Schwarzschild geodesics from Scattering Amplitudes to all orders in G

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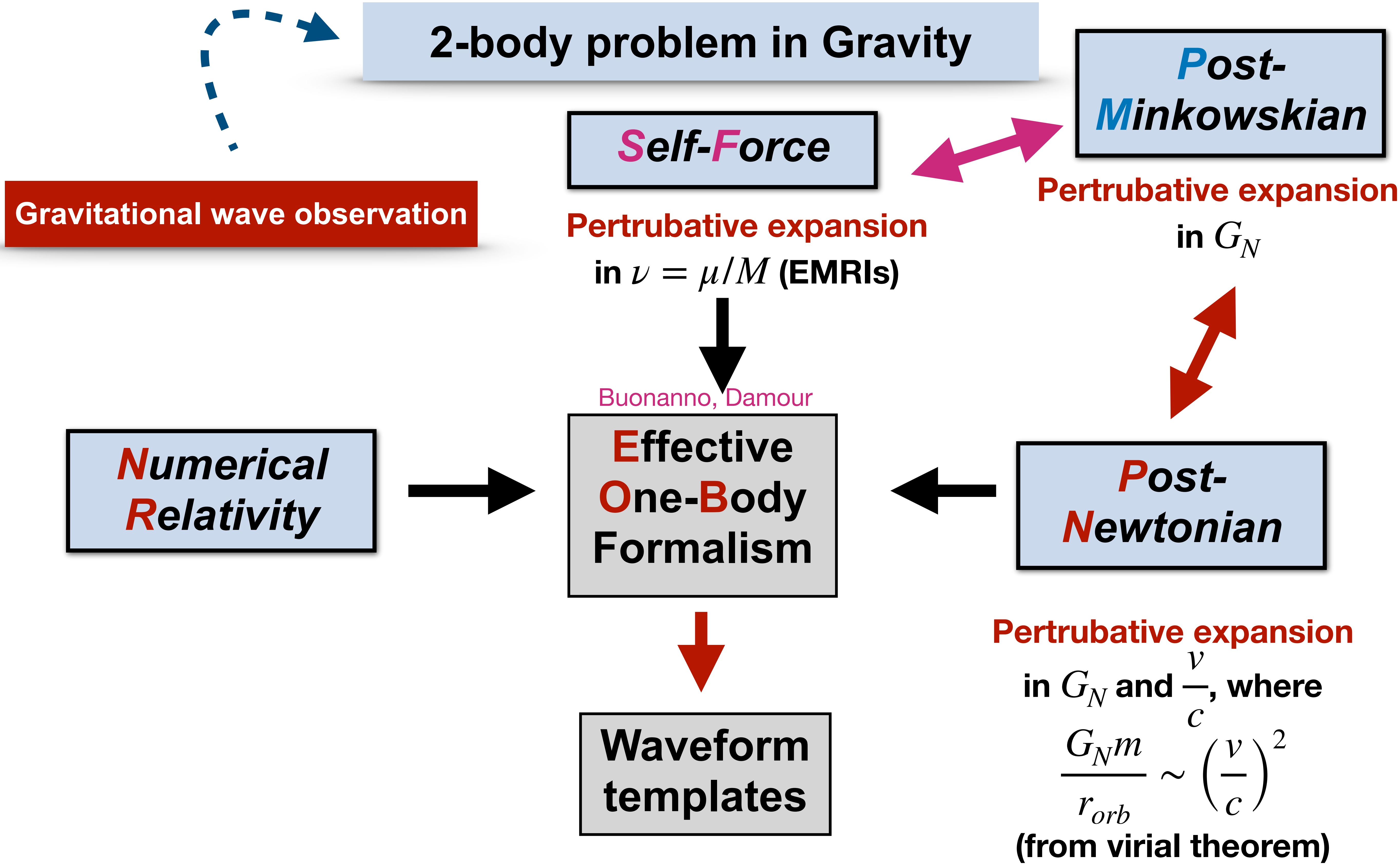
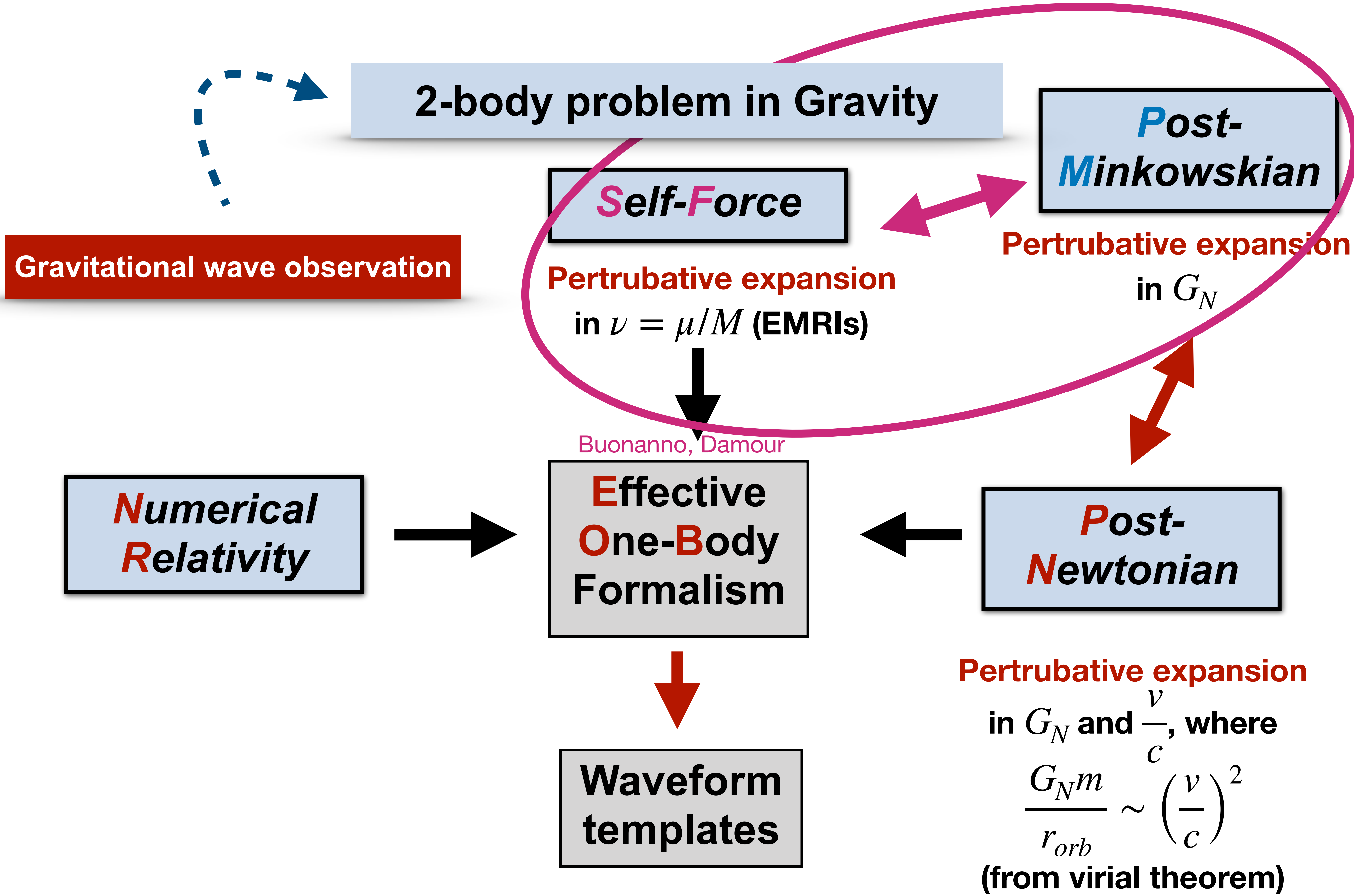
YOUNGST@RS-Loop-the-Loop MITP, online 13/11/2024

Based on works with: P.Vanhove **[2407.09448],[2405.14421]**


2-body problem in Gravity
Gravitational wave observation
Self-Force
Pertrubative expansion
in $\nu = \mu/M$ (EMRIs)
Post-Minkowskian
Pertrubative expansion
in G_N
Buonanno, Damour
Effective One-Body Formalism
Numerical Relativity
Post-Newtonian
Pertrubative expansion
in G_N and $\frac{v}{c}$, where
$\frac{G_N m}{r_{orb}} \sim \left(\frac{v}{c}\right)^2$
(from virial theorem)
Waveform templates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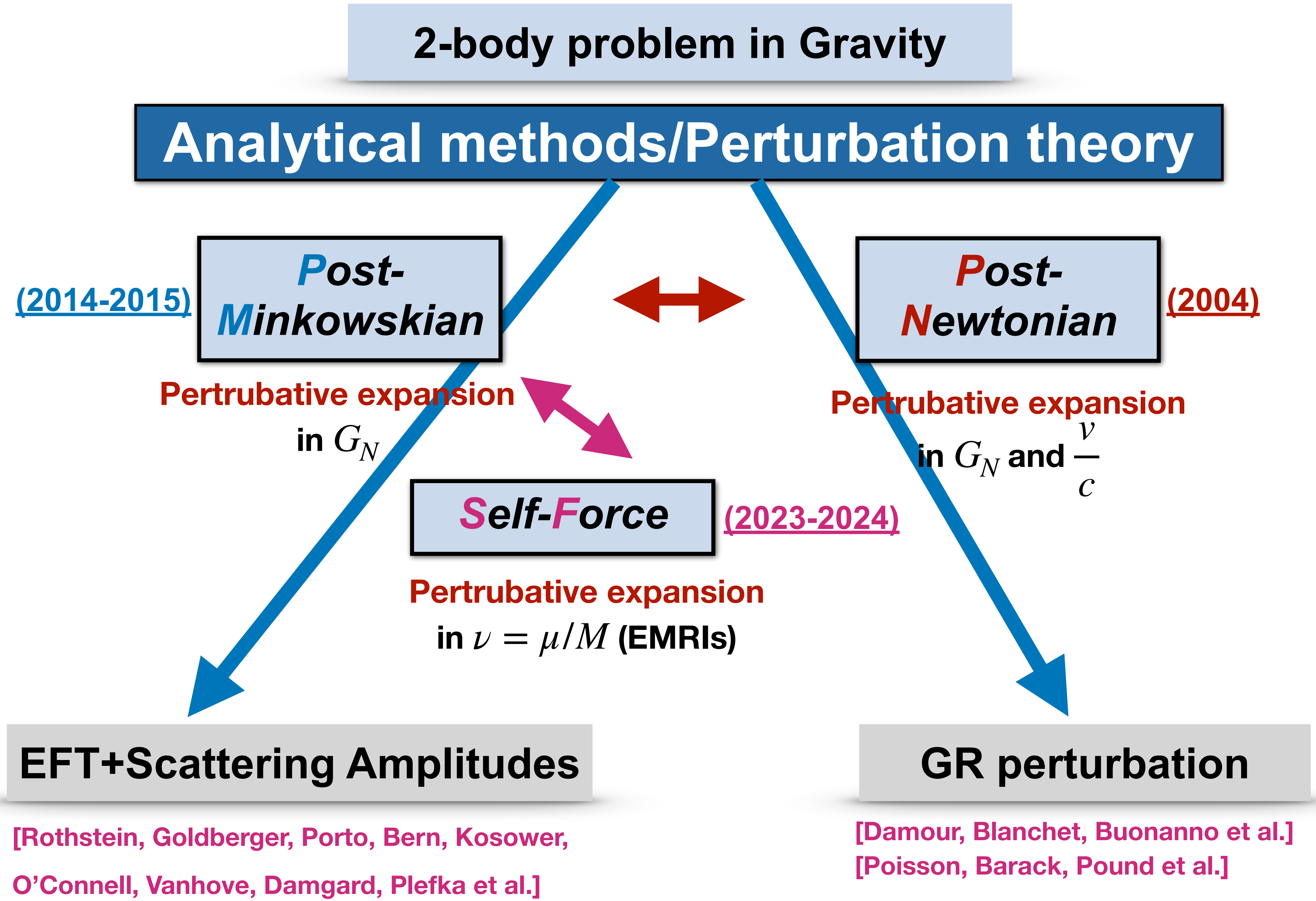
2-body problem in Gravity

Post-Minkowskian vs **Self-Force**

	1PM	2PM	3PM	4PM	5PM	...	
0SF	$\Big[(G\,m_1)$	$+(G\,m_1)^2$	$+(G\,m_1)^3$	$+(G\,m_1)^4$	$+(G\,m_1)^5$	$+\ldots\Big]$	$\times\, m_2$
1SF			$\Big[(G\,m_1)^2$	$+(G\,m_1)^3$	$+(G\,m_1)^4$	$+\ldots\Big]$	$\times\, G\, m_2^2$
2SF					$\Big[(G\,m_1)^3$	$+\ldots\Big]$	$\times\, G^2\, m_2^3$


2-body problem in Gravity
Analytical methods/Perturbation theory
(2014-2015)
Post-Minkowskian
Pertrubative expansion
in G_N
Post-Newtonian
(2004)
Pertrubative expansion
in G_N and $\frac{v}{c}$
Self-Force
(2023-2024)
Pertrubative expansion
in $\nu = \mu/M$ (EMRIs)
EFT+Scattering Amplitudes
[Rothstein, Goldberger, Porto, Bern, Kosower, O'Connell, Vanhove, Damgard, Plefka et al.]
GR perturbation
[Damour, Blanchet, Buonanno et al.]
[Poisson, Barack, Pound et al.]

2-body problem in Gravity

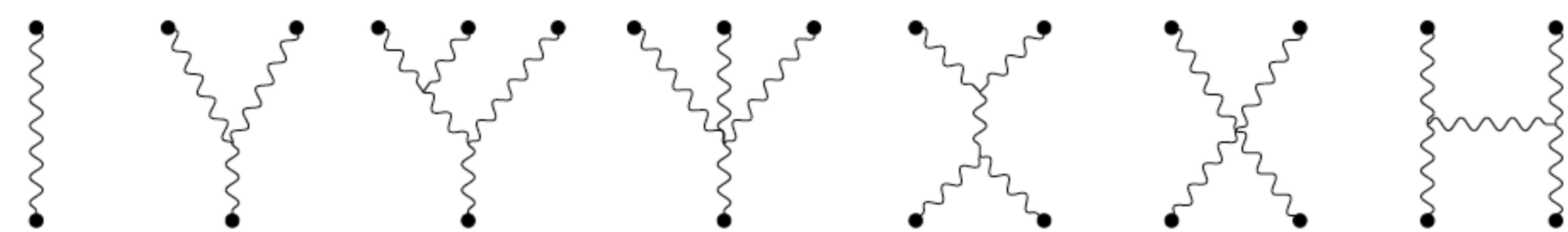
EFT+Scattering Amplitudes (in a nutshell)

$$e^{iS_{\text{eff}}[x_a]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\text{EH}}[h]+iS_{\text{GF}}[h]+iS_{\text{pp}}[x_a,h]}$$

→ **Point particle approximation +weak metric perturbations**

Feynman rules

integrate-out graviton (classical=NO grav. loops)



(a) (b) (c) (d) (e) (f) (g)

Schwarzschild from Amplitudes to all orders in G

[2407.09448],[2405.14421] S.M., P. Vanhove

[2308.14832,2406.14770] Cheung, Parra-Martinez et al.
[2308.15304] Kosmopoulos, Solon

Towards SF-EFT, it is necessary to have a self-consistent framework to all orders in G

Simplest case: *Schwarzschild metric*

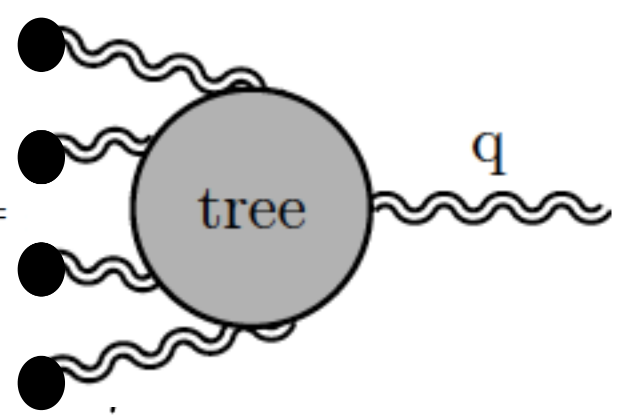
Schwarzschild from Amplitudes to all orders in G

[2407.09448],[2405.14421] S.M., P. Vanhove

GREFT ⟷ **General Relativity**

Equivalence up to:

1. **Choice of DOFs** → $g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} h_{\mu\nu}^{(n)}$
2. **Choice of Gauge** → **harmonic gauge**

$\mathcal{M}_3(p_1, q) =$ 

$$i\mathcal{M}_3^{(l)}(p_1,q) = -\frac{i\sqrt{32\pi G_N}}{2}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu}$$ Duff (1974)

$$h_{\mu\nu}^{(l+1)}(\vec{x}) = -16\pi G_N\int\frac{d^d\vec{q}}{(2\pi)^d}e^{i\vec{q}\cdot\vec{x}}\frac{1}{\vec{q}^2}\left(\langle T_{\mu\nu}^{(l)}\rangle^{\text{class.}}(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle^{\text{class.}}(q^2)\right)$$

Expectation: n**-loop diagrams generate** G_N^{n+1} **terms of the metric**

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

GREFT ⟷ **General Relativity**

Equivalence up to:

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Too complicated!!

Main problems from previous attempts: [2010.08882] **S.M**, Vanhove

1. **Infinite tower of *non-minimal couplings* (due to intermediate UV-divs)**
2. **No algorithm for higher loops (3-loops was already complicated)**

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] **S.M.**, P. Vanhove

GREFT ⟷ **General Relativity**

Equivalence up to:

1. **Choice of DOFs** → $$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} h_{\mu\nu}^{(n)}$$
2. **Choice of Gauge** → **harmonic gauge**

Take a step back

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Cubic formulation of GREFT

[1705.00626] Cheung, Remmen

Choice of DOFs: **1)Gothic metric:** $\mathfrak{g}^{ab} = \sqrt{-g} g^{ab} = \eta^{ab} - \sqrt{32\pi G_N} h^{ab}$

[2403.13216] Lee, Damgaard
See for a similar derivation with the perturbiner approach

(GR people do it, we should pay more attention)

2) Extra Auxiliary field : $A^a_{bc} = \Gamma^a_{bc} - \frac{1}{2}\delta^a_{(b}\Gamma^d_{c)d}.$

(unorthodox but necessary to constrain to 3pt vertices)

Choice of Gauge: harmonic gauge*(non unique)

+ couple to worldline $\mathcal{L}_{p.p.} = -\frac{m}{2}\int d\tau\ \left(e^{-1}\, g^{\mu\nu} v_\mu v_\nu + e\right) = -\frac{m}{2}\int d\tau\ \left(\frac{\mathfrak{g}^{\mu\nu} v_\mu v_\nu}{(\sqrt{-\mathfrak{g}})^{\frac{2}{D-2}}} + 1\right)$

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Cubic formulation of GREFT

Complete ansatz:

Single multi-loop Master Integral

$$\sqrt{32\pi G_N}\, h_{\mu\nu}^{(n)}(\mathbf{x}) = \int d^{D-1}x\; e^{i\mathbf{kx}}\; J_{\mu\nu}^{(n)}(\mathbf{k}),$$

$$\sqrt{32\pi G_N}\, A_{bc}^{a\ (n)}(\mathbf{x}) = \int d^{D-1}x\; e^{i\mathbf{kx}}\; Y_{bc}^{a\ (n)}(\mathbf{k}),$$

$$J_{\mu\nu}^{(n)}(\mathbf{k}) = \rho(|\mathbf{k}|, D, n)\left(\chi_1^{(n)}\delta_\mu^0\delta_\nu^0 + \chi_2^{(n)}\eta_{\mu\nu} + \chi_3^{(n)}\frac{k_\mu k_\nu}{\mathbf{k}^2}\right)$$

$$\rho(|\mathbf{k}|, D, n) = \frac{\Gamma\left(\frac{2-(D-3)(n-1)}{2}\right)}{\Gamma\left(\frac{n(D-3)}{2}\right)}\frac{\left(\Gamma\left(\frac{D-3}{2}\right)G_N m\right)^n}{(|\mathbf{k}|/(2\sqrt{\pi}))^{2-(D-3)(n-1)}};$$

$$Y_{bc}^{a\ (n)}(\mathbf{k}) = -i\rho(|\mathbf{k}|, D, n)\left(k_{(b}\left(\chi_7^{(n)}\delta_{c)}^0\delta_0^a + \chi_8^{(n)}\delta_{c)}^a\right)\right.$$
$$\left. + k^a\left(\chi_4^{(n)}\delta_b^0\delta_c^0 + \chi_5^{(n)}\eta_{bc} + \chi_6^{(n)}\frac{k_b k_c}{\mathbf{k}^2}\right)\right).$$

form factors

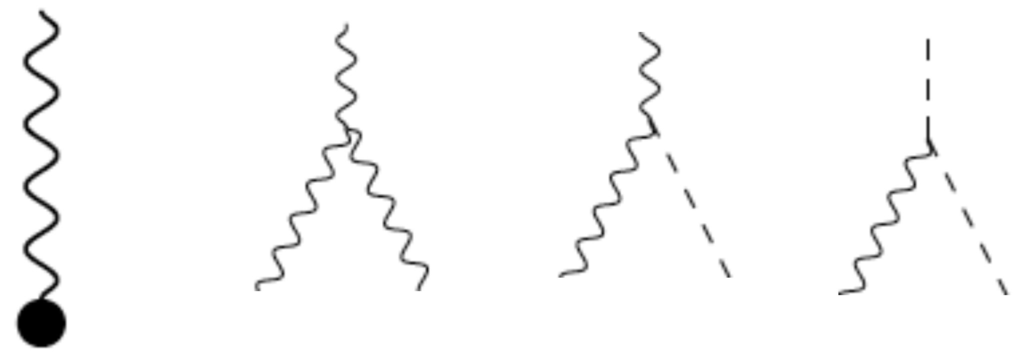
$$\chi^{(n)}(D) = (\chi_1^{(n)}, \ldots, \chi_8^{(n)})$$

Schwarzschild from Amplitudes

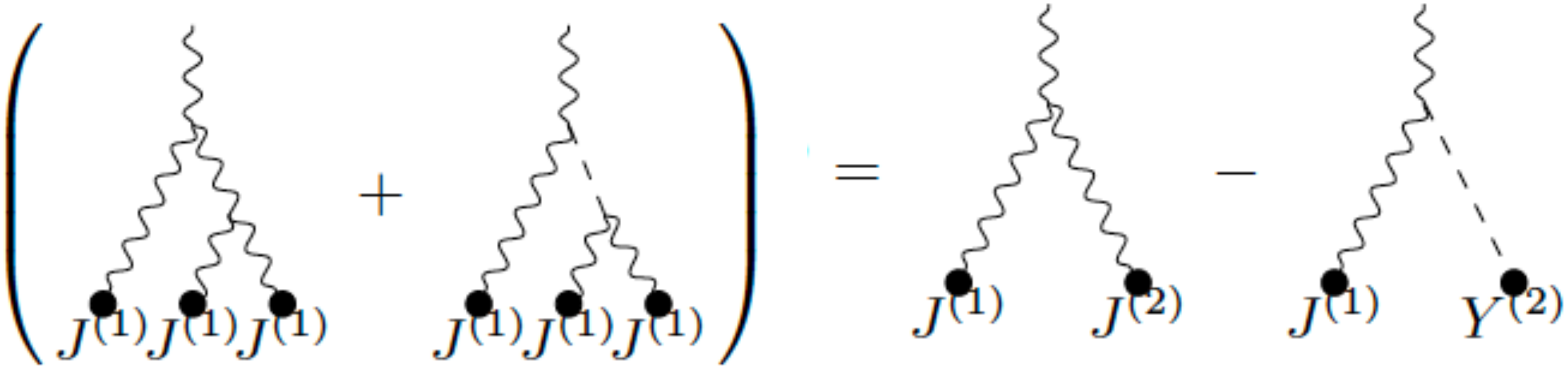
[2407.09448],[2405.14421] S.M., P. Vanhove

Cubic formulation of GREFT

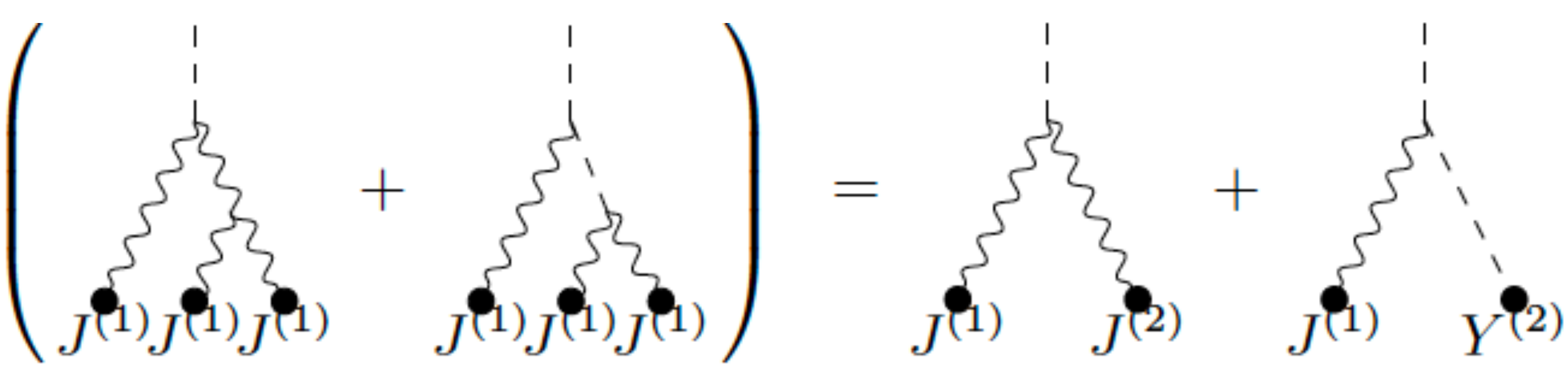
Feynman rules:



$$J^{(3)}_{\mu\nu} = \frac{1}{2}$$



$$Y^{a\ (3)}_{bc} = \frac{1}{2}$$

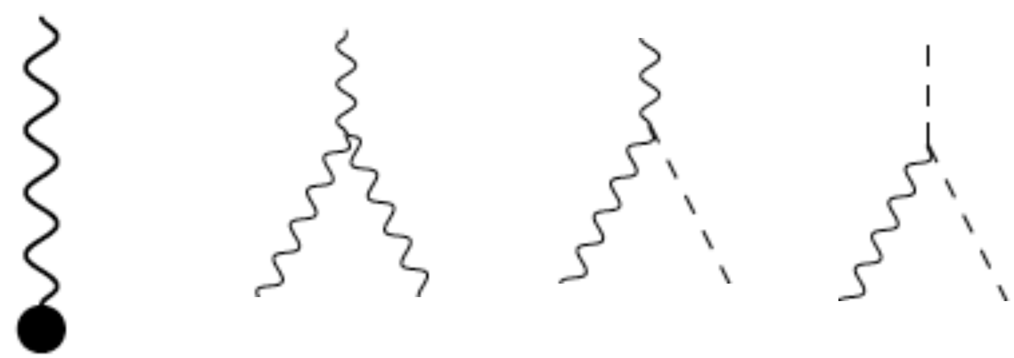


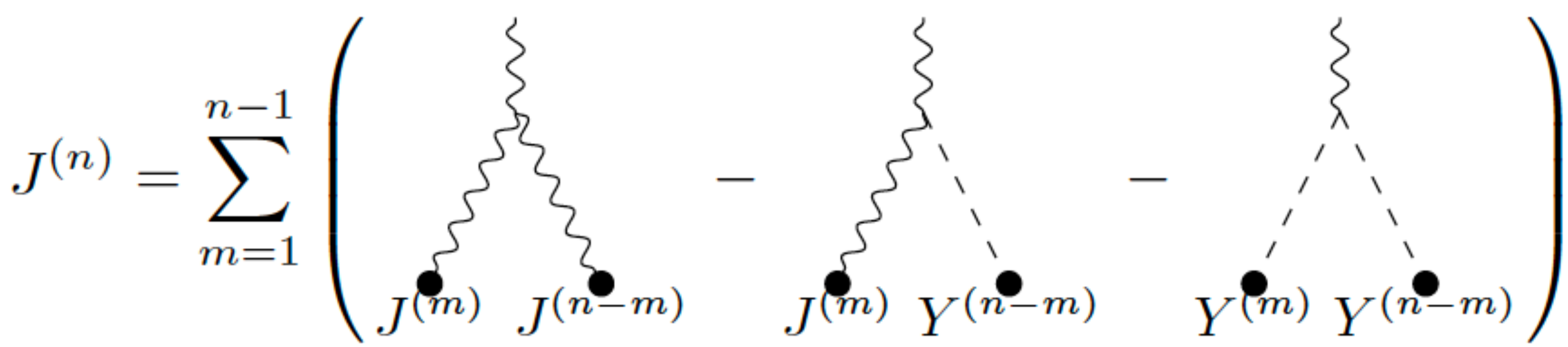
Schwarzschild from Amplitudes

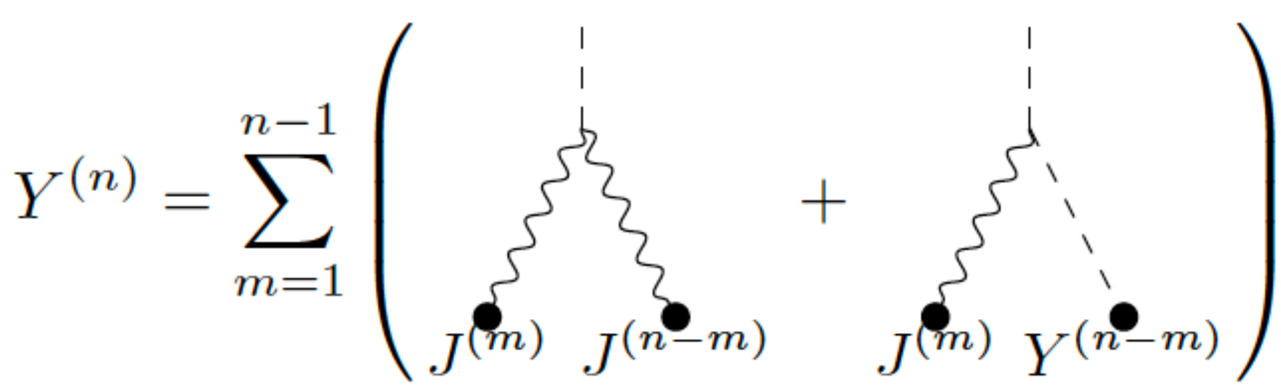
[2407.09448],[2405.14421] **S.M.**, P. Vanhove

Cubic formulation of GREFT

Feynman rules:







Iterative structure to all orders!!!
due to 3pt interactions

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] **S.M.**, P. Vanhove

Metric to all orders in G

Recursion relations

$$\chi_k^{(n)}(D) = \sum_{i,j=1}^{8} \sum_{m=1}^{n-1} \chi_i^{(m)}(D)\chi_j^{(n-m)}(D) M_k^{ij}(D)$$

Solvable at D=4 →

$$\chi^{(n)}(4) = \Big(8, 0, 0, 4 + n(-1)^n + \frac{1+3(-1)^n}{2n(n+2)}, 2 + \frac{1+3(-1)^n}{2n(n+2)}, \frac{1+3(-1)^n}{2n(n+2)}(n-3), \frac{1}{n} - 4 - \frac{1+3(-1)^n}{2n(n+2)}(n+1), \frac{1+3(-1)^n}{2n(n+2)}\Big)$$

no UV-divs!!!

→ **Resums to GR solution**

(**GREFT computation "picks" the simplest harmonic gauge**)

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Geodesic motion (0SF)

[2308.14832,2406.14770]Cheung, Parra-Martinez, Rothstein, Shah, Wilson-Gerow
[2308.15304] Kosmopoulos, Solon

SF expansion: $\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_l + \mathcal{S}_H$; $\quad \mathcal{S}_H = -\frac{M}{2}\int d\tau_H \left(\frac{\mathfrak{g}^{\mu\nu} v_{H\mu} v_{H\nu}}{(\sqrt{-\mathfrak{g}})^{\frac{2}{D-2}}} + 1 \right)$

***n-point graviton vertices* contrary to PM-expansion**

$$e^{i\mathcal{S}_{\text{eff}}[x_l, x_H]} = \int \mathcal{D}h\, \mathcal{D}A\, e^{i\mathcal{S}_{EH}[h,A] + i\mathcal{S}_{GF}[h] + i\mathcal{S}_l[x_l,h] + i\mathcal{S}_H[x_H,h]}$$

integrate-out via diagrams →

$$\mathcal{S}_{\text{eff}} = -\frac{M}{2}\int d\tau_H\, \eta^{\mu\nu} v_{H\mu} v_{H\nu} + M\sum_{n=0}^{\infty}\int d\tau_l \left(\frac{m}{M}\right)^{n+1} \mathcal{L}_n[x_l(\tau_l), x_H(\tau_H)]$$

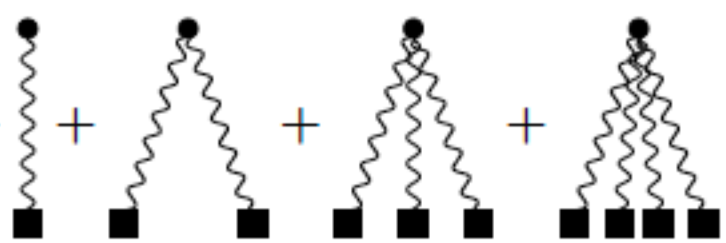
+SF expand trajectories $\quad x_l^\mu(\tau_l) \equiv x^\mu(\tau) = \sum_{n=0}^{\infty}\left(\frac{m}{M}\right)^n \delta x^{(n)\,\mu}(\tau),\; x_H^\mu(\tau_H) = u_H^\mu \tau_H + \sum_{n=1}^{\infty}\left(\frac{m}{M}\right)^n \delta x_H^{(n)\,\mu}(\tau_H)$

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Geodesic motion (0SF)

<u>**SF expansion:**</u> $$\mathcal{S}_{\text{eff}} = -\frac{M}{2}\int d\tau_H\ \eta^{\mu\nu} v_{H\mu} v_{H\nu} + M \sum_{n=0}^{\infty} \int d\tau_l\ \left(\frac{m}{M}\right)^{n+1} \mathcal{L}_n[x_l(\tau_l), x_H(\tau_H)]$$

$\mathcal{L}_0[x^\mu(\tau), u^\mu_H \tau_H] = \bullet +$  $+ \cdots$

1) <u>**Infinite**</u> **tower of n-graviton worldline vertices**

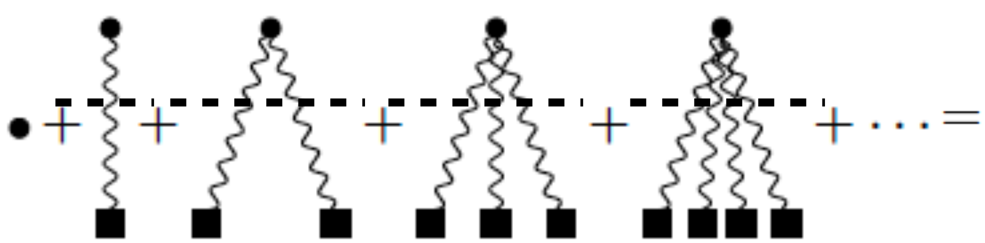
where $J_{\mu\nu}(\mathbf{k}) =$  $= \sum_{n=1}^{\infty} J^{(n)}_{\mu\nu}(\mathbf{k})$

2) Effective 1pt contains an <u>infinite</u> series -*<u>dressed graviton emission</u>*-, <u>already known</u>

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Geodesic motion (0SF)

$$\mathcal{L}_0[x^\mu(\tau), u_H^\mu \tau_H] =$$  $$= \frac{1}{2} v_\mu(\tau) v_\nu(\tau) \left(-\eta^{\mu\nu} + \sum_{L=1}^{\infty} \mathcal{I}^{(L)}_{\alpha_1\beta_1,\dots,\alpha_L\beta_L} \mathcal{T}^{\mu\nu\alpha_1\beta_1,\dots,\alpha_L\beta_L}_{(L)} \right)$$

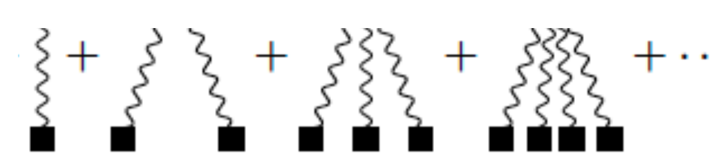
1) **Infinite tower of n-graviton worldline vertices**

$$\mathcal{T}^{\mu\nu\alpha_1\beta_1,\dots,\alpha_n\beta_n}_{(n)} = \eta^{\mu\alpha_n}\eta^{\nu\beta_n} \mathcal{P}^{\alpha_1\beta_1,\dots,\alpha_{n-1}\beta_{n-1}}_{(n-1)} - \eta^{\mu\nu} \mathcal{P}^{\alpha_1\beta_1,\dots,\alpha_n\beta_n}_{(n)} =$$ 

$$\frac{1}{(\sqrt{-\mathfrak{g}})^{\frac{2}{D-2}}} = 1 + \sum_{n=1}^{\infty} (32\pi G_N)^{\frac{n}{2}} \mathcal{P}^{\alpha_1\beta_1,\dots,\alpha_n\beta_n}_{(n)} h_{\alpha_1\beta_1} \times \cdots \times h_{\alpha_n\beta_n}$$

2) **Effective 1pt contains an infinite series**
-dressed graviton emission-

Fourier+ ∞ Loops

$$\mathcal{I}^{(L)}_{\alpha_1\beta_1,\dots,\alpha_L\beta_L} = \int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}(\tau)} \int_{\mathbb{R}^{3L}} \prod_{i=1}^{L} \frac{d^3\mathbf{q}_i}{(2\pi)^3} \delta^{(3)}\left(\sum_{i=1}^{L} \mathbf{q}_i - \mathbf{k} \right) =$$ 

$$\times \sum_{n_1=1}^{\infty} \cdots \sum_{n_L=1}^{\infty} \rho(|\mathbf{q}_i|, n_i) \left(\chi_1^{(n_i)} \delta^0_{\alpha_i} \delta^0_{\beta_i} + \chi_2^{(n_i)} \left(\eta_{\alpha_i\beta_i} - \frac{q_{i,\alpha_i} q_{i,\beta_i}}{\mathbf{q}_i^2} \right) \right)$$

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] **S.M.**, P. Vanhove

Geodesic motion (0SF)

2) Effective 1pt contains an <u>infinite</u> series

-dressed graviton emission-

$$\int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}(\tau)} \int_{\mathbb{R}^{3L}} \prod_{i=1}^{L} \frac{d^3\mathbf{q}_i}{(2\pi)^3} \delta^{(3)}\left(\sum_{i=1}^{L} \mathbf{q}_i - \mathbf{k}\right)$$

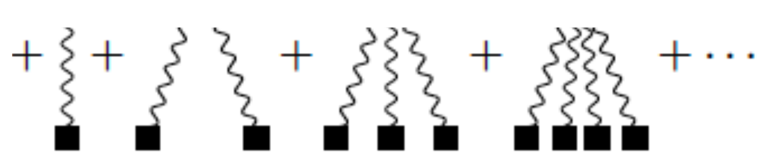
Fourier+ ∞ Loops

decouples →

$$\delta^{(3)}\left(\sum_{i=1}^{L} \mathbf{q}_i - \mathbf{k}\right) = \int_{\mathbb{R}^3} e^{i\mathbf{u}\cdot(\sum_{i=1}^{L}\mathbf{q}_i - \mathbf{k})} \frac{d^3\mathbf{u}}{(2\pi)^3}$$

$$\int_{\mathbb{R}^{3L}} \prod_{i=1}^{L} \frac{d^3\mathbf{q}_i}{(2\pi)^3} e^{i\mathbf{x}(\tau)\cdot\mathbf{q}_i}$$

∞ decoupled Fourier



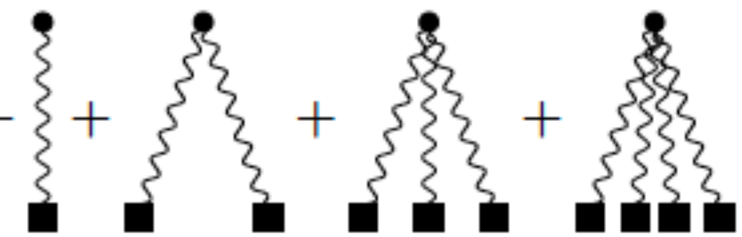
$$= \mathcal{I}^{(L)}_{\alpha_1\beta_1,\ldots,\alpha_L\beta_L} = \rho^{2L} \prod_{i=1}^{L} \left(\delta^0_{\alpha_i}\delta^0_{\beta_i}\left(\frac{4(1+\rho)}{\rho(1-\rho)} - 1\right) + n_{\alpha_i} n_{\beta_i}\right)$$

Schwarzschild from Amplitudes

[2407.09448],[2405.14421] S.M., P. Vanhove

Geodesic motion (0SF)

$$\mathcal{L}_0[x^\mu(\tau), u_H^\mu \tau_H] = \bullet + \ldots$$



$$+ \cdots = \frac{1}{2} v_\mu(\tau) v_\nu(\tau) \left(-\eta^{\mu\nu} + \sum_{L=1}^{\infty} \mathcal{I}^{(L)}_{\alpha_1\beta_1,\ldots,\alpha_L\beta_L} \mathcal{T}^{\mu\nu\alpha_1\beta_1,\ldots,\alpha_L\beta_L}_{(L)} \right)$$

recursion relations under control

$$\sum_{L=1}^{\infty} \mathcal{P}^{\alpha_1\beta_1,\ldots,\alpha_L\beta_L}_{(L)} \mathcal{I}^{(L)}_{\alpha_1\beta_1,\ldots,\alpha_L\beta_L} = \frac{1}{(1+\rho)^2} - 1.$$

resums →

$$= -\frac{1}{2} v_\mu(\tau) v_\nu(\tau) g^{\mu\nu}(|\mathbf{x}|(\tau))$$

trivially gives geodesic eq.

Rediscovered America!

1)Trivial from GR perspective but highly non-trivial from EFT
2)Crucial stepping stone to go beyond 0-SF order

Conclusion

1. Derivation from flat to curved background with usual techniques. What about Kerr?
2. Interplay between Fourier transforms and loop integrals
3. SF-EFT setup can-in principle- be extended
4. Non-perturbative, EFT-based approach can be used for questions regarding BHs (Love numbers, quantum effects etc.)