Solving Einstein equation using recursions

Kanghoon Lee

(APCTP JRG)

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@Loop-the-Loop

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apctp asia pacific center for theoretical physics

Gravitational wave from Binary BH mergers

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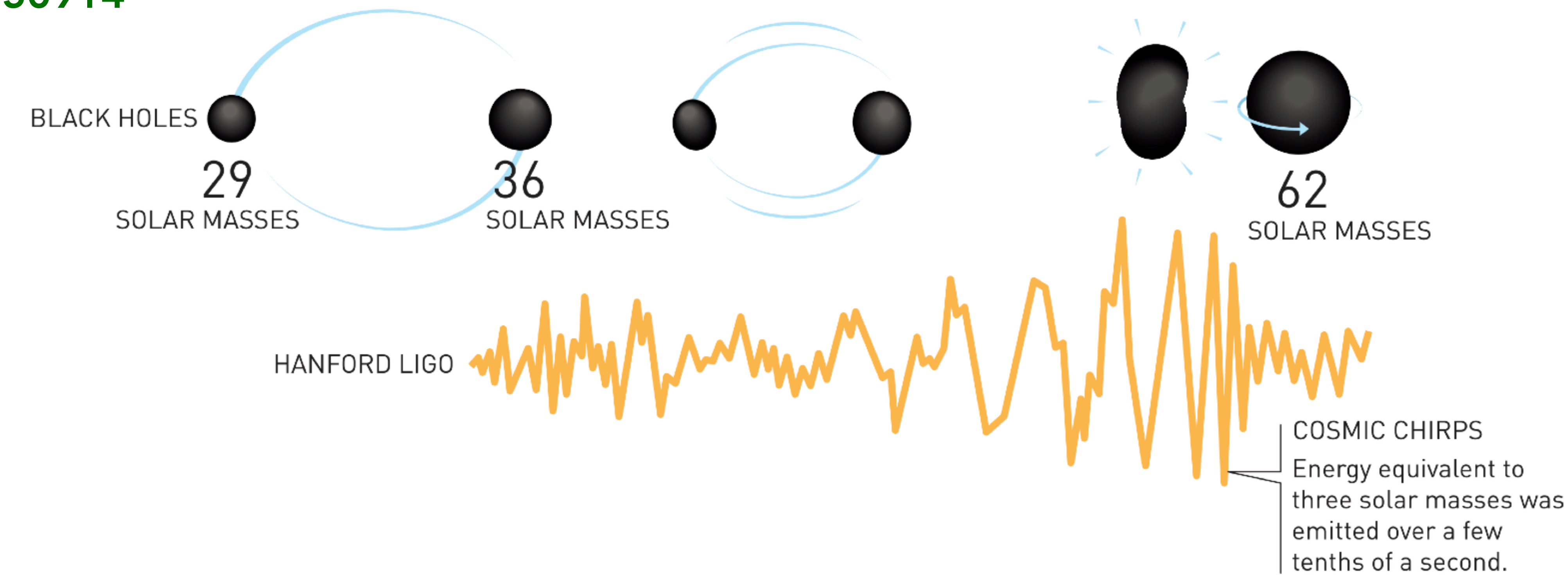


- **Gravitational wave: new window to probe our Universe**
- How do we describe this system? $\Longrightarrow$ **Solve Einstein equation** (perturbatively)
- ***Are the theoretical tools we have powerful enough to solve this problem?***

Toy model — Schwarzschild BH solution

Solving perturbative Einstein Equation

1. **Solve Einstein Equation directly** (old and brute force approach)
    - Green function method
    - Perturbative GR is notorious for its complexity
    - Leading order correction is the practical limit [Florides, Synge 61] [Westpfahl, 85]
2. **Scattering amplitude Approach** (since 2018)
    - Modern techniques in **QFT/Quantum Gravity**

      Generalized unitarity Bern, Dixon, Dunbar, Kosower, hep-ph/9403226

      On-shell recursion Britto, Cachazo, Feng, Witten, hep-th/0501052

      Color-kinematics duality and double copy Bern, Carrasco, Johansson, 0805.3993, 1004.0476
    - Issues — convergence of the series, loop integrals, etc
3. **Go back to the Einstein equation again** (armed with new techniques)

[Damgaard, KL `24]

In this talk

- returning to the solving Einstein equation explicitly
- **Two main ideas**
  - **good variable** — By doubling the fields, the perturbative Einstein equation is drastically simplified. We can hide the ugly infinite expansion.
  - **off-shell recursion** — A new methodology for solving perturbative Einstein Equation Remarkably, all the "higher-loop integrals" are represented by iterations of one-loop **bubble integrals**.
- For the Schwarzschild BH case, we derived **all-order results** — first derivation!
  - **Efficiency** — fixed number of terms, recursions and simple loop integrals…
  - **Universality** — binary black holes & rotating black holes, branes etc
- Recently, the similar results are derived from the amplitude point of view

[Mougiakakos, Vanhove `24]

Perturbative GR
and
doubling prescription

Tensor density representation

- **Two sources** of the infinite expansion: g^{-1} and $\sqrt{-g}$
- **Field redefinition - tensor density** [Landau & Lifshitz book]:

$$\sigma^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}\,, \qquad \sigma_{\mu\nu} = \frac{1}{\sqrt{-g}}\, g_{\mu\nu}\,,$$

- Why? There is no $\sqrt{-\sigma}$. The number of σ^{-1} is always greater than σ due to derivatives.
- **EH action** (up to total derivative) in terms of the **tensor density**

$$S_{\rm EH} = \int \mathrm{d}^D x \left[\frac{1}{4}\sigma^{\mu\nu}\partial_\mu \sigma^{\rho\sigma}\partial_\nu \sigma_{\rho\sigma} - \frac{1}{2}\sigma^{\mu\nu}\partial_\mu \sigma^{\rho\sigma}\partial_\rho \sigma_{\nu\sigma} + (D-2)\sigma^{\mu\nu}\partial_\mu \hat{d}\partial_\nu \hat{d} \right], \quad \partial_\mu \hat{d} = -\frac{1}{4}\sigma^{\rho\sigma}\partial_\mu \sigma_{\rho\sigma}$$

- Substitute the metric perturbation [Cheung, Remmen 18], [Deser, 70], [Capper, Leibbrandt, Medrano, 73]

$$\sigma^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}\,, \qquad \sigma_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \kappa^n (h^n)_{\mu\nu}\,.$$

- Provides **the simplest form** of the perturbative GR [Cho, Kim, Lee, 23]
  - general n-th order terms of the EH action and Einstein eq.
  - Three minimal building blocks

Doubling prescription

- **Idea:** do not substitute metric perturbations from the beginning!
- Let us treat the **metric** (σ) and the **inverse metric** (σ^{-1}) on **equal footing**

[Gomez, Lipinski Jusinskas, Lopez-Arcos, Quintero Velez `22]

- **Remove metric** (σ) and **introduce an auxiliary field $\tilde{\sigma}$. on-shell value of $\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu}$**
- Impose a **constraint:**

$$\tilde{\sigma}_{\mu\nu}\sigma^{\nu\rho} = \delta_\mu{}^\rho$$

- perturbative expansions:

$$\sigma^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,, \quad \tilde{\sigma}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}\,.$$

- Then $\tilde{h}$ satisfies the constraint $\Longrightarrow \tilde{h}^{\mu\nu} = h^{\mu\nu} + \tilde{h}^\mu{}_\rho h^{\rho\nu}\,,$

$$\tilde{h}^{\mu\nu}_{(n)} = h^{\mu\nu}_{(n)} + \sum_{m=1}^{n-1} \tilde{h}^{\mu\rho}_{(n-m)} h^{\rho\nu}_{(m)}$$

Source of the Schwarzschild BH

- Consider pure gravity with a matter

$$S = \int d^4x \left[\frac{1}{2\kappa^2}\sqrt{-g}R + \frac{1}{2} j_{\mu\nu}(x) g^{\mu\nu}(x) \right]$$

where $j_{\mu\nu}(x)$ is an external source (density) without metric dependence,

- Relation to the energy-momentum tensor $T_{\mu\nu}$

$$\sqrt{-g}T_{\mu\nu} = j_{\mu\nu}$$

- Schwarzschild BH is not a vacuum solution — point mass source

- **Energy-momentum tensor** for a point mass traveling on a worldline $x^\mu(\tau)$

$$T^{\mu\nu}\left(y^\sigma\right) = 8\pi GM \int \left[\frac{\delta^{(4)}\left(y^\sigma - x^\sigma(\tau)\right)}{\sqrt{-g}} \right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

- Source of Schwarzschild BH — a **static point mass** placed at the origin, $\boldsymbol{x} = 0$

$$j_{\mu\nu}(x) = 8\pi GM v_\mu v_\nu \delta^3(\boldsymbol{x}) , \qquad v^\mu = \frac{dx^\mu}{d\tau} = (-1,0,0,0) .$$

Field equation

- **Einstein tensor (density)**

$$\mathscr{G}^{\mu\nu} = \frac{1}{2}\sigma^{\rho\sigma}\left[\partial_\rho\partial_\sigma\sigma^{\mu\nu} + \partial_\rho\sigma^{\kappa\mu}\partial_\sigma\tilde{\sigma}_{\kappa\lambda}\sigma^{\nu\lambda}\right] - \sigma^{\rho(\mu}\left[\partial_\rho\partial_\sigma\sigma^{\nu)\sigma} + \partial_\rho\sigma^{|\kappa\lambda}\partial_\kappa\tilde{\sigma}_{\lambda\sigma}\sigma^{\sigma|\nu)}\right]$$
$$+\sigma^{\mu\kappa}\sigma^{\nu\lambda}\left[\frac{1}{4}\partial_\kappa\sigma^{\rho\sigma}\partial_\lambda\tilde{\sigma}_{\rho\sigma} + (D-2)\partial_\kappa\hat{d}\partial_\lambda\hat{d}\right]$$
$$+\frac{1}{2}\left[\partial_\rho\sigma^{\rho\sigma}\partial_\sigma\sigma^{\mu\nu} - \partial_\sigma\sigma^{\rho\mu}\partial_\rho\sigma^{\sigma\nu}\right] + \sigma^{\mu\nu}\left[\partial_\kappa\left(\sigma^{\kappa\lambda}\partial_\lambda\hat{d}\right)\right],$$

- **Einstein equation**

$$\sqrt{-\sigma}\,\mathscr{G}^{\mu\nu} = j^{\mu\nu}$$

$$\mathscr{G}^{\mu\nu} = \sum_{n=1}^{\infty} G^n \mathscr{G}^{\mu\nu}_{(n)}, \qquad j^{\mu\nu}(x) = G j^{\mu\nu}_{(1)}(x)$$

- $j^{\mu\nu}$ **contributes to the** G^1**-order only**

$$\mathscr{G}^{\mu\nu}_{(1)} = -\frac{1}{2}\Box\, h^{\mu\nu}_{(1)} = j^{\mu\nu}$$
$$\mathscr{G}^{\mu\nu}_{(n)} = 0\,, \qquad n > 1$$

Harmonic vs de Donder gauge

- One of the most straightforward gauge choices is the **harmonic** or **de Donder** **gauge**

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \qquad \text{or} \qquad \partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h^{\rho}{}_{\rho} = 0 \qquad \text{for } g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

harmonic gauge   de Donder gauge

- **Linearized harmonic gauge** = **de Donder gauge**, but not in higher orders
- However, in the tensor density perturbations, these are equivalent

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = \partial_\mu\left(\sqrt{-g}\,g^{\mu\rho}\right) = \partial_\mu \sigma^{\mu\rho} = \partial_\mu h^{\mu\rho} = 0$$

- In our perturbation convention,

harmonic gauge = de Donder gauge

- If we obtained a solution using amplitude, in what coordinates do we get the result?

  $\Longleftrightarrow$ **gauge choice**

- **However, it is not obvious in actual computation…**

Schwarzschild metric in harmonic coordinates

- The usual form of the Schwarzischild metric

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

- In the **harmonic coordinates**, the metric

$$\mathrm{d}s^2 = -\frac{r-GM}{r+GM}\mathrm{d}t^2 + \frac{r+GM}{r-GM}\mathrm{d}r^2 + (r+GM)^2\,\mathrm{d}\Omega\,, \qquad \text{obtained by } r \to r+GM$$

- The **tensor density $\sigma^{\mu\nu}$** for this metric ($\sigma^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$),

$$\sigma^{\mu\nu}\partial_\mu\partial_\nu = -\frac{(r+GM)^3}{r^2\,(r-GM)}\partial_t^2 + \left(\delta^{ij} - \frac{G^2M^2x^ix^j}{r^4}\right)\partial_i\partial_j\,.$$

- The corresponding metric perturbations $h^{\mu\nu}$

$$h^{00} = -1 + \frac{(r+GM)^3}{r^2\,(r-GM)} = \frac{4GM}{r} + \frac{7G^2M^2}{r^2} + \frac{8G^3M^3}{r^3} + \frac{8G^4M^4}{r^4} + \cdots\,,$$

$$h^{ij} = \frac{G^2M^2x^ix^j}{r^4}\,.$$

Coefficients of h^{00} is fixed by "8" while h^{ij} truncates at the second order

Remained ambiguity

[Fromholz, Poisson, Will `13
The Schwarzschild metric: It's the coordinates, stupid!]

- Even after the harmonic gauge, the form of the metric is not fixed yet
- Most general solution in harmonic coordinate — a new parameter C

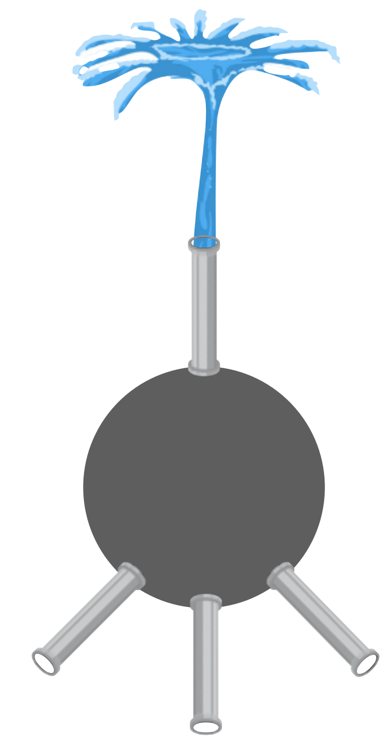
$$\sigma^{00} = -1 - \frac{4M}{r} - \frac{7M^2}{r^2} - \frac{8M^3}{r^3} - \frac{8M^4 - 2CM/3}{r^4} + \mathcal{O}\left(r^{-5}\right)$$

$$\sigma^{ij} = \left(1 - \frac{C}{3r^3} - \frac{2CM^2}{5r^5} + O\left(r^{-6}\right)\right)\delta^{ij} + \left(-\frac{G^2M^2}{r^2} + \frac{C}{r^3} + \frac{2G^2M^2C}{3r^3} + \mathcal{O}\left(r^{-6}\right)\right)\frac{x^i x^j}{r^2}$$

- Solving the de Donder gauge, $\partial_\mu h^{\mu\nu} = 0$, admits an **integration constant C**
- If we turn off C, the solution returns to the previous metric expansion.
- The existence of the parameter has recently been observed in the differential equation.
- *How can we interpret this ambiguity in **the field theory context**?*

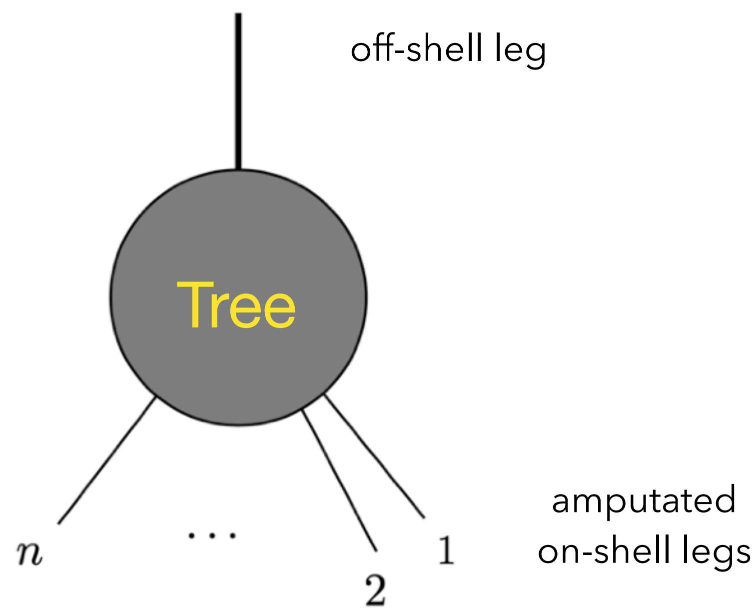
Recursion Relation
for perturbative GR solutions

Off-shell Currents



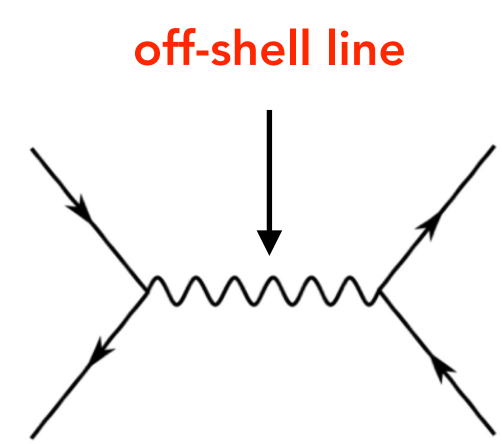
- **Off-Shell recursions**: recursions for **off-shell currents** [Berends, Giele '87] for gluon amplitude at **tree-level**
- **Rank-n Off-shell currents: sum of all $(n+1)$-point Feynman diagrams**
- Diagrammatic representation

$$\Phi_{12\cdots n} =$$



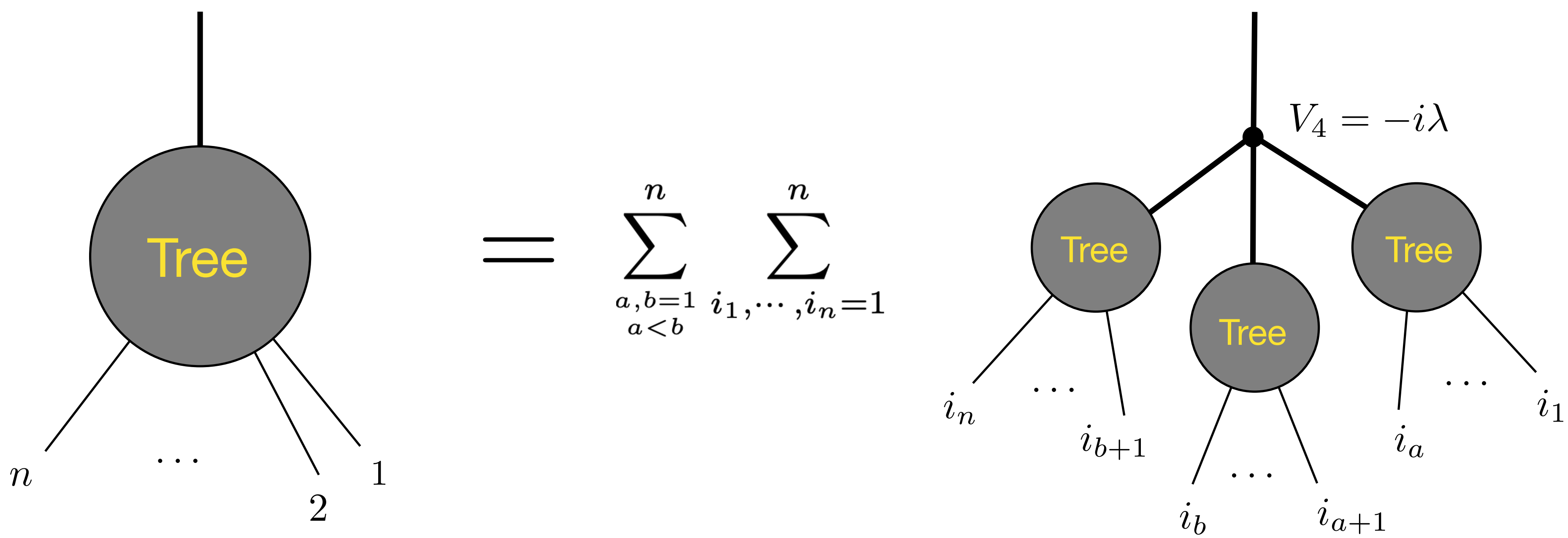
The off-shell line satisfy the conservation law
$\partial_\mu J^\mu_{12\cdots n} = 0$ without EoM — **Ward identity**

- Off-shell lines can be **glued** in a specific way (interaction vertices)
  Intermediate states are off-shell



Off-shell Recursion [Berends, Giele '87]

- **Recursions**: **hidden self-similarity** — finite number of interaction vertices (**patterns**)
- Identifying the **Hierarchy** for off-shell currents: **# of on-shell legs**
- ϕ^4 theory:



- **Efficiency**:
  - Do not treat individual diagrams
  - **Recycling** calculations - never repeat the same calculations!

- **Gravity** — **infinite number of vertices** (No patterns)

Perturbiner expansion [Rosly, Selivanov '96,'97], [KL '22]

- **Modern derivation:** substituting the **perturbiner expansion** into the classical **EoM**
  $\Longrightarrow$ connects **solutions of EoM** and **tree-level amplitudes**

- **The classical field** in the quantum effective action formalism — 1-point function in the presence of the source $j^{\mu\nu}$

$$h^{\mu\nu}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{y_1, y_2, \cdots, y_n} \left\langle 0 \middle| T\left[h_x^{\mu\nu} h_{y_1}^{\kappa_1\lambda_1} \cdots h_{y_n}^{\kappa_n\lambda_n}\right] \middle| 0 \right\rangle_c \frac{i j_{y_1}^{\kappa_1\lambda_1}}{\hbar} \cdots \frac{i j_{y_n}^{\kappa_n\lambda_n}}{\hbar} .$$

- The field corresponds to a different physical quantity depending on the sources:
  - **Inverse propagator:** $j_x^{\mu\nu} = \sum_{i=1}^{N} \int_{y_i} \mathbf{K}_{xy_i}^{\mu\nu,\rho\sigma} e^{-ik_i \cdot y_i}$ $\Longrightarrow$ scattering amplitude.
  - **Plane-wave:** $j_x^{\mu\nu} = \sum_{i=1}^{N} \int_{y_i} e^{-ik_i \cdot y_i}$ $\Longrightarrow$ **Correlation function.**
  - **Point-mass source:** $j_{\boldsymbol{x}}^{\mu\nu} = M v^\mu v^\nu \int_{\ell} e^{-i\ell \cdot x}$ $\Longrightarrow$ **solutions of EoM**. $v^\mu = \frac{dx^\mu}{d\tau} = (-1,0,0,0)$.

Perturbiner expansion for classical solutions

- Substituting the external sources: $h^{\mu\nu}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\ell_1,\ell_2,\cdots,\ell_n} J^{\mu\nu}_{\ell_1\ell_2\cdots\ell_n} e^{-i\ell_{12\cdots n}\cdot x}$,

- It is convenient to shift the loop momenta, $\ell_1 \to -\ell_{12\cdots n}$

$$h^{\mu\nu}(x) = \sum_{n=1}^{\infty} \int_{\ell_1} e^{i\ell_1\cdot x} \int_{\ell_2,\cdots,\ell_n} \frac{1}{(n-1)!} J^{\mu\nu}_{-\ell_{12\cdots n}\ell_2\cdots\ell_n} = \sum_{n=1}^{\infty} \int_{\ell_1} e^{i\ell_1\cdot x} J^{\mu\nu}_{(n)|\ell_1},$$

$$J^{\mu\nu}_{(n)|\ell_1} = \int_{\ell_2,\cdots,\ell_n} \frac{1}{(n-1)!} J^{\mu\nu}_{-\ell_{12\cdots n}\ell_2\cdots\ell_n}$$

- Compare with the **amplitude perturbiner — A continuous limit**

  finite # of particles cannot generate the classical solutions

$$h^{\mu\nu} = \sum_{\mathscr{P}} J^{\mu\nu}_{\mathscr{P}} e^{-ik_{\mathscr{P}}\cdot x}$$

- We call the number of the loop momenta of an off-shell current as **rank**.

  Here the rank is equivalent to the powers of coupling G

$$h^{\mu\nu} = \sum_{n=0}^{\infty} G^n h^{\mu\nu}_{(n)} \quad \text{and} \quad h^{\mu\nu}_{(n)} = \int_{\ell} J^{\mu\nu}_{(n)|\ell} e^{i\ell\cdot x}$$

Structure of loop integrals

- Substituting the perturbiner expansion into the EoMs

$$h^{\mu\nu}_{(n)} = \int_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{x}} J^{\mu\nu}_{(n)|\boldsymbol{\ell}} \quad \text{and} \quad \tilde{h}^{\mu\nu}_{(n)} = \int_{\boldsymbol{\ell}} e^{i\boldsymbol{\ell}\cdot\boldsymbol{x}} \tilde{J}^{\mu\nu}_{(n)|\boldsymbol{\ell}}$$

- **Perturbative Einstein eq**

$$h_1(x)h_2(x)\cdots h_n(x) \Longrightarrow \int_{\ell_1} e^{i\ell_{12\cdots n}\cdot x} \int_{\ell_2,\ell_3,\cdots\ell_n} J_{1|\ell_1} J_{2|\ell_2} \cdots J_{n|\ell_n} = \int_{\ell_1} e^{-i\ell_1\cdot x} \int_{\ell_2,\ell_3,\cdots\ell_n} J_{1|-\ell_{12\cdots n}} J_{2|\ell_2} \cdots J_{n|\ell_n}$$

$$\int_{\ell_3,\cdots\ell_n} \left(\underline{\int_{\ell_2} J_{1|-\ell_{12\cdots n}} J_{2|\ell_2}} \right) J_{3|\ell_3} \cdots J_{n|\ell_n}$$

One-loop bubble integral $J'_{1|\ell_1\ell_3\cdots\ell_n}$

$$\int_{\ell_4,\cdots\ell_n} \left(\int_{\ell_3} J'_{1|-\ell_{13\cdots n}} J_{3|\ell_3} \right) J_{4|\ell_4} \cdots J_{n|\ell_n}$$

- **Fourier integrals $\Longleftrightarrow$ loop integrals: number of loops = number of fields - 1**
- **Integral Factorization —** iterative structure of loop integrals.
- This implies that **only bubble integrals are required**

Deriving and Solving the recursions

Recursions and currents at rank 1

- Rank-1 EoM — **Poisson equation**

$$\Delta h^{\mu\nu}_{(1)} = -2j^{\mu\nu} = -2Mv^\mu v^\nu \int_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

- Substituting the perturbiner expansion $h^{\mu\nu}_{(1)} = \int_{\boldsymbol{\ell}} J^{\mu\nu}_{\boldsymbol{\ell}} e^{-i\boldsymbol{\ell}\cdot\boldsymbol{x}}$, we obtain the initial condition of the off-shell recursion relation

$$J^{\mu\nu}_{(1)|\boldsymbol{\ell}} = \frac{2\kappa^2 M}{|\boldsymbol{\ell}|^2} v^\mu v^\nu = \frac{16\pi GM}{|\boldsymbol{\ell}|^2} v^\mu v^\nu \,,$$

Or equivalently

$$J^{00}_{(1)|\boldsymbol{\ell}} = \frac{16\pi GM}{|\boldsymbol{\ell}|^2} \,, \qquad J^{0i}_{(1)|\boldsymbol{\ell}} = 0 \,, \qquad J^{ij}_{(1)|\boldsymbol{\ell}} = 0 \,.$$

- Since we are assuming an asymptotically flat metric, J^{ij} cannot be a plane wave.

- After Fourier transformation, we have the Newton potential — consistent with the metric expansion

$$h^{00}_{(1)} = \frac{4GM}{r} \qquad h^{0i}_{(1)} = 0 \qquad h^{ij}_{(1)} = 0$$

Recursions and currents at rank 2

- The corresponding recursion is

$$J^{00}_{(2)|-\ell_1} = \frac{\kappa}{|\ell_1|^2}\int_{\ell_2}\left[\frac{5}{4}|\ell_2|^2 - \frac{7}{8}\ell_{12}\cdot\ell_2\right] J^{00}_{(1)|-\ell_{12}} J^{00}_{(1)|\ell_2},$$

$$J^{ij}_{(2)|-\ell_1} = \frac{\kappa}{|\ell_1|^2}\int_{\ell_2}\left[\frac{\ell_{12}^{(i}\ell_2^{j)}}{4} - \frac{\delta^{ij}\ell_{12}\cdot\ell_2}{8}\right] J^{00}_{(1)|-\ell_{12}} J^{00}_{(1)|\ell_2}.$$

- 1-loop **bubble integrals**

$$J^{00}_{(2)|-\ell_1} = \frac{(16\pi GM)^2}{|\ell_1|^2}\int_{\ell_2}\frac{1}{|\ell_2|^2|\ell_{12}|^2}\left[\frac{5}{4}|\ell_2|^2 - \frac{7}{8}\ell_{12}\cdot\ell_2\right] = \frac{14\pi^2G^2M^2}{|\ell_1|}.$$

$$J^{ij}_{(2)|-\ell_1} = \frac{(16\pi GM)^2}{8|\ell_1|^2}\int_{\ell_2}\left[\frac{2\ell_1^{(i}\ell_2^{j)} + 2\ell_2^i\ell_2^j - \delta^{ij}\ell_1^k\ell_2^k}{|\ell_2|^2|\ell_{12}|^2} + \frac{\delta^{ij}}{2}\frac{1}{|\ell_{12}|^2}\right] = \pi^2G^2M^2\left[-\frac{\ell_1^i\ell_1^j}{|\ell_1|^3} + \frac{\delta^{ij}}{|\ell_1|}\right]$$

- The Fourier transformation gives the correct perturbed metric

Recursions and currents at rank 3

- **Rank-3 recursion**

$$|\boldsymbol{\ell}_1|^2 J^{00}_{(3)|-\boldsymbol{\ell}_1} = -(GM)^3\left[\ell^i_1 X^i_{(3)|-\boldsymbol{\ell}_1} + \ell^i_1\int_{\ell_2}\ell^j_{12}J^{00}_{(1)|-\boldsymbol{\ell}_{12}}J^{ij}_{(2)|\boldsymbol{\ell}_2}\right].$$

$$|\boldsymbol{\ell}_1|^2 J^{ij}_{(3)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2}\left[8d^{(i}_{(2)|-\boldsymbol{\ell}_{12}}d^{j)}_{(1)|\boldsymbol{\ell}_2} - 2h^{ij}_{(2)}\ell^k_2 d^k_{(1)|\boldsymbol{\ell}_2}\right] + 2\delta^{ij}\ell^k_1 d^k_{(3)} + \frac{1}{2}W^{ij}_{(3)}$$

$$X^i_{(n)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2}\ell^i_2\sum_{m=1}^{n-1}\tilde{J}^{00}_{(n-m)|-\boldsymbol{\ell}_{12}}J^{00}_{(m)|\boldsymbol{\ell}_2}, \qquad Y^i_{(n)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2}\ell^i_2\tilde{J}^{kl}_{(n-2)|-\boldsymbol{\ell}_{12}}J^{kl}_{(2)|\boldsymbol{\ell}_2},$$

- Again, we need only 1-loop bubble integrals.
- In dimensional regularization

Scaleless integral vanishes in dim. Reg.

$$J^{00}_{(3)|-\boldsymbol{\ell}_1} = \frac{(GM)^3}{|\boldsymbol{\ell}_1|^{d-3}}2^{d+1}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-3}{2}\right), \qquad J^{ij}_{(3)|\boldsymbol{\ell}} = -\int_{\boldsymbol{\ell}_2}\frac{16\pi^3\delta^{ij}}{3|\boldsymbol{\ell}_1|^2|\boldsymbol{\ell}_2|} = 0.$$

- The only place where divergences arise!
- **other regularization scheme** — it does not vanish, and the solution should be modified!
- This explains the ambiguity, C factor

All-order Currents

- From $n \geq 5$ cases, the forms of the EoM/Recursion are fixed.

- In the harmonic gauge, the Landau-Lifshitz variables are extremely simple

$$h^{00} = -1 + \frac{(r+GM)^3}{r^2(r-GM)} = \frac{4GM}{r} + \frac{7G^2M^2}{r^2} + \frac{8G^3M^3}{r^3} + \frac{8G^4M^4}{r^4} + \cdots ,$$

$$h^{ij} = \frac{G^2M^2x^ix^j}{r^4} .$$

- One can read off the currents arbitrary order in G from the Fourier transformation

$$J^{00}_{(1)|\boldsymbol{\ell}} = \frac{4(GM)2^{D-1}\pi^{\frac{D}{2}}\Gamma\left[\frac{D-1}{2}\right]}{\Gamma[\frac{1}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-1}} ,$$

$$J^{00}_{(2)|\boldsymbol{\ell}} = 7(GM)^2 2^{D-2}\pi^{\frac{D}{2}}\Gamma[\tfrac{D-2}{2}]\frac{1}{|\boldsymbol{\ell}|^{D-2}} ,$$

$$J^{00}_{(n)|\boldsymbol{\ell}} = \frac{8(GM)^n\pi^{\frac{D}{2}}\Gamma\left[\frac{D-n}{2}\right]}{2^{n-D}\Gamma[\frac{n}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-n}} , \qquad \text{for } n \geq 3$$

$$J^{ij}_{(2)|\boldsymbol{\ell}} = (GM)^2\pi^{\frac{D}{2}}2^{D-3}\left[-\frac{2\Gamma[\frac{D}{2}]\ell^i\ell^j}{|\boldsymbol{\ell}|^D} + \frac{\Gamma\left[\frac{D-2}{2}\right]\delta^{ij}}{|\boldsymbol{\ell}|^{D-2}}\right] .$$

- One can show the followings by using the **induction**

Arbitrary rank $n \geq 5$ — J^{00}

- We can show that the off-shell currents at an arbitrary order n by **induction**.
- The corresponding recursion: $J^{00}_{(n)|\ell} = \mathcal{E}^{[1]}_{(n)|\ell} - \mathcal{E}^{[2]}_{(n)|\ell}$,

even

$$\mathcal{E}^{[1]}_{(2n)|-\ell_1} = (GM)^{2n}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\Big(-X^i_{(2n)|-\ell_1} + Y^i_{(2n)|-\ell_1}\Big),$$

$$\mathcal{E}^{[2]}_{(2n)|-\ell_1} = (GM)^{2n}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\Big(-X^j_{(2n-2)|-\ell_{12}} + Y^j_{(2n-2)|-\ell_{12}} - \ell_{12}^j J^{00}_{(2n-2)|-\ell_{12}}\Big)J^{ij}_{(2)|\ell_2}.$$

odd

$$\mathcal{E}^{[1]}_{(2n+1)|-\ell_1} = -(GM)^{2n+1}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}X^i_{(2n+1)|-\ell_1},$$

$$\mathcal{E}^{[2]}_{(2n+1)|-\ell_1} = (GM)^{2n+1}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\Big(-X^j_{(2n-1)|-\ell_{12}} - \ell_{12}^j J^{00}_{(2n-1)|-\ell_{12}}\Big)J^{ij}_{(2)|\ell_2}.$$

- Performing the bubble integrals, we have

$$J^{00}_{(2n)} = \frac{8(GM)^{2n}\pi^{\frac{D}{2}}2^{D-2n}\Gamma[\frac{D}{2}-n]}{\Gamma[n]}\frac{1}{|\boldsymbol{\ell}|^{D-2n}},$$

$$J^{00}_{(2n+1)} = \frac{8(GM)^{2n}\pi^{\frac{D}{2}}2^{D-2n-1}\Gamma[\frac{D-2n-1}{2}]}{\Gamma[n+\frac{1}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-2n-1}},$$

Arbitrary rank $n \geq 5$ — J^{ij}

- The EoM for the spatial components

$$\Delta h^{ij}_{(n)} = \sum_{m=1}^{n-1} 4d^i_{(n-m)}d^j_{(m)} + 2\sigma^{ij}\partial_k d^k + Z^{k(i}_{(n)\ k}{}^{j)} - 2Z^{(i|k|}_{(n)\ \ k}{}^{j)} + \frac{1}{2}Z^{(i|k|j)}_{(n)\ \ \ \ k} + \frac{1}{2}W^{(ij)}_{(n)}$$
$$- \left(Z^{k(i}_{(n-2)kl} - 2Z^{(i|k|}_{(n-2)kl} + \frac{1}{2}Z^{(i|k|}_{(n-2)lk} + \frac{1}{2}W^{(i|l|}_{(n-2)} \right) h^{j)l}_{(2)}$$

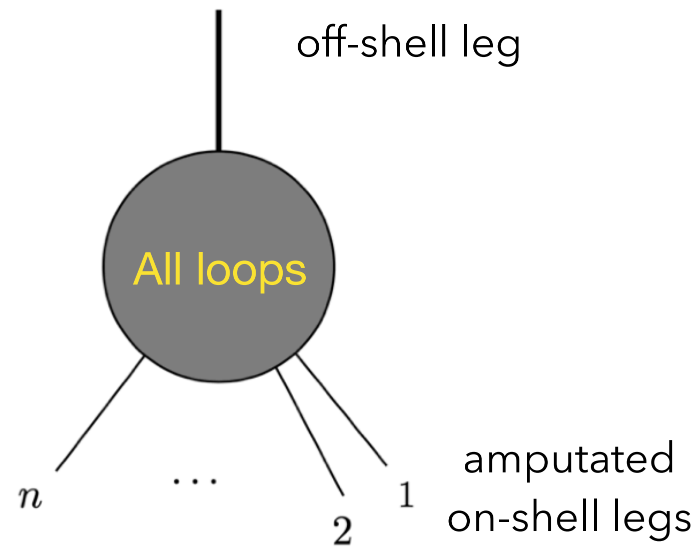
- Divide the EoM into **3 sectors**: d-sector, W-sector and Z-sector
- Interestingly, these three sectors vanish individually (**induction**).
- This implies $J^{ij}_{(n)|\ell} = 0$, as we expected
- This shows the all-order perturbative expansion satisfies the Einstein equation.

Quantum Generalization

Quantum Perturbiner Method [KL '22]

- **Quantum off-shell currents: sum of all $(n+1)$-point all-loop Feynman diagrams**

$$\Phi_{12\cdots n} =$$



Inverse two-point function

$$\Phi_{i_1\cdots i_n} = \tilde{G}_c(-k_{i_1\cdots i_n}, k_{i_1}, \cdots, k_{i_n}) \frac{i\tilde{\mathbf{K}}(-k_{i_1})}{\hbar} \cdots \frac{i\tilde{\mathbf{K}}(-k_{i_n})}{\hbar} \, .$$

- **Fields** in quantum effective action formalism — **1pt function** with the external source $j_x \equiv j(x)$

$$\varphi_x = \frac{\delta W[j]}{\delta j_x} = \sum_{n=1}^{\infty} \frac{i^n}{\hbar^n n!} \int_{y_1, y_2, \cdots, y_n} G_c\left(x, y_1, y_2 \cdots, y_n\right) j_{y_1} j_{y_2} \cdots j_{y_n}$$

- Choice of the external source for **amplitude** $j_x = \sum_{i=1}^{N} \int_{y_i} K_{xy_i} e^{-ik_i \cdot x} = \sum_{i=1}^{N} \tilde{K}(-k_i) e^{-ik_i \cdot x}$

- **Quantum perturbiner expansion:** $\varphi_x = \sum_{\mathscr{P}} \Phi_{\mathscr{P}} e^{-ik_{\mathscr{P}} \cdot x}$

Substituting into the "quantum" EoM?

Dyson-Schwinger equation

- Quantum analogous of classical EoM: **Dyson-Schwinger equation**
- **Quantization** $\Longleftrightarrow$ **deformation** of a field to an **operator**

$$\phi_x \mapsto \hat{\varphi}_x = \varphi_x + \frac{\hbar}{i}\frac{\delta}{\delta j_x}$$

- **DS equation** for phi-4 theory:

$$\int_y K(x,y)\varphi_y + \frac{\lambda}{3!}\varphi_x^3 = j_x - \frac{\lambda}{2}\frac{\hbar}{i}\varphi_x\frac{\delta\varphi_x}{\delta j_x} + \hbar^2\frac{\lambda}{3!}\frac{\delta^2\varphi_x}{\delta j_x \delta j_x}.$$

- **Strategy**: Treat the functional derivatives φ_x as **new independent field variables**
- **Descendant fields:** higher point functions with external sources, multiple off-shell legs

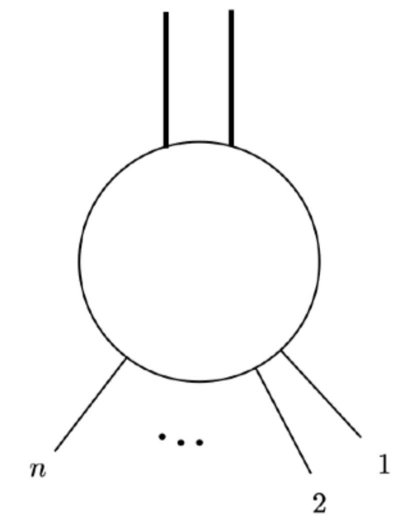
$$\text{1st}: \psi_{xy} = \frac{\delta\varphi_x}{\delta j_x} = \frac{\delta^2 W[j]}{\delta j_x \delta j_y}, \qquad \text{2nd}: \psi'_{xyz} = \frac{\delta^2\varphi_x}{\delta j_x \delta j_x} = \frac{\delta^3 W[j]}{\delta j_x \delta j_y \delta j_z}, \qquad \cdots,$$

- **Derive the perturbiner expansion for the descendant fields**

$$\psi_{x,y} = \int_p \Psi_{p|\varnothing} e^{ip\cdot(x-y)} + \sum_{\mathscr{P}} \int_p \Psi_{p|\mathscr{P}} e^{ip\cdot(x-y)} e^{-ik_{\mathscr{P}}\cdot x},$$

$$\psi'_{x,y,z} = \sum_{\mathscr{P}} \int_{p,q} \Psi'_{p,q|\mathscr{P}} e^{ip\cdot(x-y)+iq\cdot(x-z)} e^{-ik_{\mathscr{P}}\cdot x},$$

$\Psi_{p|12\cdots n}$:



Descendant equations [KL '22]

- Derive the **descendant equations:** acting $\dfrac{\delta}{\delta j_x}$ on the DS eq.

$$\psi_{x,z} = D_{xz} - \frac{\lambda}{2}\int_y D_{xy}\phi_y^2\psi_{y,z} + i\hbar\frac{\lambda}{2}\int_y D_{xy}\left(\phi_y\psi'_{y,y,z} + \psi_{y,z}\psi_{y,y}\right)$$
$$+ \hbar^2\frac{\lambda}{3!}\int_y D_{xy}\psi''_{y,y,y,z}\,,$$
$$\psi'_{x,z,w} = -\frac{\lambda}{2}\int_y D_{xy}\left(2\phi_y\psi_{y,w}\psi_{y,z} + \phi_y^2\psi'_{y,z,w}\right)$$
$$+ i\hbar\frac{\lambda}{2}\int_y D_{xy}\left(\psi_{y,w}\psi'_{y,y,z} + \phi_y\psi''_{y,y,z,w} + \psi'_{y,z,w}\psi_{y,y} + \psi_{y,z}\psi'_{y,y,w}\right)$$
$$+ \hbar^2\frac{\lambda}{3!}\int_y D_{xy}\psi'''_{y,y,y,z,w}\,.$$

- However, new descendant fields arise ψ'' and ψ'''
- How to truncate them?

$\hbar$ expansion and recursions

- Up to now, all the equations are exact
- $\hbar$ **expansion**

$$\varphi_x = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \varphi_x^{(n)}, \qquad \psi_{x,y} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi_{x,y}^{(n)}, \qquad \psi'_{x,y,z} = \sum_{n=0}^{\infty} \left(\frac{\hbar}{i}\right)^n \psi'^{(n)}_{x,y,z}$$

- We can truncate the new descendant fields **because these are from higher $\hbar$-order terms**
- **1-loop DS equations and tree-level descendant equation**

$$\phi_x^{(1)} = \int_y D_{xy} \left[j_y^{(1)} - \frac{\lambda}{2} \left(\left(\phi_y^{(0)}\right)^2 \phi_y^{(1)} + \phi_y^{(0)} \psi_{y,y}^{(0)} \right) \right]$$

$$\psi_{x,z}^{(0)} = D_{xz} - \frac{\lambda}{2} \int_y D_{xy} \left(\phi_y^{(0)}\right)^2 \psi_{y,z}^{(0)}$$

- Substitute the perturbiner expansion into the DS equation

$$\Phi_{\mathscr{P}}^{(1)} = -\frac{\lambda}{2} \frac{1}{\left(k_{\mathscr{P}}\right)^2 + m^2} \left(\sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}\cup\mathscr{S}} \Phi_{\mathscr{Q}}^{(0)} \Phi_{\mathscr{R}}^{(0)} \Phi_{\mathscr{S}}^{(1)} + \sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}} \int_p \Phi_{\mathscr{Q}}^{(0)} \Psi_{p|\mathscr{R}}^{(0)} \right)$$

$$\Psi_{p|\mathscr{P}}^{(0)} = -\frac{\lambda}{2} \sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}\cup\mathscr{S}} \frac{1}{\left(p - k_{\mathscr{P}}\right)^2 + m^2} \Phi_{\mathscr{Q}}^{(0)} \Phi_{\mathscr{R}}^{(0)} \Psi_{p|\mathscr{S}}^{(0)}$$

reduction of higher loop integrals

2-loop recursion

$$\Phi^{(2)}_{\mathscr{P}} = -\frac{\lambda}{2}\frac{1}{k^2_{\mathscr{P}}+m^2}\Bigg(\sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}\cup\mathscr{S}}\left(\Phi^{(0)}_{\mathscr{Q}}\Phi^{(0)}_{\mathscr{R}}\Phi^{(2)}_{\mathscr{S}} + 2\Phi^{(0)}_{\mathscr{Q}}\Phi^{(1)}_{\mathscr{R}}\Phi^{(1)}_{\mathscr{S}}\right) + \sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}}\int_p\left(\Phi^{(0)}_{\mathscr{Q}}\Psi^{(1)}_{p|\mathscr{R}} + \Phi^{(1)}_{\mathscr{Q}}\Psi^{(0)}_{p|\mathscr{R}}\right) + \frac{1}{3}\int_{p,q}\Psi'^{(0)}_{p,q|\mathscr{P}}\Bigg),$$

$$\Psi^{(1)}_{p|\mathscr{P}} = -\frac{\lambda}{2}\frac{1}{(p-k_{\mathscr{P}})^2+m^2}\sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}\cup\mathscr{S}}\left(2\Phi^{(0)}_{\mathscr{Q}}\Phi^{(1)}_{\mathscr{R}}\Psi^{(0)}_{p|\mathscr{S}} + \Phi^{(0)}_{\mathscr{Q}}\Phi^{(0)}_{\mathscr{R}}\Psi^{(1)}_{p|\mathscr{S}}\right) - \frac{\lambda}{2}\frac{1}{(p-k_{\mathscr{P}})^2+m^2}\sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}}\int_q\left(\Phi^{(0)}_{\mathscr{Q}}\Psi'^{(0)}_{p,q|\mathscr{R}} + \Psi^{(0)}_{p|\mathscr{Q}}\Psi^{(0)}_{q|\mathscr{R}}\right),$$

$$\Psi'^{(0)}_{p,q|\mathscr{P}} = -\frac{\lambda}{2}\frac{1}{(p+q-k_{\mathscr{P}})^2+m^2}\sum_{\mathscr{P}=\mathscr{Q}\cup\mathscr{R}\cup\mathscr{S}}\left(2\Phi^{(0)}_{\mathscr{Q}}\Psi^{(0)}_{p|\mathscr{R}}\Psi^{(0)}_{q|\mathscr{S}} + \Phi^{(0)}_{\mathscr{Q}}\Phi^{(0)}_{\mathscr{R}}\Psi'^{(0)}_{p,q|\mathscr{S}}\right).$$

General structure

- Theories with n-point vertex — $(n-2)$ loop reducible
- pure YM/pure GR — 3pt vertices using the first-order formalism (1-loop reducible)

Steps of deriving the recursions

1. Write down the EoM
2. Constructing the **Dyson-Schwinger equation** from the EoM by the deformation
3. Substituting the **perturbiner expansion**
4. $\hbar$-expansion and truncate the higher $\hbar$ order terms
5. Deriving the off-shell recursion relation
6. Solve them!

Applied to

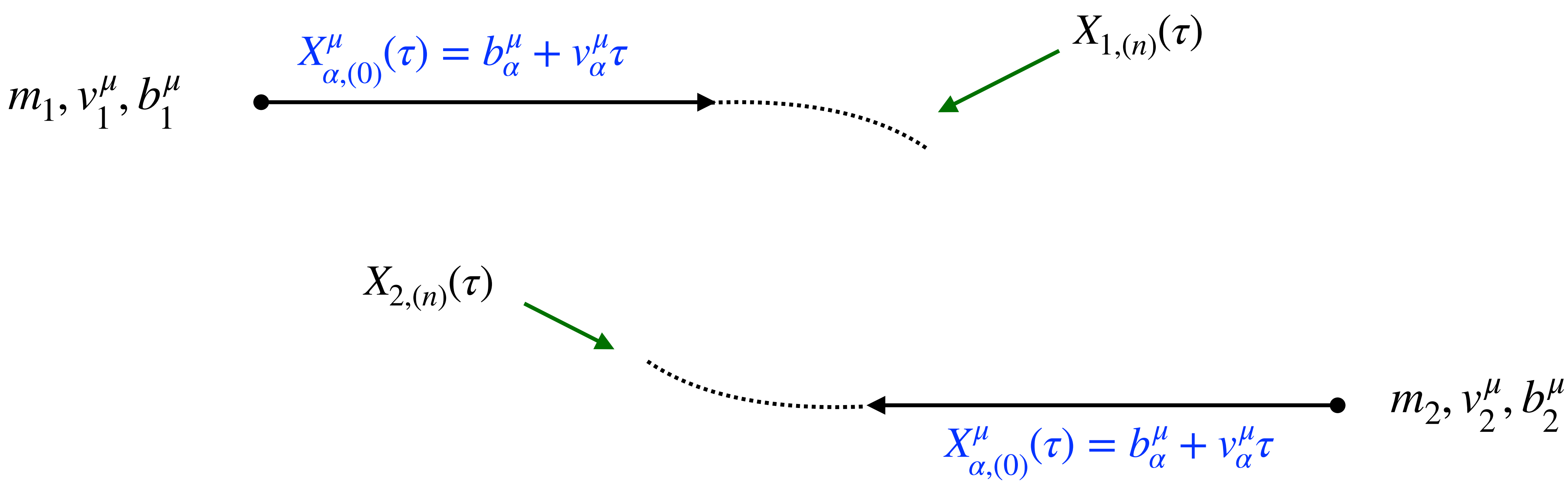
- phi-4 theory: 2-loop
- Pure Yang-Mills theory: 1-loop 4pt
- Einstein-scalar theory (for binary BH system): 2-loop 4pt

Generalization to
Binary black holes

Kinematics

- In the inspiral phase, we may treat the spinless BHs as **point particles**
- Considering two body motions (massive point particles)

$$X^\mu_\alpha(\tau) = X^\mu_{\alpha,(0)}(\tau) + \sum_{n=1}^{\infty} G^n X^\mu_{\alpha,(n)}(\tau) \qquad \alpha = 1,2$$



- We will consider the conservative potential — leading order
- **Goal**: Compute the **momentum kick** order by order in G, $\Delta P^\mu_{1,2} = \int_{-\infty}^{\infty} \mathrm{d}\tau \ddot{X}^\mu_{1,2}(\tau)$

Action/EoM for two point masses

- Change the notation:

$$\mathfrak{g}^{\mu\nu} := \sqrt{-g}\, g^{\mu\nu}, \qquad \mathfrak{g}_{\mu\nu} := \frac{1}{\sqrt{-g}} g_{\mu\nu}$$

- Action:

$$S[\mathfrak{g}, j] = S_{\mathrm{EH}}[\mathfrak{g}] + \frac{1}{16\pi G} \int \mathrm{d}^4 x j^{\mu\nu}(x) \frac{1}{\sqrt{-\mathfrak{g}}} \mathfrak{g}_{\mu\nu}(x)$$

The external source/Energy momentum tensor

$$j^{\mu\nu}(x) = 8\pi G \sum_{\alpha=1}^{2} m_\alpha \int \mathrm{d}\tau \frac{dX_\alpha^\mu(\tau)}{d\tau} \frac{dX_\alpha^\nu(\tau)}{d\tau} \delta^4\big(x^\mu - X_\alpha^\mu(\tau)\big)$$

- Einstein equation:

$$\delta_{\mathfrak{g}} S = \frac{1}{16\pi G} \int \mathrm{d}^D x\, \delta \mathfrak{g}_{\mu\nu} \left[-\mathscr{G}^{\mu\nu} + \frac{1}{\sqrt{-\mathfrak{g}}} \left(j^{\mu\nu} - \frac{1}{2} \mathfrak{g}^{\mu\nu} j^{\rho\sigma} \mathfrak{g}_{\rho\sigma} \right) \right]$$

- Geodesic equation:

$$\frac{d}{d\tau} \left[\frac{1}{\sqrt{-\mathfrak{g}}} \tilde{\mathfrak{g}}_{\rho\sigma} \dot{X}_\alpha^\sigma \right] = \frac{1}{2} \partial_\rho \left[\frac{1}{\sqrt{-\mathfrak{g}}} \tilde{\mathfrak{g}}_{\mu\nu} \right]_{x \to X_\alpha} \dot{X}_\alpha^\mu \dot{X}_\alpha^\nu$$

Perturbiner expansion

- The n-th order metric fluctuations $h^{\mu\nu}_{(n)}$ is a function of coordinates x^μ, as well as the impact parameter b_α **implicitly**

$$h^{\mu\nu}_{(n)}(x) = h^{\mu\nu}_{(n)}(x; b_1, b_2)$$

- **Perturbiner expansion (**$x^\mu_\alpha = x^\mu - b^\mu_\alpha, \quad \alpha = 1,2,$ ℓ_α are the Fourier dual of x_α**)**

$$n = 1: \quad h^{\mu\nu}_{(1)}(x_\alpha) = \int_{\ell_1} e^{i\ell_1 \cdot x_1} J^{\mu\nu}_{(1)|[\ell_1,0]} + \int_{\ell_2} e^{i\ell_2 \cdot x_2} J^{\mu\nu}_{(1)|[0,\ell_2]}$$

$$n > 1: \quad h^{\mu\nu}_{(n)}(x_\alpha) = \int_{\ell_1} e^{i\ell_1 \cdot x_1} J^{\mu\nu}_{(n)|[\ell_1,0]} + \int_{\ell_2} e^{i\ell_2 \cdot x_2} J^{\mu\nu}_{(n)|[0,\ell_2]} + \int_{\ell_1,\ell_2} e^{i\ell_1 \cdot x_1 + i\ell_2 \cdot x_2} J^{\mu\nu}_{(n)|[\ell_1,\ell_2]} \, .$$

$$n = 1: \quad X^\rho_{\alpha,(1)}(\tau) = \int_{\ell_1} X^\mu_{\alpha,(1)|[\ell_1,0]} e^{-i\ell_1 \cdot X_{\alpha,1,(0)}} + \int_{\ell_2} X^\mu_{\alpha,(1)|[0,\ell_2]} e^{-i\ell_2 \cdot X_{\alpha,2,(0)}} \, ,$$

$$n > 1: \quad X^\rho_{\alpha,(n)}(\tau) = \int_{\ell_1} X^\mu_{\alpha,(n)|[\ell_1,0]} e^{-i\ell_1 \cdot X_{\alpha,1,(0)}} + \int_{\ell_2} X^\mu_{\alpha,(n)|[0,\ell_2]} e^{-i\ell_2 \cdot X_{\alpha,2,(0)}} + \int_{\ell_1,\ell_2} X^\mu_{\alpha,(n)|[\ell_1,\ell_2]} e^{-i\ell_1 \cdot X_{\alpha,1,(0)} - i\ell_2 \cdot X_{\alpha,2,(0)}} \, ,$$

$$X^\mu_{\alpha,\beta,(0)} = b^\mu_\alpha - b^\mu_\beta + v^\mu_\alpha \tau \, .$$

Summary and future directions

- **Established a new computational framework for perturbative GR**
  - defined a "good" variable — tensor density & doubled metric
  - derived a recursion relation in a remarkably simple form — no infinite expansion
  - Showed the integral factorization occurs — only bubble integrals arise
  - Derived Schwarzschild BH solution all order in Newton constant
- **Applications**
  - Extension to binary black holes — two moving point masses
  - Kerr BH — massive higher spin or worldline SUSY
  - Finding interesting unknown solutions — physically intuitive setup.
  - Computing scattering amplitudes for QCD/SM

Thank you!