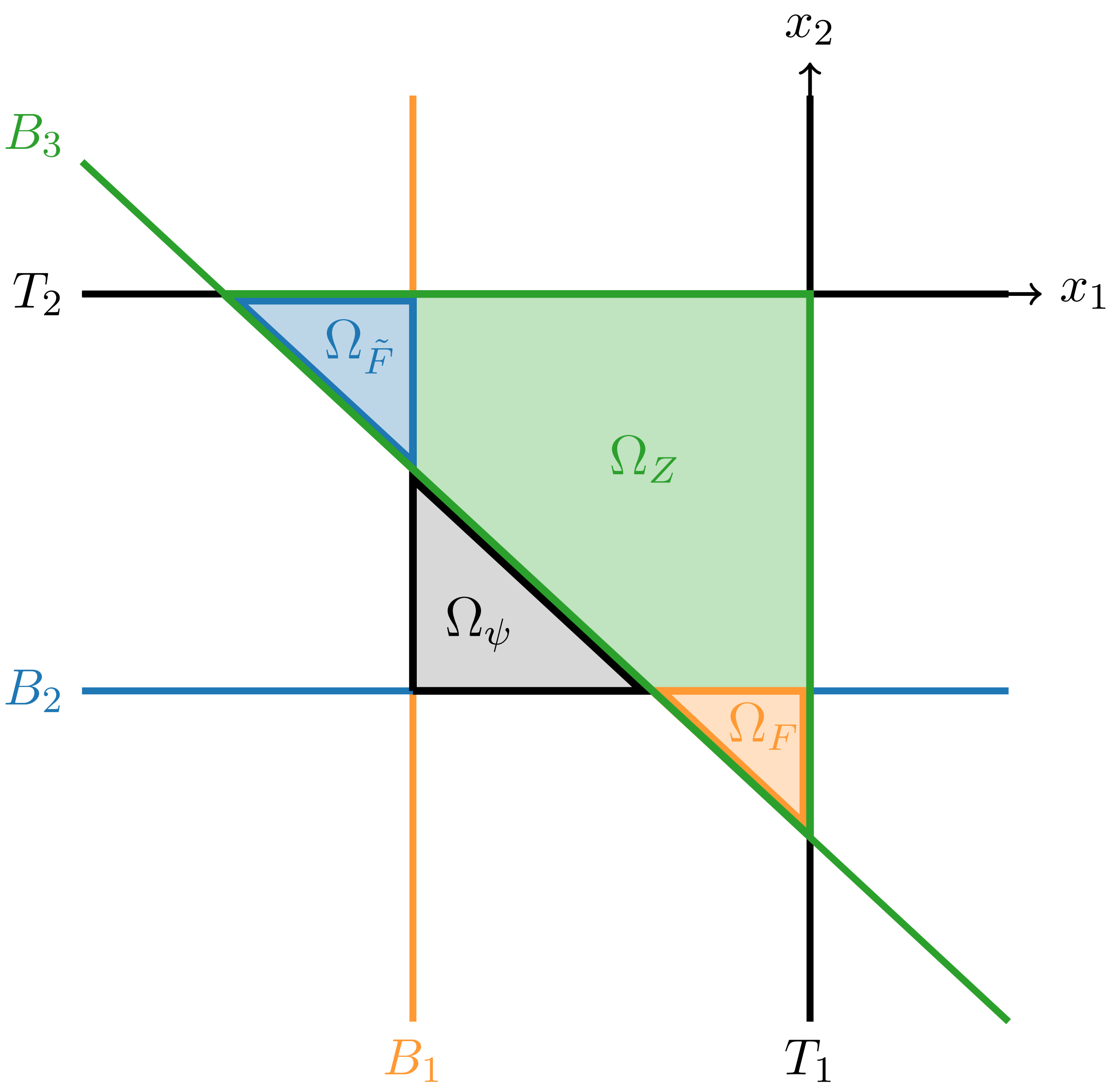


$$T_1 = x_1$$
$$T_2 = x_2$$
$$B_1 = x_1 + X_1 + Y$$
$$B_2 = x_2 + X_2 + Y$$
$$B_3 = x_1 + x_2 + X_1 + X_2$$

De, Pokraka [2023]
Arkani-Hamed, Baumann, Hillman, Joyce, HL, Pimentel [2023]

Master Integrals

A good basis choice is given by the **canonical forms** of the bounded regions.



$$\vec{I} = \begin{bmatrix} \psi \\ F \\ \tilde{F} \\ Z \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_\psi \\ \Omega_F \\ \Omega_{\tilde{F}} \\ \Omega_Z \end{bmatrix}$$

$$\Omega_{\text{can}}[\Delta_{L_1 L_2 L_3}] = \mathrm{d}\log\left(\frac{L_1}{L_3}\right) \wedge \mathrm{d}\log\left(\frac{L_2}{L_3}\right)$$

De, Pokraka [2023]

Arkani-Hamed, Baumann, Hillman, Joyce, HL, Pimentel [2023]

Differential Equations

Taking the differential of the basis vector and performing IBP gives

$$\mathrm{d}\vec{I} = \varepsilon A \vec{I}$$

$$\mathrm{d} = \sum_i \mathrm{d}X_i \frac{\partial}{\partial X_i} \qquad A = \sum_i \alpha_i \, \mathrm{d}\log R_i$$

letters

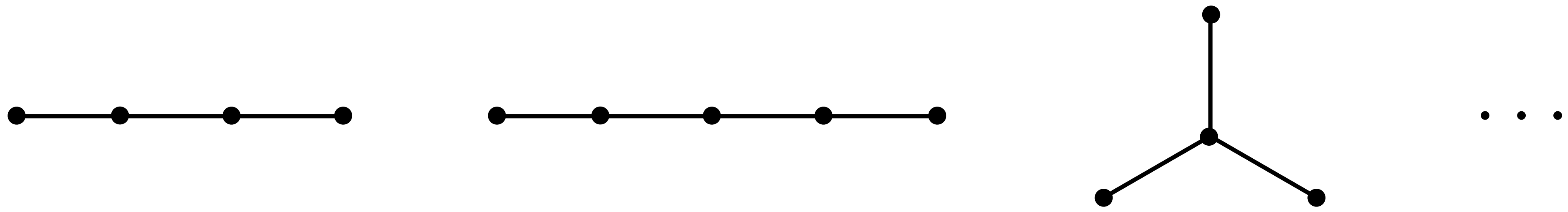
with

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathrm{d}\log(X_1 + X_2) + \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 + Y) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 - Y)$$

$$+ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 + Y) + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 - Y)$$

General Tree Graphs

Unfortunately, this direct approach breaks down for more complicated graphs.



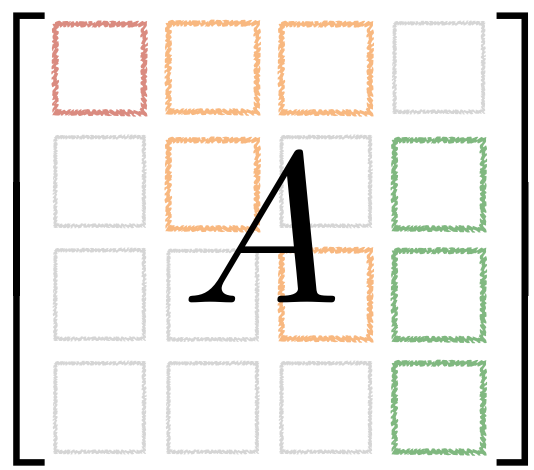
- **Twisted cohomology** naively gives an **over-complete basis**.
- Deriving equations using **IBP relations** is **highly technical**.

Remarkably, these are solved by simple graphical rules.

Differential Equations

We've derived the system of differential equations for the two-site chain.

$$\mathrm{d}\begin{bmatrix}\psi\\ F\\ \tilde{F}\\ Z\end{bmatrix} = \varepsilon\begin{bmatrix} & & & \\ & & A & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}\psi\\ F\\ \tilde{F}\\ Z\end{bmatrix}$$

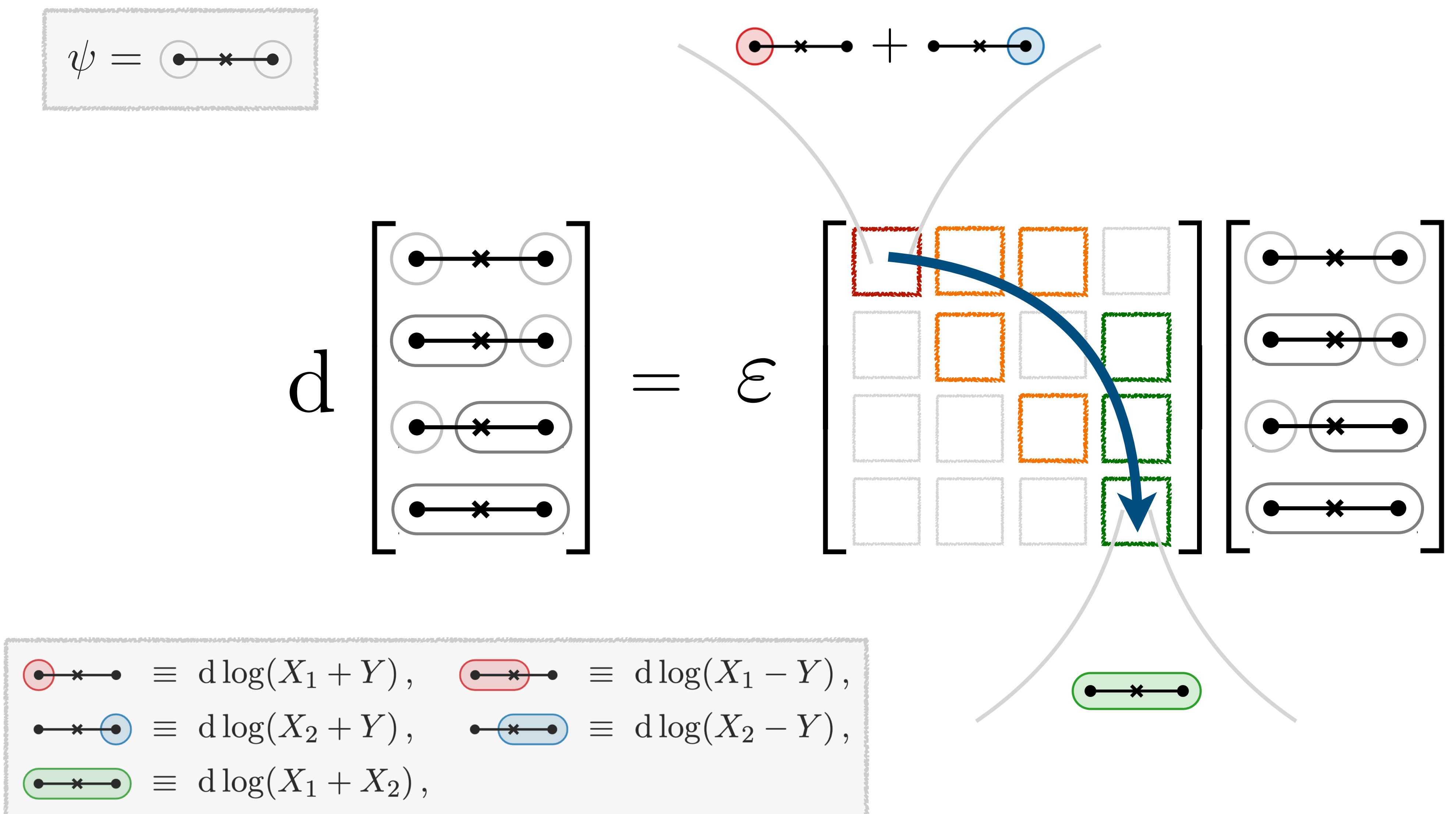


$$A = \sum_i \alpha_i \,\mathrm{d}\log R_i$$

However, the explicit result isn't very illuminating and is hard to generalize.

A Hidden Pattern

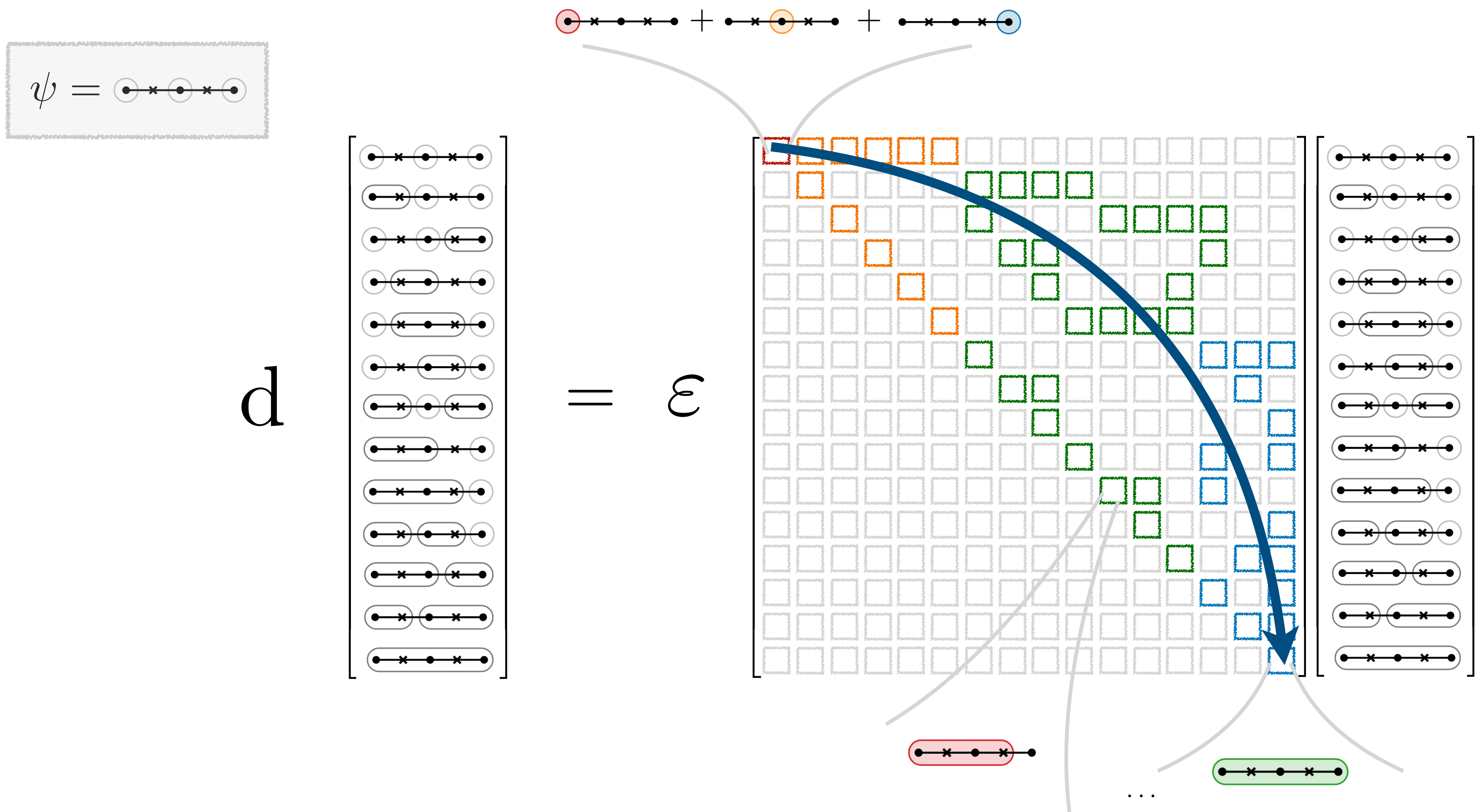
A **hidden pattern** was revealed when we drew pictures of the results!



The tubings **grow**, and the **system closes** when all vertices are enclosed.

A Hidden Pattern

The same pattern is found for arbitrary **n-site graphs** at tree level.



Remarkably, we can predict all entries with **simple graphical rules.**

Graphical Representation

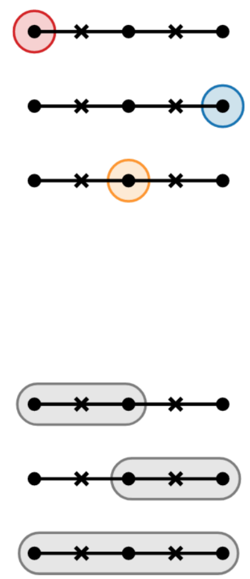
Letters:

connected

(activated)

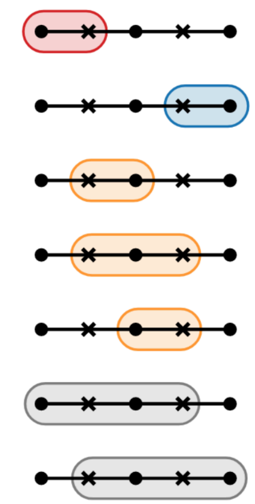
tubings

13 (~~19~~) letters



$$\equiv \mathrm{d}\log(X_1+Y)\,,\quad \equiv \mathrm{d}\log(X_3+Y')\,,\quad \equiv \mathrm{d}\log(X_2+Y+Y')\,,$$

$$\equiv \mathrm{d}\log(X_1+X_2+Y')\,,\quad \equiv \mathrm{d}\log(X_2+X_3+Y)\,,\quad \equiv \mathrm{d}\log(X_1+X_2+X_3)\,.$$



$$\equiv \mathrm{d}\log(X_1-Y)\,,\quad \equiv \mathrm{d}\log(X_3-Y')\,,\quad \equiv \mathrm{d}\log(X_2-Y+Y')\,,\quad \equiv \mathrm{d}\log(X_2-Y-Y')\,,$$

$$\equiv \mathrm{d}\log(X_2+Y-Y')\,,\quad \equiv \mathrm{d}\log(X_1+X_2-Y')\,,\quad \equiv \mathrm{d}\log(X_2+X_3-Y')\,,$$

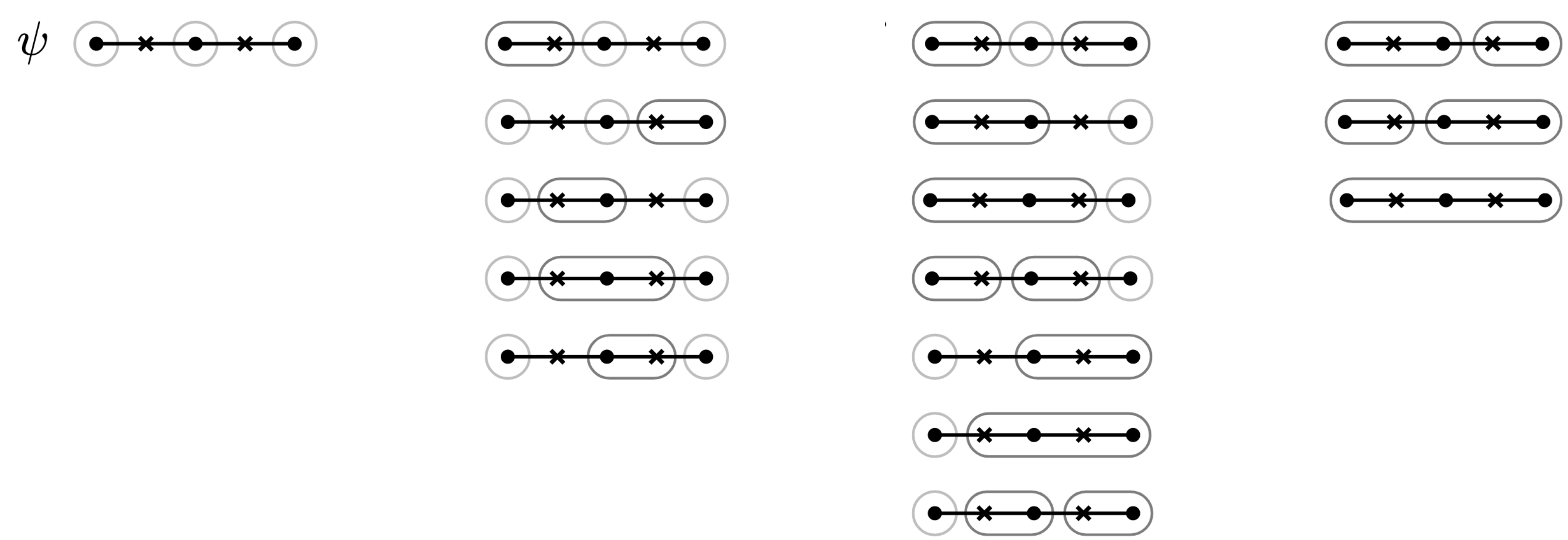
Functions:

complete

(disconnected)

tubings

16 (~~25~~) functions

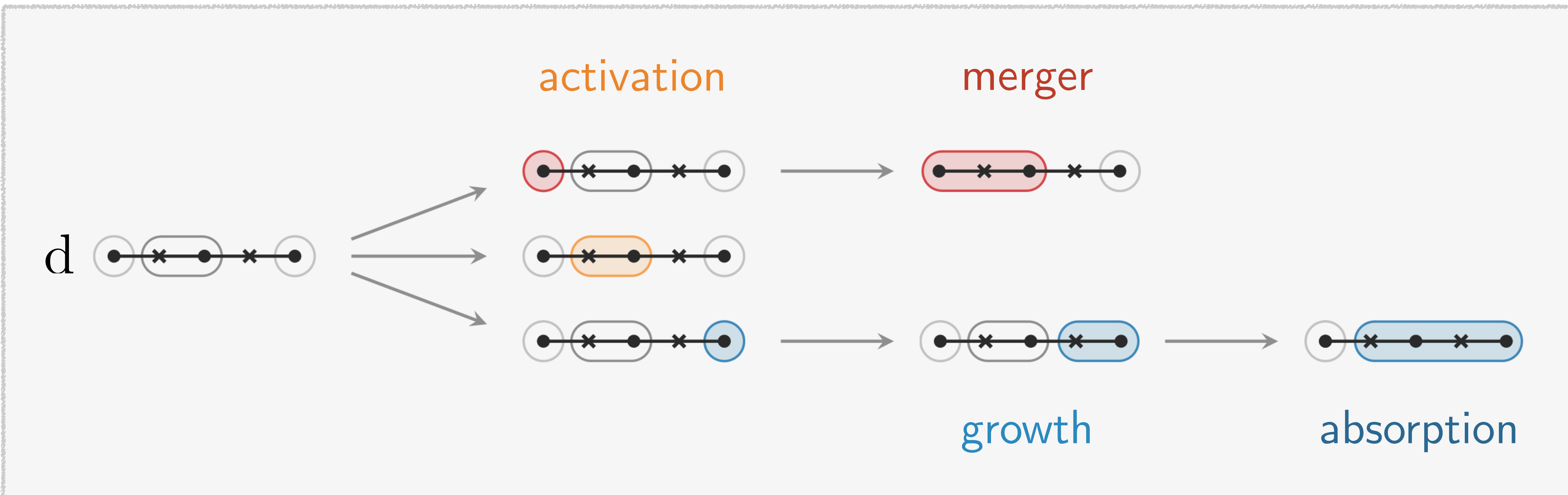


naive counting from twisted cohomology (64 vs. 201 for n_{site}=4)

[See also S. De's talk]

Kinematic Flow

Upon differentiation, graph tubings evolve according to simple graphical rules.



These rules allow us to predict (by hand!) the equations for **all tree graphs**.

This reformulates bulk time evolution as a flow in kinematic space.

Arkani-Hamed, Baumann, Hillman, Joyce, HL, Pimentel [2023]

Loop Integrands

The FRW wavefunction for the **loop integral** can be written as

$$\psi_{\mathrm{FRW}}^{\mathrm{loop}}(\mathbf{X},\mathbf{Y}) = \int \prod_{\ell} \mathrm{d}^d y_\ell \, \hat{\psi}_{\mathrm{FRW}}^{\mathrm{loop}}(\mathbf{X},\mathbf{y})$$

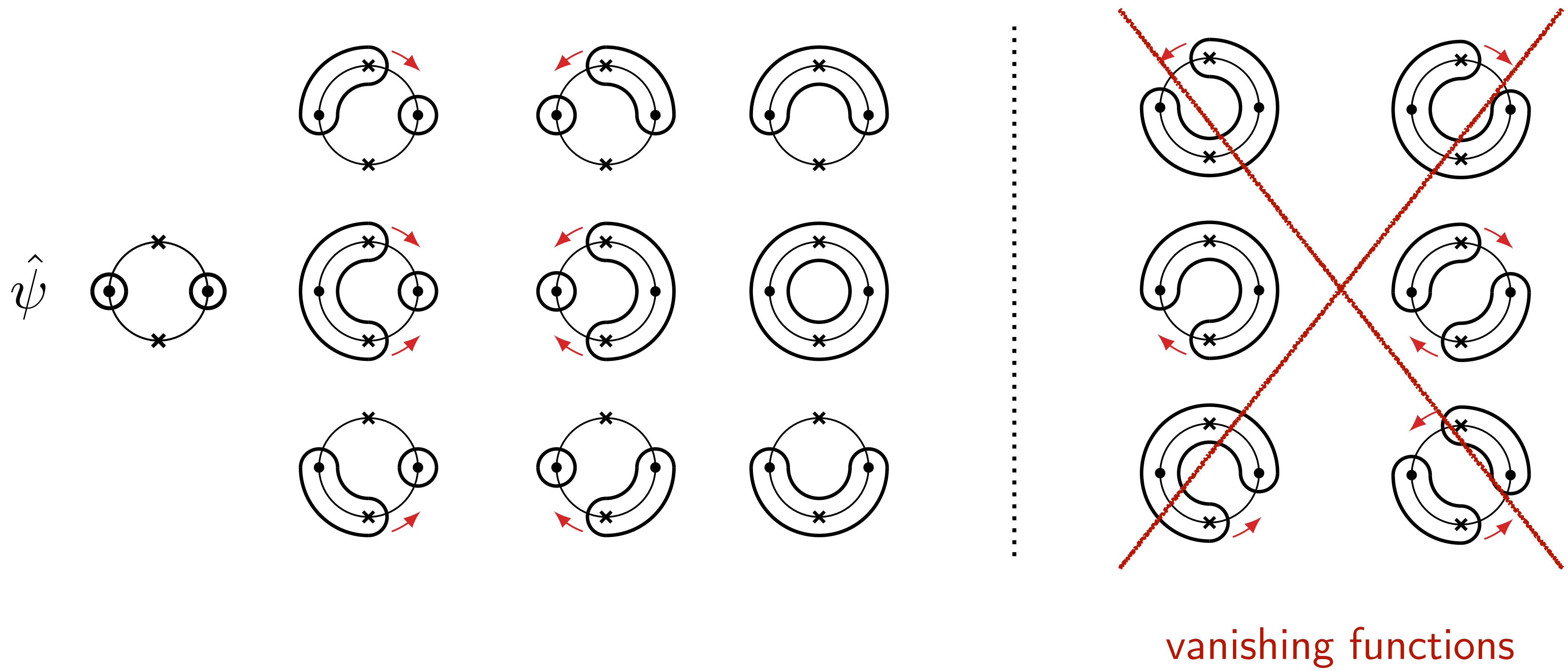
$$\hat{\psi}_{\mathrm{FRW}}^{\mathrm{loop}}(\mathbf{X},\mathbf{y}) = \int_0^\infty \left(\prod_v \mathrm{d}x_v \, x_v^{\varepsilon}\right) \hat{\psi}_{\mathrm{flat}}^{\mathrm{loop}}(\mathbf{X}+\mathbf{x},\mathbf{y})$$

Just like in the tree-level case, we can write the **loop integrand/integral** as a twisted integral over the flat-space wavefunction integrand/integral.

[See also F. Vazão's talk]

Loop Integrands

The **same graphical rules** apply to loop integrands, except one modification:



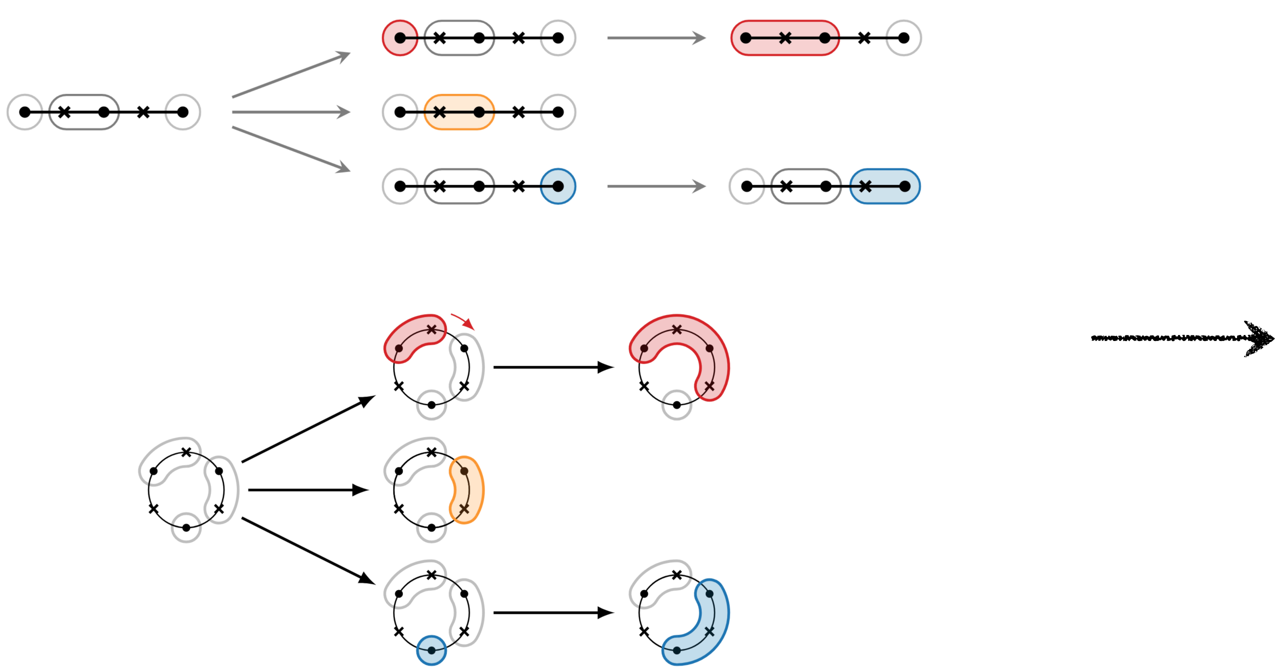
A loop graph **vanishes** if a tube can return to itself by following the arrows.

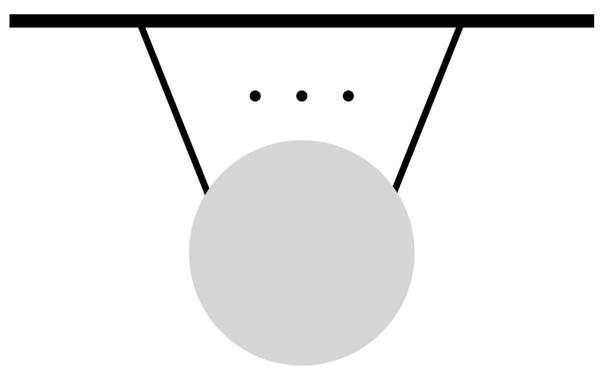
He, Jiang, Liu, Yang, Zhang [2024]
Baumann, Goodhew, Lee [2024]

Conclusion

We've found a **hidden pattern** in the differential equations

for the FRW wavefunction of conformally coupled scalars.





$$\mathrm{d}\vec{I} = \varepsilon A \vec{I}$$

Simple rules give the differential equations for **all trees and loop integrands**.