Effective Field Theory in a Mass Ratio Expansion

Loop-the-Loop

13 Nov 2024

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Based on:

Cheung, NS, Solon (2010.08568);

Cheung, Wilson-Gerow, Parra-Martinez, Rothstein, NS (2308.14832, 2406.14770)

Point Particle Effective Field Theory

Separation of Scales

Dimensional regularization

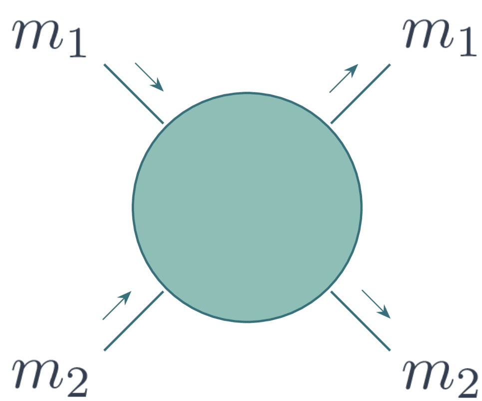
Differential equations

Integration-by-parts identities

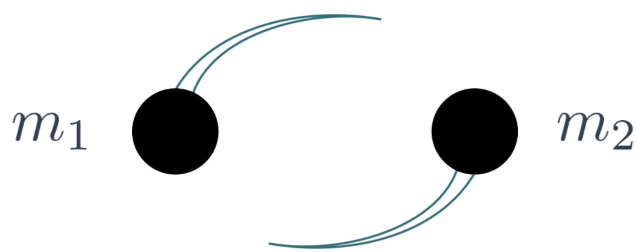
Recursion relations

Generalized unitarity

Double Copy



Relativistic
Perturbative in G

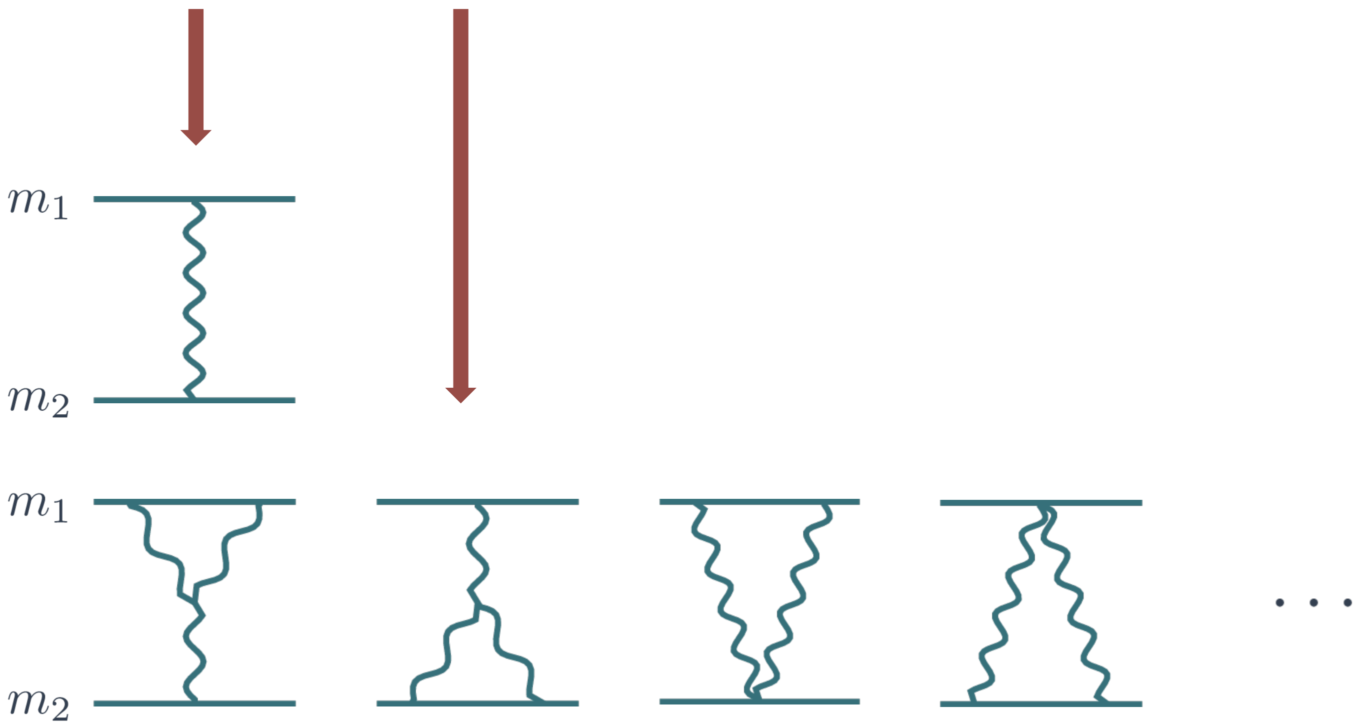


Post-Minkowskian Perturbation Theory

$$V = c_1 \left(\frac{G}{r}\right) + c_2 \left(\frac{G}{r}\right)^2 + \cdots$$

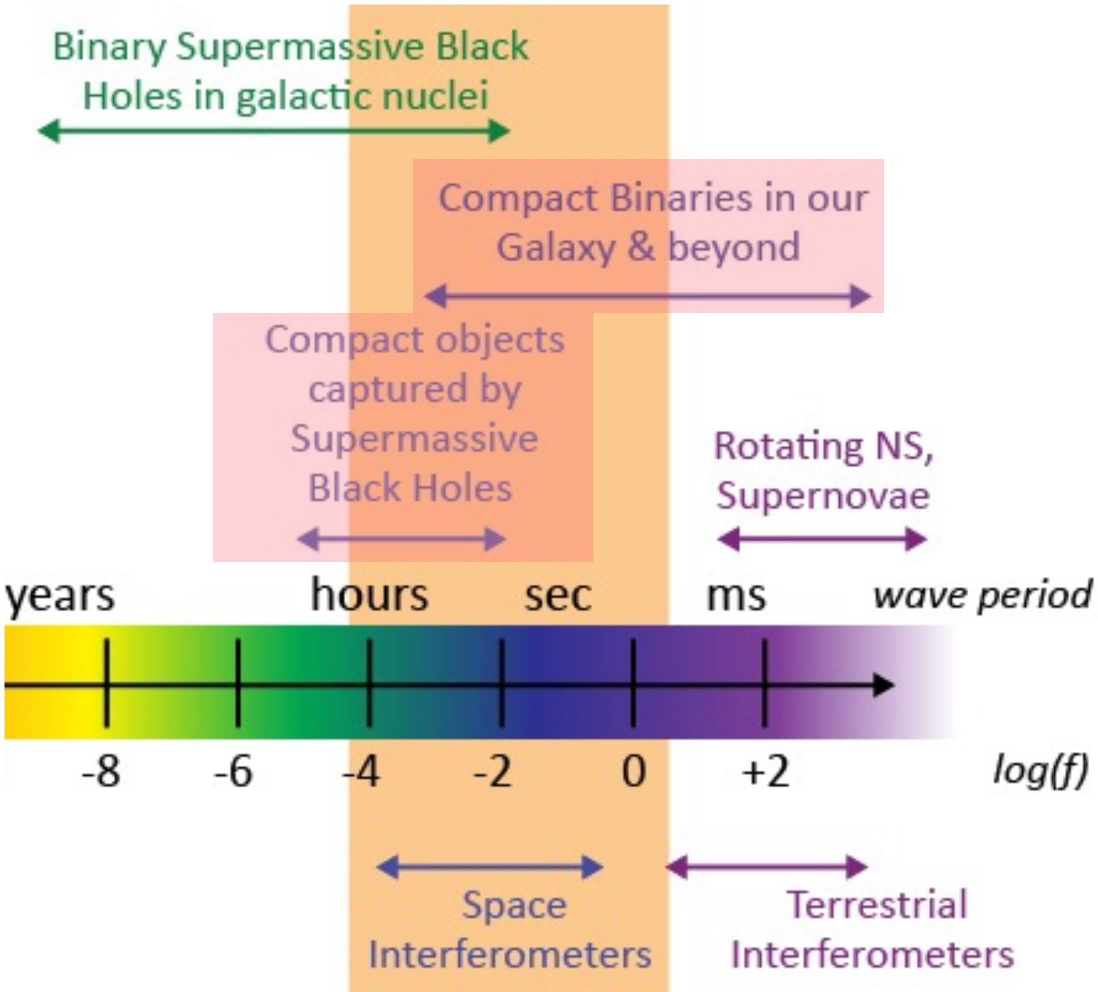
Generically: $c_n(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{r})$

Isotropic: $c_n(\mathbf{p}^2)$



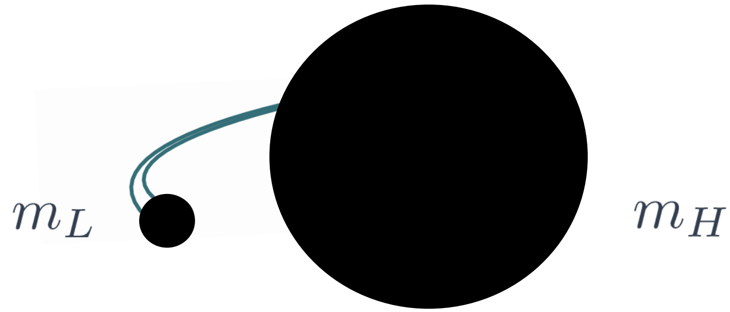
An expansion in loops with all orders in velocity at each loop order

Gravitational Wave Astronomy



https://lisa.nasa.gov

Extreme Mass Ratio Inspirals (EMRIs)



Strong field and relativistic

LISA is expected to detect ~100 Extreme Mass Ratio Inspirals per year

Gair, Barack, Creighton, Cutler, Larson, Phinney, Vallisner (2004); Gair, Mandel, Wen (2008)

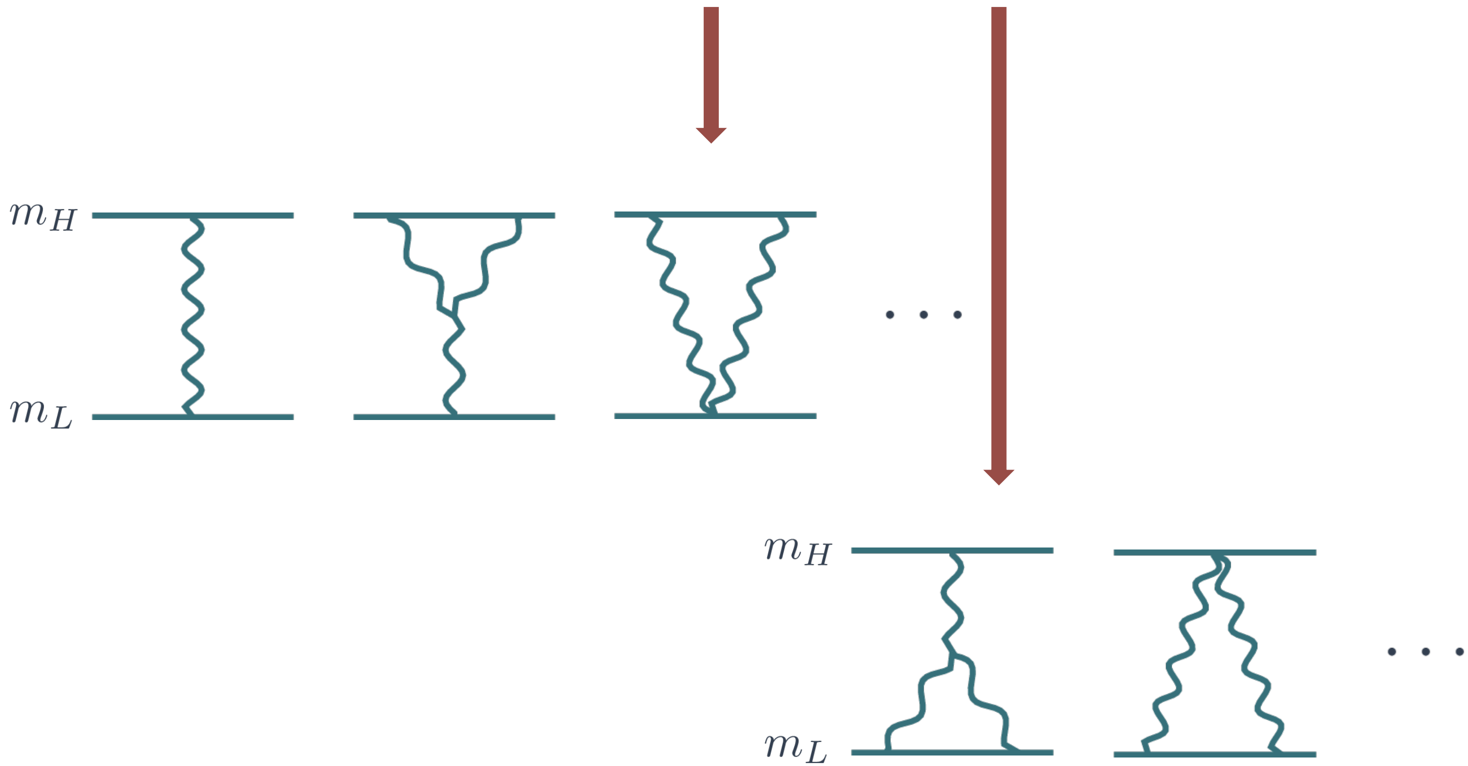
Self force expansion: $\lambda = \frac{m_1}{m_2} = \frac{m_L}{m_H}$

When $\lambda \to 0$, we can think of the lighter body as a probe or test particle while the larger static black hole generates a Schwarzschild background

At higher orders, m_L will cause m_H to recoil which will in turn affect m_L in an effect known as the self force

Reorganization in a Self Force (SF) Expansion

Classical Amplitude: $\mathcal{M} = \lambda^0 \mathcal{M}_{0\mathrm{SF}} + \lambda^1 \mathcal{M}_{1\mathrm{SF}} + \lambda^2 \mathcal{M}_{2\mathrm{SF}} + \cdots$



An infinite set of diagrams at each order

Interplay of PM and SF Expansions

$$\mathcal{M}_{1\mathrm{PM}} \sim G m_L^2 m_H^2 \, (\, 1 \,)$$

$$\mathcal{M}_{2\mathrm{PM}} \sim G^2 m_L^2 m_H^2 (m_H + m_L)$$

0SF

$$\mathcal{M}_{3\mathrm{PM}} \sim G^3 m_L^2 m_H^2 (m_H^2 + m_H m_L + m_L^2)$$

$$\mathcal{M}_{4\mathrm{PM}} \sim G^4 m_L^2 m_H^2 (m_H^3 + m_H^2 m_L + m_H m_L^2 + m_L^3)$$

1SF

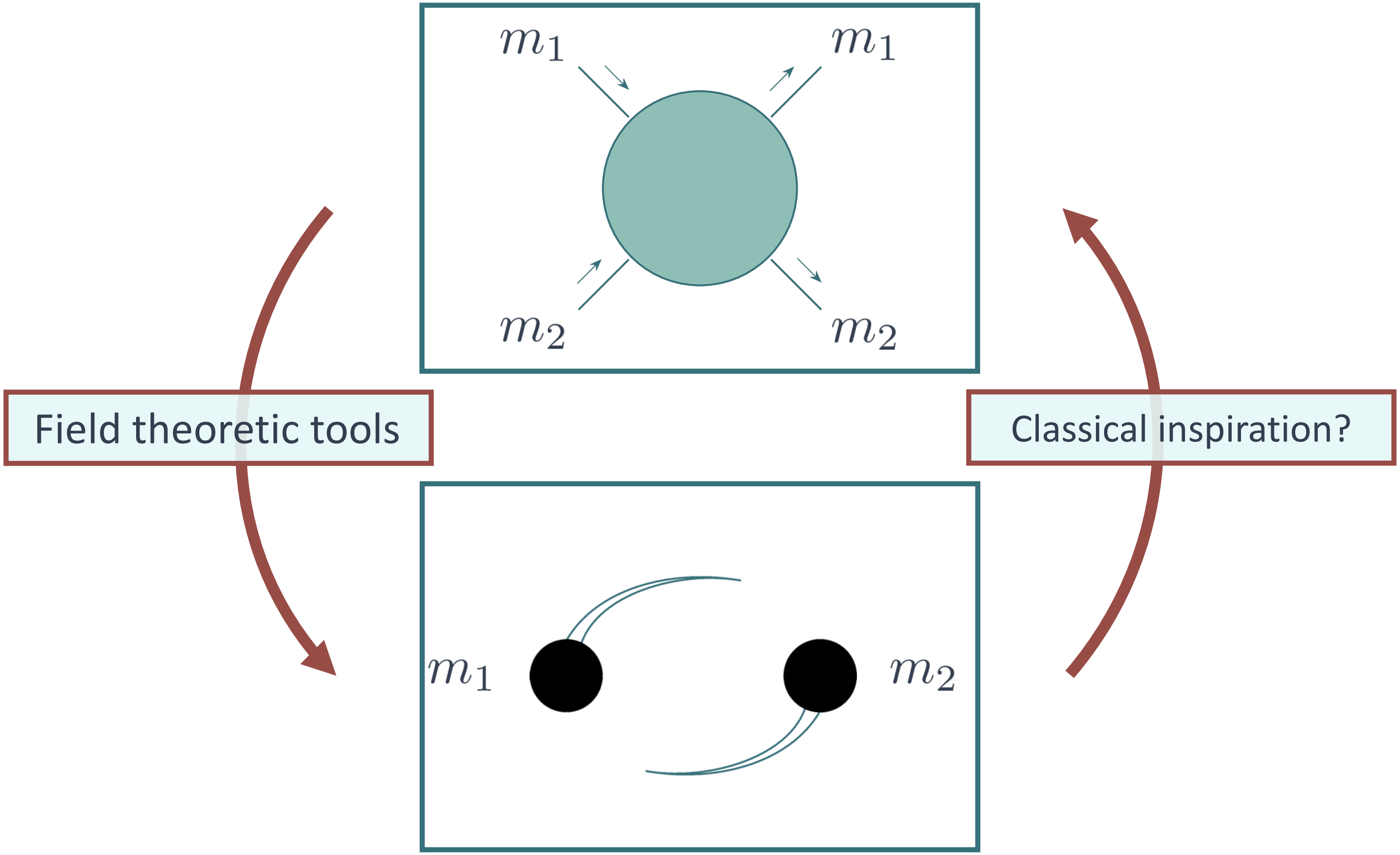
$$\mathcal{M}_{5\mathrm{PM}} \sim G^5 m_L^2 m_H^2 (m_H^4 + m_H^3 m_L + m_H^2 m_L^2 + m_H m_L^3 + m_L^4)$$

0SF 1SF 2SF 3SF 4SF

At each post-Minkowskian order, the amplitude has a simple polynomial expression in terms of the masses

Vines, Steinhoff, Buonanno (2019); Siemonsen, Vines (2019); Bini, Damour, Geralico (2019, 2020); Damour (2020); Antonelli, Kavanagh, Khalil, Steinhoff, Vines (2020)

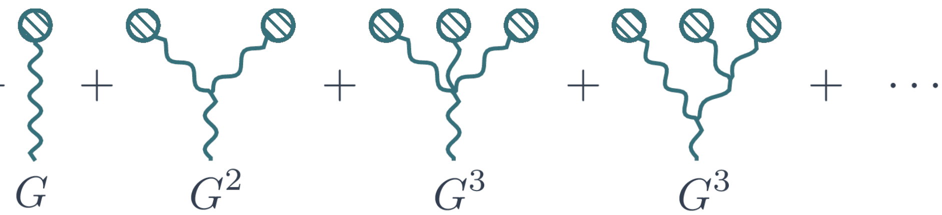
Best of both worlds?



Classical Solutions at All Orders in G

The metric encodes all order PM data as a sum of infinite flat space diagrams:

$$g_{\mu\nu} = \eta_{\mu\nu} + \ldots$$



Duff (1972)

$$g^{00} = -1 - \frac{2GM}{r} - \frac{2(GM)^2}{r^2} + O(G^3)$$

$$g^{ij} = \left(1 - \frac{2GM}{r} + \frac{3(GM)^2}{r^2}\right)\eta^{ij} - \frac{(GM)^2}{r^2}\frac{x^i x^j}{r^2} + O(G^3)$$

Resummation to all orders reproduces the Schwarzschild metric

Damgaard, Lee (2024); Mougiakakos, Vanhove (2024)

Classical Solutions at All Orders in G

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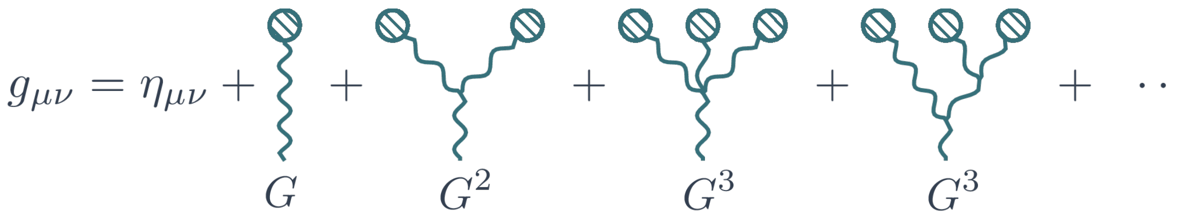
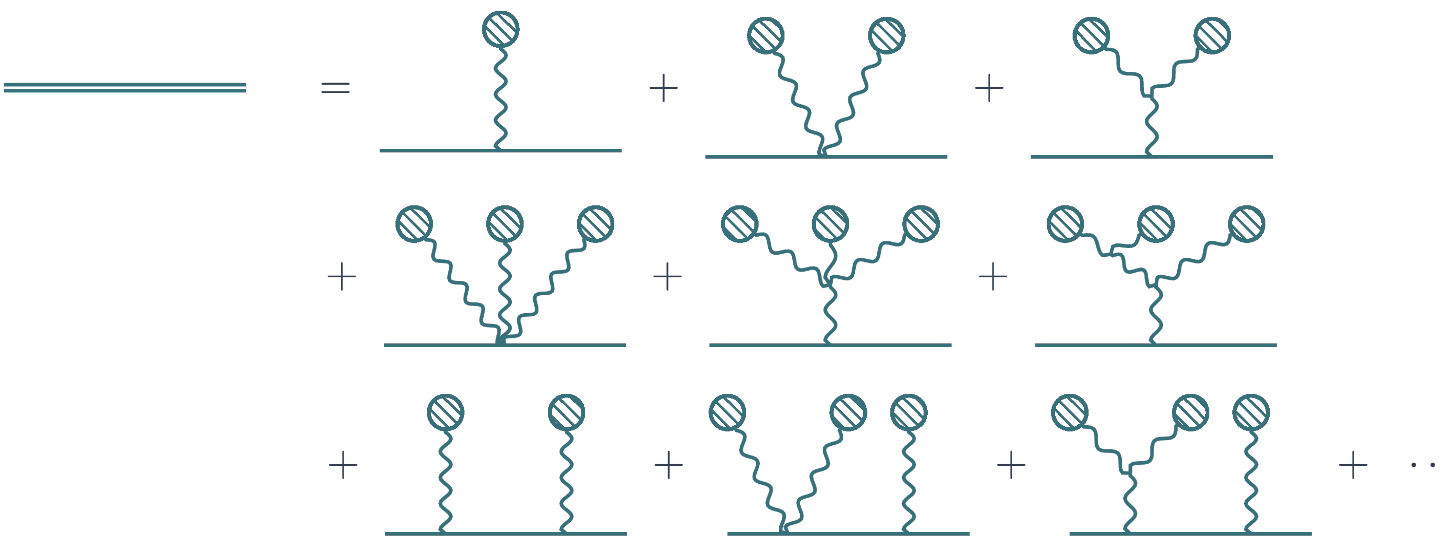


Duff (1972)

The geodesic equation encodes data for a test particle in a background:



Mougiakakos, Vanhove (2024)

Expand known solutions

Perturbations Away From a Probe in Schwarzschild

1. At 0SF:

perturbative corrections away from a non-spinning black hole binary system

- for arbitrary mass ratios in the leading PM order
- in the probe limit at all PM orders

Examples:
tidal distortions
higher derivative corrections to general relativity
interactions with a charged body

Cheung, NS, Solon (2020)

2. Beyond 0SF:

Effective field theory in a mass ratio expansion

Cheung, Parra-Martinez, Rothstein, NS, Wilson-Gerow (2023)

Schematic Procedure: OSF

Geodesic equation, Hamiltonian

$$g^{\mu\nu} P_\mu P_\nu + m^2 - \alpha \mathcal{O}(g, P) = 0\,, \quad P_\mu = (H, p_r, p_\theta, p_\phi)$$

$$H(p, r, J) \quad \rightarrow \quad H^{\text{iso}}(p, r)$$

Diffeomorphism: $r \quad \rightarrow \quad f(r)$

Amplitude

$$\mathcal{M}(p, r) = \frac{1}{2\sqrt{(p^2 + m^2)}} (\overline{p}(r)^2 - p^2) + \text{iterations}$$

$$H^{\text{iso}}(\overline{p}(r), r) = E = \sqrt{p^2 + m^2}$$

The extraction procedure is purely algebraic thanks to the *impetus formula*

Bern, Cheung, Roiban, Shen, Solon, Zeng (2018, 2019);
Kälin, Porto (2020); Bjerrum-Bohr, Cristofoli, Damgaard (2020)

Schematic Procedure: Arbitrary Mass Ratios at 1PM

Geodesic equation, Hamiltonian

$$g^{\mu\nu}P_\mu P_\nu + m^2 - \alpha\mathcal{O}(g,P) = 0\,,\quad P_\mu = (H, p_r, p_\theta, p_\phi)$$

$$H(p,r,J) \quad \rightarrow \quad H^{\text{iso}}(p,r)$$

$$\frac{J^{2k}}{r^n} \;\rightarrow\; \frac{p^{2k}r^{2k}}{r^n} \times \frac{\text{Poch}(\frac{n}{2}-\frac{1}{2}-k,k)}{\text{Poch}(\frac{n}{2}-k,k)}$$

Amplitude

$$\mathcal{M}(p,r) = \frac{1}{2\sqrt{(p^2+m^2)}}(\overline{p}(r)^2 - p^2) + \text{iterations}$$

$$H^{\text{iso}}(\overline{p}(r),r) = E = \sqrt{p^2+m^2}$$

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An EFT Formalism for Extreme Mass Ratios

1. Starting point: probe motion in curved spacetime, $\bar{x}_L$ and $\bar{g}_{\mu\nu}(x)$

2. Accounting for heavy particle motion with x_H

3. A systematic expansion in $\lambda = \frac{m_L}{m_H}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$
$$x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu$$

4. Integrate out the heavy particle fluctuation, δx_H

5. Derive effective action: background field + *corrective operators*

6. Calculate quantities of interest: e.g. on-shell radial action which is a generating function for the angle of scattering, $\Delta\phi = -\frac{d\bar{S}}{dJ}$

Electromagnetism

Background Configuration: Test Particle in an Electromagnetic Field

Coulomb potential: $\bar{A}_\mu(x) = \frac{z_H m_H u_{H\mu}}{4\pi r}$ where $r = \sqrt{(u_H x)^2 - x^2}$

and $\ddot{\bar{x}}^\mu_L - z_L \bar{F}^{\mu\nu}(\bar{x}_L)\dot{\bar{x}}_{L\nu} = 0$ is the light particle equation of motion

Assume $z_i = \frac{q_i}{m_i}$ of equal size so that forces scale with mass

For the heavy particle: $\ddot{\bar{x}}^\mu_H - z_H \bar{F}^{\mu\nu}(\bar{x}_H)\dot{\bar{x}}_{H\nu} = 0$

$\left.\frac{1}{r}\right|_0$ Divergence!

vanish in dimensional regularization

Background motion of the heavy particle is simply $\bar{x}^\mu_H(\tau) = u^\mu_H \tau$

Mass Ratio Expanded Action

Electromagnetically interacting charged massive scalars:

$$S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\dot{x}_H^2 - z_H \dot{x}_H^\mu A_\mu(x_H) - \lambda \left(\tfrac{1}{2}\dot{x}_L^2 + z_L \dot{x}_L^\mu A_\mu(x_L) \right) \right]$$
$$+ \int d^4x \left[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

light particle action is suppressed by λ

$$A_\mu = \bar{A}_\mu + \delta A_\mu$$

$$x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu$$

dimensional regularization sets terms with $\bar{A}_\mu(x_H)$ to zero

$$S_{\rm EM} = \bar{S}_{\rm EM} + \delta S_{\rm EM}$$

$$\delta S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) - \lambda z_L \dot{\bar{x}}_L^\mu \delta A_\mu(\bar{x}_L) \right]$$
$$+ \int d^4x \left[-\tfrac{1}{4}\delta F_{\mu\nu} \delta F^{\mu\nu} \right] + \cdots$$

Effective Action and Recoil Operator: 1SF

$$\delta S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) - \lambda z_L \dot{\bar{x}}_L^\mu \delta A_\mu(\bar{x}_L)\right]$$
$$+ \int d^4x \left[-\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu}\right] + \cdots$$

Integrate out δx_H using $\delta\ddot{x}_H^\mu - z_H \delta F^{\mu\nu}(\bar{x}_H)\dot{\bar{x}}_{H\nu} = 0$

$$\delta S_{\rm EM}^{\rm eff} = \mathcal{R}_{\rm EM} + \int d^4x \left(-\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} - \delta A_\mu \bar{J}_L^\mu\right)$$

$$\mathcal{R}_{\rm EM} = -\frac{1}{2} z_H^2 m_H \int d\tau\, \dot{\bar{x}}_H^\alpha \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^\beta \delta F_\beta{}^\mu(\bar{x}_H)$$

The non-local in time recoil operator accounts for the wobble of the heavy particle

Effective Action and Recoil Operator: 2SF

For light particle fluctuation:

$$m_L \int d\tau \Big[- \tfrac{1}{2}\delta\dot{x}_L^2 - z_L \delta x_L^\mu \dot{\bar{x}}_L^\nu \delta F_{\mu\nu}(\bar{x}_L)$$

$$-\tfrac{1}{2} z_L \delta x_L^\mu \delta\dot{x}_L^\nu \bar{F}_{\mu\nu}(\bar{x}_L) - \tfrac{1}{2} z_L \delta x_L^\rho \delta x_L^\nu \dot{\bar{x}}_L^\mu \partial_\rho \bar{F}_{\mu\nu}(\bar{x}_L) \Big]$$

The light particle experiences recoil and deviation from its trajectory
but continues to propagate in the background

Heavy particle recoil:

$$\mathcal{R}_{\mathrm{EM}}^{(2\mathrm{SF})} = \frac{z_H^2 m_H}{2} \int d\tau \Bigg[\delta E^\alpha(\bar{x}_H) \frac{1}{\overleftarrow{\partial}_\tau^2} \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\overrightarrow{\partial}_\tau} \delta E^\mu(\bar{x}_H)$$

$$+ \delta E^\alpha(\bar{x}_H) \frac{1}{\overleftarrow{\partial}_\tau^2} \partial_\mu \delta E_\alpha(\bar{x}_H) \frac{1}{\overrightarrow{\partial}_\tau^2} \delta E^\mu(\bar{x}_H) \Bigg]$$

(where $\delta E^\mu = \dot{\bar{x}}_{H\nu} \delta F^{\mu\nu}$)

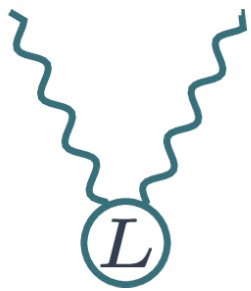
Types of Interactions

1SF:

m_L background trajectory source

Heavy particle recoil operator

2SF:



Deviation and propagation

Heavy particle recoil operator

Gravity

Effective Action at 1SF

Gravitationally interacting massive scalars:

$$S_{\mathrm{GR}} = \sum_{i=H,L} m_i \int d\tau \left[-\tfrac{1}{2}\dot{x}_i^\mu \dot{x}_i^\nu g_{\mu\nu}(x_i)\right] + \int d^4x \sqrt{-g}\left[-\tfrac{1}{16\pi G} R\right]$$

Self-energy contributions are dropped and the action at 1SF is

$$\delta S^{\mathrm{eff}}_{\mathrm{GR}} = \mathcal{R}_{\mathrm{GR}} - \int d^4x \sqrt{-\bar{g}}\left[\tfrac{1}{32\pi G}\left(\tfrac{1}{2}\bar{\nabla}_\rho \delta g_{\mu\nu} \bar{\nabla}^\rho \delta g^{\mu\nu} + \cdots + \tfrac{1}{2}\delta g_{\mu\nu}\bar{T}_L^{\mu\nu}\right]\right.$$

$$\mathcal{R}_{\mathrm{GR}} = -\frac{1}{2} m_H \int d\tau\, \dot{\bar{x}}_H^\alpha \dot{\bar{x}}_H^\beta \delta\Gamma^\mu{}_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^\gamma \dot{\bar{x}}_H^\delta \delta\Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

The action describes interactions of a probe particle with fluctuations of the background with heavy particle recoil encoded in a nonlocal operator

Expanding curved spacetime solutions provides simplification

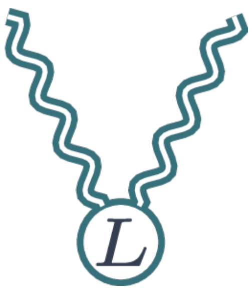
Types of Interactions

1SF:

Geodesic source

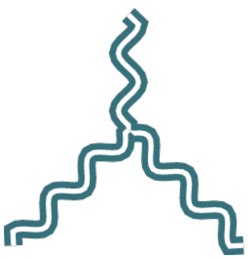
Recoil operator

2SF:



Geodesic deviation

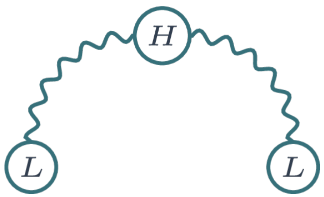
Recoil operator



Self-interactions

Computing the Radial Action

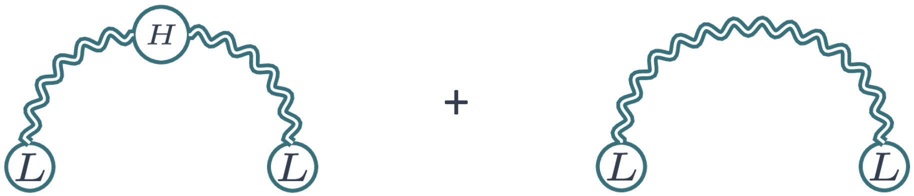
Radial action (in the potential region) at 1SF:



Electromagnetism

Computing the Radial Action

Radial action (in the potential region) at 1SF:



Gravitation

Toy Models

Toy Models: Additional scalar or vector particle that interacts directly with the light particle but only gravitationally with the heavy particle

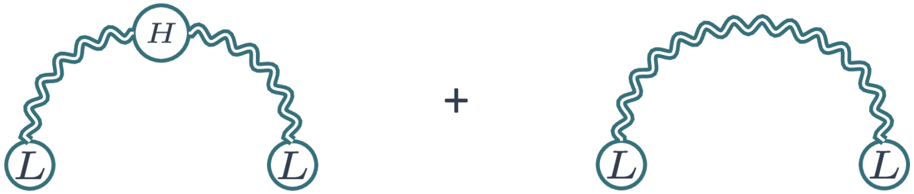
Scalar field coupled to the light particle:

$$S_{\text{scalar}} = \int d^4x \sqrt{-\bar{g}} \left[\tfrac{1}{2} \bar{\nabla}_\mu \Phi \bar{\nabla}^\mu \Phi + \tfrac{1}{2} \xi \bar{R} \Phi^2 - \Phi J \right]$$

where $$J(x) = y_L m_L \int d\tau \frac{\delta^4(x - \bar{x}_L)}{\sqrt{-\bar{g}}}$$

Computing the Radial Action

Radial action (in the potential region) at 1SF:



Gravitation

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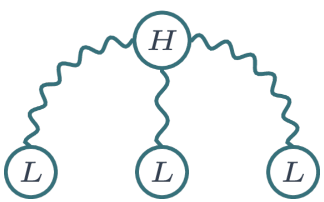
Radial action (in the potential region) at 2SF to 3PM:



Electromagnetism

Computing the Radial Action

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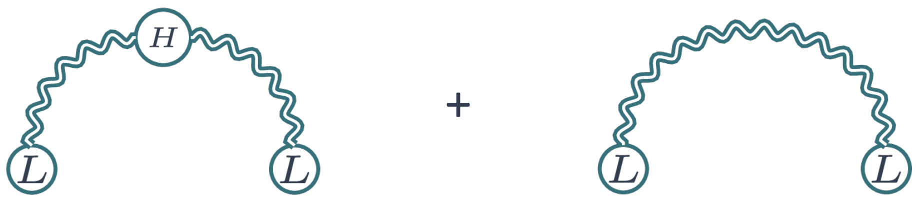

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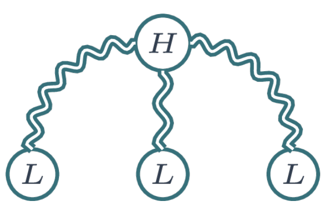
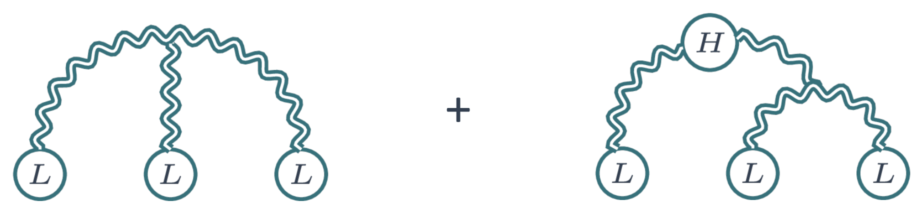


Electromagnetism



Gravitation

Contributions from the Geodesic Trajectory

The exact solution for the geodesic motion of a light body in the background generated by a heavier body is known in four dimensions but we want expressions in D dimensions

Option 1

Perturbatively solve the second order equation, $\ddot{\bar{x}}^\mu_L + \bar{\Gamma}^\mu{}_{\rho\sigma}(x_L)\dot{\bar{x}}^\rho_L\dot{\bar{x}}^\sigma_L = 0$

using $\bar{x}^\mu_L = \sum_{k=0}^{\infty} \bar{x}^\mu_k$ and $\bar{\Gamma}^\mu_{\alpha\beta} = \sum_{k=1}^{\infty} \bar{\Gamma}^\mu_{k\ \alpha\beta}$

Option 2

Use conserved quantities to reformulate into first order equations

$$\dot{t} = A(E, J, r) \qquad \dot{x} = B(E, J, r) \qquad \dot{y} = C(E, J, r)$$

$$\partial_\tau^{(-1)} \qquad \rightarrow \qquad (u_L.l)^{(-1)}$$

Many propagators are linearized and tensor structures are reduced

Results

Verifications:

	1SF	2SF
Electromagnetism	3PL	3PL
Gravitation	3PM	3PM
Scalar toy model	3PM	

New Results at 1SF:

Radial action for the scattering of dyonic charges to 3PL

Vector toy model radial action to 3PM

Nonminimally coupled scalar radial action to 3PM

Checks: Probe action in a black hole background with scalar charge and a Reissner–Nordström background

In Conclusion

What are the best ways to harness the power of both approaches?

The resummation provided by classical solutions can be fed into the field theoretic computational pipeline to maximize benefits using an effective field theory in mass ratios

Old results were verified and new results obtained using this method

Are there pathways to all order computations?

Thank You!