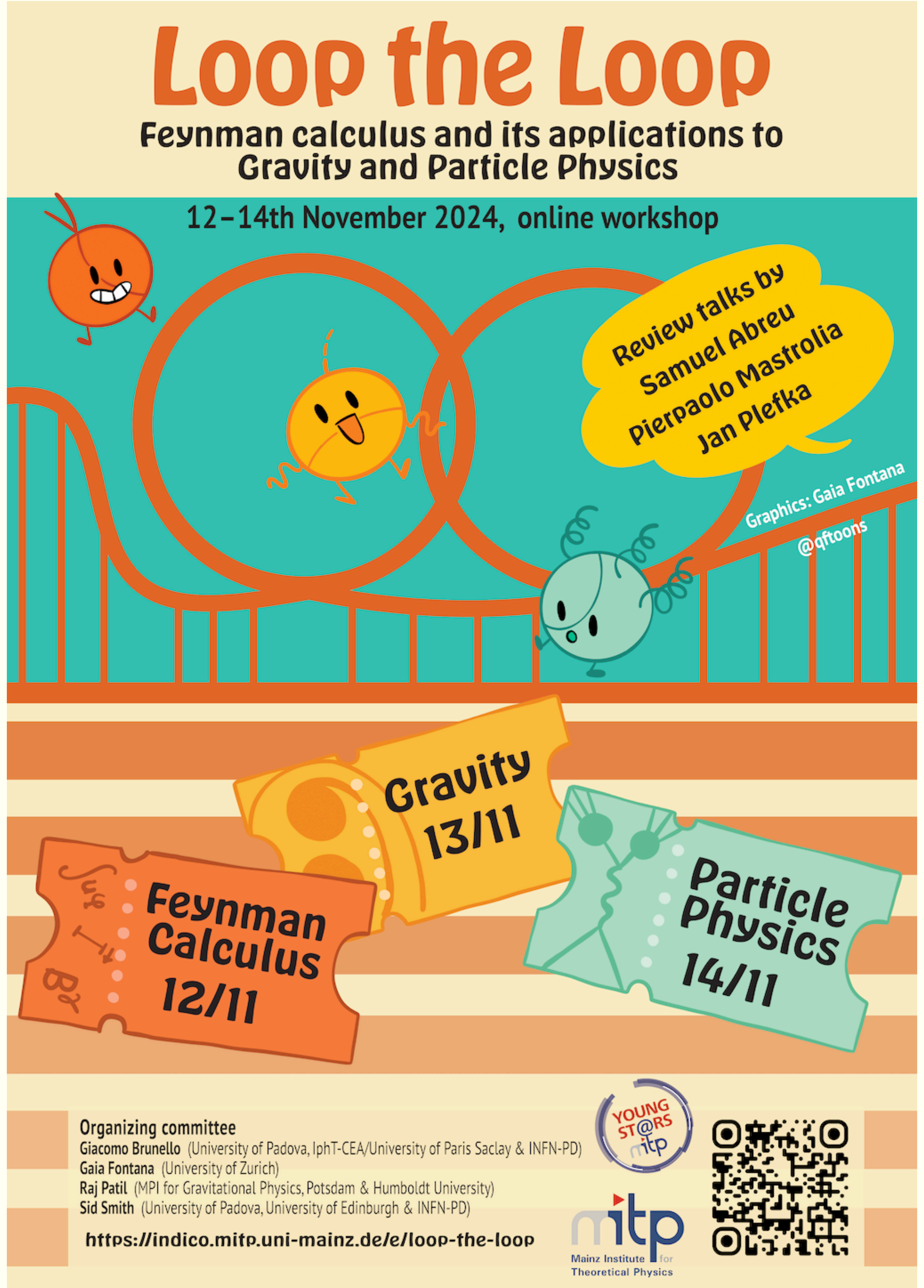
Progress in Landau Analysis

Mathieu Giroux (McGill)

with Simon Caron-Huot and Miguel Correia [2406.05241]

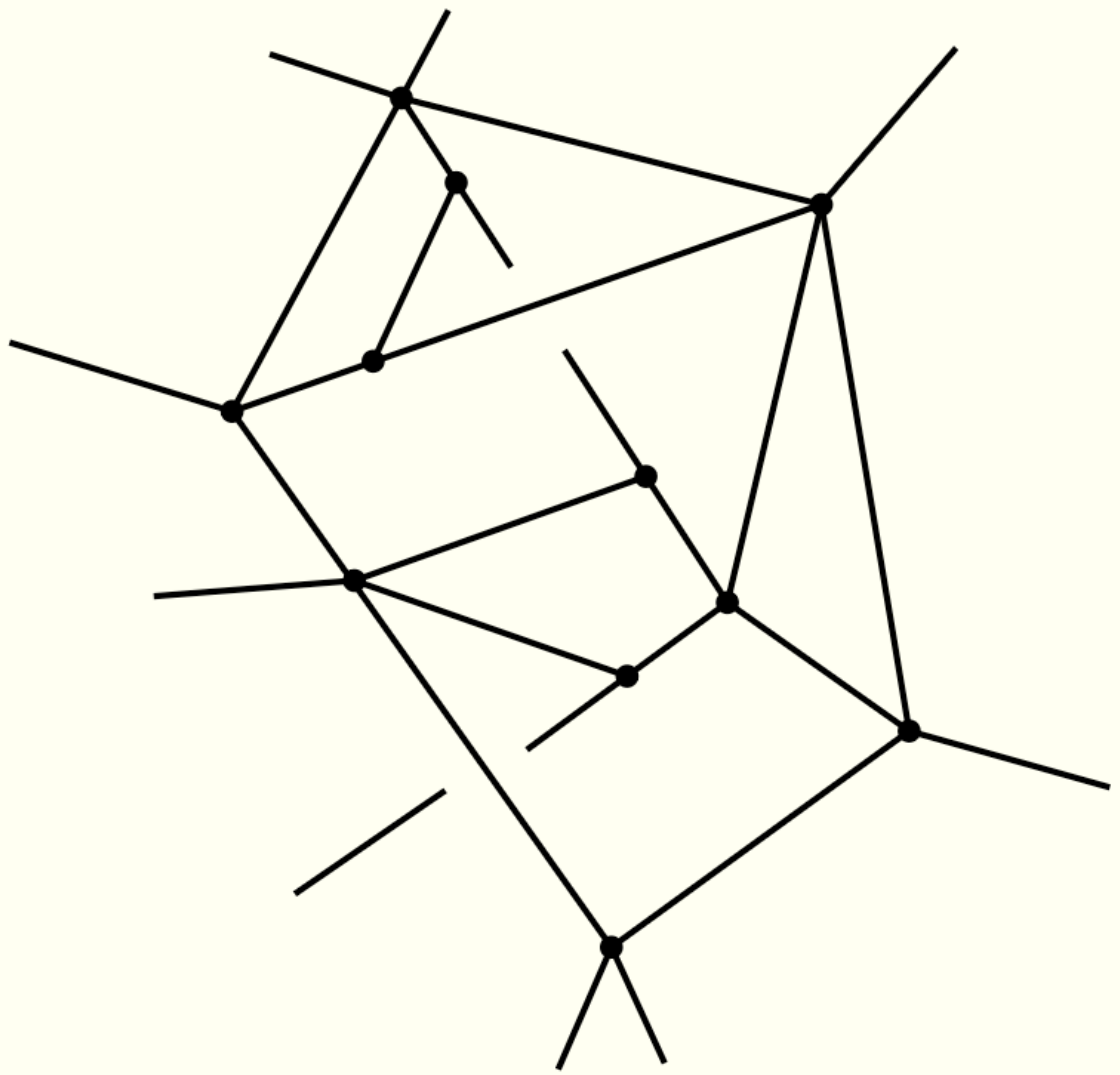
+ work in progress with Sebastian Mizera


ias.edu/amplitudes2024
Amplitudes 2024
Conference
Institute for Advanced Study, Princeton, NJ, United States
10 - 14 June 2024
Speakers: Samuel Abreu, Zvi Bern, Alessandra Buonanno, Lucile Cangemi, Mariana Carrillo González, François Charton, Christian Copetti, Kevin Costello, Stefano De Angelis, Federica Devoto, Carolina Figueiredo, Mathieu Giroux, Tobias Hansen, Yifei He, Johannes Henn, Aidan Herderschee, Mina Himwich, Mikhail Ivanov, Diksha Jain, Renata Kallosh, David Kosower, Hayden Lee, Juan Maldacena, Andrew McLeod, Ian Moult, Shruti Paranjape, Franziska Porkert, Oliver Schlotterer, Yael Shadmi, Stephen Sharpe, Chiara Signorile-Signorile, Marcus Spradlin, George Sterman, Iain Stewart, Bernd Sturmfels, Lorenzo Tancredi, Natalia Toro, Lauren Williams, Xiaofeng Xu, Zahra Zahraee
Panelists: Nima Arkani-Hamed, Lance Dixon, Graham Farmelo, Rachel Rosen
Organizers:
Nima Arkani-Hamed, Jacob Bourjaily, Hofie Hannesdottir, Sebastian Mizera
Image credits: Gaia Fontana
Carl P. Feinberg Program in Cross-Disciplinary Innovation
IAS INSTITUTE FOR ADVANCED STUDY

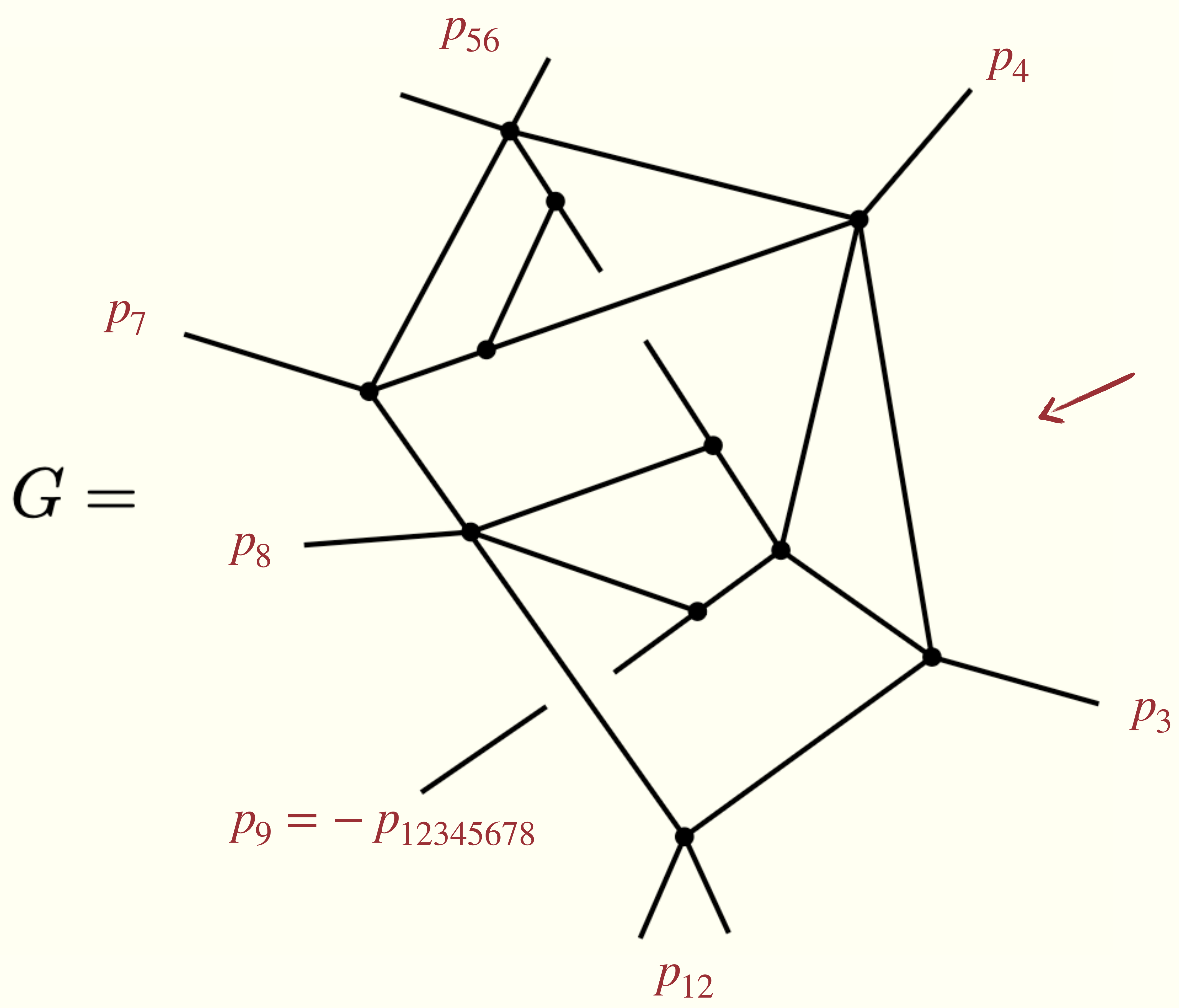

Loop the Loop
Feynman calculus and its applications to Gravity and Particle Physics
12–14th November 2024, online workshop
Review talks by Samuel Abreu Pierpaolo Mastrolia Jan Plefka
Graphics: Gaia Fontana
@qftoons
Gravity 13/11
Feynman Calculus 12/11
Particle Physics 14/11
Organizing committee
Giacomo Brunello (University of Padova, IphT-CEA/University of Paris Saclay & INFN-PD)
Gaia Fontana (University of Zurich)
Raj Patil (MPI for Gravitational Physics, Potsdam & Humboldt University)
Sid Smith (University of Padova, University of Edinburgh & INFN-PD)
https://indico.mitp.uni-mainz.de/e/loop-the-loop
YOUNG ST@RS mitp
mitp Mainz Institute for Theoretical Physics

Let's first set up the stage

$G =$



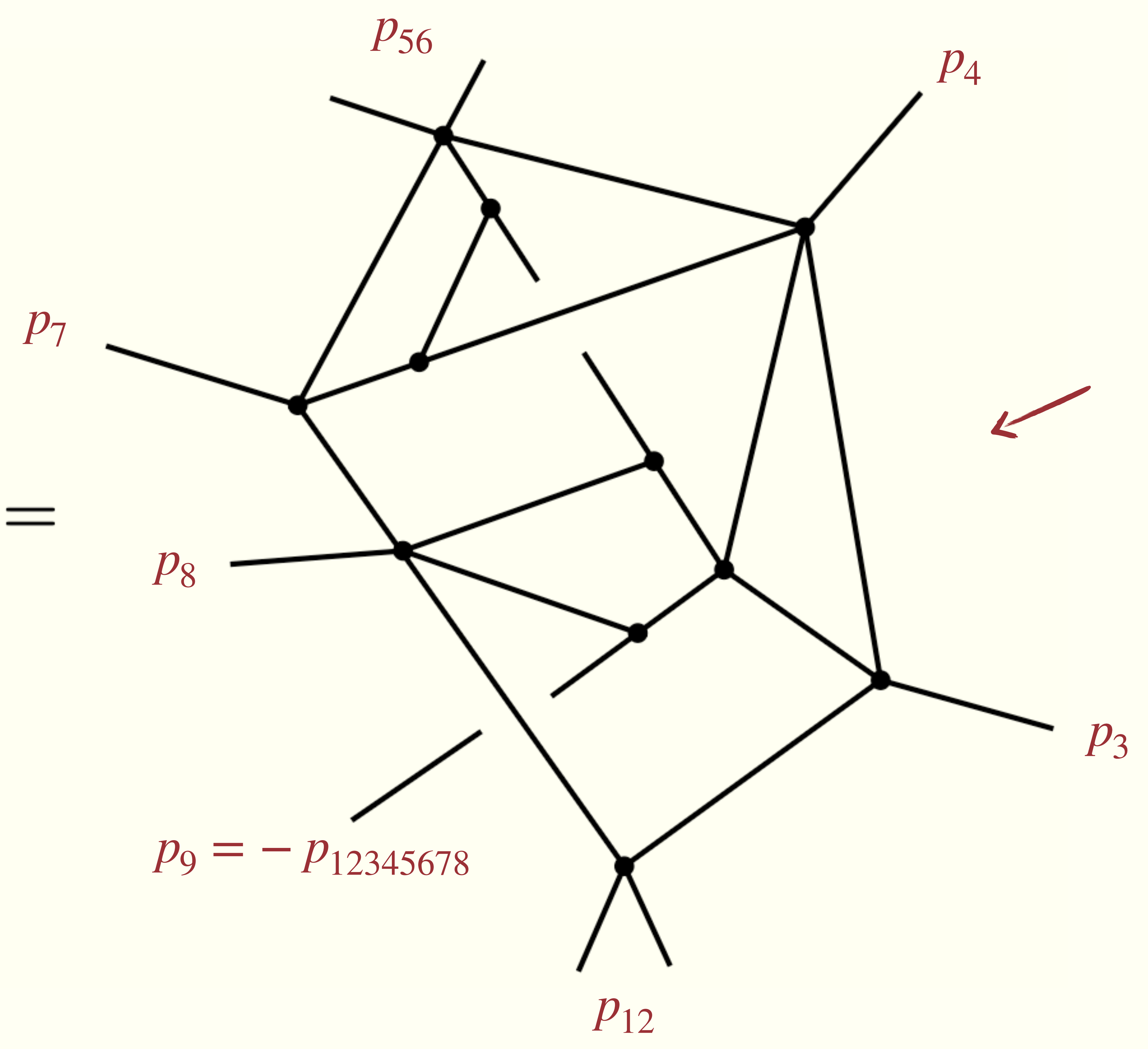
Let's first set up the stage



A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

$$p_I \equiv \sum_{i \in I} p_i$$
$$s_I \equiv p_I^2$$

Let's first set up the stage

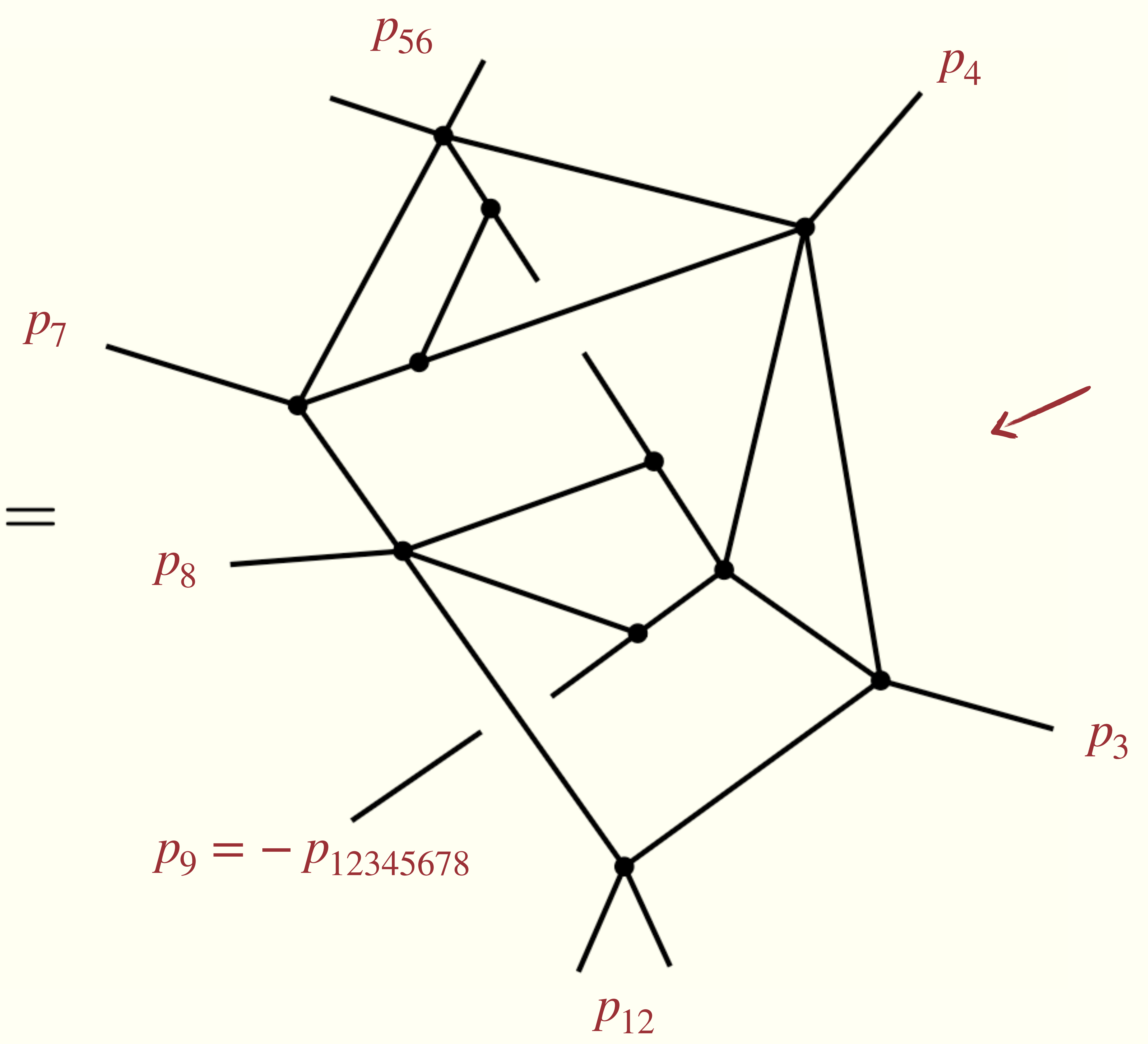
$G =$ 

A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

$$p_I \equiv \sum_{i \in I} p_i$$
$$s_I \equiv p_I^2$$

What's the analytic structure of G ?

Let's first set up the stage

$G =$ 

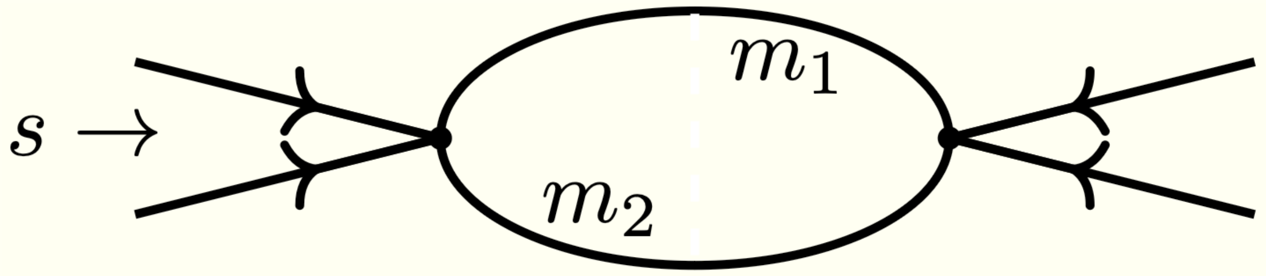
A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

$p_I \equiv \sum_{i \in I} p_i$

$s_I \equiv p_I^2$

In other words, where are its *kinematic* singularities ?

Let us make sure we are on the same page



$$\stackrel{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{s}} \log\left[\frac{m_1+m_2+\sqrt{s}}{m_1+m_2-\sqrt{s}}\right]$$

Normal threshold
(± branches)

Second type
singularity

Well understood at one-loop; can be much *harder* beyond!

Having good control over this question would be enormously useful for

Differential equations and numerical integration of Feynman integrals
(boundary conditions, analytic continuation and contour deformations)

[See Simone's talks]

Symbol calculus and bootstrap of Feynman integrals
(singularities constrain the letters)

[See Maria's talks]

Knowing singularities *beforehand* has proven central for state-of-the-art phenomenological applications — e.g.,

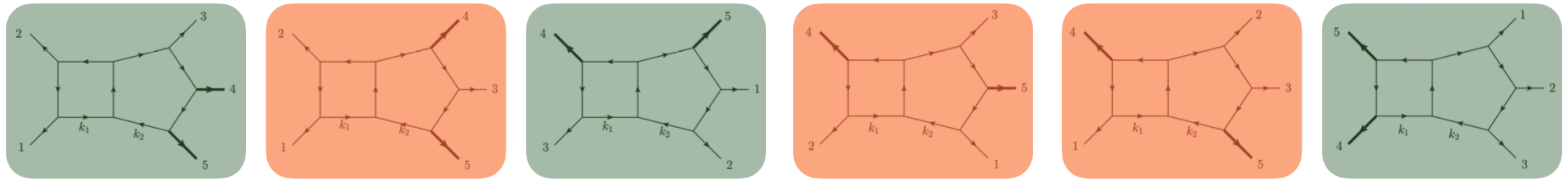
[Samuel Abreu's slide]

Computing Feynman Integrals: Alphabets and Letters

12

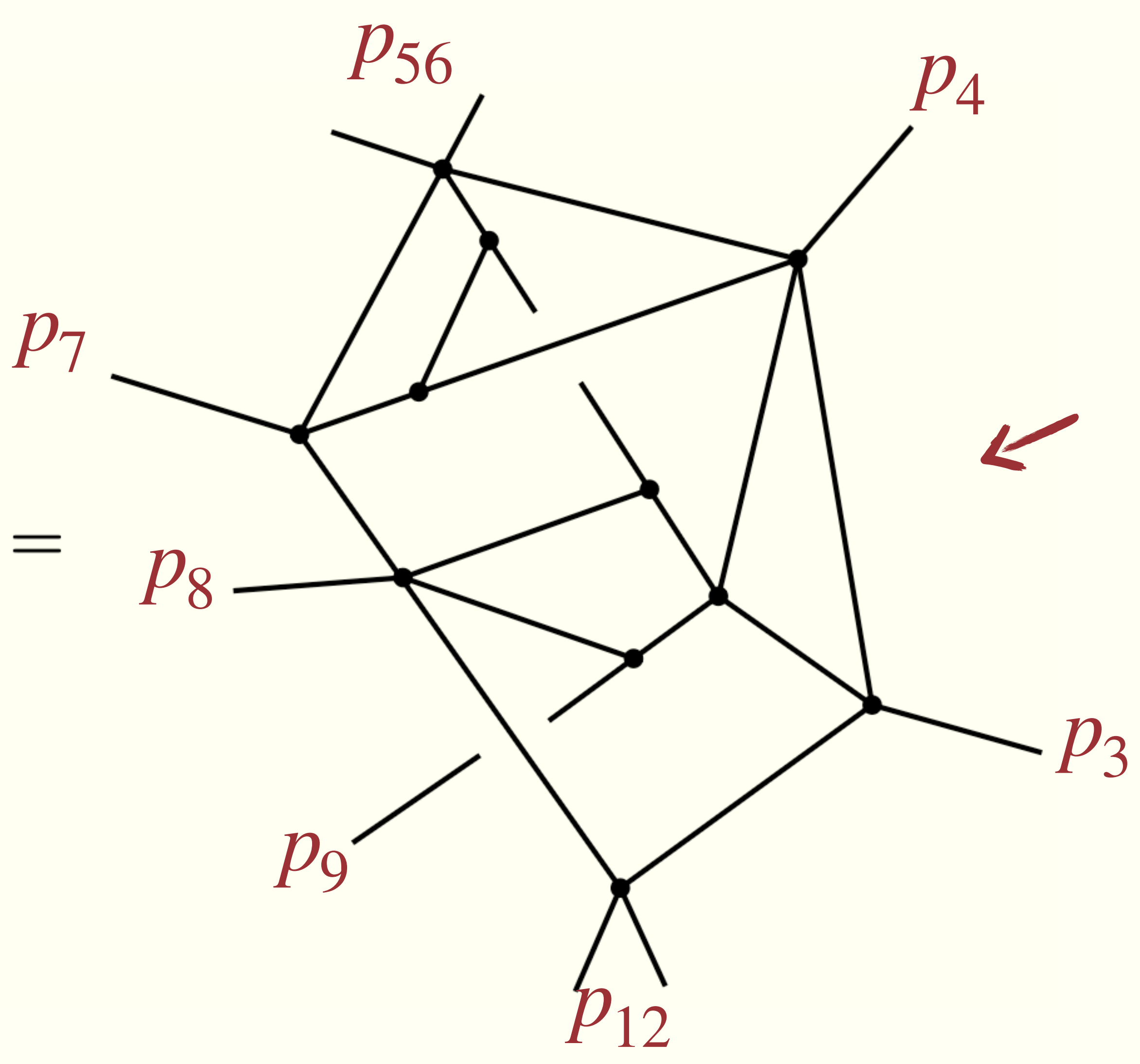
$$d\vec{\mathcal{J}}(x,\epsilon) = \epsilon\Big(\sum_i A_i\, d\log W_i(x)\Big)\vec{\mathcal{J}}(x,\epsilon)$$

- Getting diff. eq. relies on IBPs: difficult to do analytically…
- If the W_i are known, determine the A_i from numerical IBPs!
  - ✓ removes the IBP bottleneck, allows to attack multi-scale problems
- The W_i give singularities of Feynman integrals ⇒ Landau conditions
  - ✓ Factorisation of work: determine W_i without computing the differential equation!
  - ✓ Active area of research in Amplitudes area: coactions, solving Landau conditions, principal A-determinants, Gram determinants, Schubert problem, …
  - ✓ Two highlights: **[2311.14669, Fevola, Mizera, Telen], [2401.07632, Jiang, Liu, Xu, Yang, 24]**
- Baikovletter [2401.07632] misses one of the new five-point roots
  - ✓ Not really an issue, we know it's there



+ related work by [Abreu, Caron-Huot, Chicherin, Dixon, Gehrmann, Henn, Ita, McLeod, Mitev, Moriello, Page, Presti, Sotnikov, Tschernow, von Hippel, Wasser, Wilhelm, Zhang, Zoia, …]

WHAT'S OUR GOAL ?

$G =$ 

A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

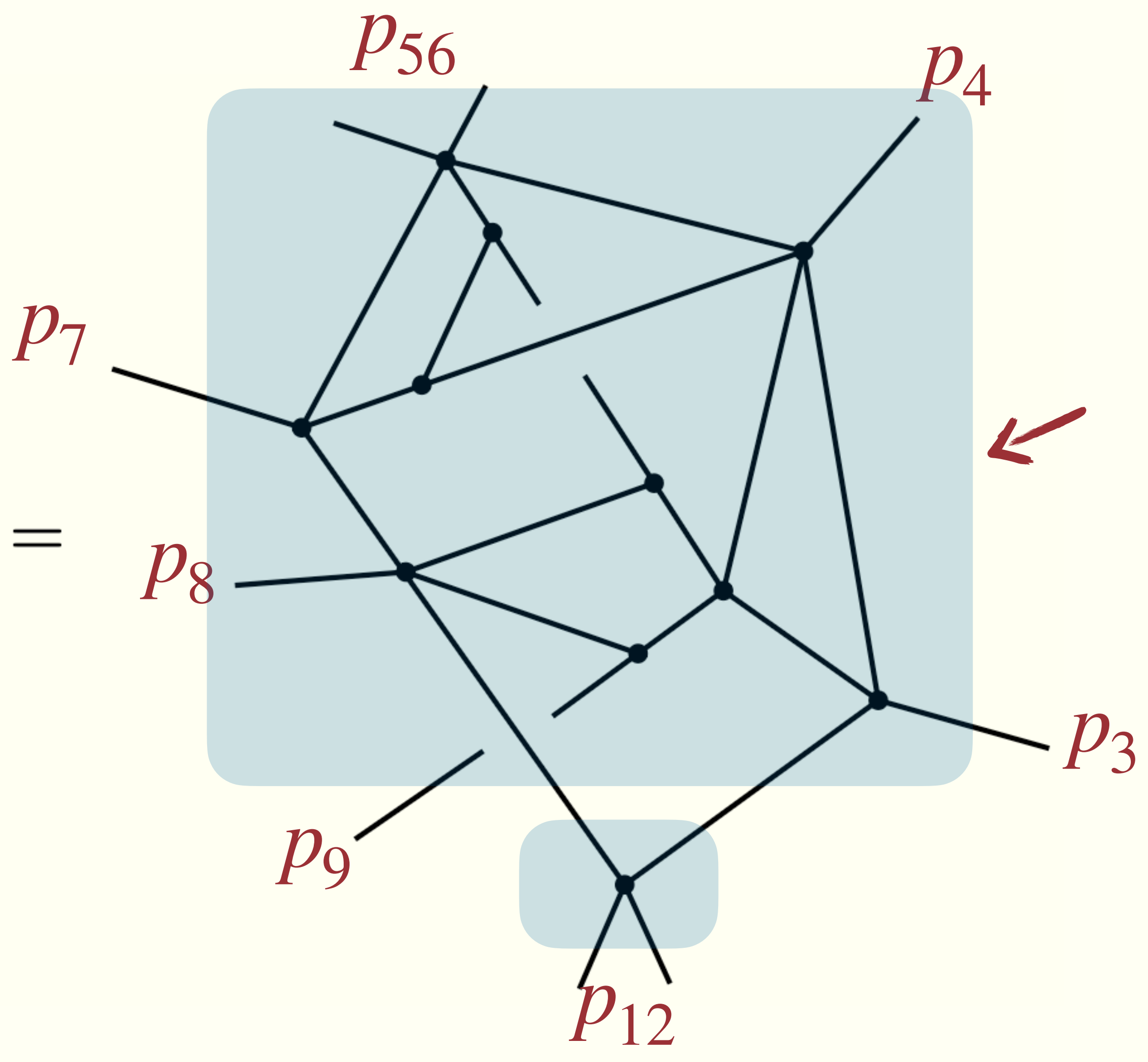
Singularities are written as a list $\mathcal{L}(G)$ of polynomials in X_G

$$\mathcal{L}(G)_i = 0$$

The product over i is called the *Landau discriminant*

[Fevola, Mizera, Telen (2023)]

What's our goal ?

$G =$ 

A function of $X_G = \{p_i \cdot p_j\}_{i,j=1}^{n-1}$ and internal masses on the kinematic space

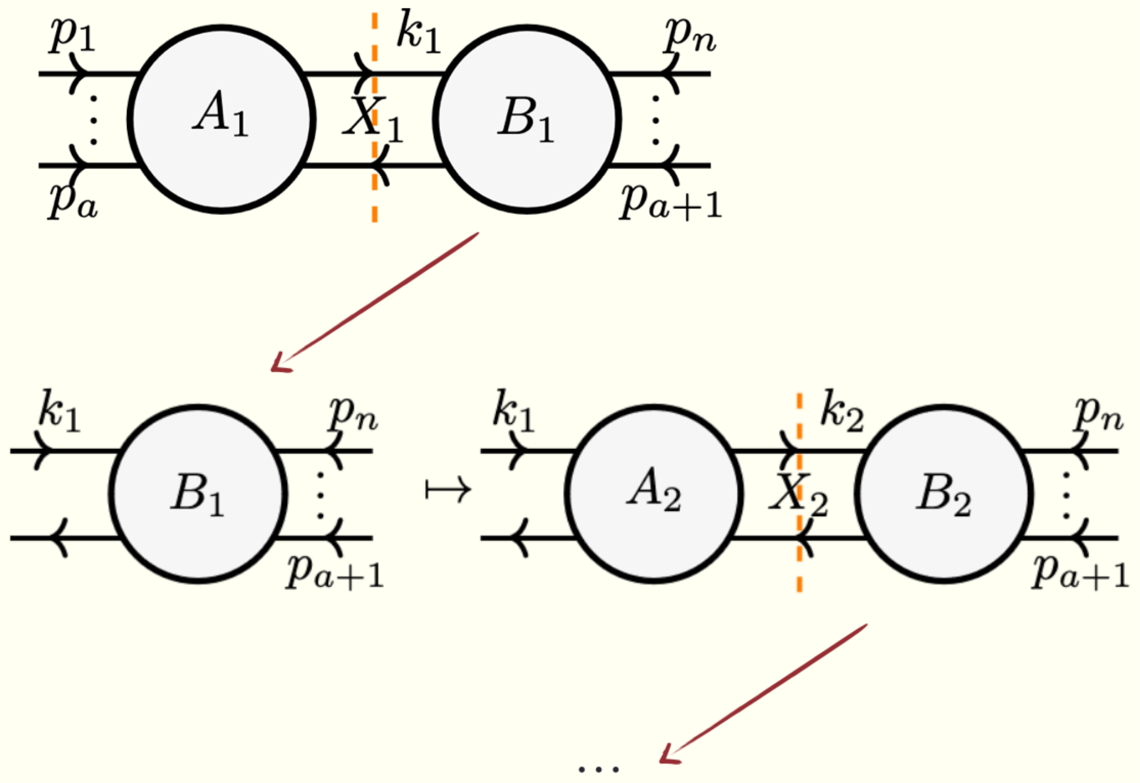
Singularities are written as a list $\mathcal{L}(G)$ of polynomials in X_G

$$\mathcal{L}(G)_i = 0$$

The goal of this talk is to learn how to compute these polynomials *recursively* in terms of those of subgraphs (we'll see that this is *surprisingly* efficient!)

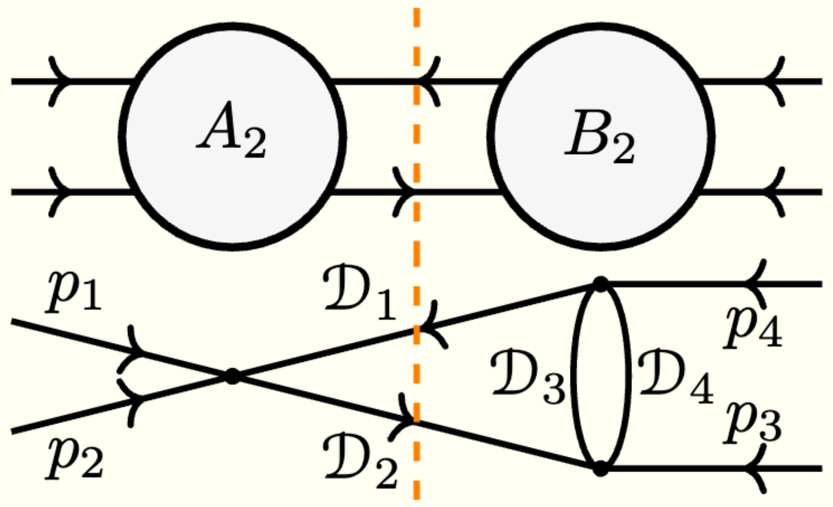
Outline

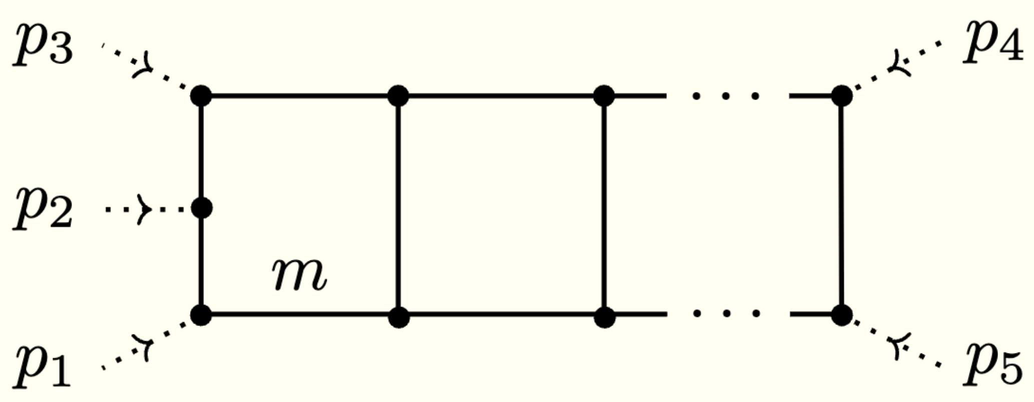
Recursion via unitarity



Proof of principle examples:

Recursively finding singularities

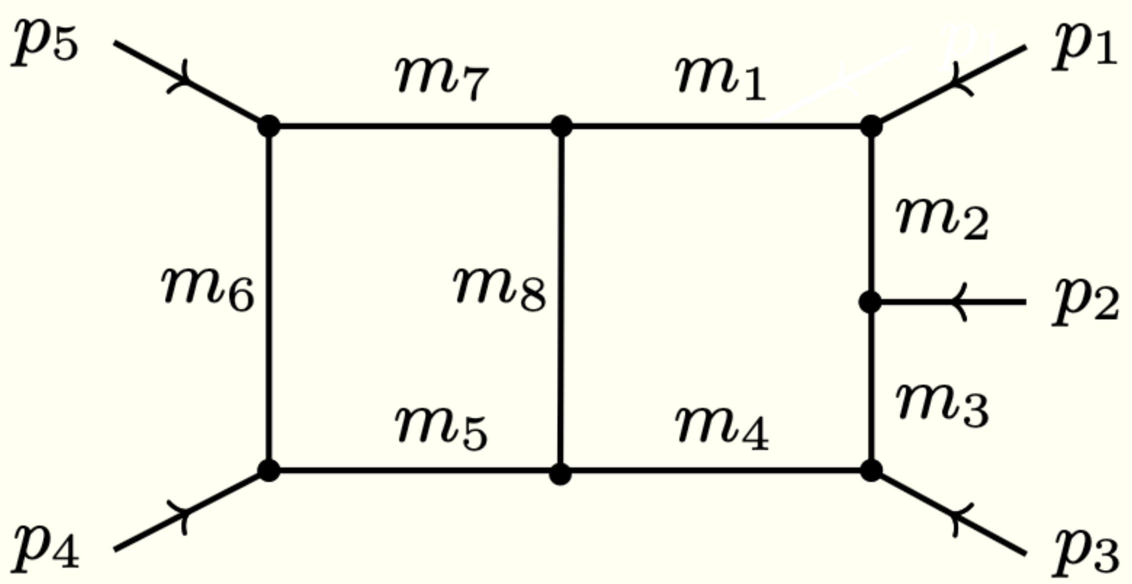




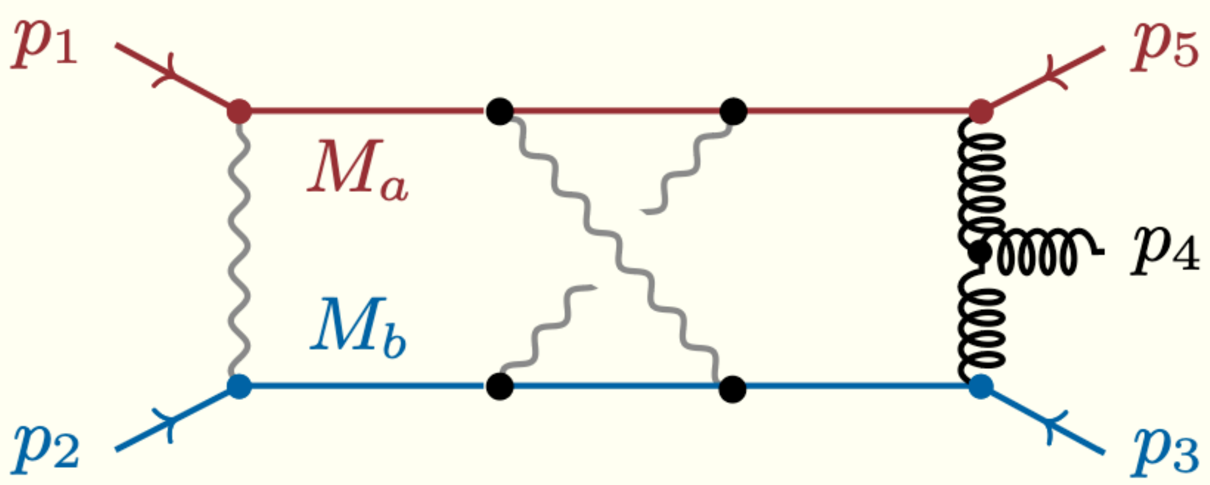
Checks and new analytic predictions:

Leading singularities

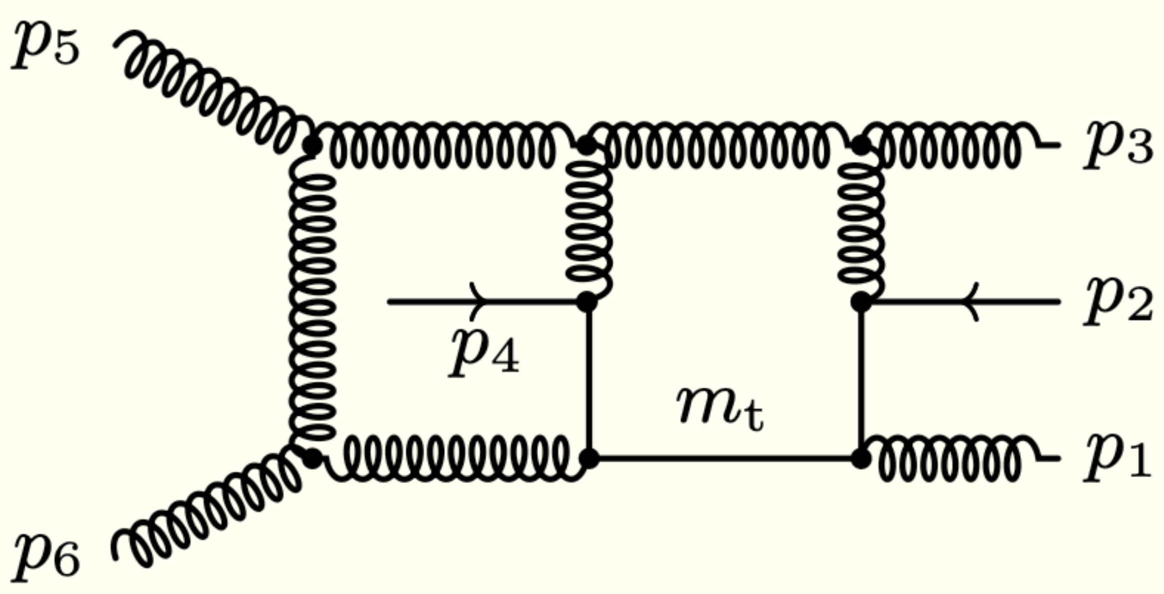
(Generic kinematic pentabox)



(Three-loop QED+QCD boX)

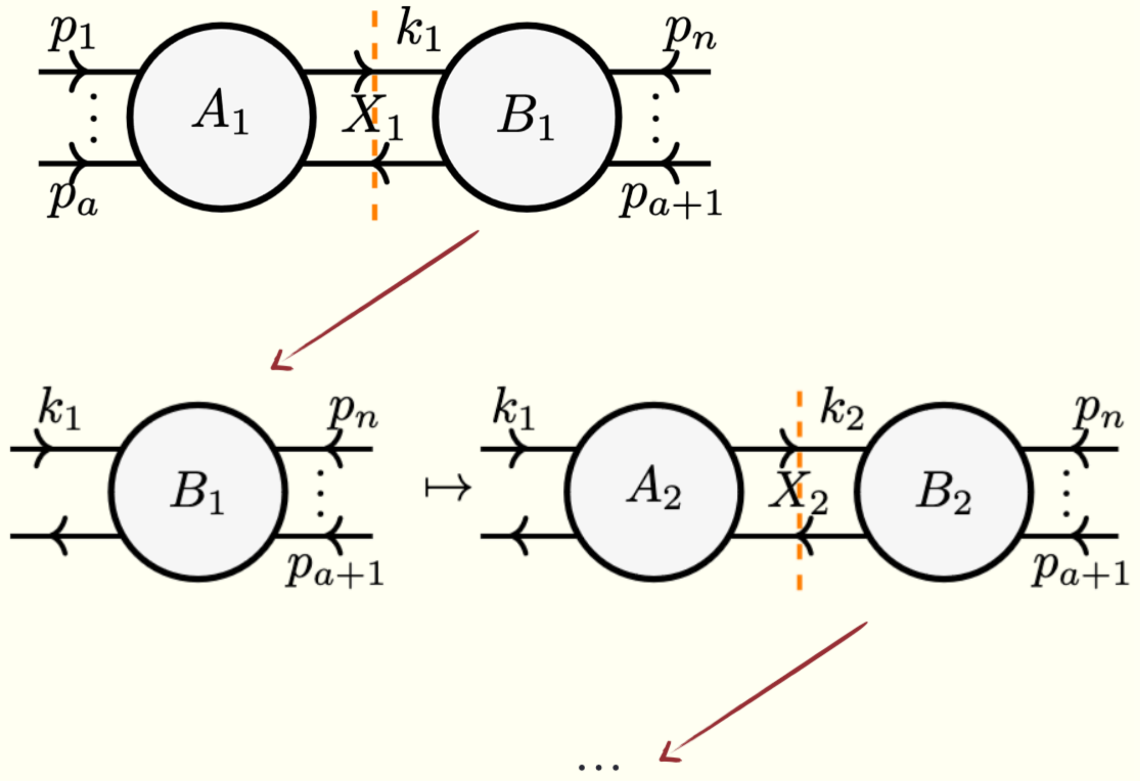


(Non-planar massive hexabox)



Outline

Recursion via unitarity



(Generic kinematic pentabox)

(Three-loop QED+QCD boX)

(Non-planar massive hexabox)

Unitarity and thresholds

Unitarity of the S-matrix implies that

$$SS^\dagger = \mathbb{1}$$

$$S = \mathbb{1} + iT \quad \Longrightarrow \quad \frac{1}{2i}(T - T^\dagger) = \frac{1}{2}TT^\dagger$$

Separation between free and interacting parts

Unitarity and thresholds

Unitarity of the S-matrix implies that

$$SS^\dagger = \mathbb{1}$$

$$S = \mathbb{1} + iT$$

$$\Longrightarrow$$

$$\operatorname{Im} T = \frac{1}{2} T T^\dagger$$

For the experts:
Assuming (for now) reality of
momenta and Feynman's $i\varepsilon$

Unitarity and thresholds

Unitarity of the S-matrix implies that

$$SS^\dagger = \mathbb{1}$$
$$S = \mathbb{1} + iT \qquad \Longrightarrow \qquad \operatorname{Im} T = \frac{1}{2} T T^\dagger$$

Positivity manifests, but singularities are not

[Hannesdóttir, Mizera (2022)]

Unitarity and thresholds

Unitarity of the S-matrix implies that

$$SS^\dagger = \mathbb{1}$$

$$S = \mathbb{1} + iT$$

$$\implies \quad \mathrm{Im}\, T = \sum\!\!\!\!\!\!\!\int_X T \left| X \right\rangle \left\langle X \right| T^\dagger$$

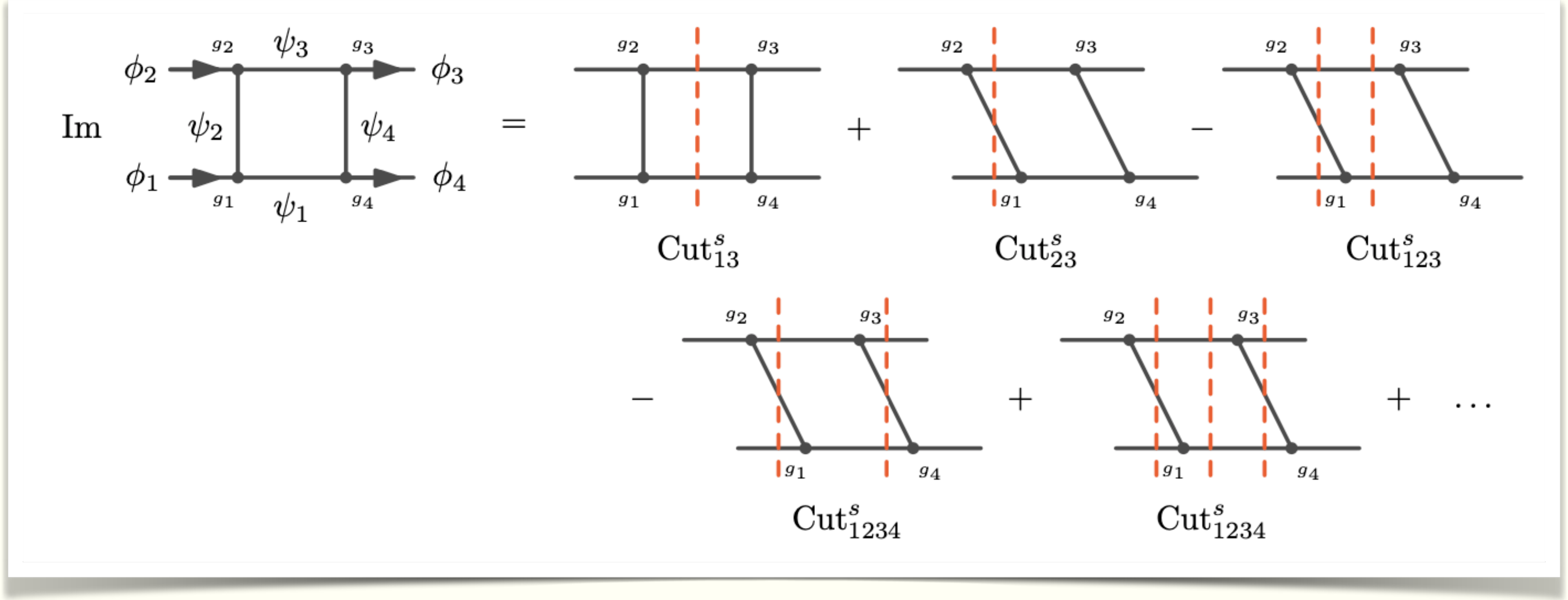
Insert a complete basis of (on-shell) states

Unitarity and thresholds

At the level of the matrix elements $\mathcal{M}_{\text{in}\to\text{out}} \equiv \langle \text{out}|T|\text{in}\rangle$

$$\operatorname{Im} \mathcal{M}_{n_A \to n_B} = \frac{1}{2} \sum\!\!\!\!\!\!\int_X \mathcal{M}_{n_A \to X}\, \mathcal{M}^*_{X \to n_B}$$

In perturbation theory, this gives the *Cutkosky equation*



[Cutkosky (1961), Hannesdóttir, Mizera (2022)]

Unitarity and thresholds

At the level of the matrix elements $\mathcal{M}_{\text{in}\to\text{out}} \equiv \langle \text{out}|T|\text{in}\rangle$

$$\text{Im}\,\mathcal{M}_{n_A\to n_B} \;=\; \text{Sum over unitarity cuts}$$

The locations at which a cut starts contributing are called *thresholds*

Takeaway point

The imaginary part has support where cuts themselves have support

Unitarity and thresholds

At the level of the matrix elements $\mathcal{M}_{\text{in}\to\text{out}} \equiv \langle \text{out}|T|\text{in}\rangle$

$$\text{Im}\,\mathcal{M}_{n_A\to n_B} \;=\; \text{Sum over unitarity cuts}$$

The locations at which a cut starts contributing are called *thresholds*

Takeaway point

The imaginary part has support where cuts themselves have support

At these locations the amplitude cannot be real analytic, and we say that it is *singular*

Necessary conditions for singularities (I)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

Necessary conditions for singularities (I)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

We will see that these can be phrased algebraically *without* reference to the reality of momenta

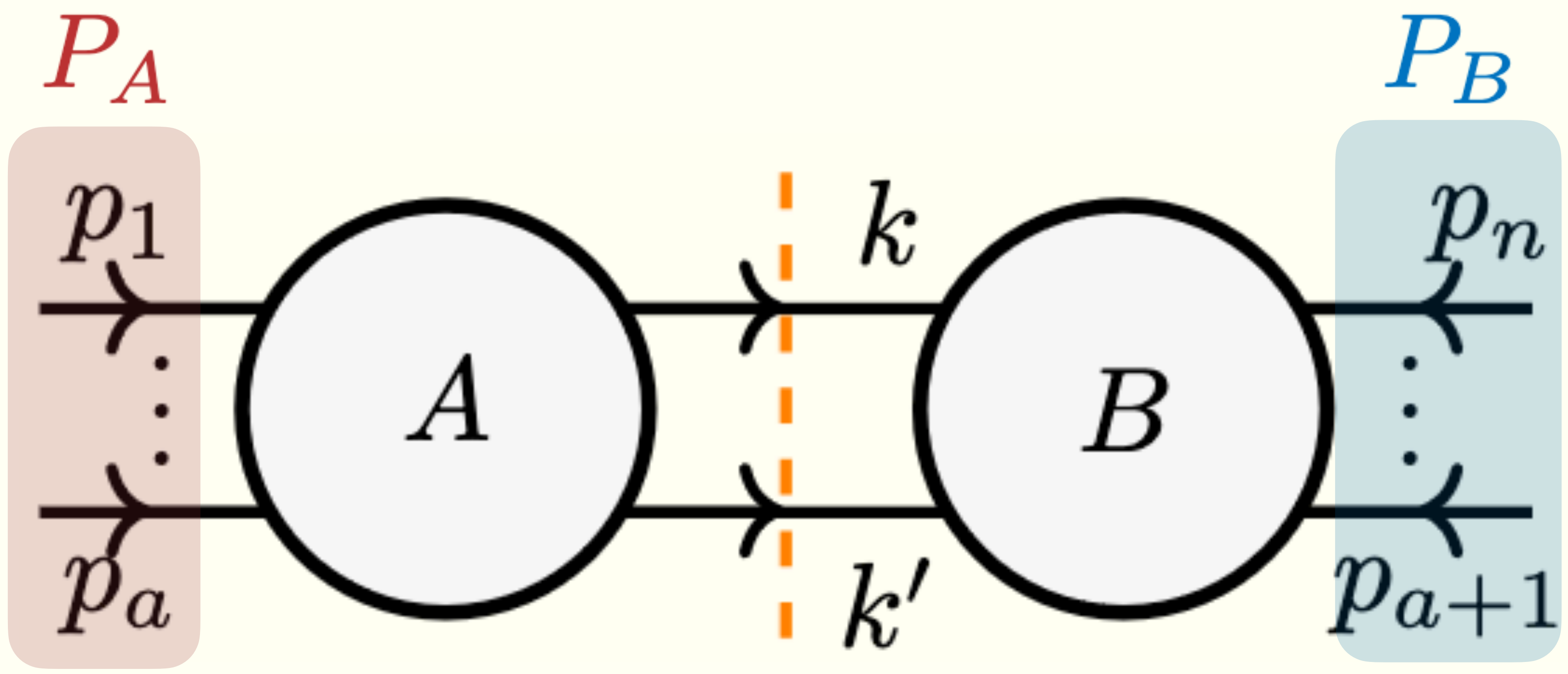
Necessary conditions for singularities (I)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

We will see that these can be phrased algebraically *without* reference to the reality of momenta

Our focus is on Feynman graphs AB that can be *disconnected* into two subgraphs A and B *two-particle cut*



The invariants on each side are

$$X_\xi = \{q_i \cdot q_j \mid q_\bullet \in \{k\} \cup P_\xi\}$$

$$(\xi = A, B)$$

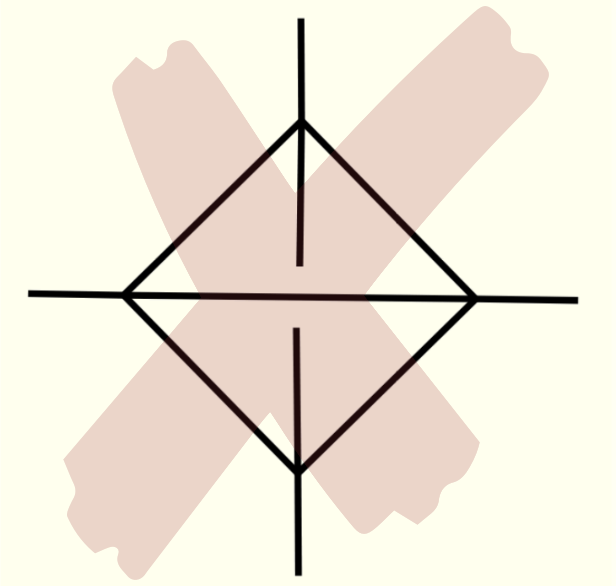
NECESSARY CONDITIONS FOR SINGULARITIES (I)

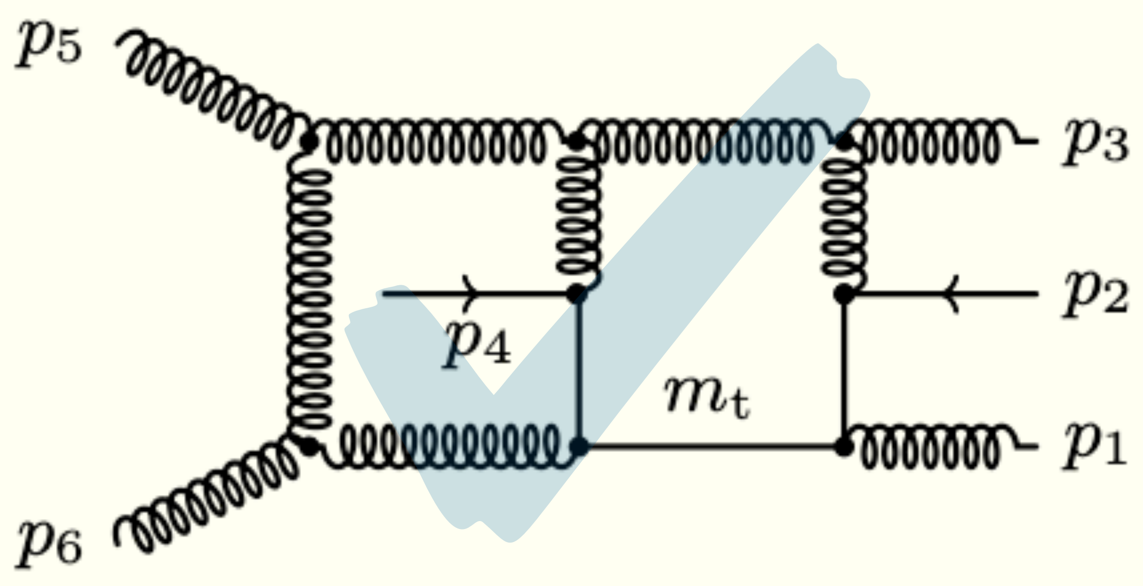
Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

We will see that these can be phrased algebraically *without* reference to the reality of momenta

Our focus is on Feynman graphs AB that can be *disconnected* into two subgraphs A and B *two-particle cut*





We'll learn how to compute singularities of Standard Model processes like this one!

Necessary conditions for singularities (I)

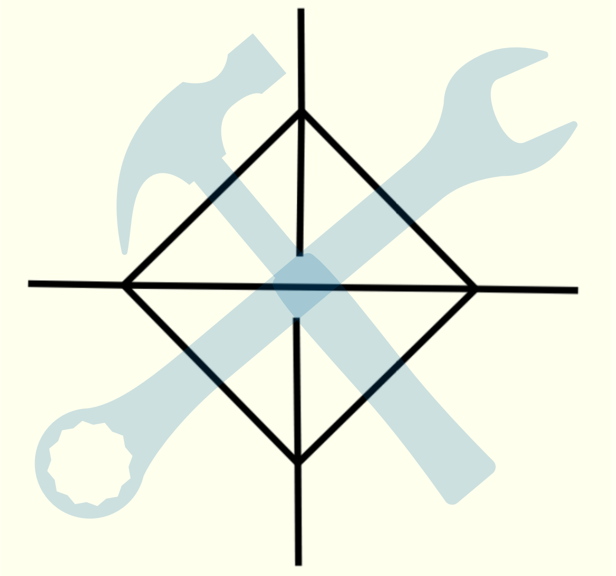
Qualitative necessary conditions

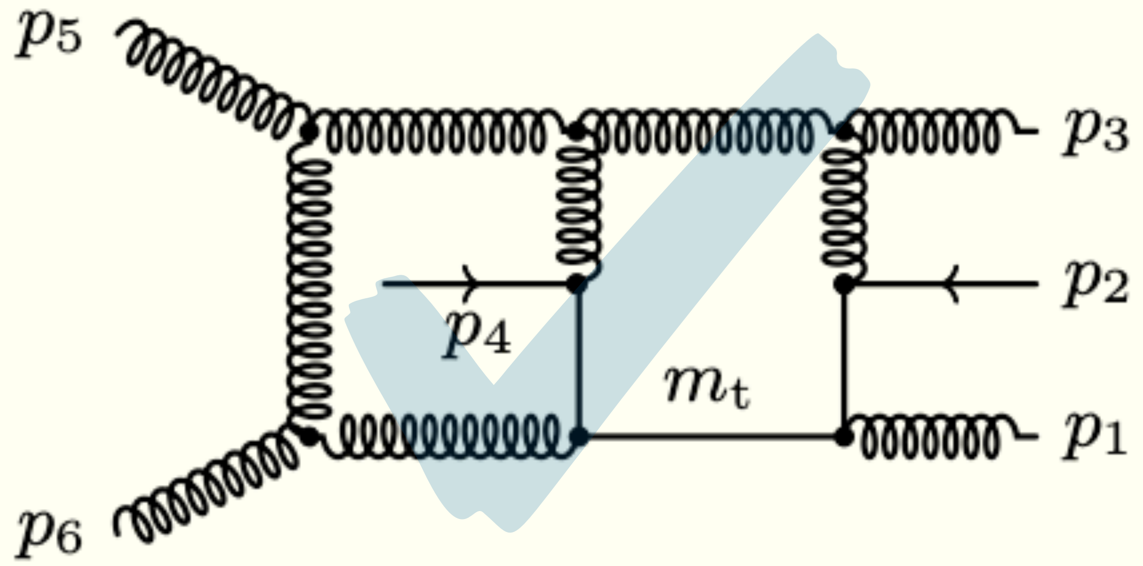
Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

We will see that these can be phrased algebraically *without* reference to the reality of momenta

Our focus is on Feynman graphs AB that can be *disconnected* into two subgraphs A and B *two-particle cut*

Generalizations that include such graphs



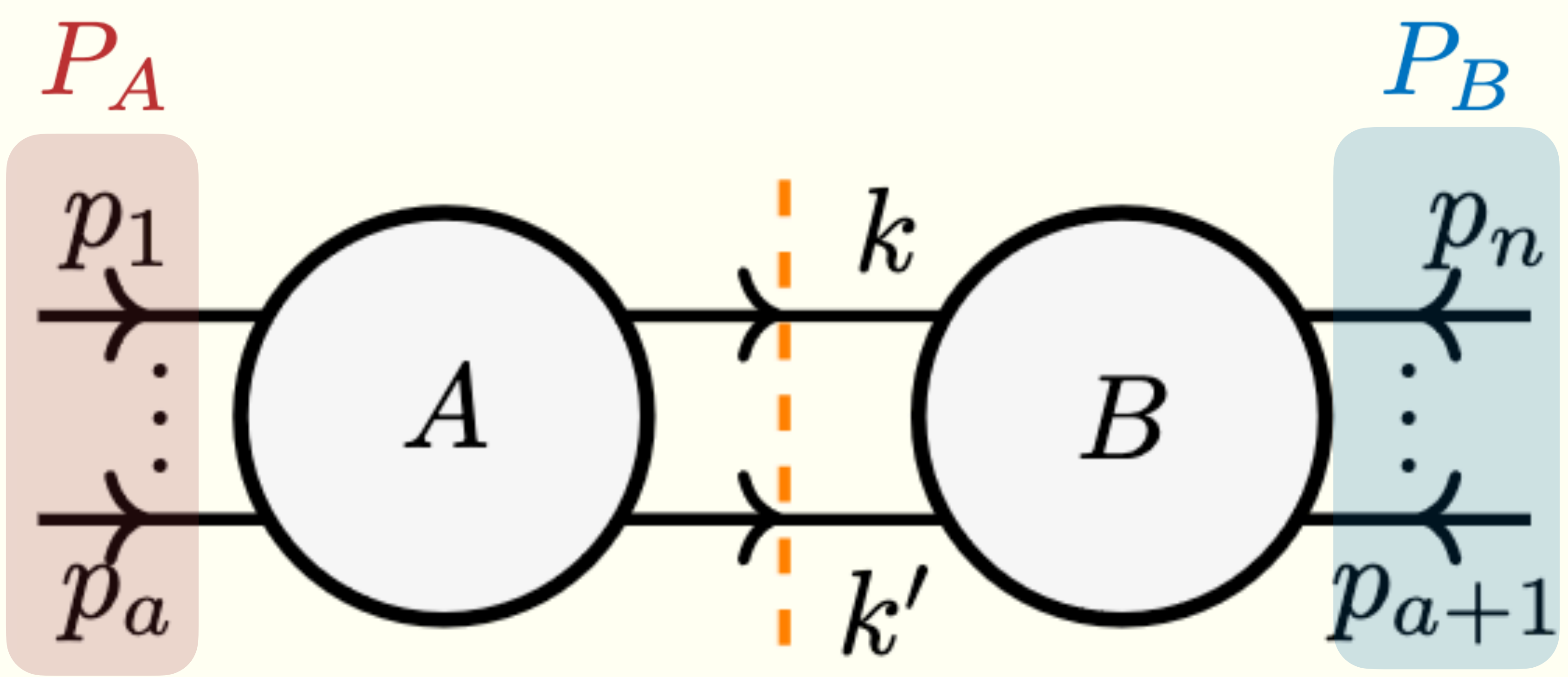


We'll learn how to compute singularities of Standard Model processes like this one!

Two-particle cuts in Baikov form

As an integral over independent the scalar products between loop and external momenta

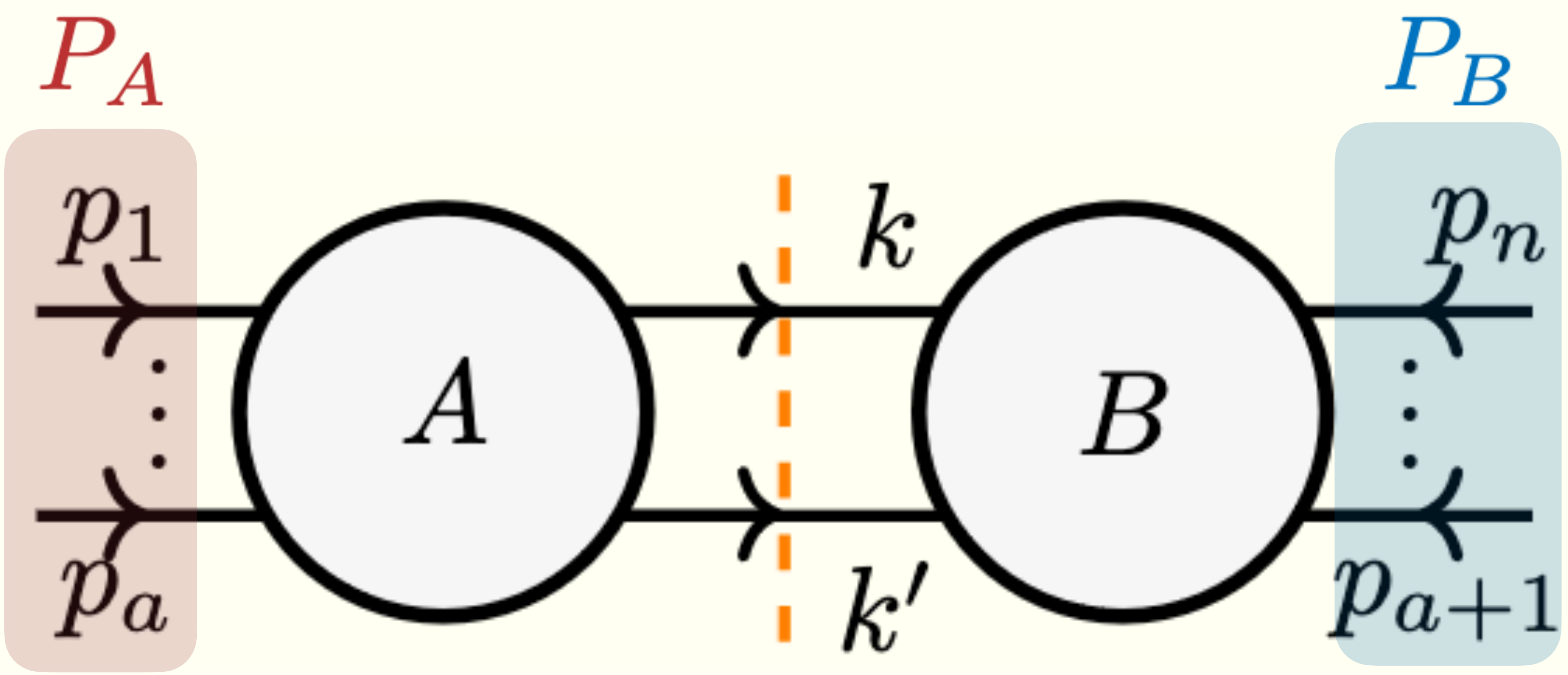
Ask me later to fill the details!



$$= C \int_\Gamma \mathrm{d}\mu \frac{A(X_A)\, B(X_B)}{\det G(Q)^{\frac{n+1-\mathrm{D}}{2}}}$$

Two-particle cuts in Baikov form

(The details I am skipping over)



$$= C \int_\Gamma \mathrm{d}\mu \frac{A(X_A)\, B(X_B)}{\det G(Q)^{\frac{n+1-D}{2}}}$$

Integration measure

$$\mathrm{d}\mu = \prod_{Q_i \in Q} \mathrm{d}(k \cdot Q_i)\ \delta[k^2 - m^2]\ \delta[(k + p_{1\ldots a})^2 - m^2]$$

Set of Baikov variables for the B-blob

$k^2,\ k \cdot P_B,\ P_B^2$

Gram determinant over

$Q = \{k\} \cup P_A \cup P_B \backslash \{p_n\}$

Normalization

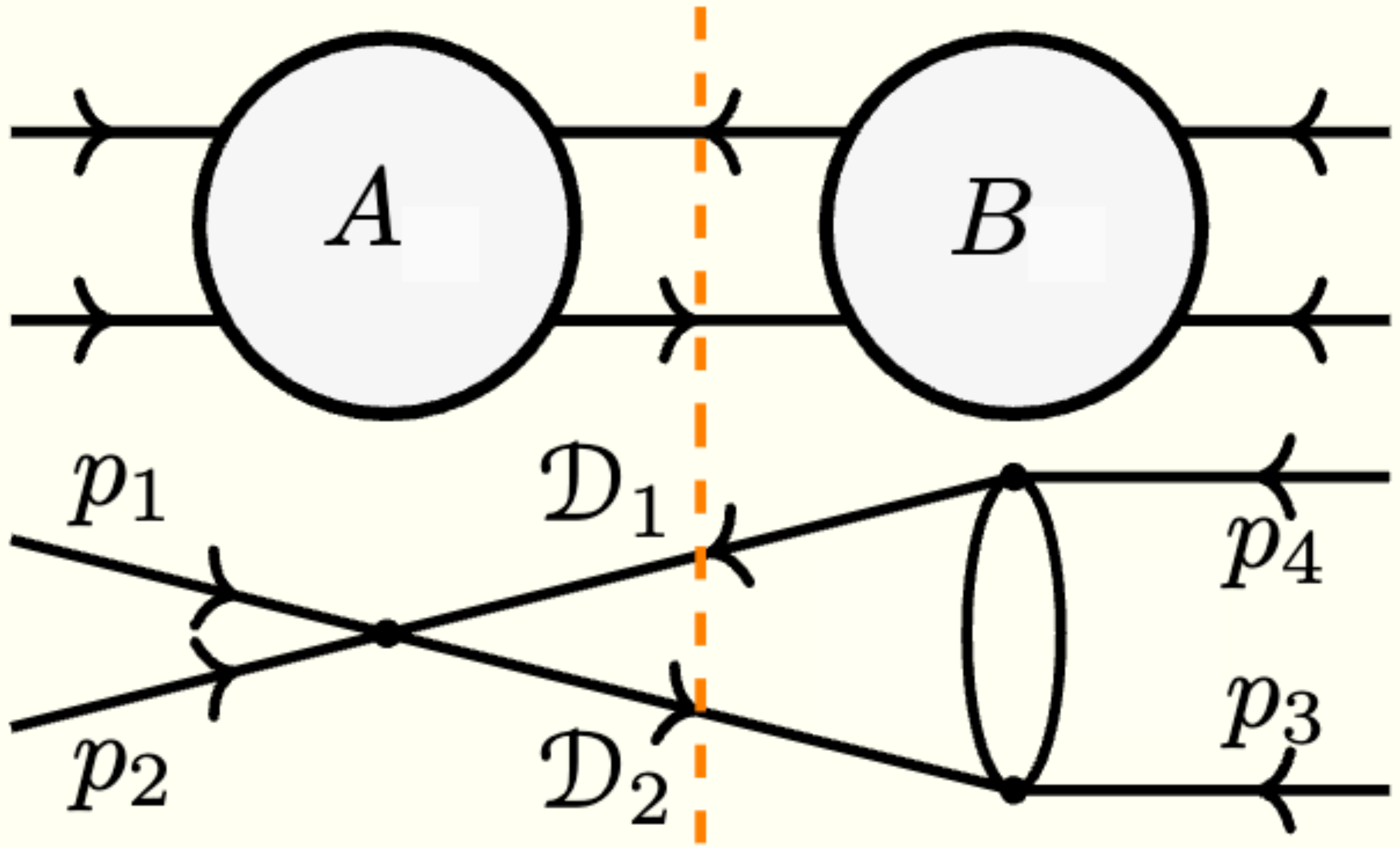
$$C = \frac{\det G(P_A \cup P_B \backslash \{p_n\})^{\frac{n-D}{2}}}{\sqrt{\pi}}$$

Integration contour

$$\Gamma = \left\{ k \cdot Q_i \,\middle|\, \frac{\det G(Q)}{\det G(P_A \cup P_B \backslash \{p_n\})} > 0, Q_i \in Q \right\}$$

Two-particle cuts in Baikov form

As an integral over independent the scalar products between loop and external momenta



$$\propto \int_\Gamma \frac{\mathrm{d}\mu \;\; A \;\; B}{(\det G \;)^{\gamma}}$$

$$\mathrm{d}\mu \;\; = \mathrm{d}(k_1 \cdot p_{12}) \wedge \mathrm{d}(k_1 \cdot p_3) \wedge \mathrm{d}(k_1^2) \;\; \delta(\mathcal{D}_1)\delta(\mathcal{D}_2)$$

$$G \;\; = \begin{bmatrix} p_{12}^2 & p_{12} \cdot k_1 & p_{12} \cdot p_3 \\ p_{12} \cdot k_1 & k_1^2 & k_1 \cdot p_3 \\ p_{12} \cdot p_3 & k_1 \cdot p_3 & p_3^2 \end{bmatrix}$$

Necessary conditions for singularities (II)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

What does it mean for two-particle cut ?

Necessary conditions for singularities (II)

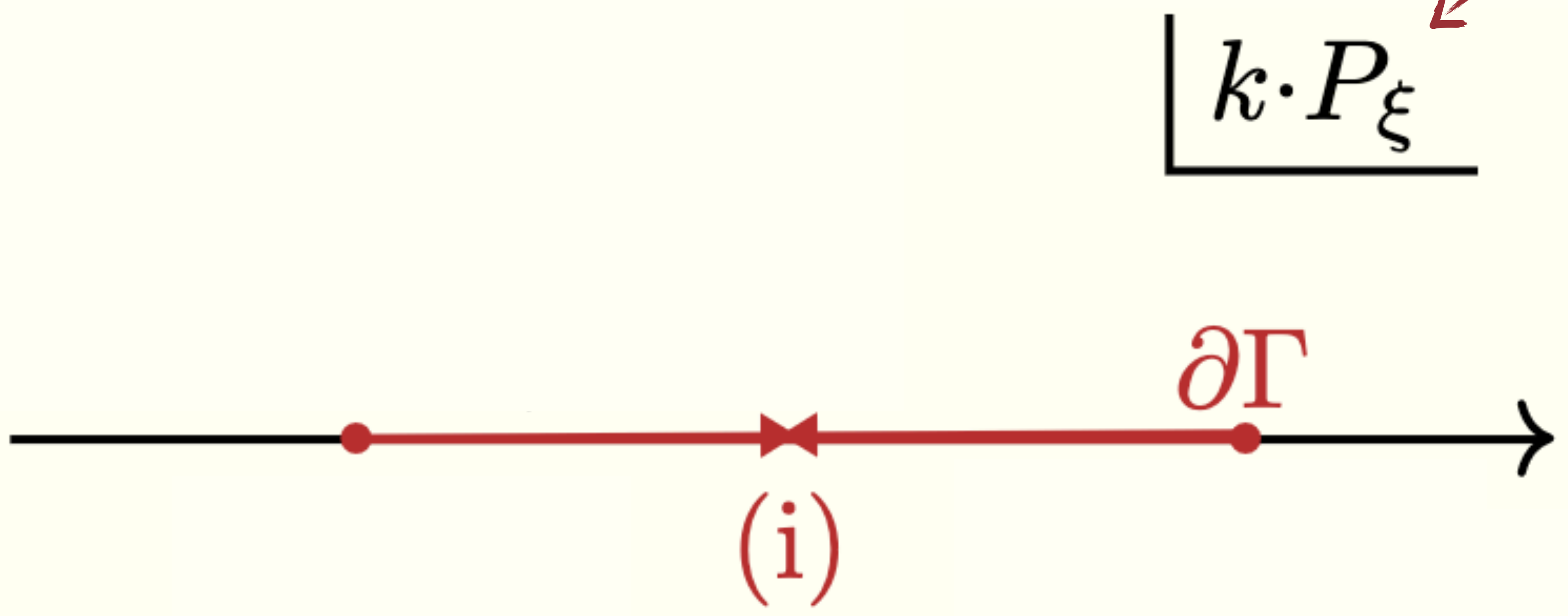
Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

What does it mean for two-particle cut ?

(i) At thresholds, the phase space Γ *closes down* to a single isolated point (only classical scattering is possible)

Left over integration variables on the two-particle cut



$(\xi = A, B)$

Boundary $\partial\Gamma = \{\det G = 0\}$ collapses to a point (i.e., from all directions)

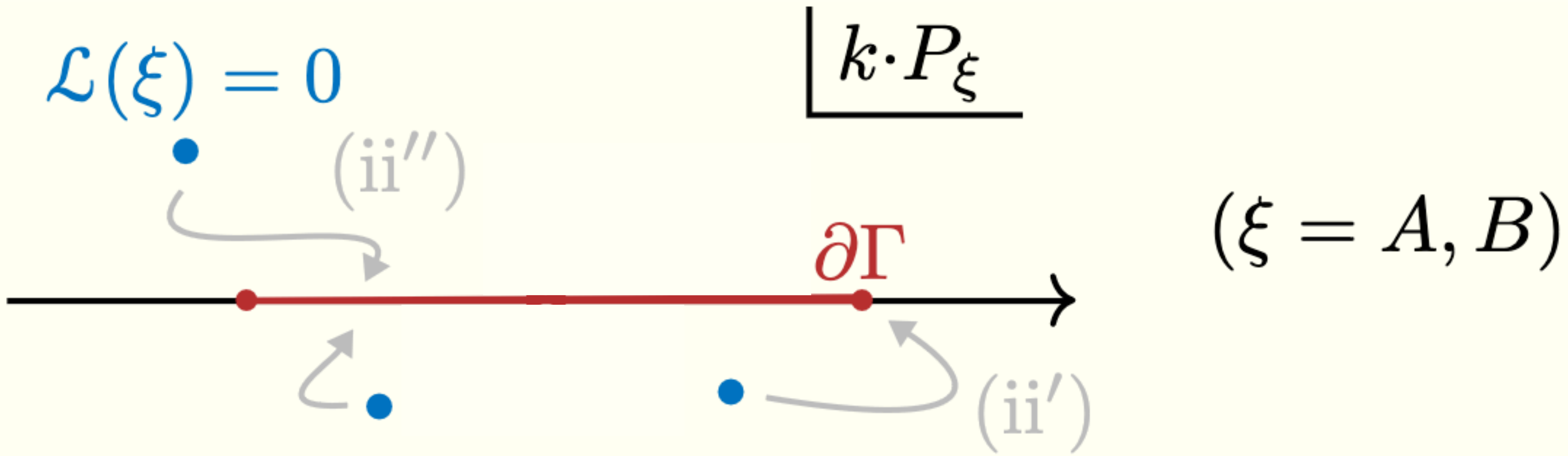
Necessary conditions for singularities (II)

Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

What does it mean for two-particle cut ?

(ii) Double discontinuities happen where the singular locus of A (or B) pinches Γ or hits $\partial\Gamma$



Necessary conditions for singularities (II)

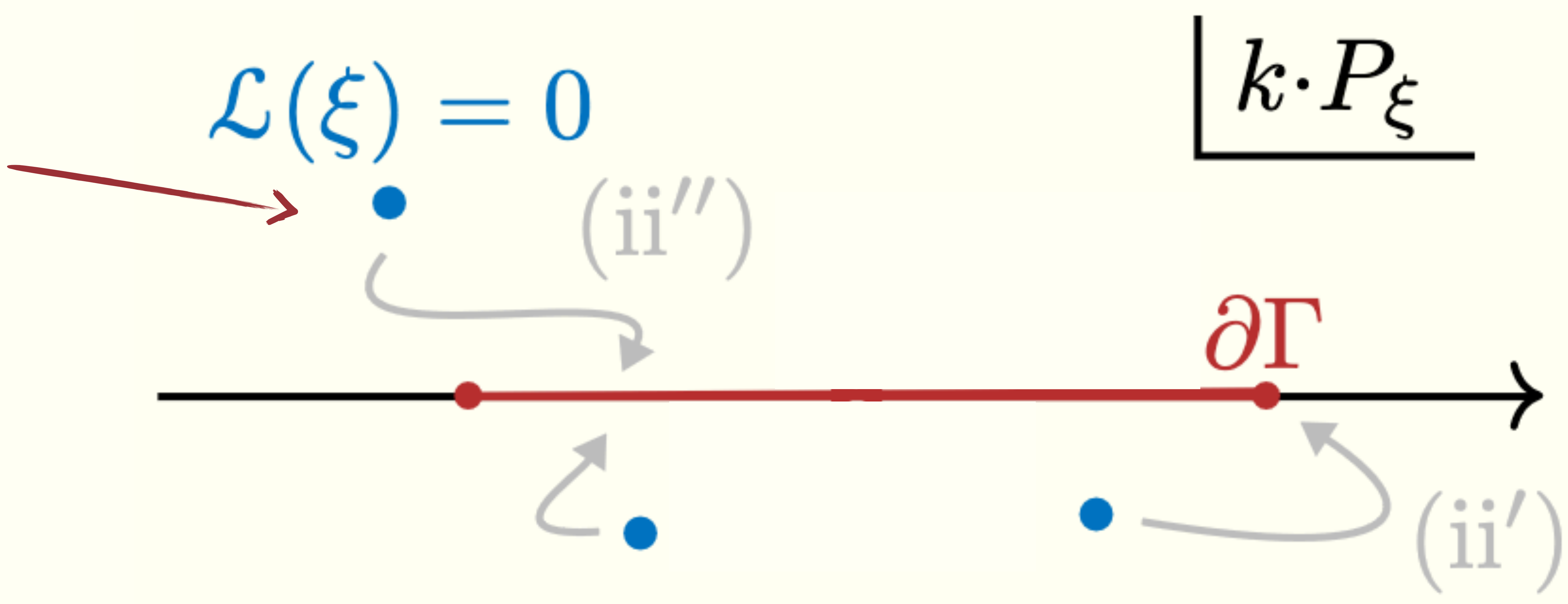
Qualitative necessary conditions

Amplitudes *can* be singular when (i) the phase space of cuts opens up, and (ii) when cuts are singular

What does it mean for two-particle cut ?

(ii) Double discontinuities happen where the singular locus of A (or B) pinches Γ or hits $\partial\Gamma$

Never expected to happen in momentum space without $\det G = 0$ (*Landau*: on a singularity k is a linear combination of external momenta)



$(\xi = A, B)$

Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

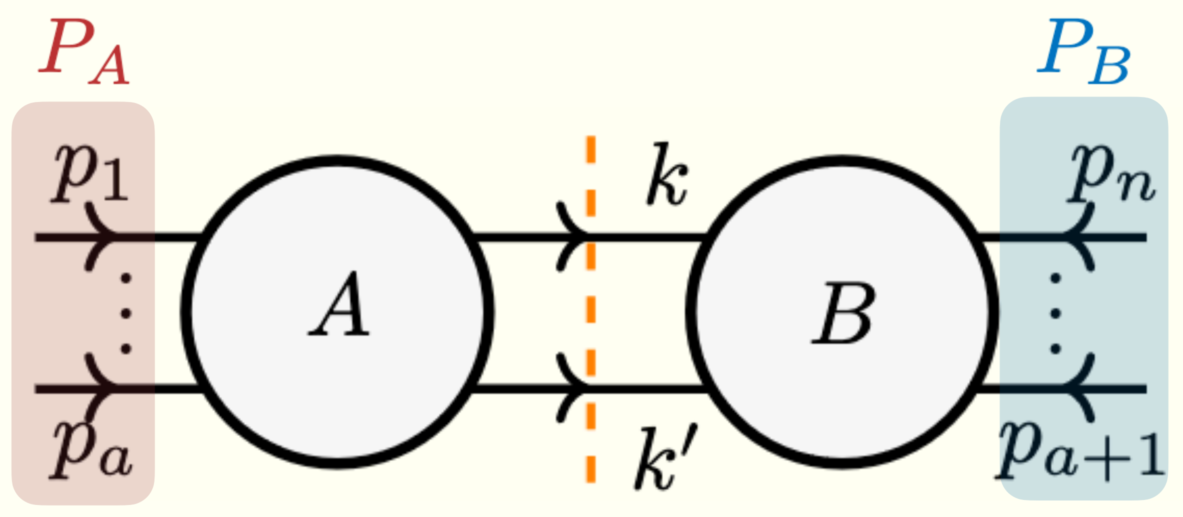
1) Pick a (possibly empty) subset $\mathcal{S} \subset \mathcal{L}(A) \cup \mathcal{L}(B)$ of singularities on the left and right

Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

1) Pick a (possibly empty) subset $\mathcal{S} \subset \mathcal{L}(A) \cup \mathcal{L}(B)$ of singularities on the left and right

2) Localizing on the two-particle cut *and* on the loci $\{\mathcal{S}_i = 0\}$ fixes $\dim(\mathcal{S})$ variables from

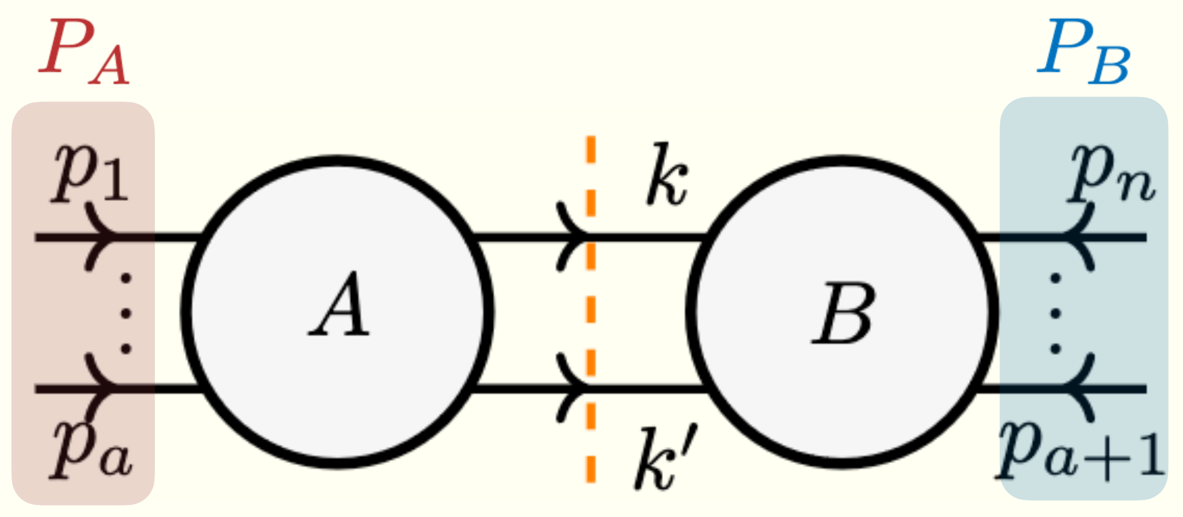


Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

1) Pick a (possibly empty) subset $\mathcal{S} \subset \mathcal{L}(A) \cup \mathcal{L}(B)$ of singularities on the left and right

2) Localizing on the two-particle cut *and* on the loci $\{\mathcal{S}_i = 0\}$ fixes $\dim(\mathcal{S})$ variables from



3) This leaves a set $X_{\mathcal{S}}$ of independent variables in terms of which $\partial\Gamma$ is

$$0 = \det \tilde{G}(X_{\mathcal{S}}) \equiv \det G|_{\{\mathcal{S}_i=0\}}$$

Necessary conditions for singularities (III)

Algebraic necessary conditions for (i) and (ii′) can be uniformly obtained as follows:

To ensure that there are no direction along which we could deform the contour to avoid the singularity, we have

$$\mathcal{L}(AB)_{\mathcal{S}} : \left[\begin{array}{l} \det \tilde{G} = 0 \\ \dfrac{\partial \det \tilde{G}}{\partial (k \cdot p_i)} = 0 \end{array} \right. \qquad \text{for} \quad k{\cdot}p_i \in X_{\mathcal{S}}$$

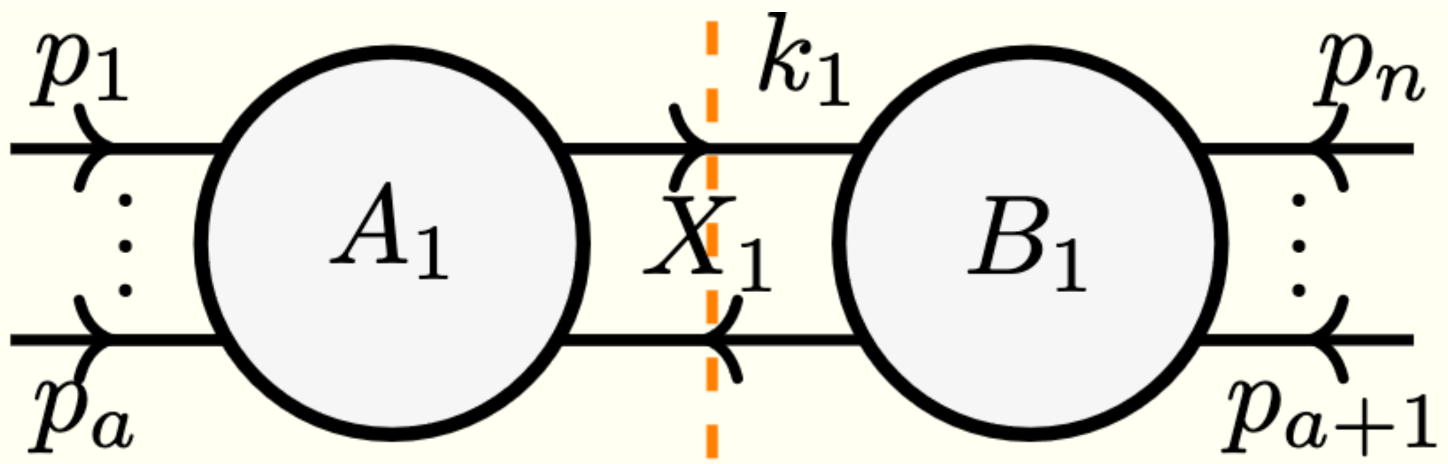
There is always one more equation than unknowns and so this system yields an algebraic constraint *on kinematic space*

$$\mathcal{L}(AB)_{\mathcal{S}} = 0$$

Necessary conditions for singularities (III)

To find *all* (leading) singularities of AB that contains a two-particle cut, it suffices to consider all sets $\mathcal{S}$ of (leading) singularities of the subamplitudes on that cut

Recursion via unitarity

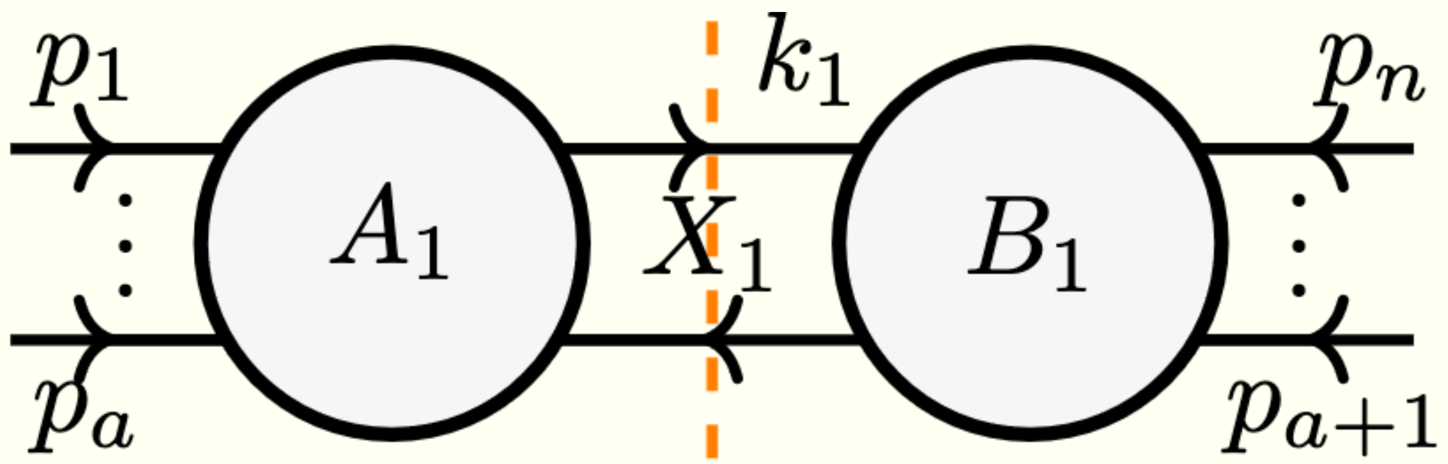


The necessary conditions for (e.g., leading) singularities require to *know*

$$\mathcal{L}(A_1) = \mathcal{L}(B_1) = 0$$

Can these be constructed recursively ?

Recursion via unitarity



The necessary conditions for (e.g., leading) singularities require to *know*

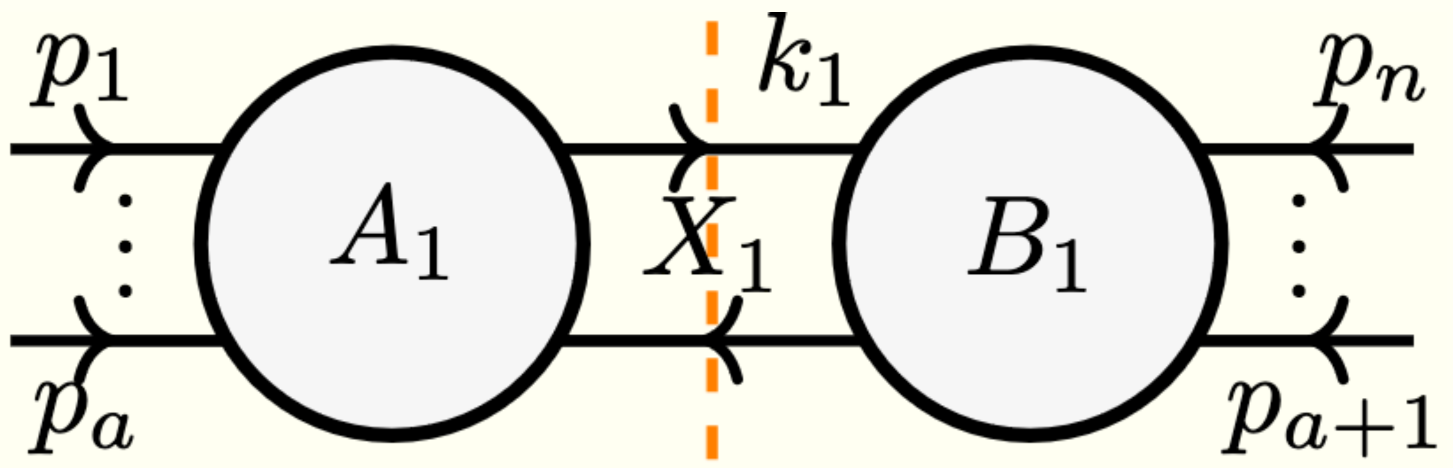
$$\mathcal{L}(A_1) = \mathcal{L}(B_1) = 0$$

Can these be constructed recursively ?

If either is two-particle-reducible, yes

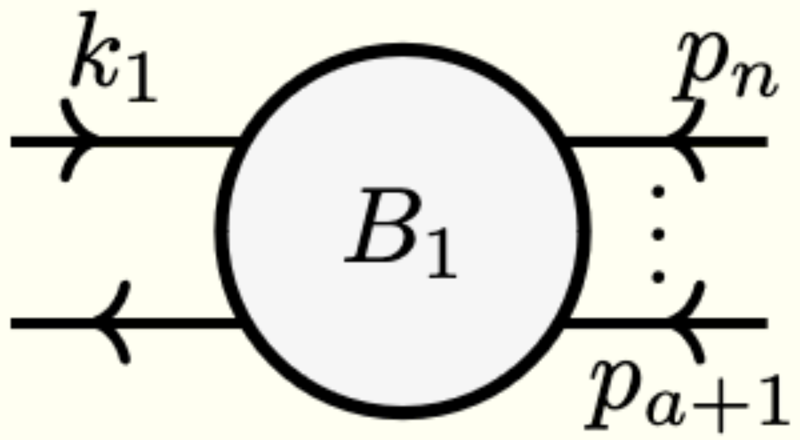
(just repeat the same argument over the blobs!)

Recursion via unitarity

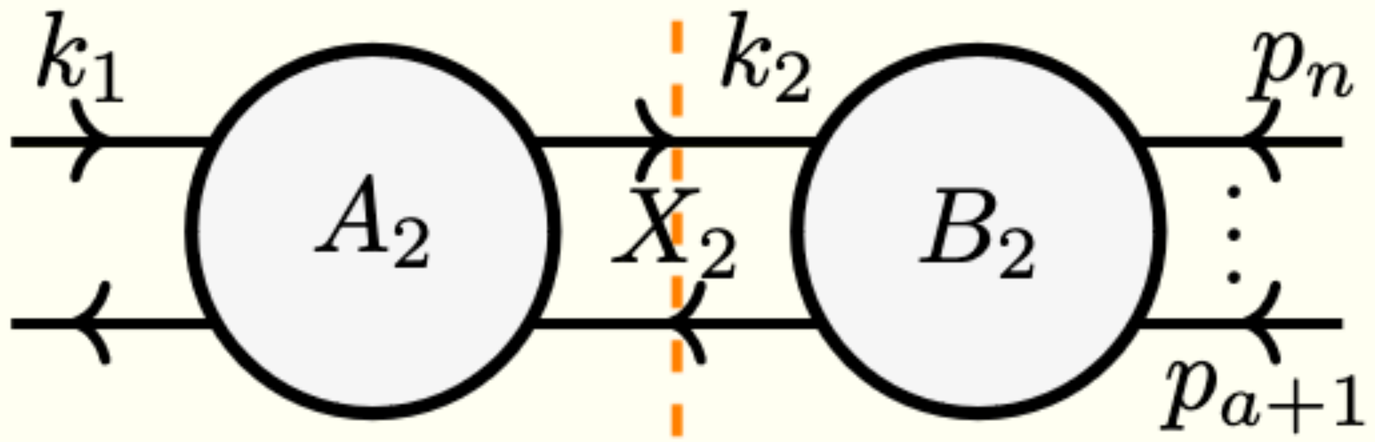


$$= C_1 \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{n+1-\mathrm{D}}{2}}}$$

If B_1 is two-particle-reducible,
just repeat the same argument



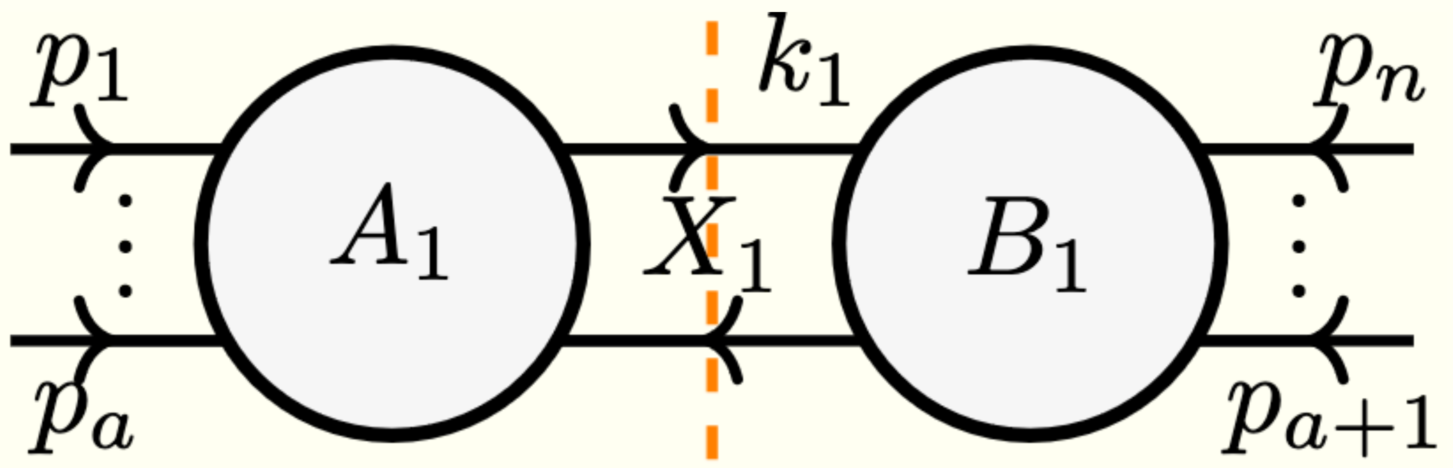
$\mapsto$



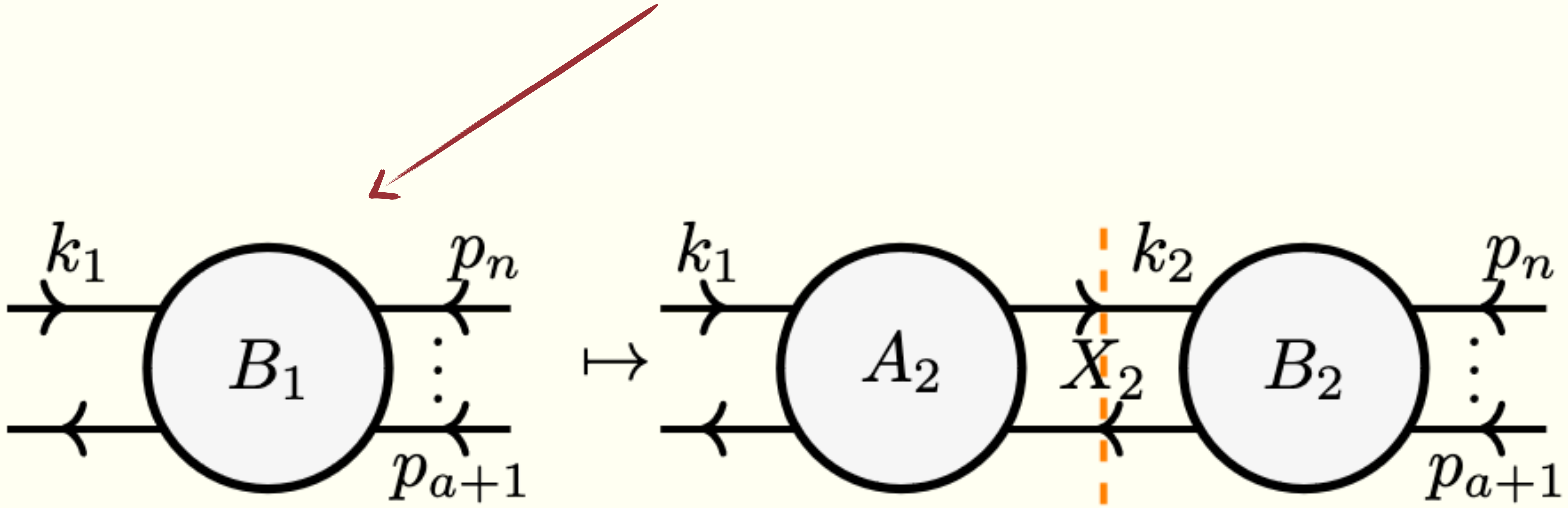
$$= C_2 \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)\;\cdots}$$

Means we take another
two-particle cut

Recursion via unitarity



$$= C_1 \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{n+1-\mathrm{D}}{2}}}$$

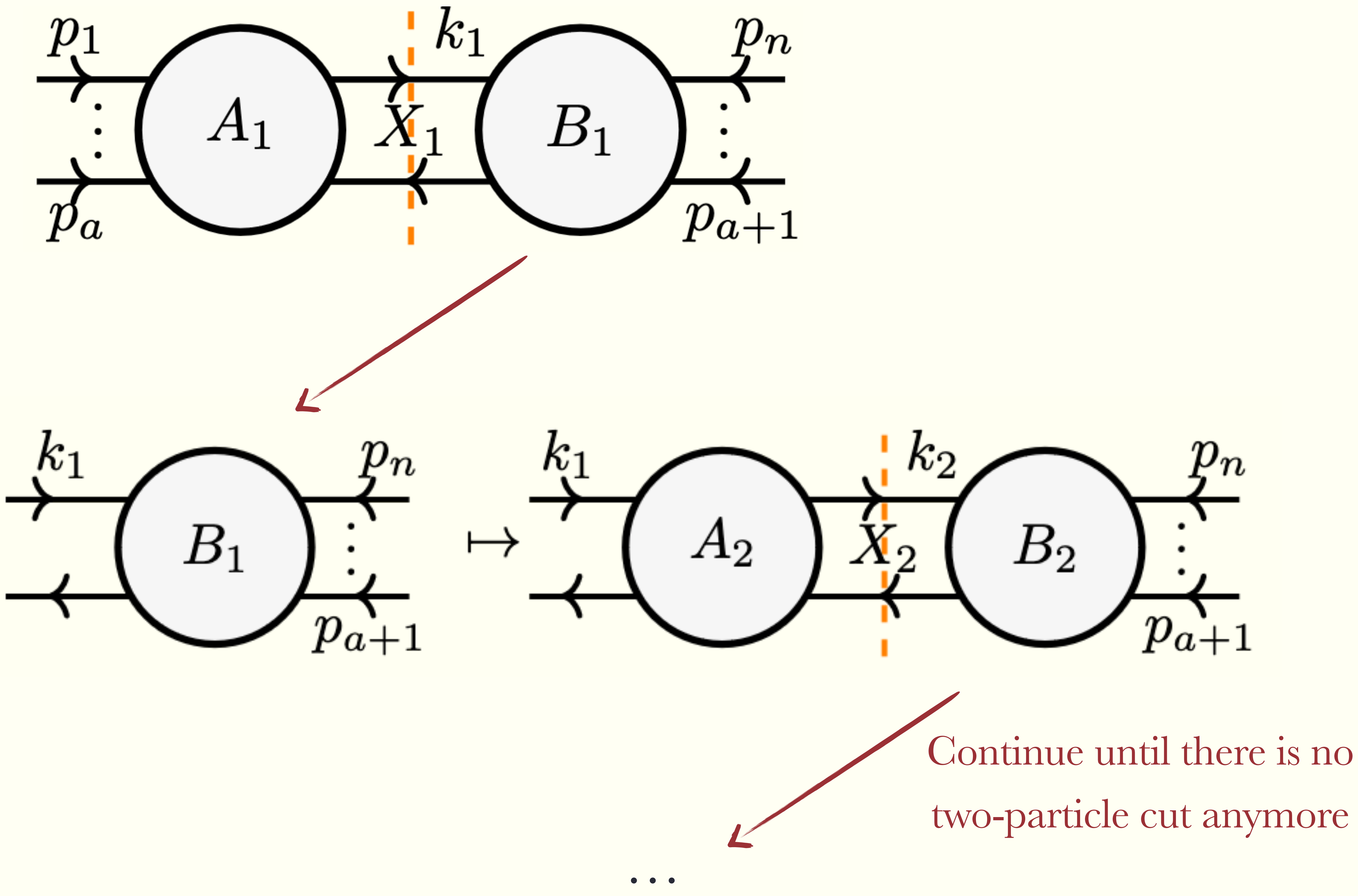


$$= C_2 \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)\ \cdots}$$

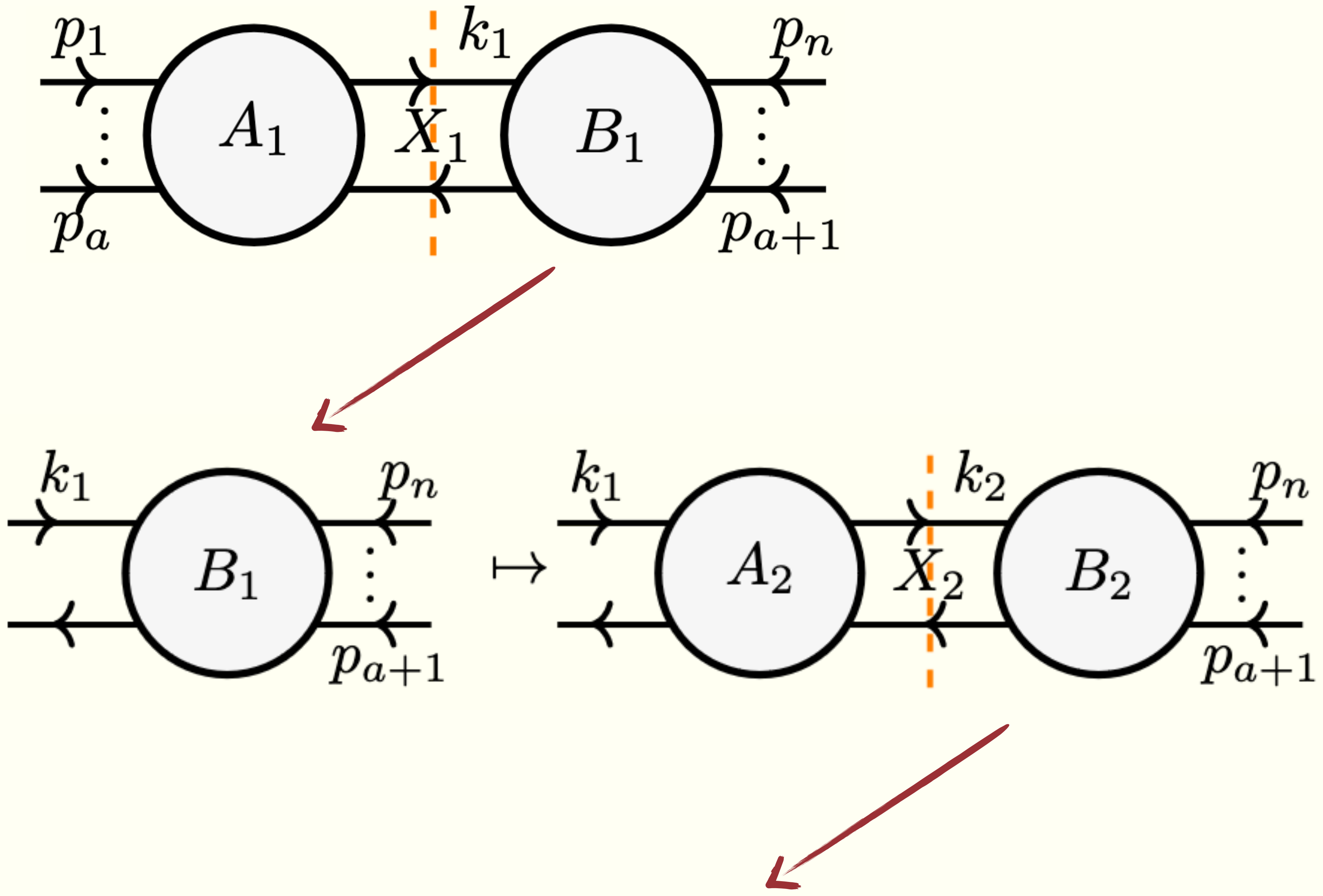
Singular locus of B_1 is given by solving

$$\mathcal{L}(B_1)_S : \left[\begin{array}{l} \det \tilde{G}_2 = 0 \\ \dfrac{\partial \det \tilde{G}_2}{\partial (k_2 \cdot p_i)} = 0 \end{array}\right. \qquad \text{for } k_2 \cdot p_i \in X_S$$

Recursion via unitarity



Recursion via unitarity



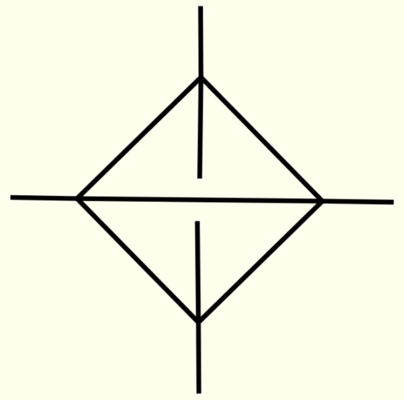
…

At the *end* of the recursion, we are left with either:

(1) A collection of tree-level subgraphs [easy/systematic]

Not the focus today

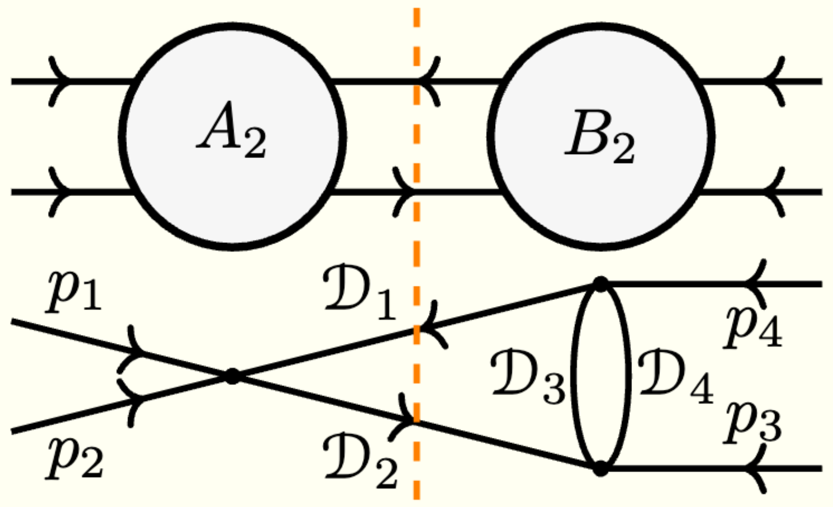
(2) A collection of subgraphs contains loop(s) [harder]
(may need external inputs for non-2PR subgraphs)

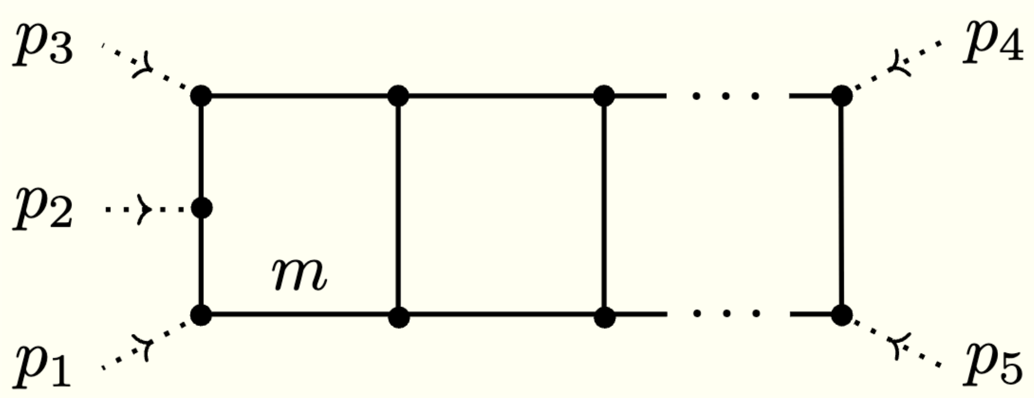


Outline

Proof of principle examples:

Recursively finding singularities





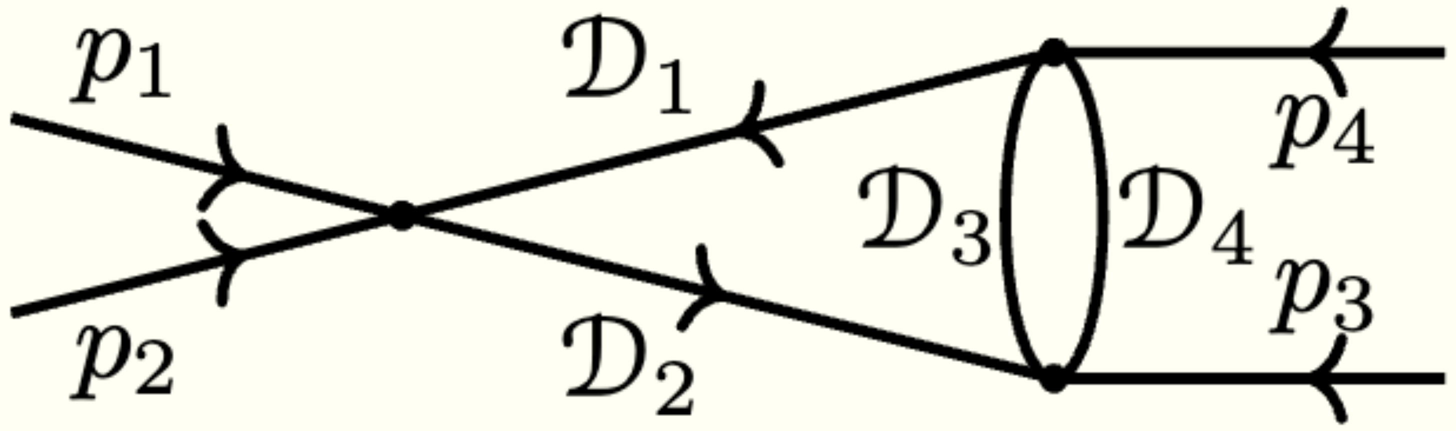
(Generic kinematic pentabox)

(Three-loop QED+QCD boX)

(Non-planar massive hexabox)

Recursively finding singularities

The generic kinematic parachute graph



$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

What are the candidate leading singularities?

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

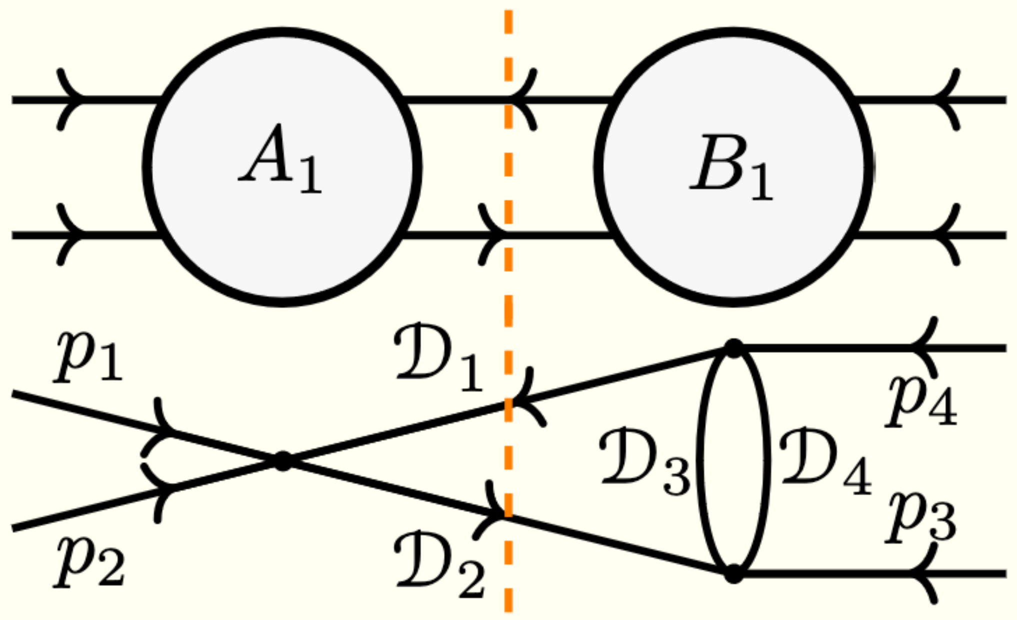
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$

$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$$\propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

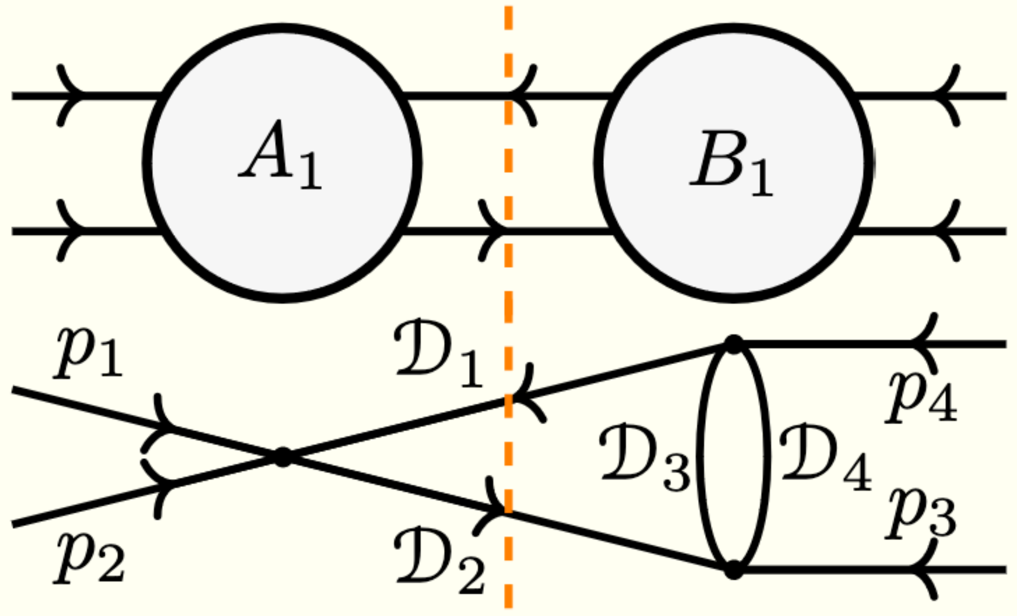
$$p_i^2 = M_i^2$$

$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Let's look at a first two-particle cut

$$= \mathrm{d}(k_1 \cdot p_{12})\mathrm{d}(k_1 \cdot p_3)\mathrm{d}(k_1^2)\delta[\mathcal{D}_1]\delta[\mathcal{D}_2]$$



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$$\propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

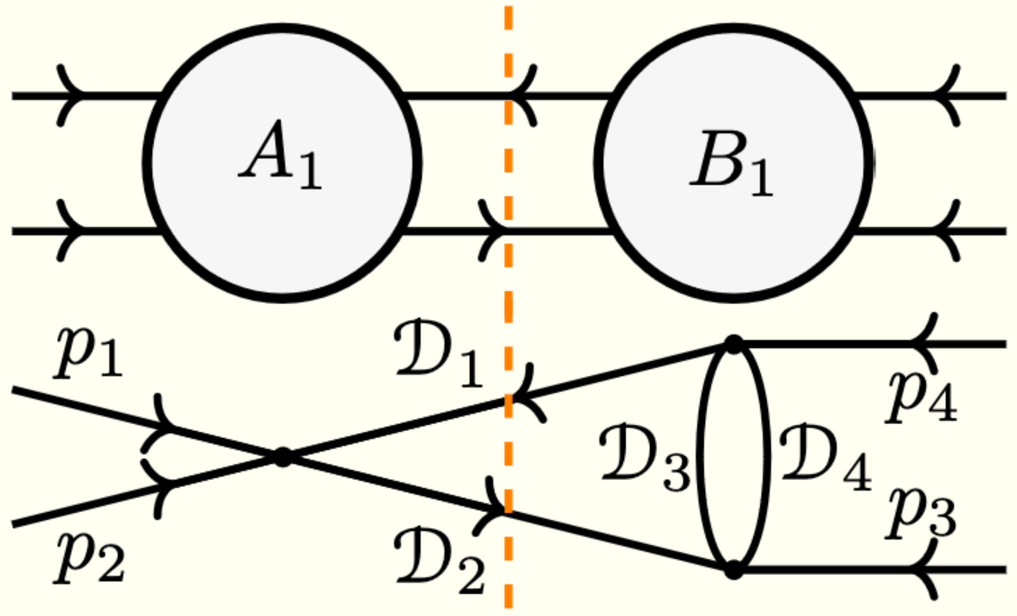
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$

$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1\, A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$\mathrm{d}\mu_1 = \mathrm{d}(k_1 \cdot p_{12})\,\mathrm{d}(k_1 \cdot p_3)\,\mathrm{d}(k_1^2)\,\delta[\mathcal{D}_1]\,\delta[\mathcal{D}_2]$

$A_1 = -i\lambda$

$C_{\text{par}} \propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$

Recursively finding singularities

$$p_i^2 = M_i^2$$

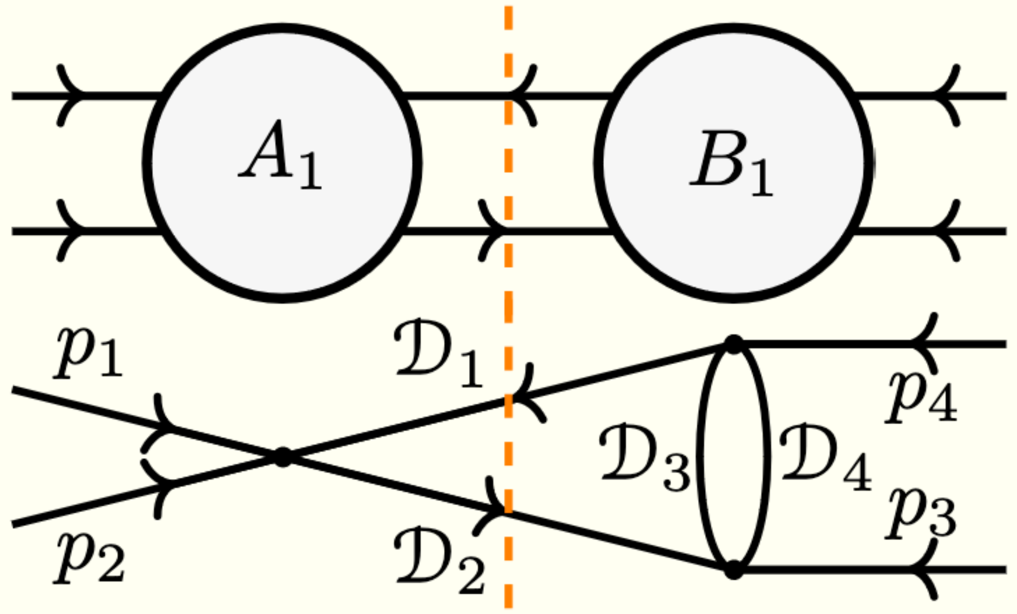
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1\, A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$$\mathrm{d}\mu_1 = \mathrm{d}(k_1 \cdot p_{12})\,\mathrm{d}(k_1 \cdot p_3)\,\mathrm{d}(k_1^2)\,\delta[\mathcal{D}_1]\,\delta[\mathcal{D}_2]$$

$$A_1 = -i\lambda$$

$$G_1 = \begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

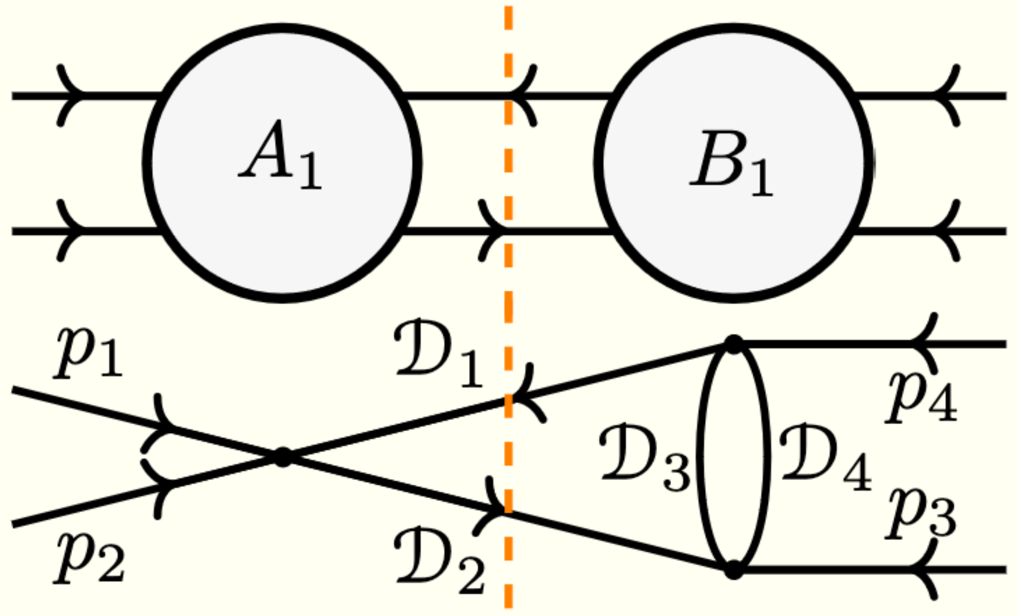
$$C_{\text{par}} \propto \left(\det[p_i \cdot p_j]_{i,j=12,3}\right)^{\frac{3-\mathrm{D}}{2}}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$\mathrm{d}\mu_1 = \mathrm{d}(k_1 \cdot p_{12})\mathrm{d}(k_1 \cdot p_3)\mathrm{d}(k_1^2)\delta[\mathcal{D}_1]\delta[\mathcal{D}_2]$

$A_1\, B_1 = -i\lambda$

$C_{\text{par}} \propto (\det[p_i \cdot p_j]_{i,j=12,3})^{\frac{3-\mathrm{D}}{2}}$

$$G_1 = \begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

Recursively finding singularities

$$p_i^2 = M_i^2$$

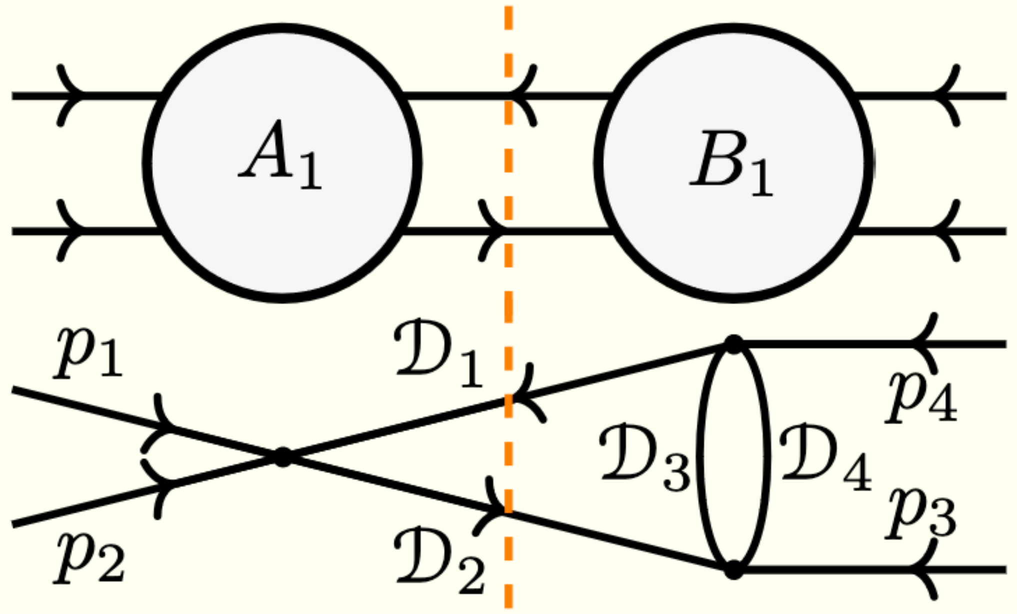
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1\, B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$\mathrm{d}\mu_1 = \mathrm{d}(k_1 \cdot p_{12})\mathrm{d}(k_1 \cdot p_3)\mathrm{d}(k_1^2)\delta[\mathcal{D}_1]\delta[\mathcal{D}_2]$

$A_1 = -i\lambda$

$C_{\text{par}} \propto (\det[p_i \cdot p_j]_{i,j=12,3})^{\frac{3-\mathrm{D}}{2}}$

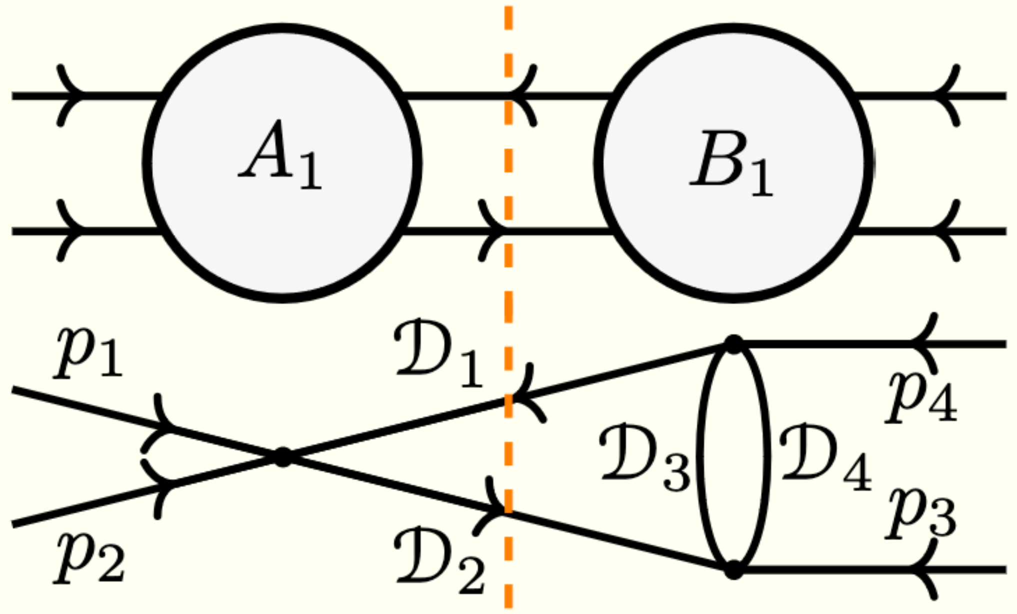
$$G_1 = \begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1 \cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Let's look at a first two-particle cut



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1 B_1}{(\det G_1)^{\frac{4-\mathrm{D}}{2}}}$$

$\mathrm{d}\mu_1 = \mathrm{d}(k_1 \cdot p_{12})\mathrm{d}(k_1 \cdot p_3)\mathrm{d}(k_1^2)\delta[\mathcal{D}_1]\delta[\mathcal{D}_2]$

$A_1 = -i\lambda$

B_1: Fixed by the singular locus of B_1

$$C_{\text{par}} \propto (\det[p_i \cdot p_j]_{i,j=12,3})^{\frac{3-\mathrm{D}}{2}}$$

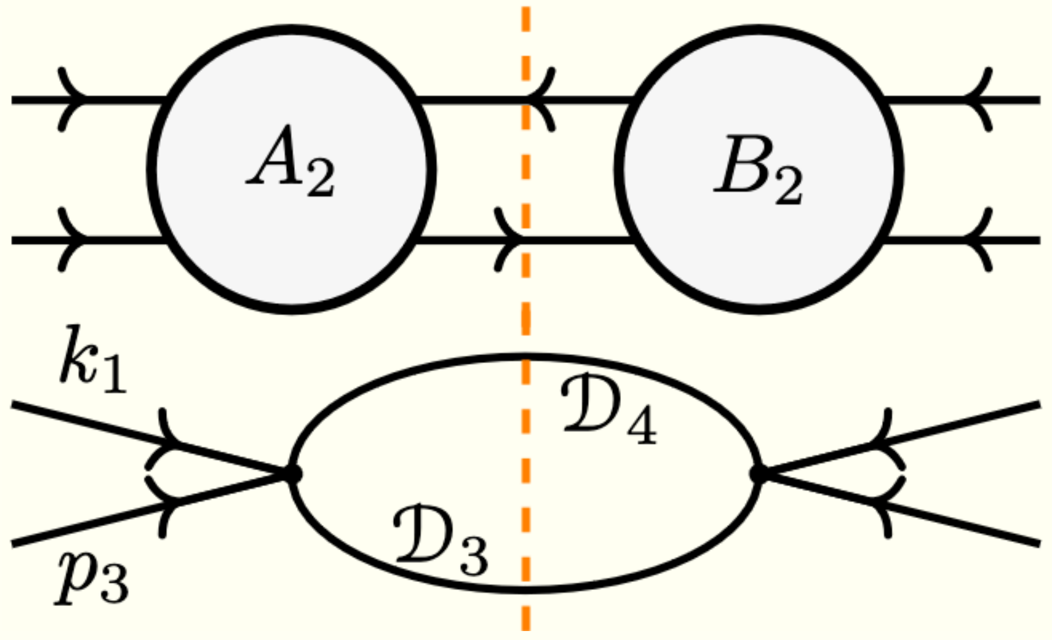
$$G_1 = \begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$C_{\text{bub}} \propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

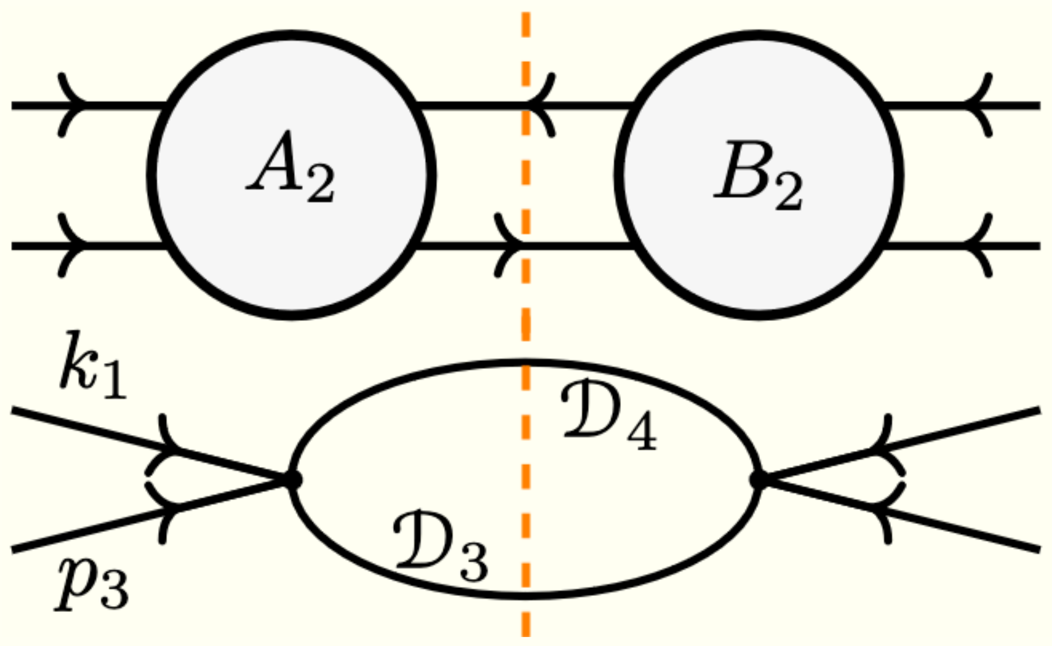
$\Lambda^\mu = (p_3 + k_1)^\mu$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$=\mathrm{d}(k_2{\cdot}\Lambda)\mathrm{d}(k_2^2)\delta[\mathcal{D}_3]\delta[\mathcal{D}_4]$

$\propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

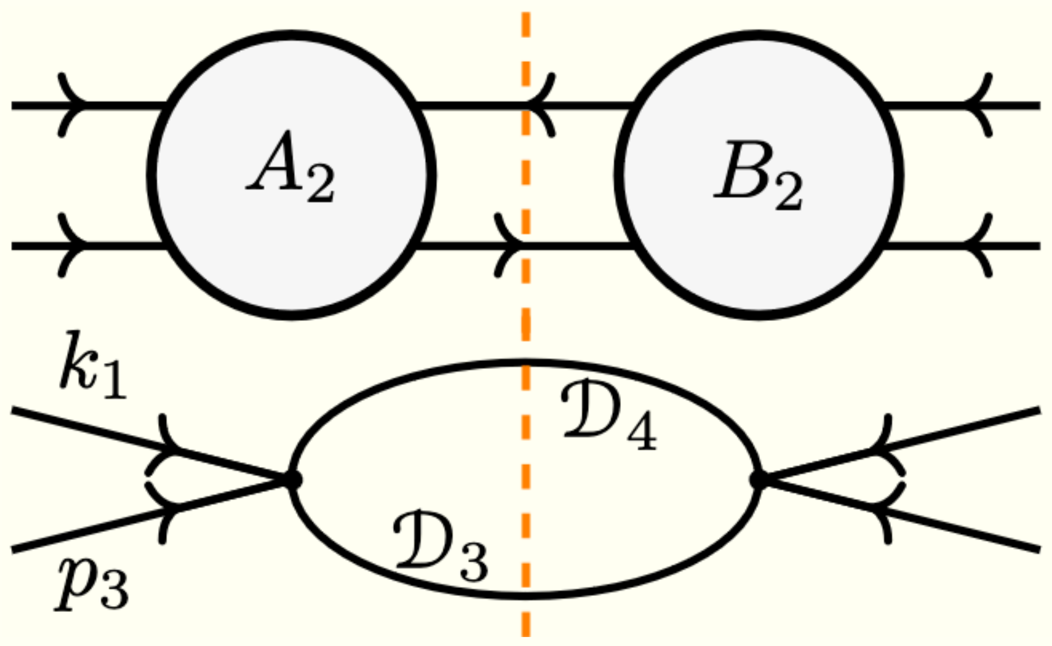
$\Lambda^\mu = (p_3 + k_1)^\mu$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$\mathrm{d}\mu_2 = \mathrm{d}(k_2{\cdot}\Lambda)\mathrm{d}(k_2^2)\delta[\mathcal{D}_3]\delta[\mathcal{D}_4]$

$C_{\text{bub}} \propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

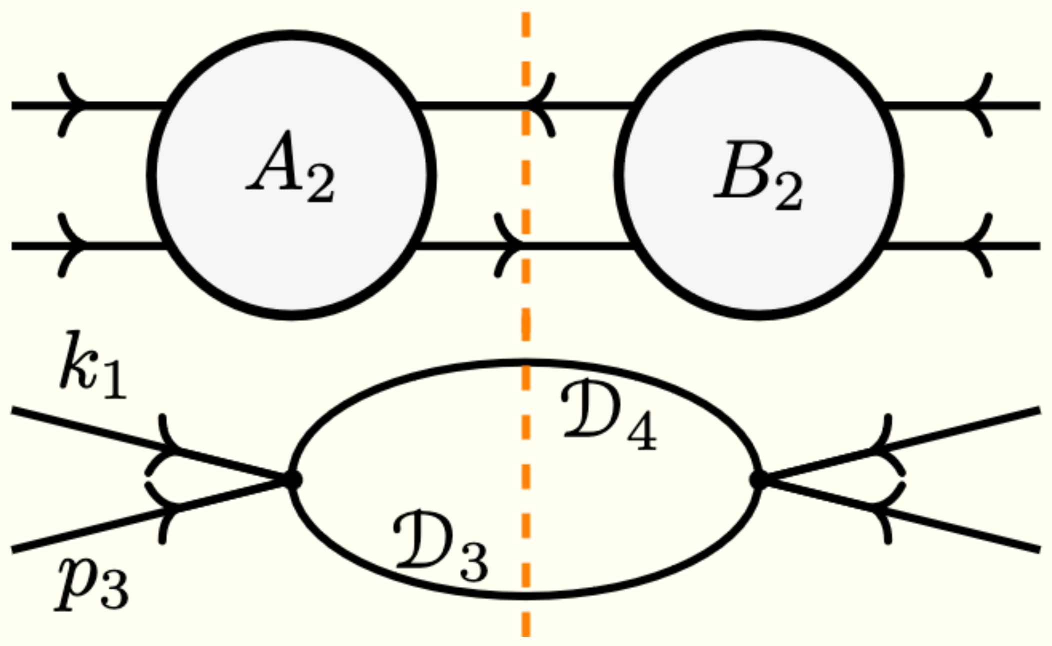
$\Lambda^\mu = (p_3 + k_1)^\mu$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$= \mathrm{d}(k_2 \cdot \Lambda)\, \mathrm{d}(k_2^2)\, \delta[\mathcal{D}_3]\, \delta[\mathcal{D}_4]$

$\propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$

$= \begin{bmatrix} \Lambda^2 & \Lambda \cdot k_2 \\ \Lambda \cdot k_2 & k_2^2 \end{bmatrix}$

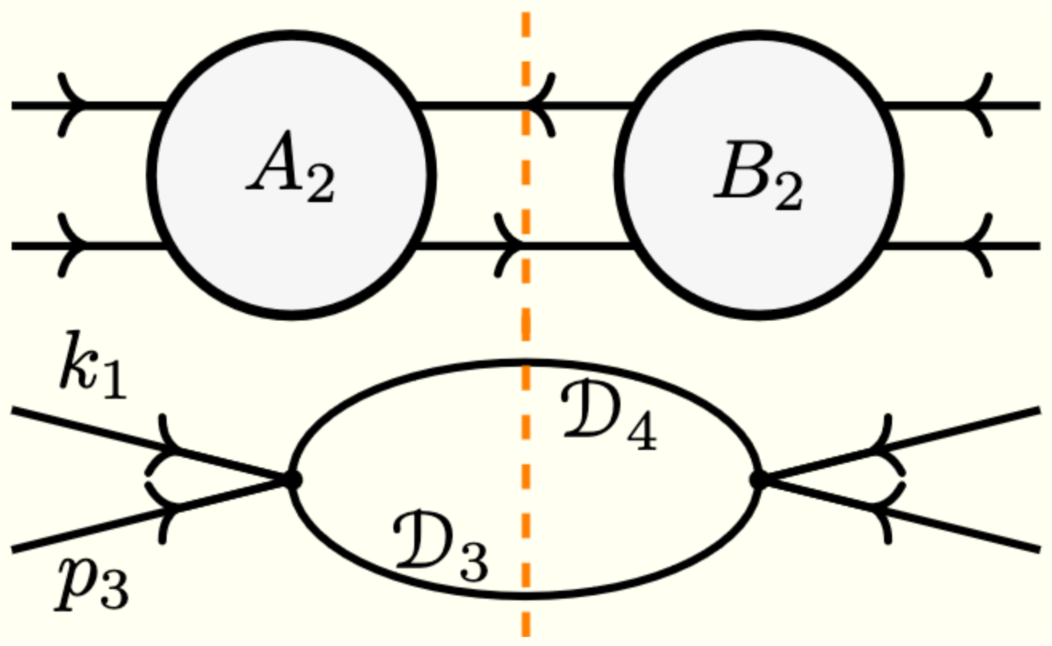
$\Lambda^\mu = (p_3 + k_1)^\mu$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-\mathrm{D}}{2}}}$$

$$= \mathrm{d}(k_2 \cdot \Lambda)\,\mathrm{d}(k_2^2)\,\delta[\mathcal{D}_3]\,\delta[\mathcal{D}_4]$$

$$\propto (\Lambda^2)^{\frac{2-\mathrm{D}}{2}}$$

$$= \begin{bmatrix} \Lambda^2 & \Lambda \cdot k_2 \\ \Lambda \cdot k_2 & k_2^2 \end{bmatrix}$$

$$\Lambda^\mu = (p_3 + k_1)^\mu$$

Recursively finding singularities

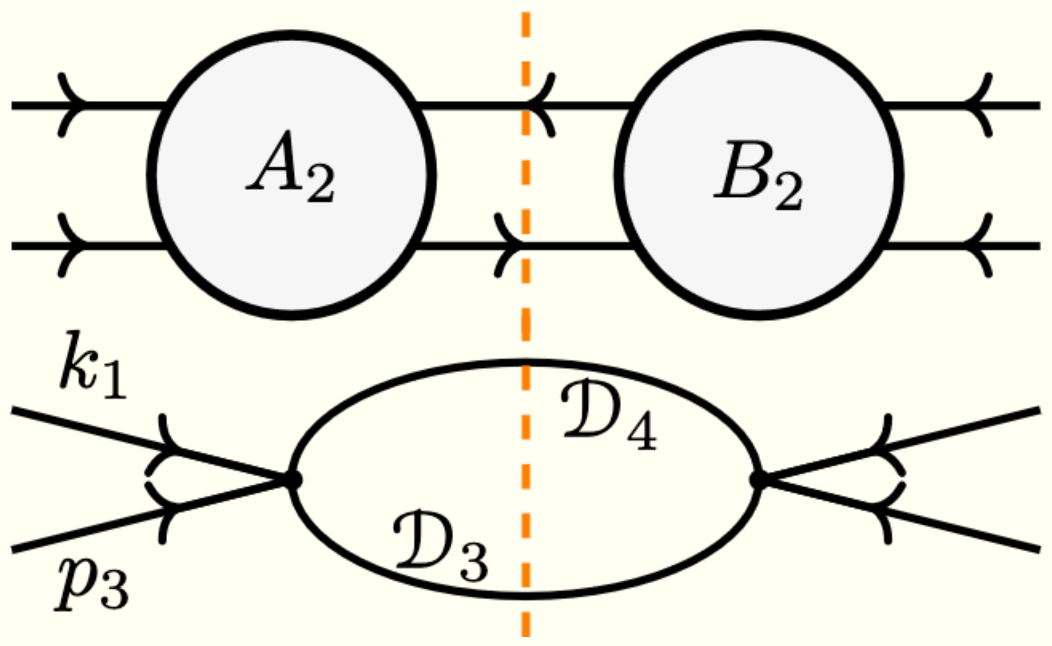
$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

Singular locus of B_1 is given by repeating the *same* argument over the bubble

Imposing $\det \tilde{G}_2 = 0$ gives $\mathcal{L}(B_1)_1 = 0$

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$



$$= C_{\text{bub}} \int_{\Gamma_2} \frac{\mathrm{d}\mu_2 A_2\, B_2}{(\det G_2)^{\frac{3-D}{2}}}$$

$\mathrm{d}\mu_2 = \mathrm{d}(k_2\cdot\Lambda)\mathrm{d}(k_2^2)\delta[\mathcal{D}_3]\delta[\mathcal{D}_4]$

$C_{\text{bub}} \propto (\Lambda^2)^{\frac{2-D}{2}}$

$$G_2 = \begin{bmatrix} \Lambda^2 & \Lambda\cdot k_2 \\ \Lambda\cdot k_2 & k_2^2 \end{bmatrix}$$

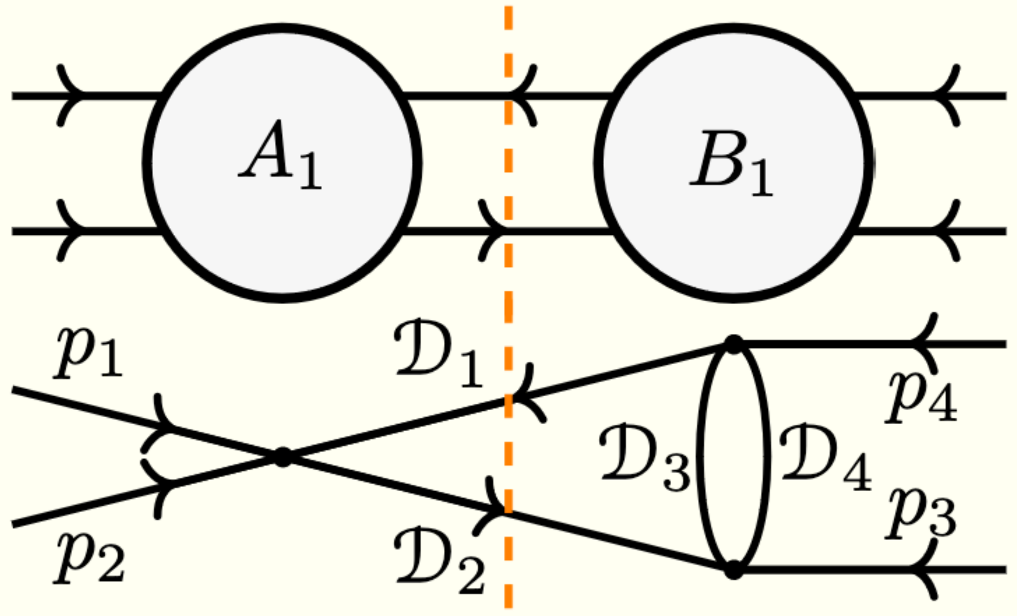
$\Lambda^\mu = (p_3 + k_1)^\mu$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

What are the candidate leading singularities ?



$$= C_{\text{par}} \int_{\Gamma_1} \frac{\mathrm{d}\mu_1 A_1 B_1}{(\det G_1)^{\frac{5-\mathrm{D}}{2}}}$$

Setting $\mathcal{S} = \{\mathcal{L}(B_1)_1 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$

$$= \begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$
$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$
$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$
$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

What are the candidate leading singularities ?

$$\begin{vmatrix} s & \frac{m_2^2-m_1^2+s}{2} & \frac{M_4^2-M_3^2-s}{2} \\ \frac{m_2^2-m_1^2+s}{2} & m_2^2 & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} \\ \frac{M_4^2-M_3^2-s}{2} & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

Recursively finding singularities

$$\mathcal{D}_1 = (k_1 - p_{12})^2 - m_1^2\,, \qquad \mathcal{D}_2 = k_1^2 - m_2^2$$

$$\mathcal{D}_3 = (k_1 + k_2 + p_3)^2 - m_3^2\,, \qquad \mathcal{D}_4 = k_2^2 - m_4^2$$

$$p_i^2 = M_i^2$$

$$p_{12}^2 = p_{34}^2 = s\,, \quad p_{13}^2 = p_{24}^2 = t$$

$$p_{14}^2 = p_{23}^2 = \sum_{i=1}^{4} M_i^2 - s - t$$

What are the candidate leading singularities ?

```
################################
# Component 5
################################

D[5] = M[3]^4*m[1]^2 - 2*M[3]^3*M[4]*m[1]^2 - 2*M[3]^3*M[4]*m[1]*m[2] + 2*M[3]^3*M[4]*m[1]*s + 2*M[3]^3*m[1]^3 - 2*M[3]^3*m[1]^2*m[2] - 2*M[3]^3*m[1]^2*m[3] - 2*M[3]^3*m[1]^2*m[4] - 2*M[3]^3*m[1]^2*s + 2*M[3]^3*m[1]*m[2]*m[3] +
2*M[3]^3*m[1]*m[2]*m[4] - 2*M[3]^3*m[1]*m[3]*s - 2*M[3]^3*m[1]*m[4]*s + M[3]^2*M[4]^2*m[1]^2 + 4*M[3]^2*M[4]^2*m[1]*m[2] - 2*M[3]^2*M[4]^2*m[1]*s + M[3]^2*M[4]^2*m[2]^2 - 2*M[3]^2*M[4]^2*m[2]*s + M[3]^2*M[4]^2*s^2 - 2*M[3]^2*M[4]*m[1]^3 -
2*M[3]^2*M[4]*m[1]^2*m[2] + 4*M[3]^2*M[4]*m[1]^2*m[3] + 4*M[3]^2*M[4]*m[1]^2*m[4] + 4*M[3]^2*M[4]*m[1]^2*s + 4*M[3]^2*M[4]*m[1]*m[2]^2 - 2*M[3]^2*M[4]*m[1]*m[2]*m[3] - 2*M[3]^2*M[4]*m[1]*m[2]*m[4] - 2*M[3]^2*M[4]*m[1]*m[2]*s -
2*M[3]^2*M[4]*m[1]*m[3]*s - 2*M[3]^2*M[4]*m[1]*m[4]*s - 2*M[3]^2*M[4]*m[1]*s^2 - 2*M[3]^2*M[4]*m[2]^2*m[3] - 2*M[3]^2*M[4]*m[2]^2*m[4] + 4*M[3]^2*M[4]*m[2]*m[3]*s + 4*M[3]^2*M[4]*m[2]*m[4]*s - 2*M[3]^2*M[4]*m[3]*s^2 - 2*M[3]^2*M[4]*m[4]*s^2 +
M[3]^2*m[1]^4 - 2*M[3]^2*m[1]^3*m[2] - 2*M[3]^2*m[1]^3*m[3] - 2*M[3]^2*m[1]^3*m[4] - 2*M[3]^2*m[1]^3*s + M[3]^2*m[1]^2*m[2]^2 + 4*M[3]^2*m[1]^2*m[2]*m[3] + 4*M[3]^2*m[1]^2*m[2]*m[4] + 4*M[3]^2*m[1]^2*m[2]*s + M[3]^2*m[1]^2*m[3]^2 -
2*M[3]^2*m[1]^2*m[3]*m[4] - 2*M[3]^2*m[1]^2*m[3]*s + M[3]^2*m[1]^2*m[4]^2 - 2*M[3]^2*m[1]^2*m[4]*s + M[3]^2*m[1]^2*s^2 - 2*M[3]^2*m[1]*m[2]^2*m[3] - 2*M[3]^2*m[1]*m[2]^2*m[4] - 2*M[3]^2*m[1]*m[2]*m[3]^2 + 4*M[3]^2*m[1]*m[2]*m[3]*m[4] -
2*M[3]^2*m[1]*m[2]*m[3]*s - 2*M[3]^2*m[1]*m[2]*m[4]^2 - 2*M[3]^2*m[1]*m[2]*m[4]*s + 4*M[3]^2*m[1]*m[3]^2*s + 8*M[3]^2*m[1]*m[3]*m[4]*s + 4*M[3]^2*m[1]*m[3]*s^2 + 4*M[3]^2*m[1]*m[4]^2*s + 4*M[3]^2*m[1]*m[4]*s^2 + M[3]^2*m[2]^2*m[3]^2 -
2*M[3]^2*m[2]^2*m[3]*m[4] + M[3]^2*m[2]^2*m[4]^2 - 2*M[3]^2*m[2]*m[3]^2*s + 4*M[3]^2*m[2]*m[3]*m[4]*s - 2*M[3]^2*m[2]*m[4]^2*s + M[3]^2*m[3]^2*s^2 - 2*M[3]^2*m[3]*m[4]*s^2 + M[3]^2*m[4]^2*s^2 - 2*M[3]*M[4]^3*m[1]*m[2] - 2*M[3]*M[4]^3*m[2]^2 +
2*M[3]*M[4]^3*m[2]*s + 4*M[3]*M[4]^2*m[1]^2*m[2] - 2*M[3]*M[4]^2*m[1]^2*m[3] - 2*M[3]*M[4]^2*m[1]^2*m[4] - 2*M[3]*M[4]^2*m[1]*m[2]^2 - 2*M[3]*M[4]^2*m[1]*m[2]*m[3] - 2*M[3]*M[4]^2*m[1]*m[2]*m[4] - 2*M[3]*M[4]^2*m[1]*m[2]*s +
4*M[3]*M[4]^2*m[1]*m[3]*s + 4*M[3]*M[4]^2*m[1]*m[4]*s - 2*M[3]*M[4]^2*m[2]^3 + 4*M[3]*M[4]^2*m[2]^2*m[3] + 4*M[3]*M[4]^2*m[2]^2*m[4] + 4*M[3]*M[4]^2*m[2]^2*s - 2*M[3]*M[4]^2*m[2]*m[3]*s - 2*M[3]*M[4]^2*m[2]*m[4]*s - 2*M[3]*M[4]^2*m[2]*s^2 -
2*M[3]*M[4]^2*m[3]*s^2 - 2*M[3]*M[4]^2*m[4]*s^2 - 2*M[3]*M[4]*m[1]^3*m[2] + 2*M[3]*M[4]*m[1]^3*m[3] + 2*M[3]*M[4]*m[1]^3*m[4] + 4*M[3]*M[4]*m[1]^2*m[2]^2 - 2*M[3]*M[4]*m[1]^2*m[2]*m[3] - 2*M[3]*M[4]*m[1]^2*m[2]*m[4] - 2*M[3]*M[4]*m[1]^2*m[2]*s
- 2*M[3]*M[4]*m[1]^2*m[3]^2 + 4*M[3]*M[4]*m[1]^2*m[3]*m[4] - 2*M[3]*M[4]*m[1]^2*m[3]*s - 2*M[3]*M[4]*m[1]^2*m[4]^2 - 2*M[3]*M[4]*m[1]^2*m[4]*s - 2*M[3]*M[4]*m[1]*m[2]^3 - 2*M[3]*M[4]*m[1]*m[2]^2*m[3] - 2*M[3]*M[4]*m[1]*m[2]^2*m[4] -
2*M[3]*M[4]*m[1]*m[2]^2*s + 4*M[3]*M[4]*m[1]*m[2]*m[3]^2 - 8*M[3]*M[4]*m[1]*m[2]*m[3]*m[4] + 12*M[3]*M[4]*m[1]*m[2]*m[3]*s + 4*M[3]*M[4]*m[1]*m[2]*m[4]^2 + 12*M[3]*M[4]*m[1]*m[2]*m[4]*s + 4*M[3]*M[4]*m[1]*m[2]*s^2 - 2*M[3]*M[4]*m[1]*m[3]^2*s -
12*M[3]*M[4]*m[1]*m[3]*m[4]*s - 2*M[3]*M[4]*m[1]*m[3]*s^2 - 2*M[3]*M[4]*m[1]*m[4]^2*s - 2*M[3]*M[4]*m[1]*m[4]*s^2 + 2*M[3]*M[4]*m[2]^3*m[3] + 2*M[3]*M[4]*m[2]^3*m[4] - 2*M[3]*M[4]*m[2]^2*m[3]^2 + 4*M[3]*M[4]*m[2]^2*m[3]*m[4] -
2*M[3]*M[4]*m[2]^2*m[3]*s - 2*M[3]*M[4]*m[2]^2*m[4]^2 - 2*M[3]*M[4]*m[2]^2*m[4]*s - 2*M[3]*M[4]*m[2]*m[3]^2*s - 12*M[3]*M[4]*m[2]*m[3]*m[4]*s - 2*M[3]*M[4]*m[2]*m[3]*s^2 - 2*M[3]*M[4]*m[2]*m[4]^2*s - 2*M[3]*M[4]*m[2]*m[4]*s^2 +
4*M[3]*M[4]*m[3]^2*s^2 + 8*M[3]*M[4]*m[3]*m[4]*s^2 + 2*M[3]*M[4]*m[3]*s^3 + 4*M[3]*M[4]*m[4]^2*s^2 + 2*M[3]*M[4]*m[4]*s^3 + 2*M[3]*m[1]^3*m[2]*s - 2*M[3]*m[1]^3*m[3]*s - 2*M[3]*m[1]^3*m[4]*s - 2*M[3]*m[1]^2*m[2]^2*s - 2*M[3]*m[1]^2*m[2]*m[3]*s
- 2*M[3]*m[1]^2*m[2]*m[4]*s - 2*M[3]*m[1]^2*m[2]*s^2 + 4*M[3]*m[1]^2*m[3]^2*s + 8*M[3]*m[1]^2*m[3]*m[4]*s + 4*M[3]*m[1]^2*m[3]*s^2 + 4*M[3]*m[1]^2*m[4]^2*s + 4*M[3]*m[1]^2*m[4]*s^2 + 4*M[3]*m[1]*m[2]^2*m[3]*s + 4*M[3]*m[1]*m[2]^2*m[4]*s -
2*M[3]*m[1]*m[2]*m[3]^2*s - 12*M[3]*m[1]*m[2]*m[3]*m[4]*s - 2*M[3]*m[1]*m[2]*m[3]*s^2 - 2*M[3]*m[1]*m[2]*m[4]^2*s - 2*M[3]*m[1]*m[2]*m[4]*s^2 - 2*M[3]*m[1]*m[3]^3*s + 2*M[3]*m[1]*m[3]^2*m[4]*s - 2*M[3]*m[1]*m[3]^2*s^2 +
2*M[3]*m[1]*m[3]*m[4]^2*s - 12*M[3]*m[1]*m[3]*m[4]*s^2 - 2*M[3]*m[1]*m[3]*s^3 - 2*M[3]*m[1]*m[4]^3*s - 2*M[3]*m[1]*m[4]^2*s^2 - 2*M[3]*m[1]*m[4]*s^3 - 2*M[3]*m[2]^2*m[3]^2*s + 4*M[3]*m[2]^2*m[3]*m[4]*s - 2*M[3]*m[2]^2*m[4]^2*s +
2*M[3]*m[2]*m[3]^3*s - 2*M[3]*m[2]*m[3]^2*m[4]*s + 4*M[3]*m[2]*m[3]^2*s^2 - 2*M[3]*m[2]*m[3]*m[4]^2*s - 8*M[3]*m[2]*m[3]*m[4]*s^2 + 2*M[3]*m[2]*m[4]^3*s + 4*M[3]*m[2]*m[4]^2*s^2 - 2*M[3]*m[3]^3*s^2 + 2*M[3]*m[3]^2*m[4]*s^2 - 2*M[3]*m[3]^2*s^3
+ 2*M[3]*m[3]*m[4]^2*s^2 + 4*M[3]*m[3]*m[4]*s^3 - 2*M[3]*m[4]^3*s^2 - 2*M[3]*m[4]^2*s^3 + M[4]^4*m[2]^2 - 2*M[4]^3*m[1]*m[2]^2 + 2*M[4]^3*m[1]*m[2]*m[3] + 2*M[4]^3*m[1]*m[2]*m[4] + 2*M[4]^3*m[2]^3 - 2*M[4]^3*m[2]^2*m[3] - 2*M[4]^3*m[2]^2*m[4]
- 2*M[4]^3*m[2]^2*s - 2*M[4]^3*m[2]*m[3]*s - 2*M[4]^3*m[2]*m[4]*s + M[4]^2*m[1]^2*m[2]^2 - 2*M[4]^2*m[1]^2*m[2]*m[3] - 2*M[4]^2*m[1]^2*m[2]*m[4] + M[4]^2*m[1]^2*m[3]^2 - 2*M[4]^2*m[1]^2*m[3]*m[4] + M[4]^2*m[1]^2*m[4]^2 - 2*M[4]^2*m[1]*m[2]^3 +
4*M[4]^2*m[1]*m[2]^2*m[3] + 4*M[4]^2*m[1]*m[2]^2*m[4] + 4*M[4]^2*m[1]*m[2]^2*s - 2*M[4]^2*m[1]*m[2]*m[3]^2 + 4*M[4]^2*m[1]*m[2]*m[3]*m[4] - 2*M[4]^2*m[1]*m[2]*m[3]*s - 2*M[4]^2*m[1]*m[2]*m[4]^2 - 2*M[4]^2*m[1]*m[2]*m[4]*s -
2*M[4]^2*m[1]*m[3]^2*s + 4*M[4]^2*m[1]*m[3]*m[4]*s - 2*M[4]^2*m[1]*m[4]^2*s + M[4]^2*m[2]^4 - 2*M[4]^2*m[2]^3*m[3] - 2*M[4]^2*m[2]^3*m[4] - 2*M[4]^2*m[2]^3*s + M[4]^2*m[2]^2*m[3]^2 - 2*M[4]^2*m[2]^2*m[3]*m[4] - 2*M[4]^2*m[2]^2*m[3]*s +
M[4]^2*m[2]^2*m[4]^2 - 2*M[4]^2*m[2]^2*m[4]*s + M[4]^2*m[2]^2*s^2 + 4*M[4]^2*m[2]*m[3]^2*s + 8*M[4]^2*m[2]*m[3]*m[4]*s + 4*M[4]^2*m[2]*m[3]*s^2 + 4*M[4]^2*m[2]*m[4]^2*s + 4*M[4]^2*m[2]*m[4]*s^2 + M[4]^2*m[3]^2*s^2 - 2*M[4]^2*m[3]*m[4]*s^2 +
M[4]^2*m[4]^2*s^2 - 2*M[4]*m[1]^2*m[2]^2*s + 4*M[4]*m[1]^2*m[2]*m[3]*s + 4*M[4]*m[1]^2*m[2]*m[4]*s - 2*M[4]*m[1]^2*m[3]^2*s + 4*M[4]*m[1]^2*m[3]*m[4]*s - 2*M[4]*m[1]^2*m[4]^2*s + 2*M[4]*m[1]*m[2]^3*s - 2*M[4]*m[1]*m[2]^2*m[3]*s -
2*M[4]*m[1]*m[2]^2*m[4]*s - 2*M[4]*m[1]*m[2]^2*s^2 - 2*M[4]*m[1]*m[2]*m[3]^2*s - 12*M[4]*m[1]*m[2]*m[3]*m[4]*s - 2*M[4]*m[1]*m[2]*m[3]*s^2 - 2*M[4]*m[1]*m[2]*m[4]^2*s - 2*M[4]*m[1]*m[2]*m[4]*s^2 + 2*M[4]*m[1]*m[3]^3*s -
2*M[4]*m[1]*m[3]^2*m[4]*s + 4*M[4]*m[1]*m[3]^2*s^2 - 2*M[4]*m[1]*m[3]*m[4]^2*s - 8*M[4]*m[1]*m[3]*m[4]*s^2 + 2*M[4]*m[1]*m[4]^3*s + 4*M[4]*m[1]*m[4]^2*s^2 - 2*M[4]*m[2]^3*m[3]*s - 2*M[4]*m[2]^3*m[4]*s + 4*M[4]*m[2]^2*m[3]^2*s +
8*M[4]*m[2]^2*m[3]*m[4]*s + 4*M[4]*m[2]^2*m[3]*s^2 + 4*M[4]*m[2]^2*m[4]^2*s + 4*M[4]*m[2]^2*m[4]*s^2 - 2*M[4]*m[2]*m[3]^3*s + 2*M[4]*m[2]*m[3]^2*m[4]*s - 2*M[4]*m[2]*m[3]^2*s^2 + 2*M[4]*m[2]*m[3]*m[4]^2*s - 12*M[4]*m[2]*m[3]*m[4]*s^2 -
2*M[4]*m[2]*m[3]*s^3 - 2*M[4]*m[2]*m[4]^3*s - 2*M[4]*m[2]*m[4]^2*s^2 - 2*M[4]*m[2]*m[4]*s^3 - 2*M[4]*m[3]^3*s^2 + 2*M[4]*m[3]^2*m[4]*s^2 - 2*M[4]*m[3]^2*s^3 + 2*M[4]*m[3]*m[4]^2*s^2 + 4*M[4]*m[3]*m[4]*s^3 - 2*M[4]*m[4]^3*s^2 -
2*M[4]*m[4]^2*s^3 + m[1]^2*m[2]^2*s^2 - 2*m[1]^2*m[2]*m[3]*s^2 - 2*m[1]^2*m[2]*m[4]*s^2 + m[1]^2*m[3]^2*s^2 - 2*m[1]^2*m[3]*m[4]*s^2 + m[1]^2*m[4]^2*s^2 - 2*m[1]*m[2]^2*m[3]*s^2 - 2*m[1]*m[2]^2*m[4]*s^2 + 4*m[1]*m[2]*m[3]^2*s^2 +
8*m[1]*m[2]*m[3]*m[4]*s^2 + 2*m[1]*m[2]*m[3]*s^3 + 4*m[1]*m[2]*m[4]^2*s^2 + 2*m[1]*m[2]*m[4]*s^3 - 2*m[1]*m[3]^3*s^2 + 2*m[1]*m[3]^2*m[4]*s^2 - 2*m[1]*m[3]^2*s^3 + 2*m[1]*m[3]*m[4]^2*s^2 + 4*m[1]*m[3]*m[4]*s^3 - 2*m[1]*m[4]^3*s^2 -
2*m[1]*m[4]^2*s^3 + m[2]^2*m[3]^2*s^2 - 2*m[2]^2*m[3]*m[4]*s^2 + m[2]^2*m[4]^2*s^2 - 2*m[2]*m[3]^3*s^2 + 2*m[2]*m[3]^2*m[4]*s^2 - 2*m[2]*m[3]^2*s^3 + 2*m[2]*m[3]*m[4]^2*s^2 + 4*m[2]*m[3]*m[4]*s^3 - 2*m[2]*m[4]^3*s^2 - 2*m[2]*m[4]^2*s^3 +
m[3]^4*s^2 - 4*m[3]^3*m[4]*s^2 + 2*m[3]^3*s^3 + 6*m[3]^2*m[4]^2*s^2 - 2*m[3]^2*m[4]*s^3 + m[3]^2*s^4 - 4*m[3]*m[4]^3*s^2 - 2*m[3]*m[4]^2*s^3 - 2*m[3]*m[4]*s^4 + m[4]^4*s^2 + 2*m[4]^3*s^3 + m[4]^2*s^4
Ï‡[5] = 18
weights[5] = [[-1, -1, -1, -1]]
computed_with[5] = ["PLD_num", "HyperInt"]
```

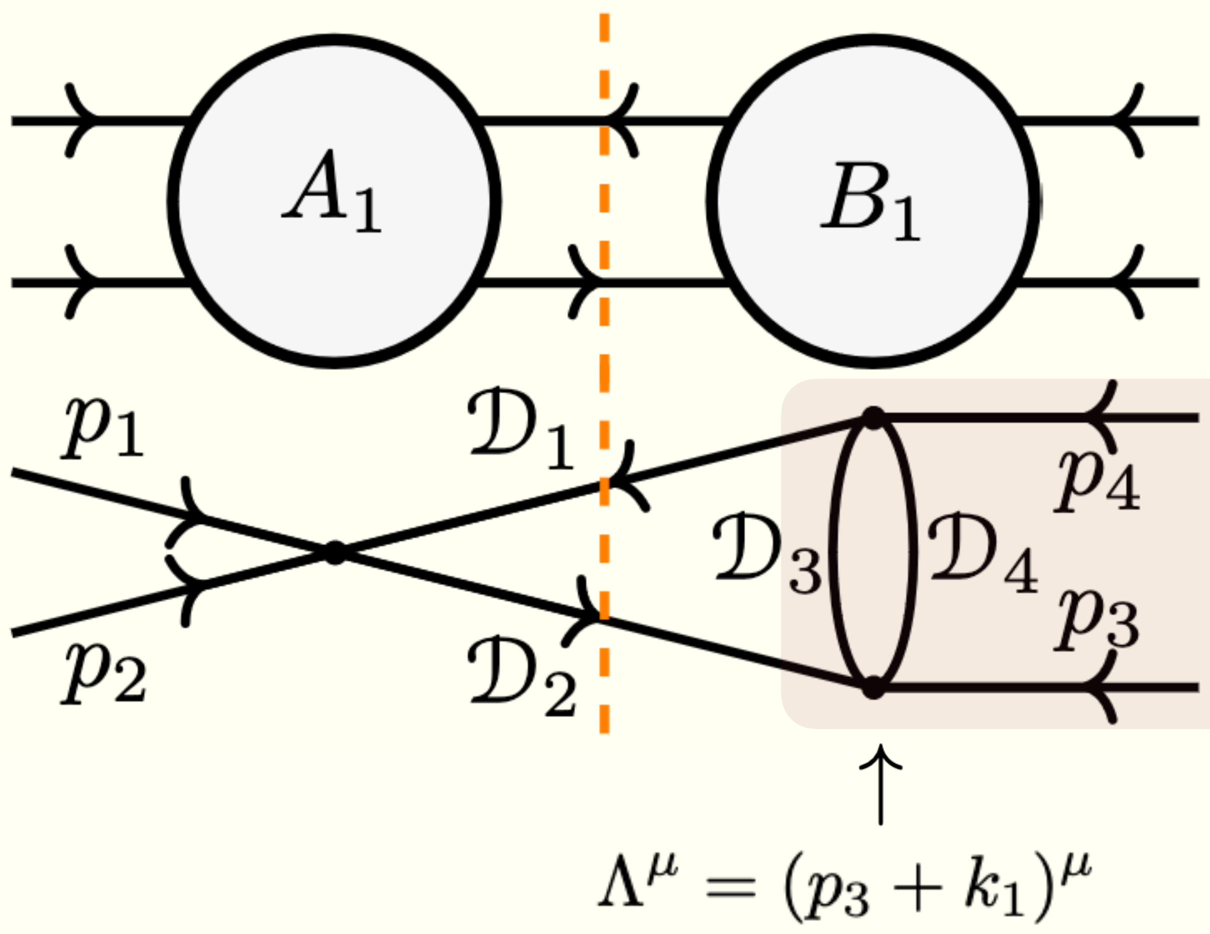
What about other singularities ?

On the previous slide, we localized G_1 on the bubble *leading* singularity

$\mathcal{S} = \{\mathcal{L}(B_1)_1 = 0\}$ fixed the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4)^2 - m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

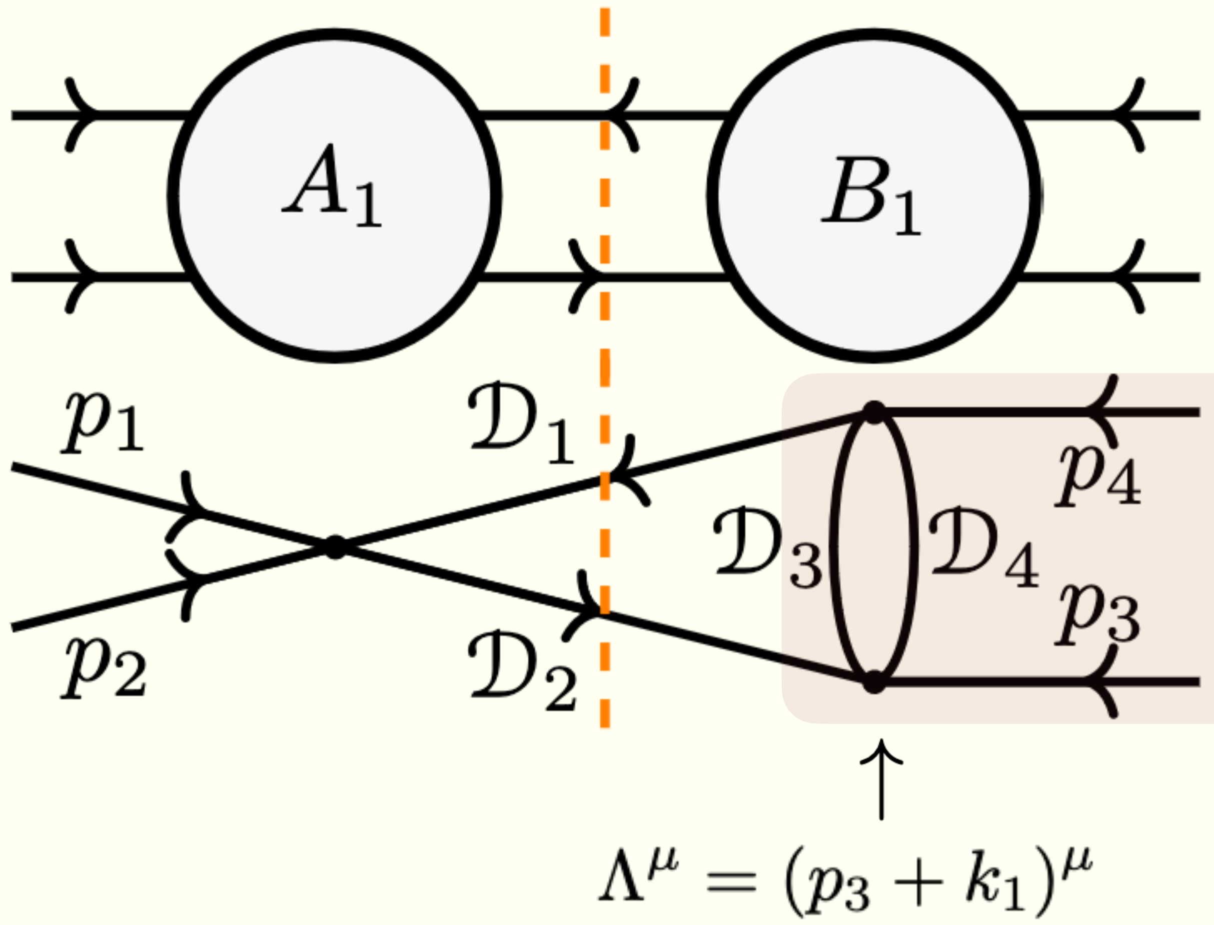
What about other singularities ?

But nothing stops us to localize on *other* singularities of B_1 (e.g., second-type singularity at $\Lambda^2 = 0$)

$\mathcal{S} = \{\mathcal{L}(B_1)_2 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[-m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

$$\begin{vmatrix} s & \frac{m_2^2-m_1^2+s}{2} & \frac{M_4^2-M_3^2-s}{2} \\ \frac{m_2^2-m_1^2+s}{2} & m_2^2 & -\frac{m_2^2+M_3^2}{2} \\ \frac{M_4^2-M_3^2-s}{2} & -\frac{m_2^2+M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

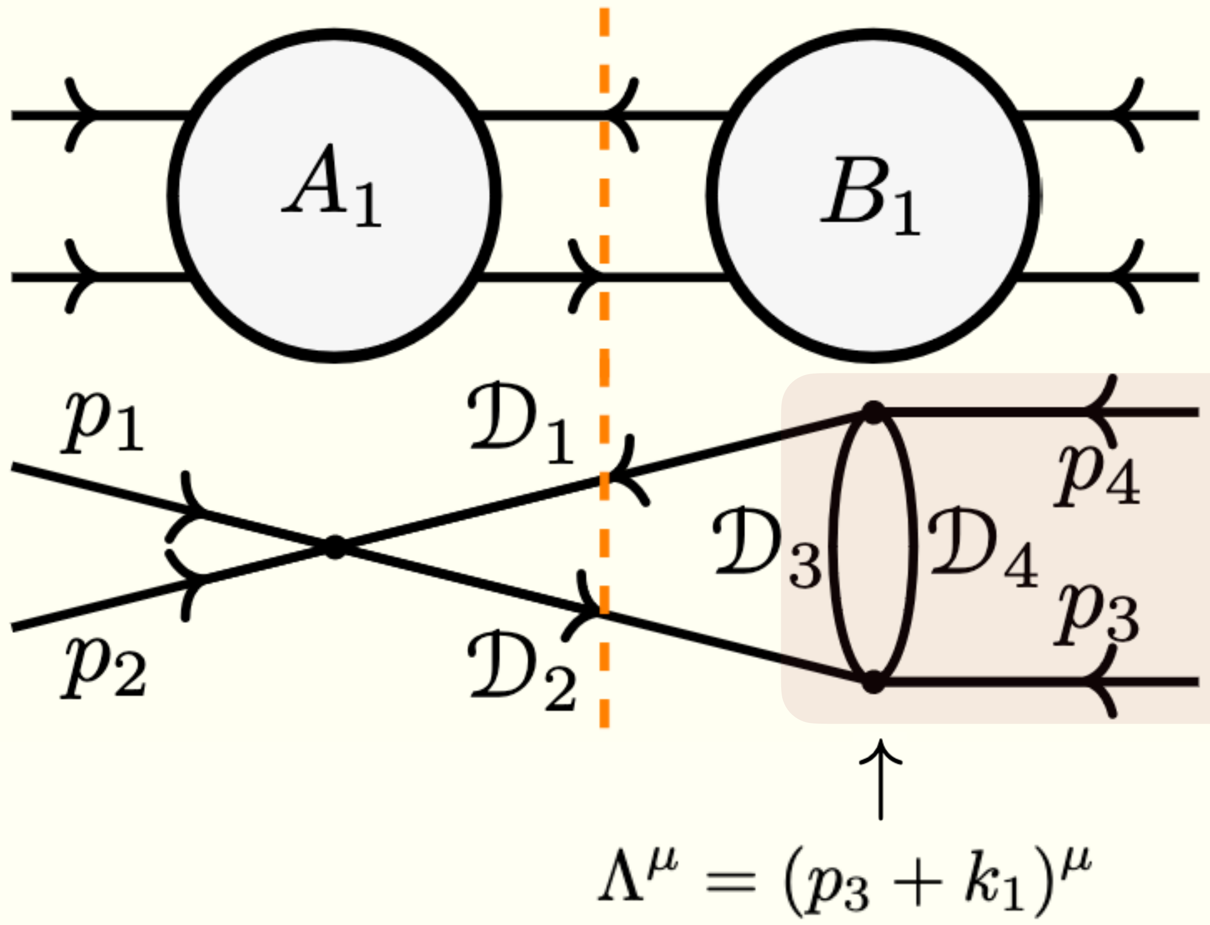
What about other singularities ?

But nothing stops us to localize on *other* singularities of B_1 (e.g., second-type singularity at $\Lambda^2 = 0$)

$\mathcal{S} = \{\mathcal{L}(B_1)_2 = 0\}$ fixes the remaining invariant:

$$k_1 \cdot p_3 = \frac{1}{2}\left[-m_2^2 - M_3^2\right]$$

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$



$$\overset{D=3}{=} \frac{\sqrt{\pi}}{\sqrt{\Lambda^2}} \log\left[\frac{m_3 + m_4 + \sqrt{\Lambda^2}}{m_3 + m_4 - \sqrt{\Lambda^2}}\right]$$

```
##################################
# Component 3
##################################

D[3] = M[3]^2*m[1] - M[3]*M[4]*m[1] - M[3]*M[4]*m[2] + M[3]*M[4]*s + M[3]*m[1]^2 - M[3]*m[1]*m[2] - M[3]*m[1]*s + M[4]^2*m[2] -
M[4]*m[1]*m[2] + M[4]*m[2]^2 - M[4]*m[2]*s + m[1]*m[2]*s
χ[3] = 18
weights[3] = []
computed_with[3] = ["HyperInt"]
```

What about other singularities ?

Same phenomenon captures subtle singularities found in state-of-the-art amplitude computations

[Submitted on 9 Aug 2024 (v1), last revised 6 Nov 2024 (this version, v3)]

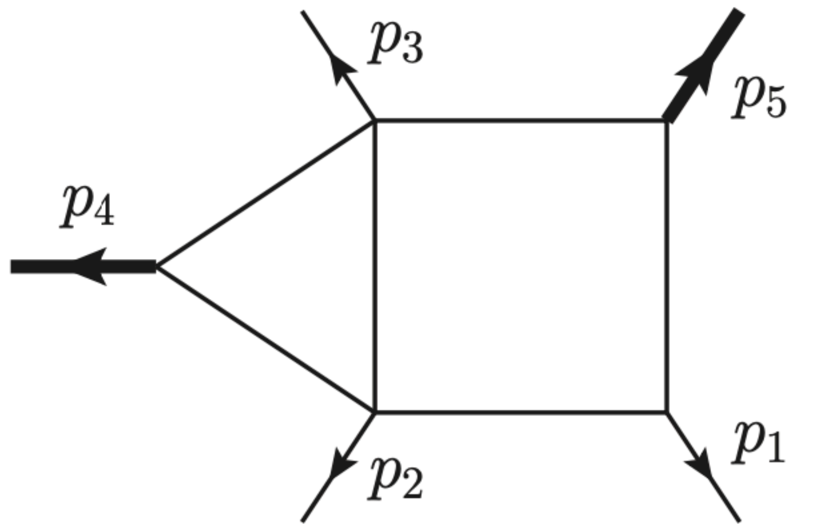
Two-Loop Five-Point Two-Mass Planar Integrals and Double Lagrangian Insertions in a Wilson Loop

Samuel Abreu, Dmitry Chicherin, Vasily Sotnikov, Simone Zoia

and it can appear in 6 permutations $r_2^{(i)}$, $i = 1, \ldots, 6$. The fourth root appears as the leading singularity of the integral in fig. 3c with unit numerator, its argument is

$$r_3^{(1)} = 4 s_4 s_{12} (s_5 - s_{15}) s_{15} + (s_5(s_{23} + s_{34}) - s_{15}(s_{34} + s_{45}))^2 \,, \tag{3.18}$$

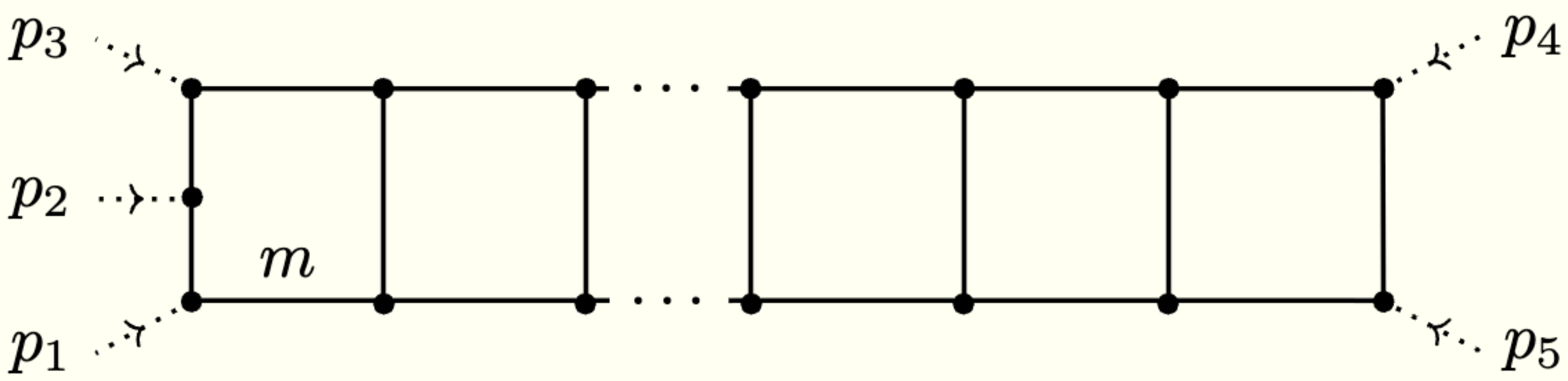
and it can appear in 12 permutations $r_3^{(i)}$, $i = 1, \ldots, 12$. This square root can be computed in a very similar way as the Σ_5 square root was computed in [31]. As mentioned previously, it is missed by the `Baikovletter` code. It is however captured by the recursive Landau approach of [16]. The package `PLD.jl` [9] also detects it when computing Euler discriminants, but fails to detect it when computing principal Landau discriminants.[3] Finally, we also find the square-root of the five-point



(c) $r_3^{(1)}$ of eq. (3.18).

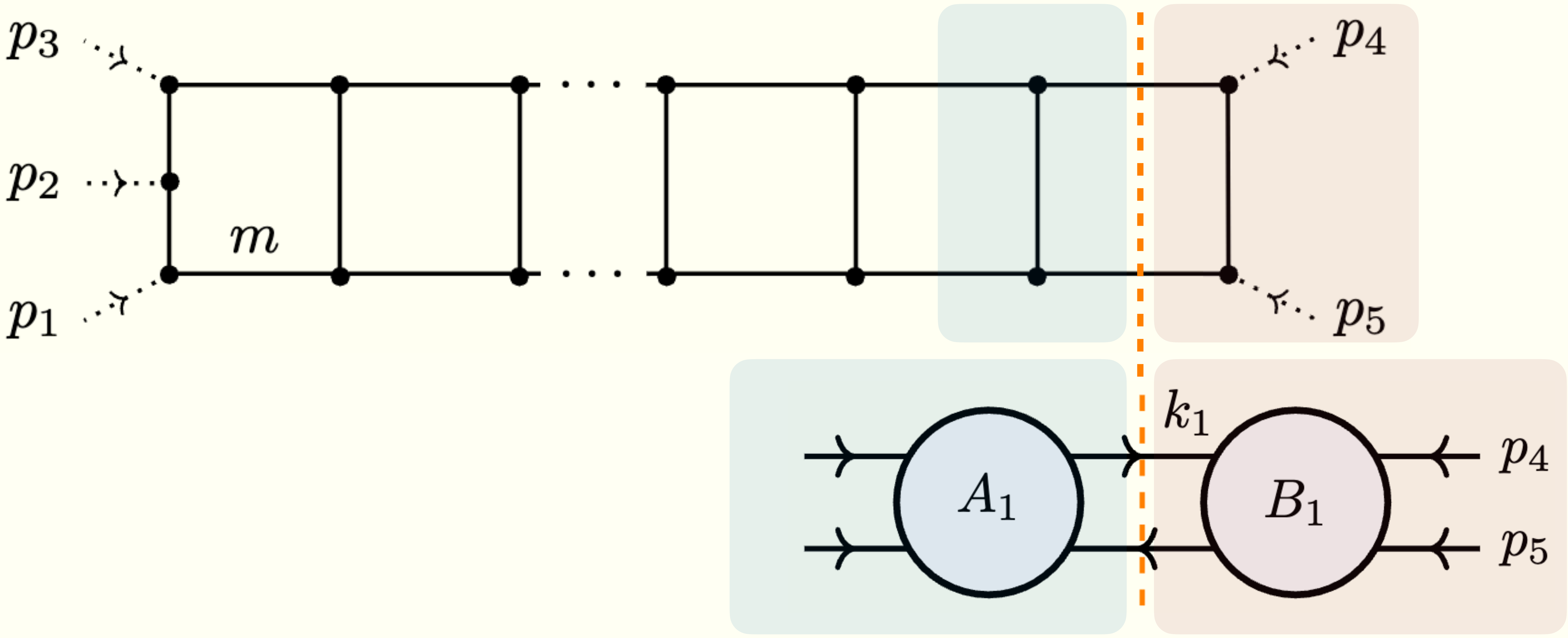
L-loop results

The massive penta-ladder



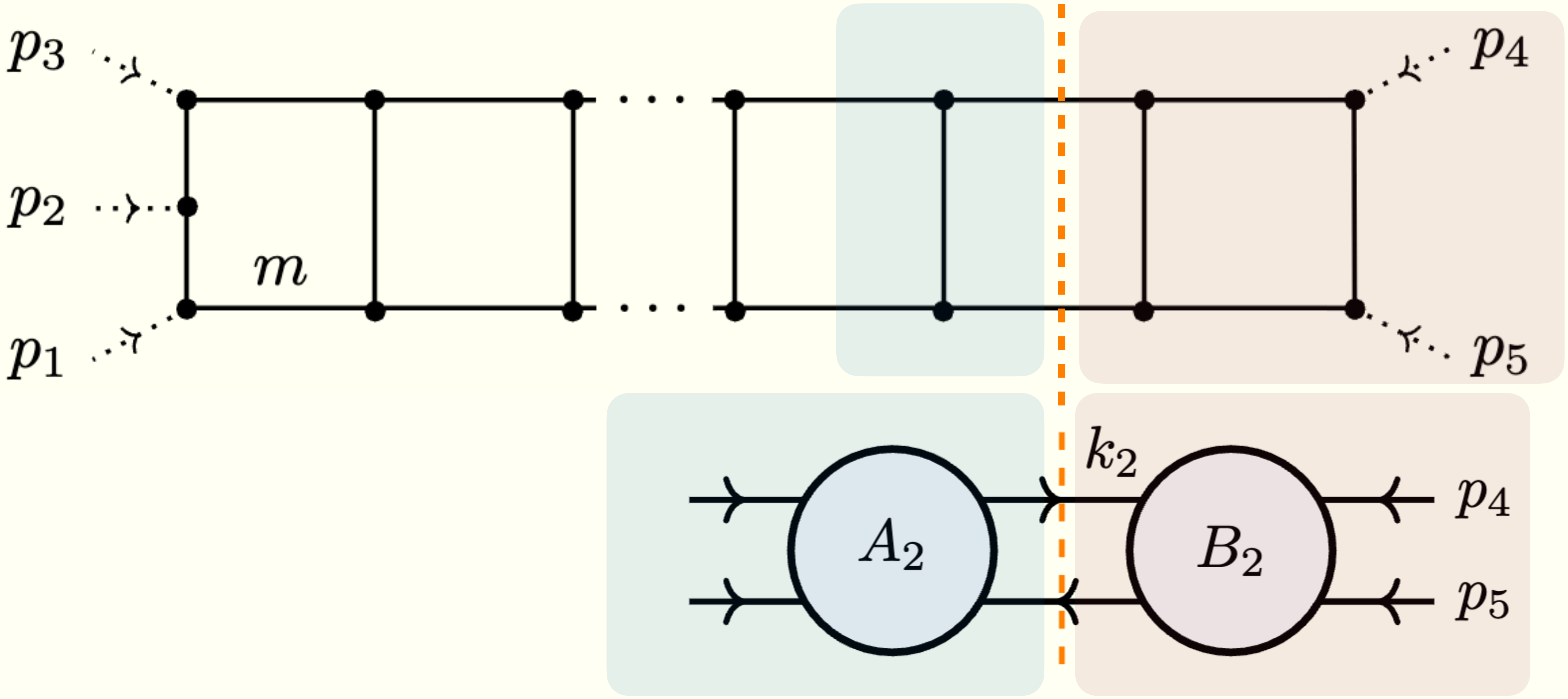
L-LOOP RESULTS

The massive penta-ladder



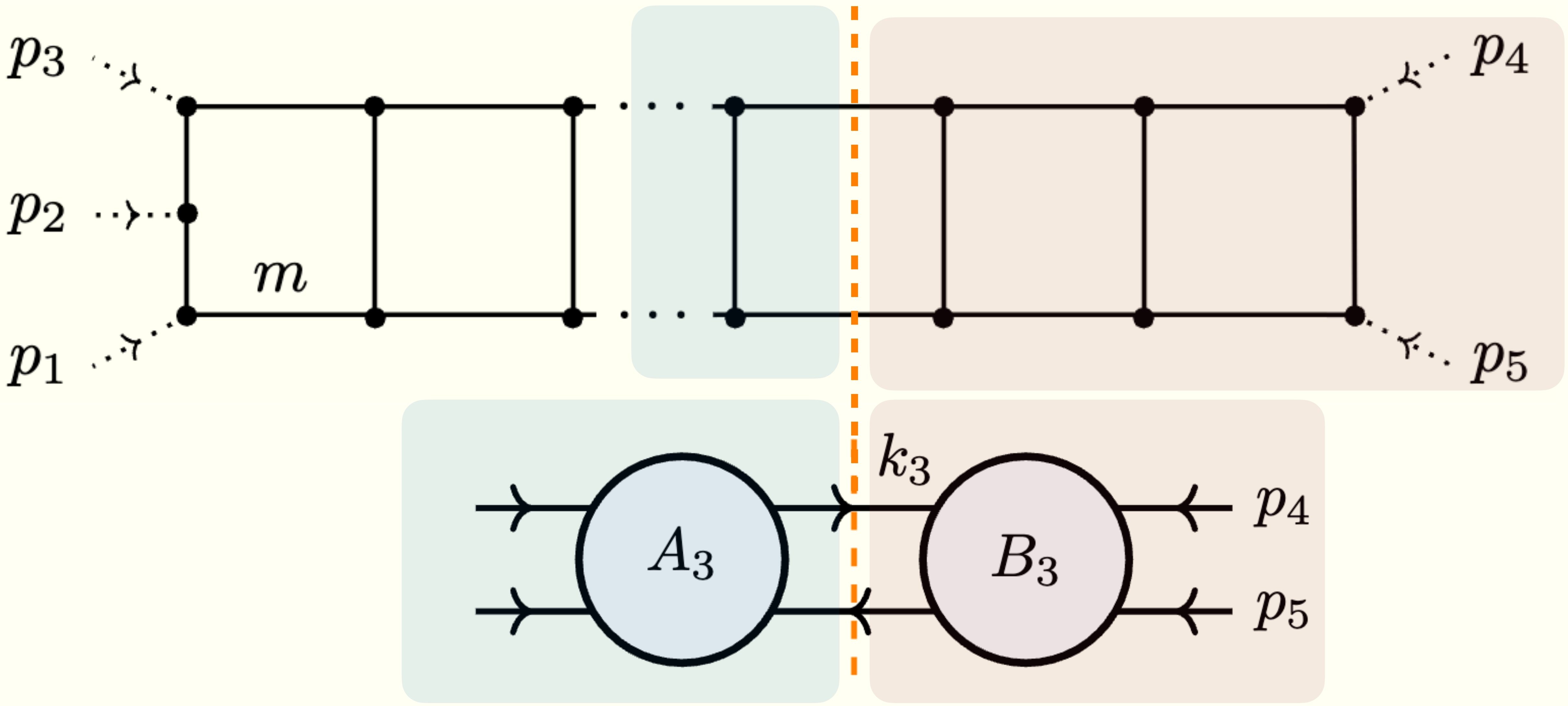
L-LOOP RESULTS

The massive penta-ladder



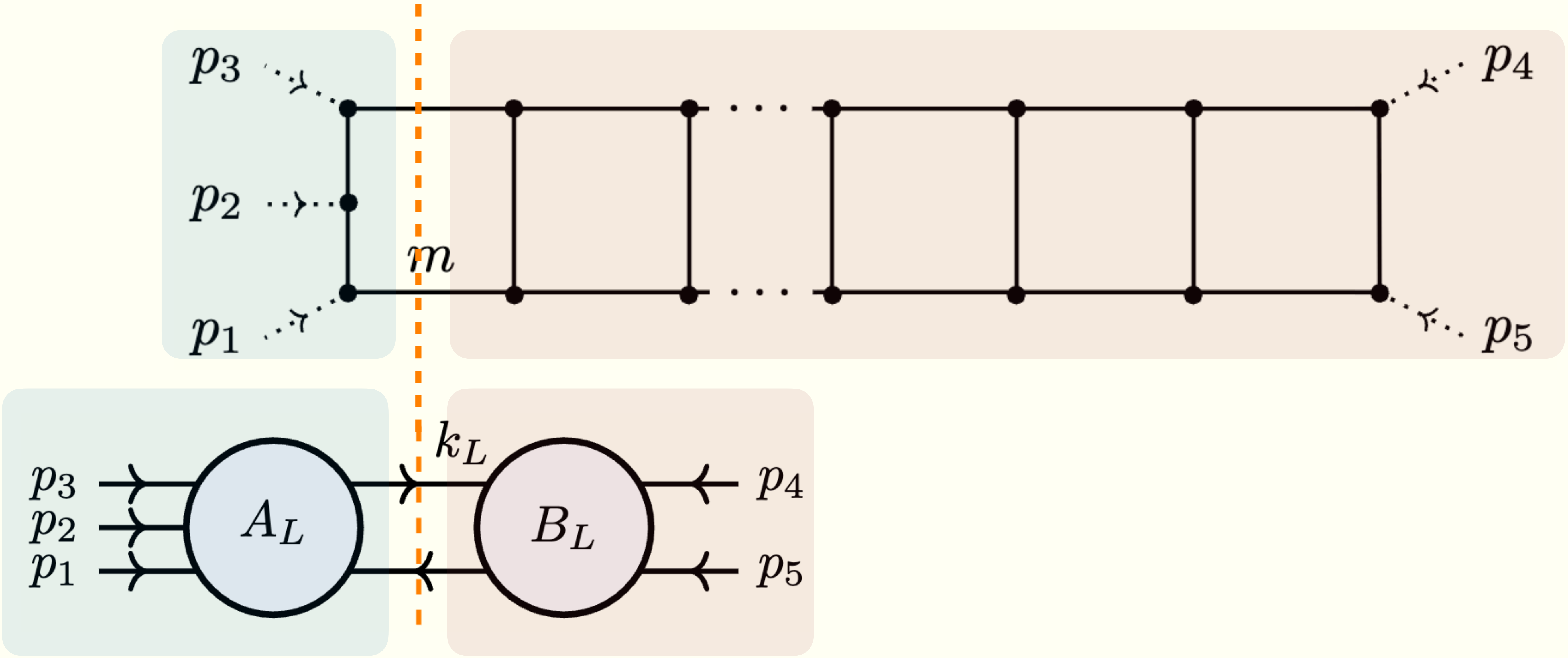
L-LOOP RESULTS

The massive penta-ladder



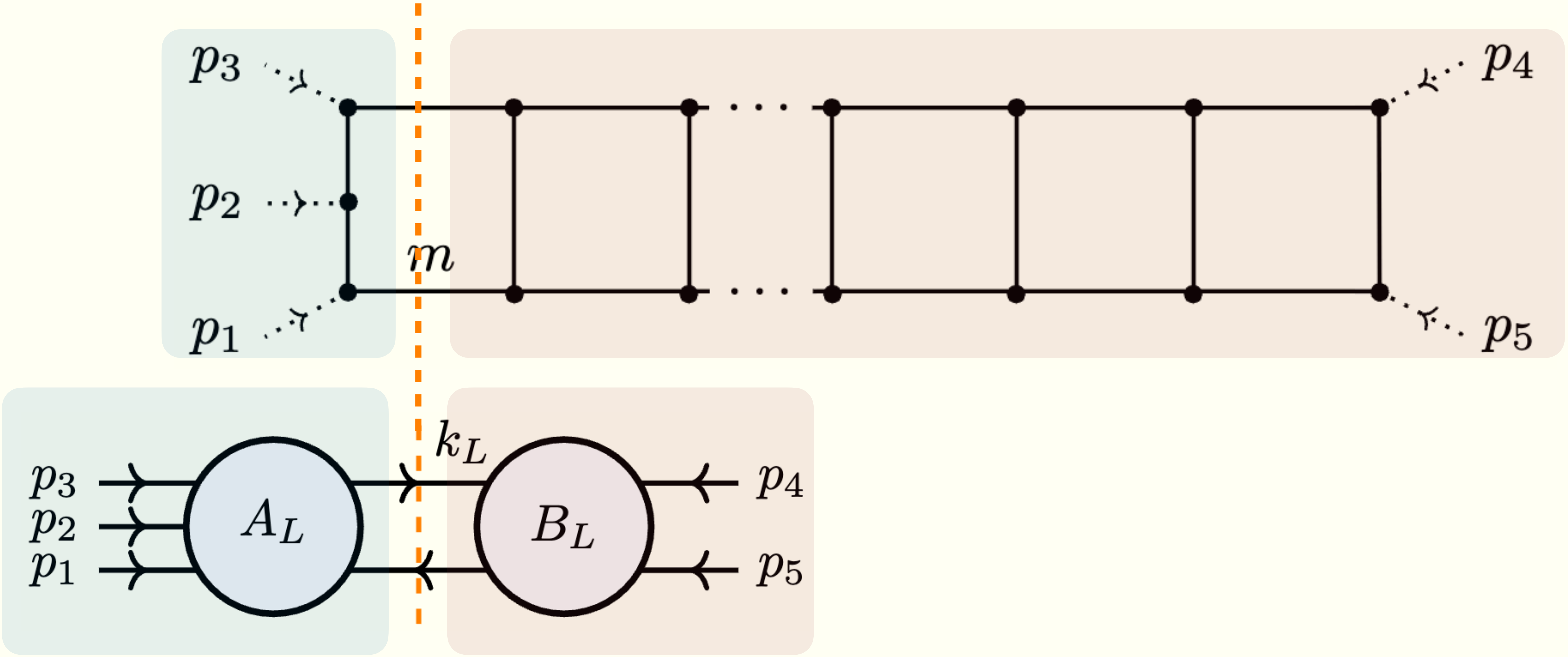
L-LOOP RESULTS

The massive penta-ladder



L-LOOP RESULTS

The massive penta-ladder



The leading singularity of the L-loop penta-ladder is the same as for the ladder when t is replaced by

$$\lambda(Z_{m,m,m,m})^{L-1}\lambda(Z_{m,0,0,m}) - \lambda(Z_{m,0,0,\sqrt{t}}) = 0$$

$$\left[\begin{array}{l} \lambda(z) = z + \sqrt{z^2-1} \\ Z_{a,b,c,d} = \dfrac{\sqrt{s_{45}}(s_{45}+2d^2-2a^2-b^2-c^2)}{\sqrt{s_{45}-4a^2}\sqrt{s_{45}-(b+c)^2}\sqrt{s_{45}-(b-c)^2}} \end{array}\right.$$

[Correia, Sever, Zhiboedov (2020)]

$$\begin{aligned} &m^4 s_{12}s_{23}(s_{12}+s_{23}-s_{45})+s_{12}s_{23}\big[t^2(s_{12}+s_{23}-s_{45}) \\ &-s_{15}s_{34}s_{45}+t(s_{12}(s_{23}-s_{15}) - s_{23}s_{34}+(s_{15}+s_{34})s_{45})\big] \\ &+m^2\big[s_{12}^2(s_{15}^2-2ts_{23}-s_{15}s_{23})+(s_{23}s_{34}+(s_{15}-s_{34})s_{45})^2 \\ &+s_{12}(s_{23}s_{34}(s_{45}-s_{23})-2ts_{23}(s_{23}-s_{45})-2s_{15}^2 s_{45} \\ &+s_{15}(2s_{34}s_{45}+s_{23}(2s_{34}+s_{45})))\big] = 0 \end{aligned}$$

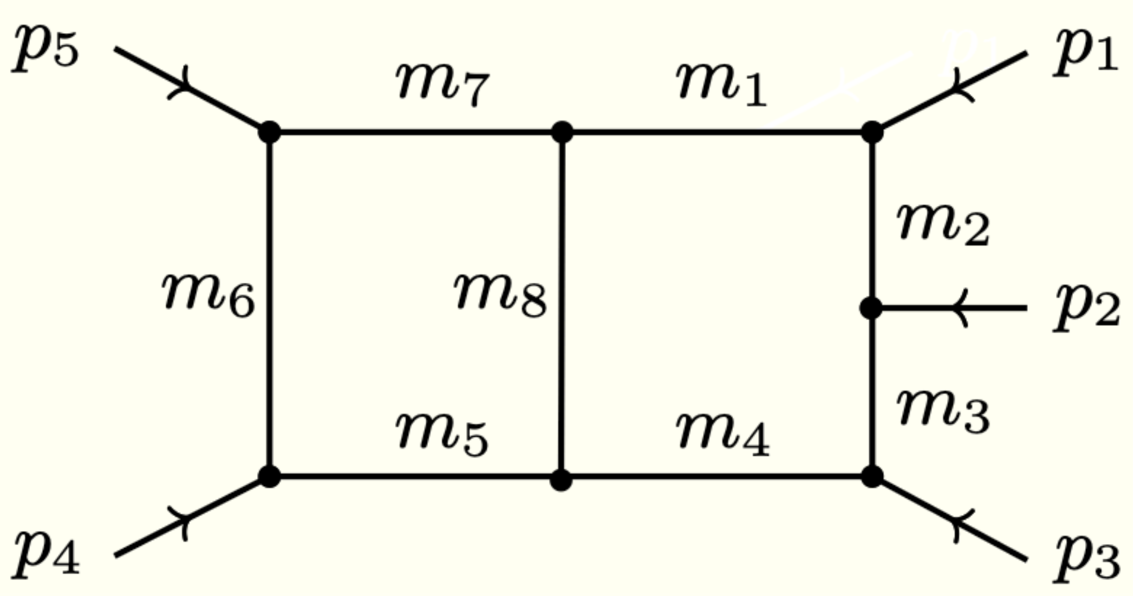
[Caron-Huot, Correia, Giroux (2024)]

Outline

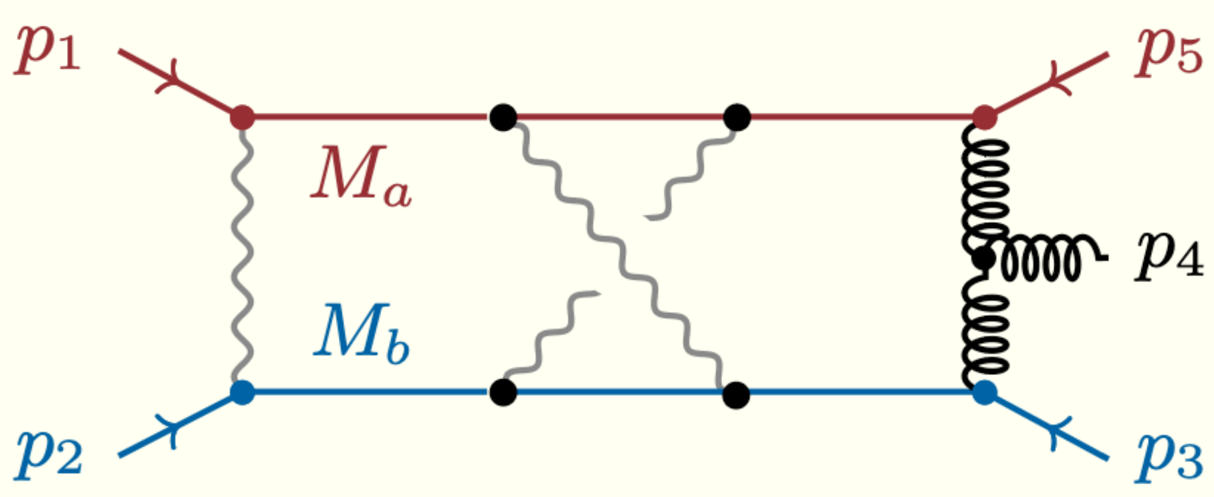
Checks and new analytic predictions:

Leading singularities

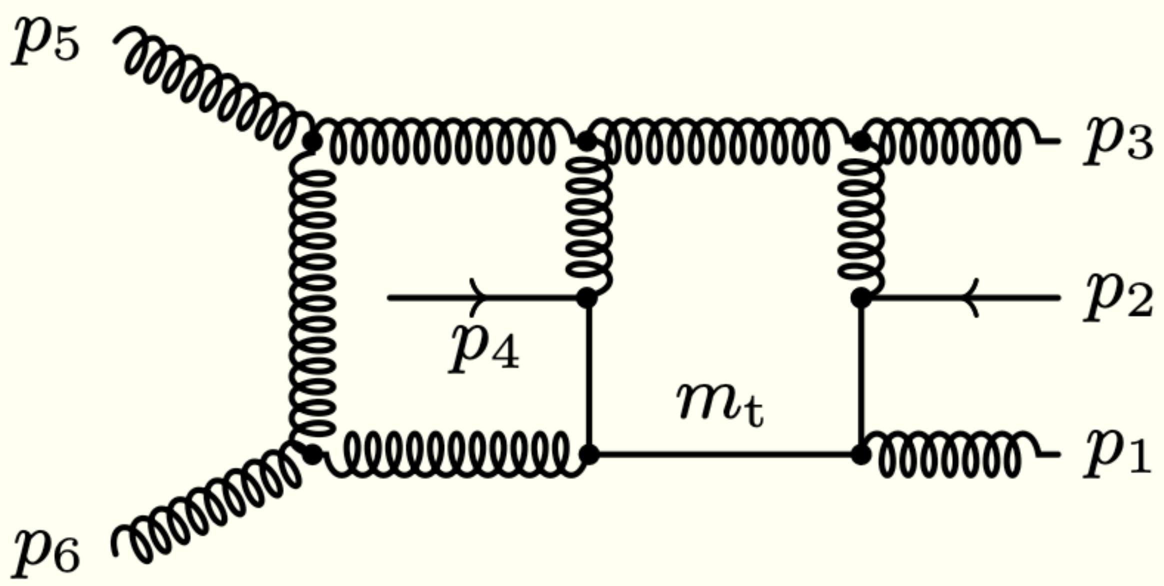
(Generic kinematic pentabox)



(Three-loop QED+QCD boX)



(Non-planar massive hexabox)



Explicit checks

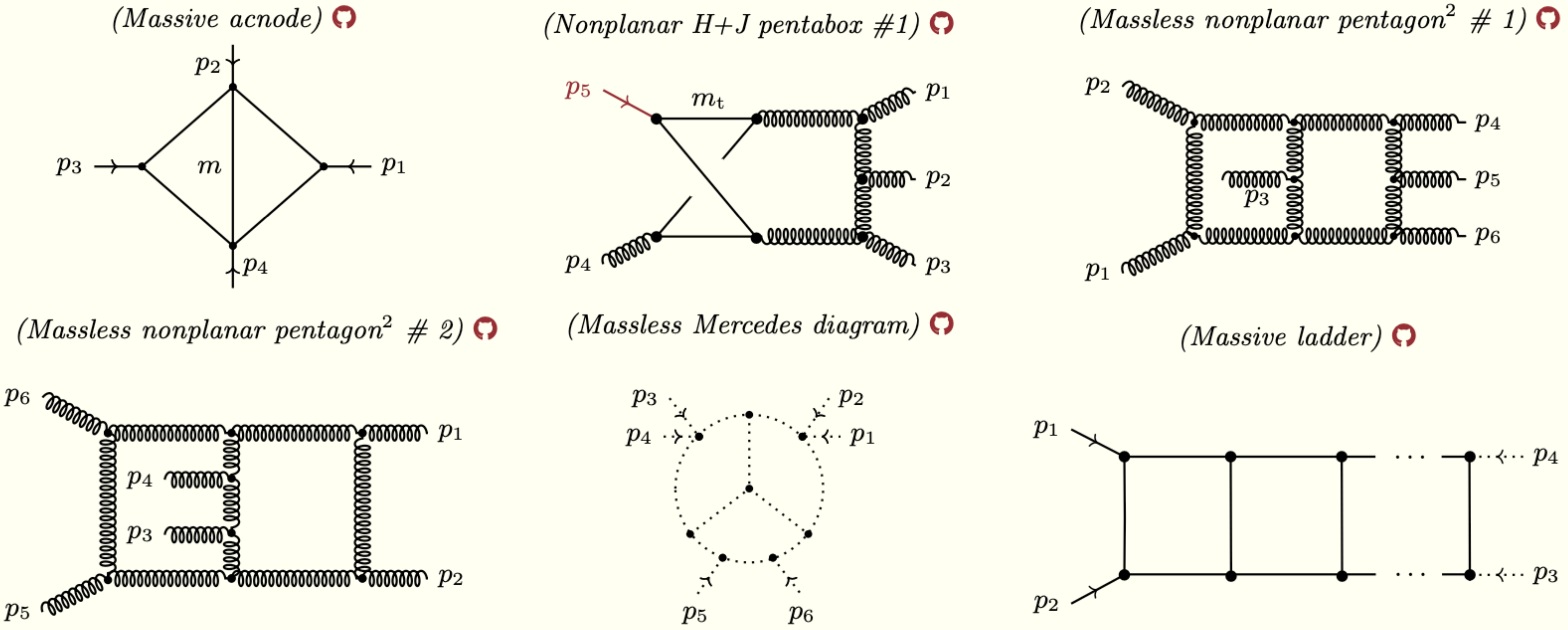


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

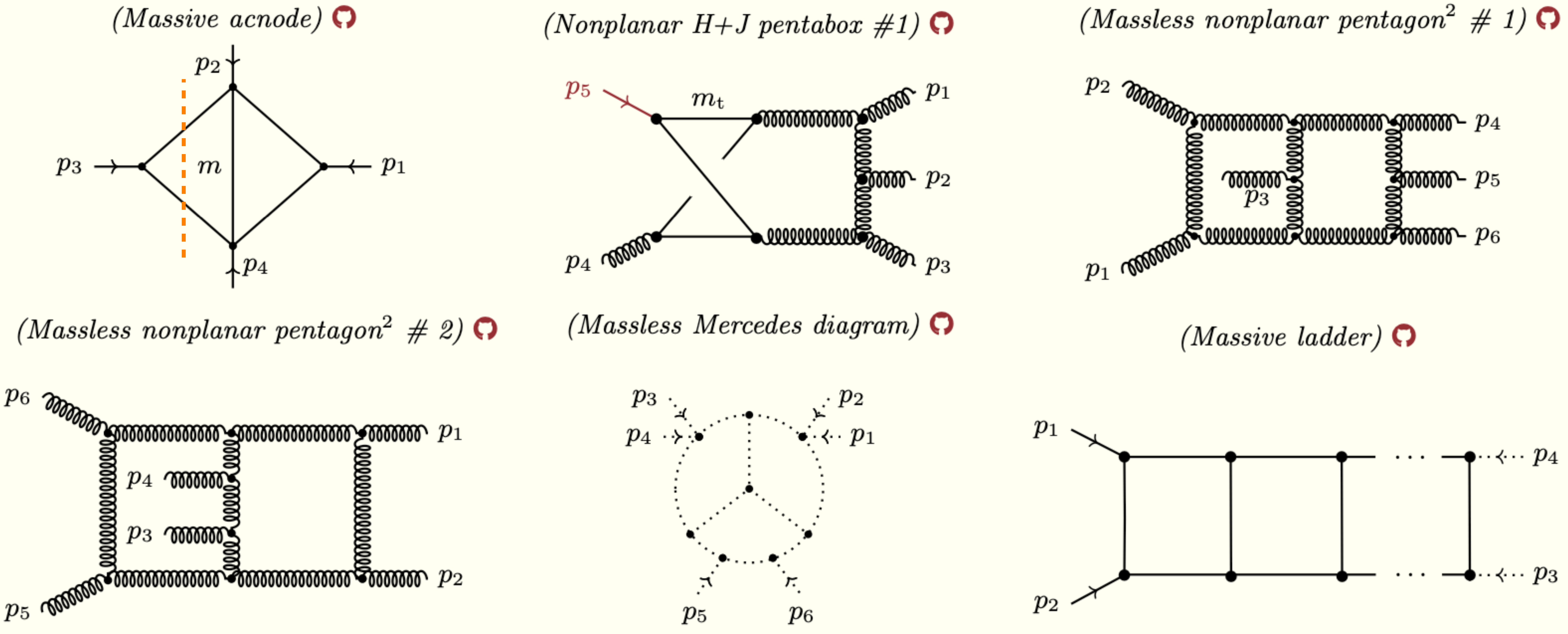


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

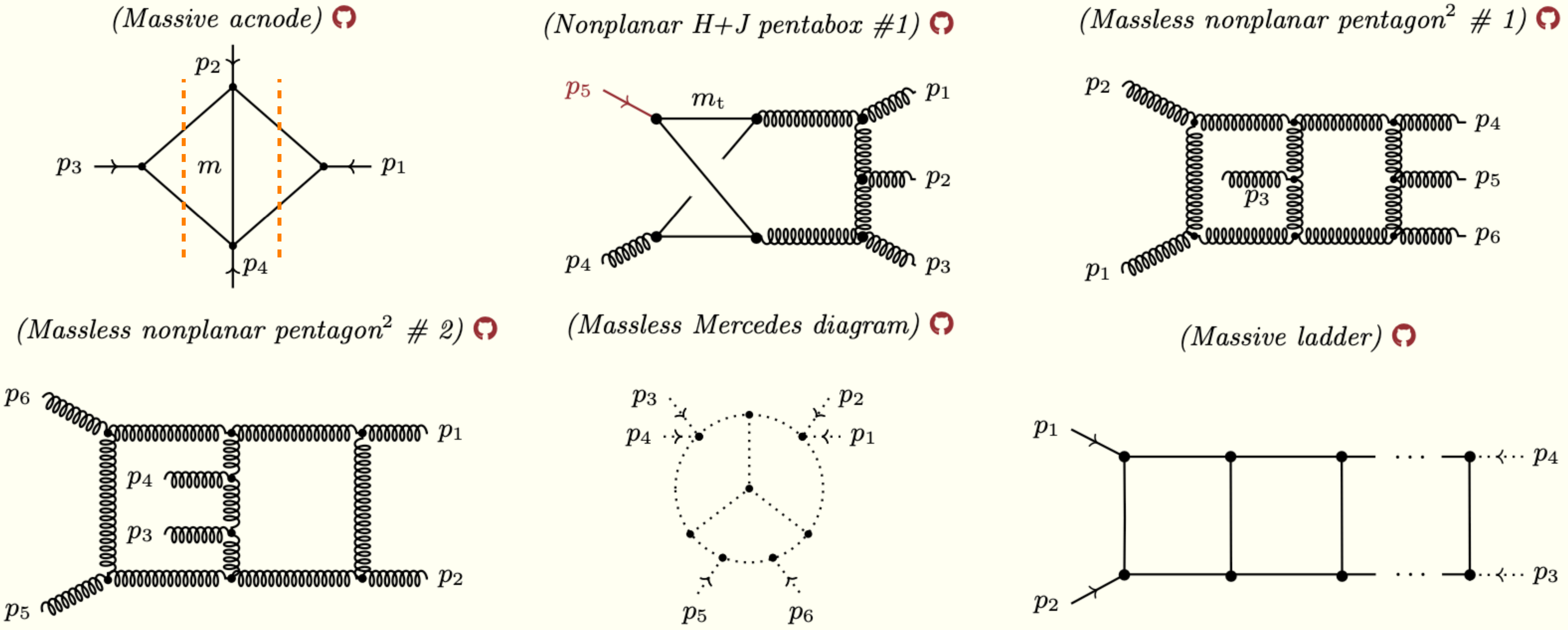


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

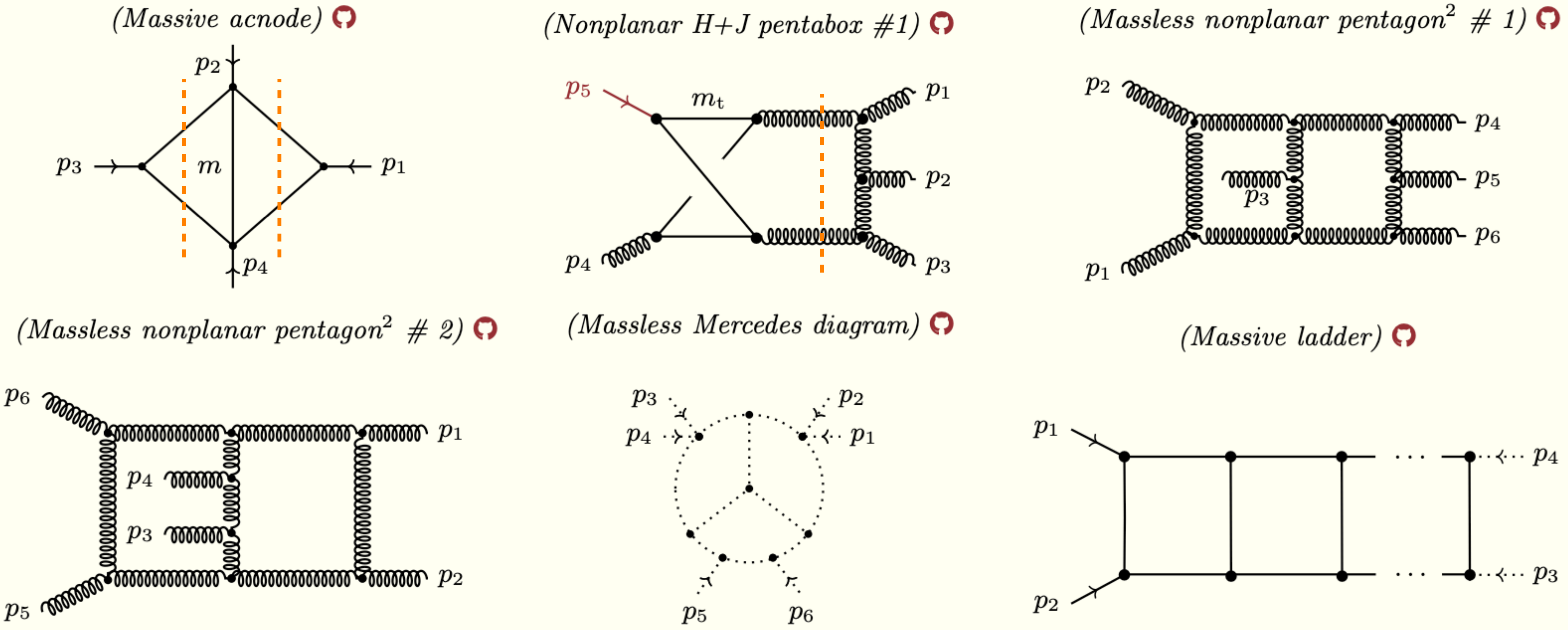


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

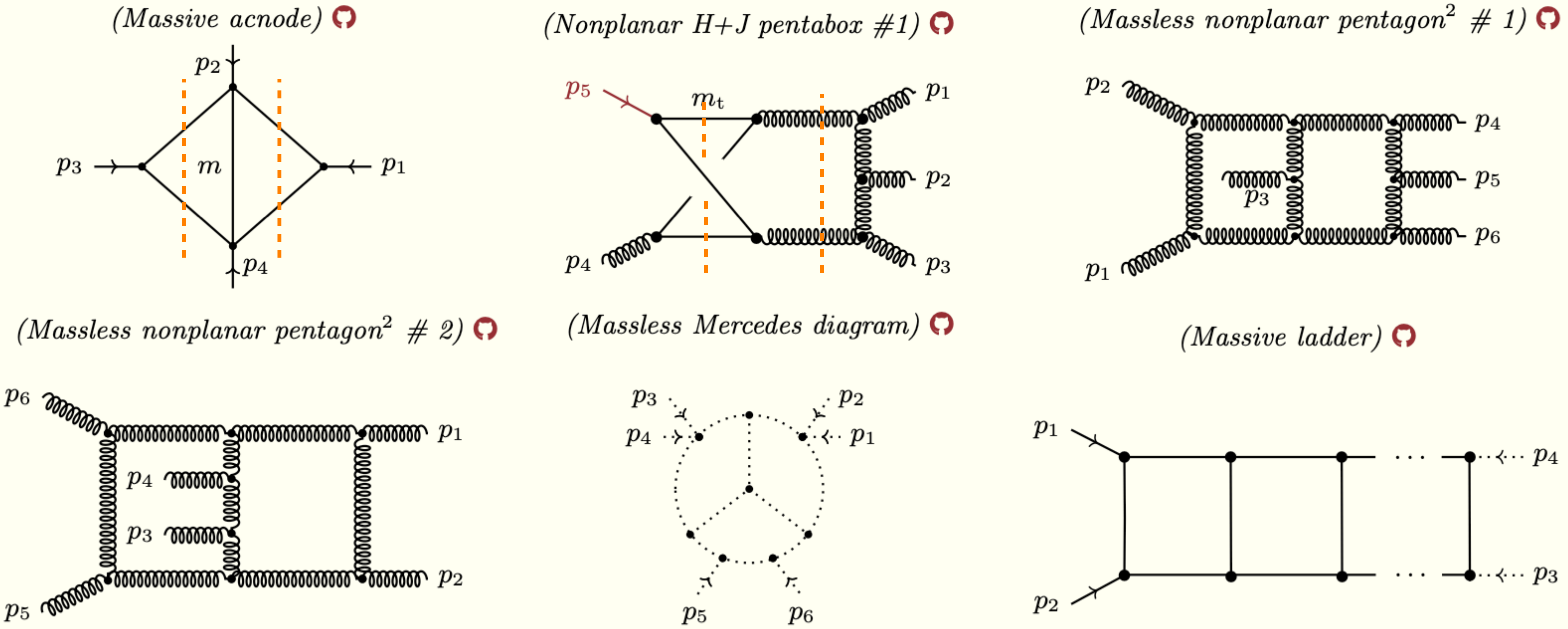


Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

Explicit checks

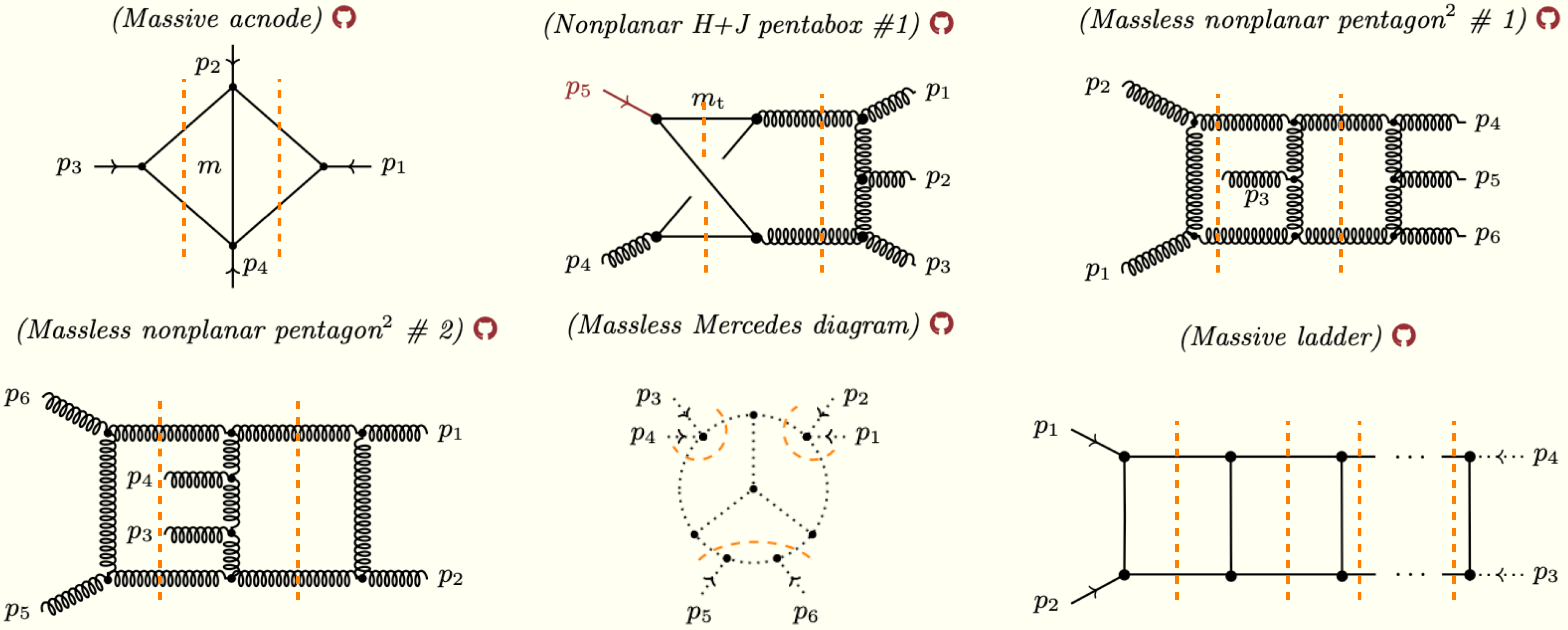
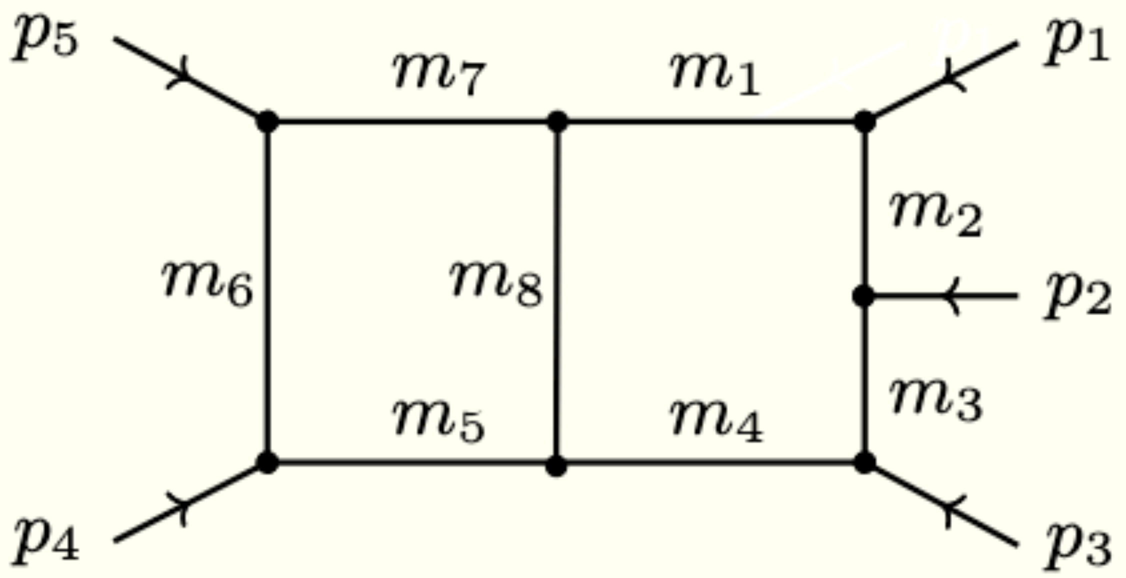


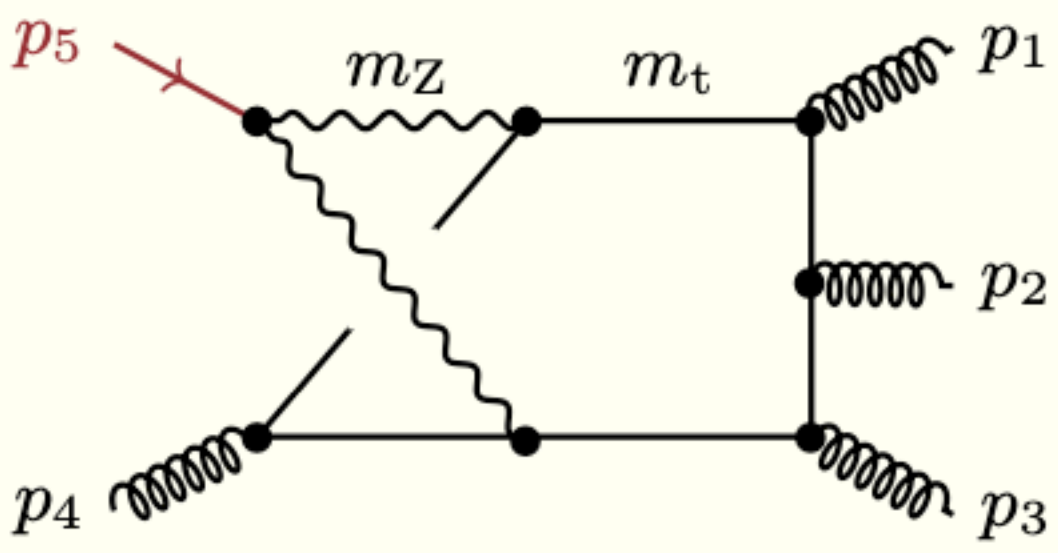
Figure 2. A list of nontrivial examples checked against `PLD.jl` and [19] (for the massive ladder).

New predictions

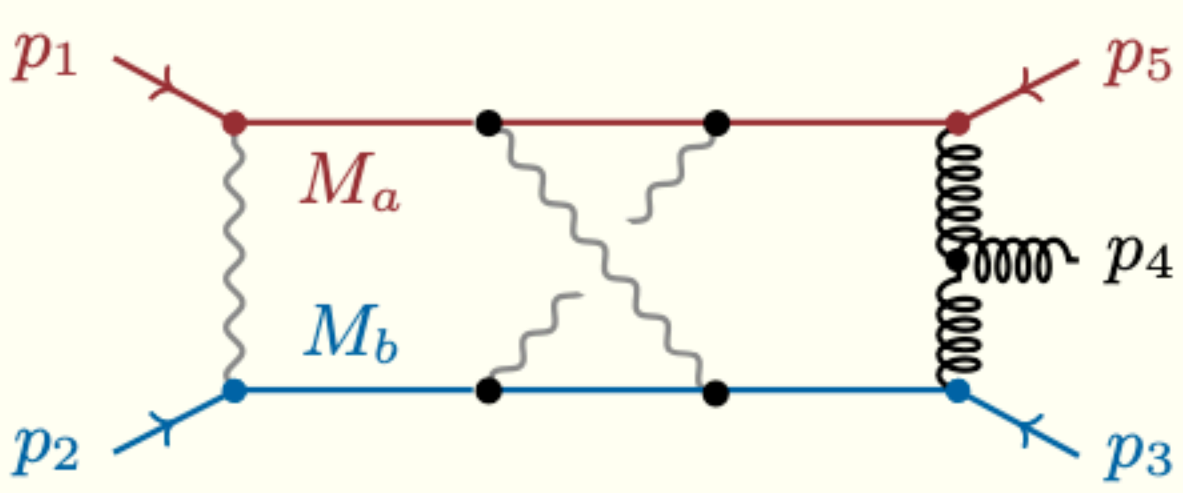
(Generic kinematic pentabox)



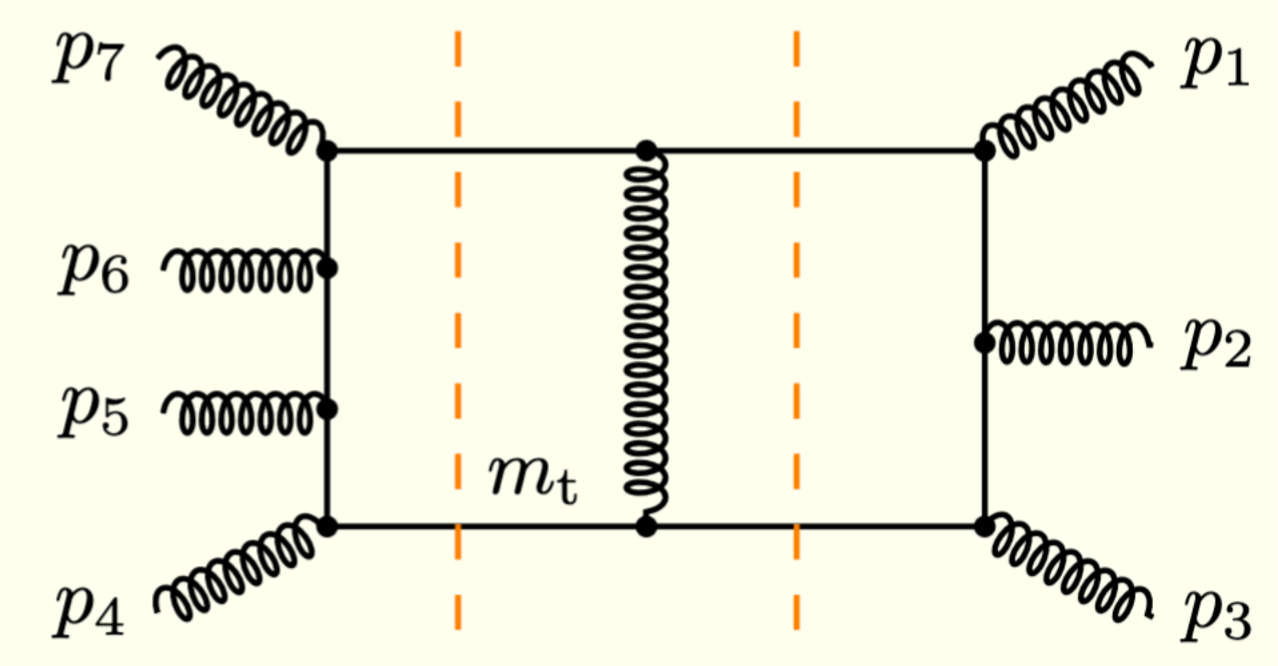
(Nonplanar H+J pentabox #2)



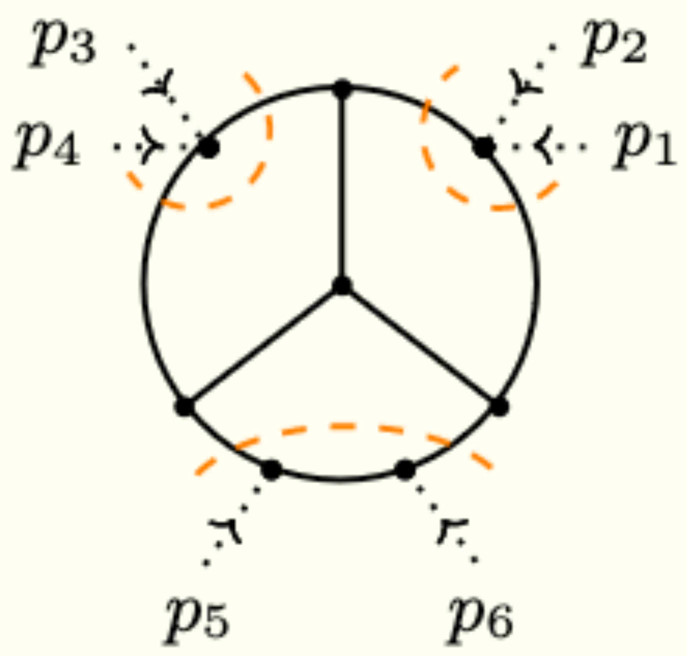
(Three-loop QED+QCD boX)



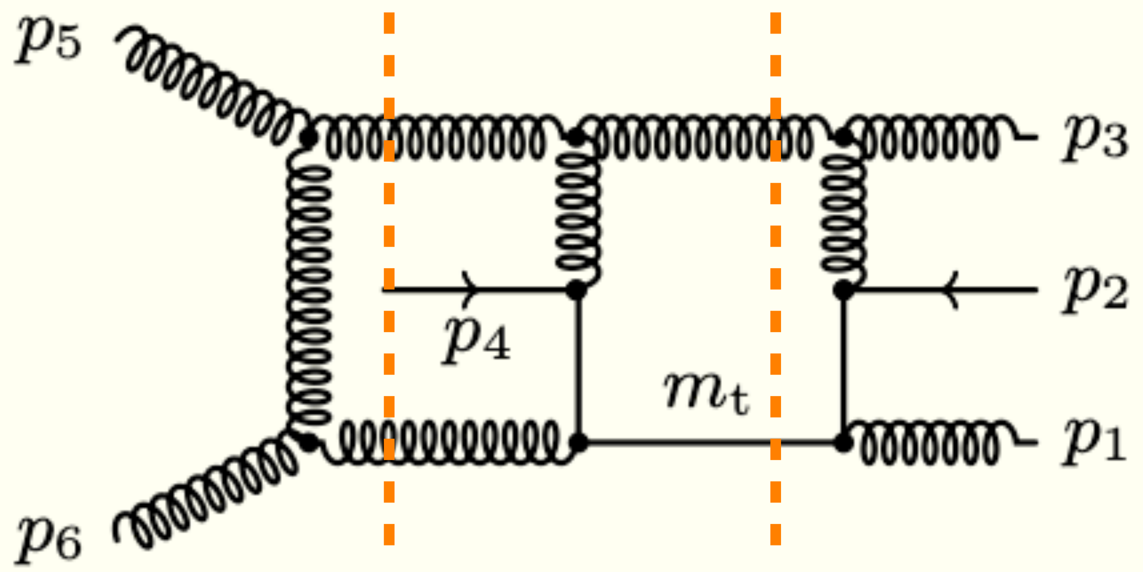
(Massless hexapentagon)



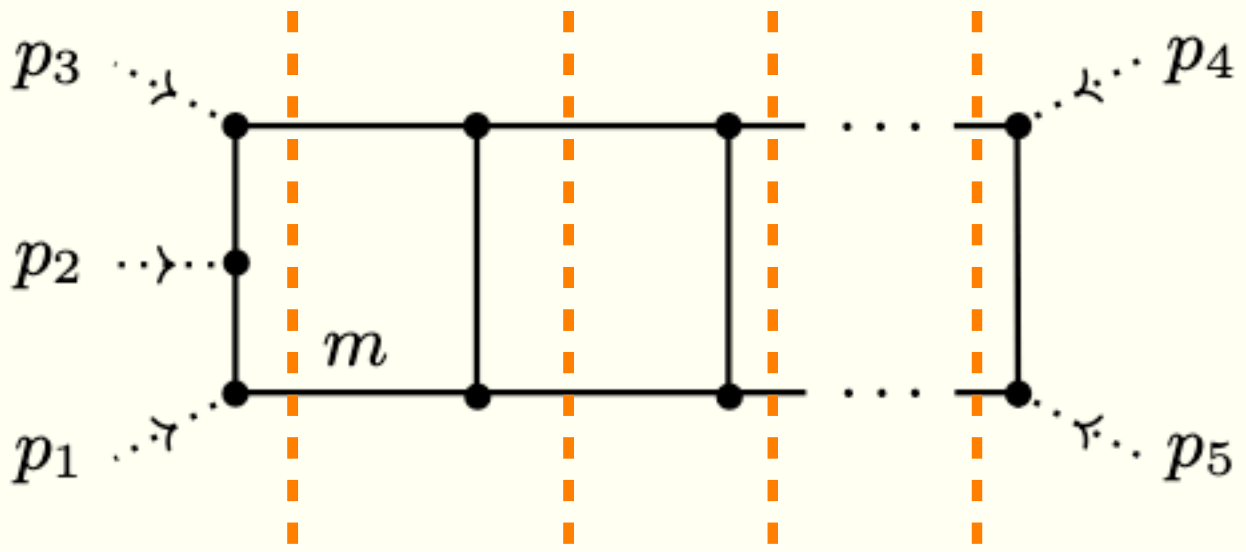
(Massive Mercedes diagram)



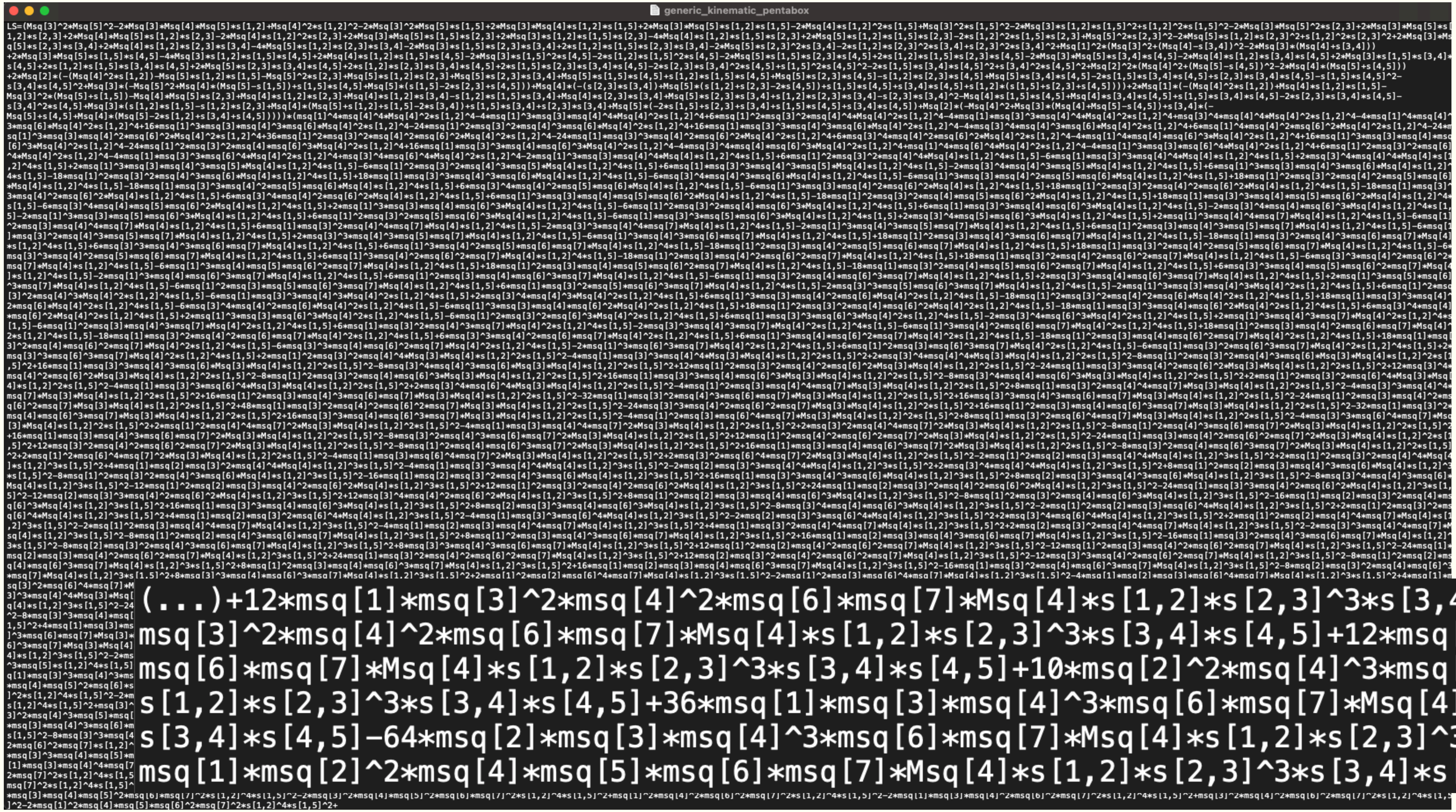
(Non-planar massive hexabox)



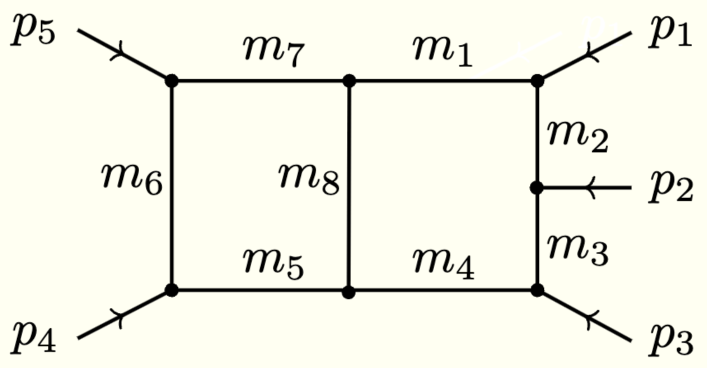
(Massive pentaladder)



Leading singularities can get quite complicated




```
(...)+12*msq[1]*msq[3]^2*msq[4]^2*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+32*msq[2]*
msq[3]^2*msq[4]^2*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+12*msq[1]*msq[2]*msq[4]^3*
msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+10*msq[2]^2*msq[4]^3*msq[6]*msq[7]*Msq[4]*
s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+36*msq[1]*msq[3]*msq[4]^3*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*
s[3,4]*s[4,5]-64*msq[2]*msq[3]*msq[4]^3*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]+12*
msq[1]*msq[2]^2*msq[4]*msq[5]*msq[6]*msq[7]*Msq[4]*s[1,2]*s[2,3]^3*s[3,4]*s[4,5]-(...)
```



$+\mathcal{O}(10^6)$ terms

[40.52 Mb polynomial]

Conclusion

We introduced an efficient unitarity-based method to extract singularities of Feynman integrals

Stress-tested the method against cutting-edge tools like `HyperInt` and `PLD.jl`

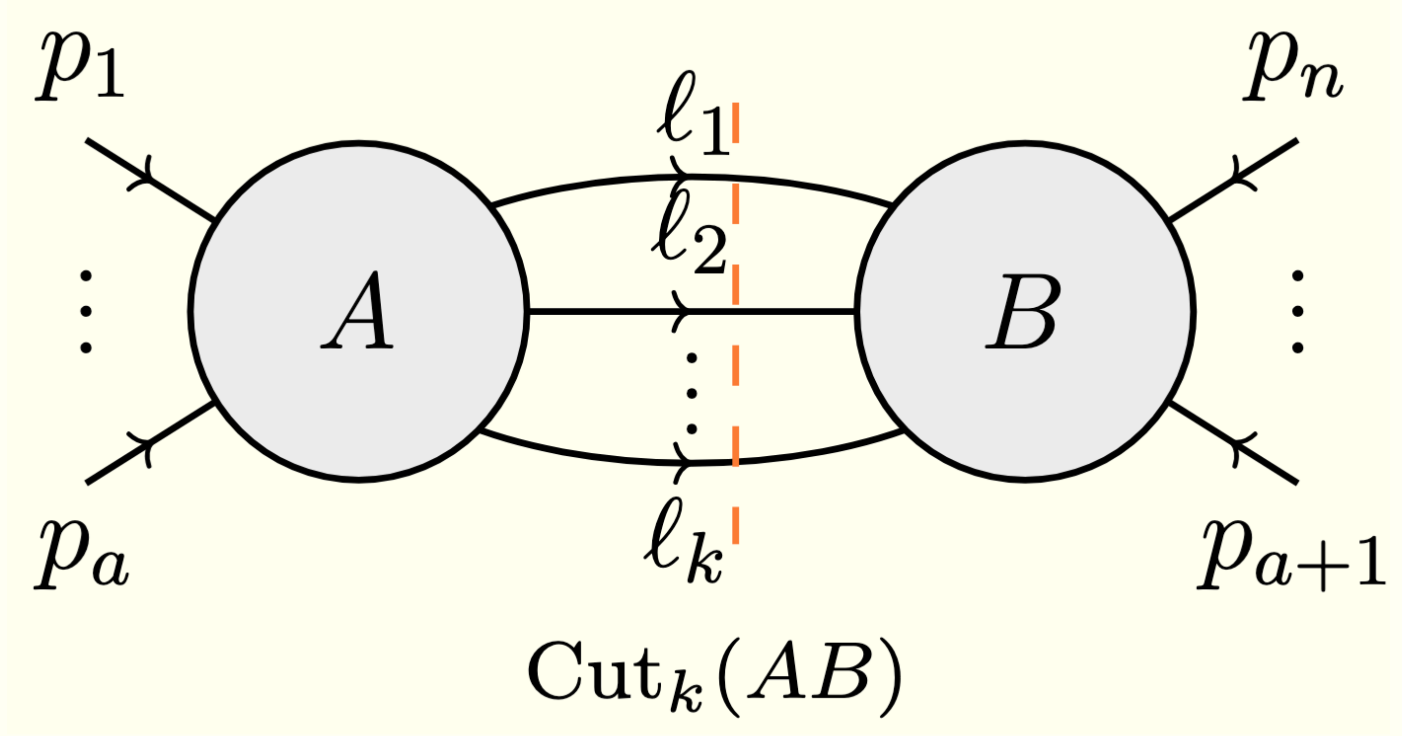
Made new predictions for multi-loop processes, including many examples in the Standard Model

Outlook

Many future directions… here are some we are working on with Caron-Huot, Correia and Mizera

Systematic way to include higher-cut subgraphs into the recursion without knowing *a priori* their singularities ?

We now know[*,**] *how to deal with higher cuts*



**Current computational limitation lies in your ability to solve high-degree coupled polynomial systems*

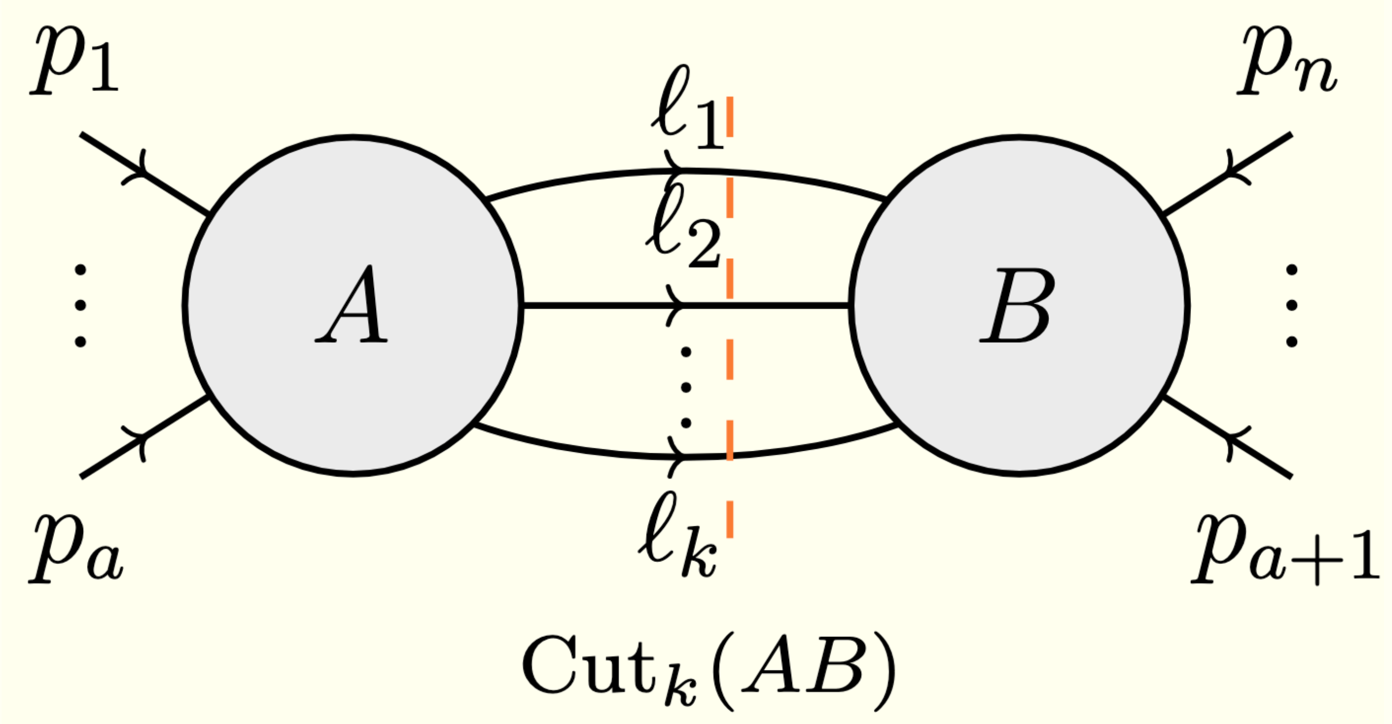
***There are few different working prescriptions: which one is the best ?*

Outlook

Many future directions… here are some we are working on with Caron-Huot, Correia and Mizera

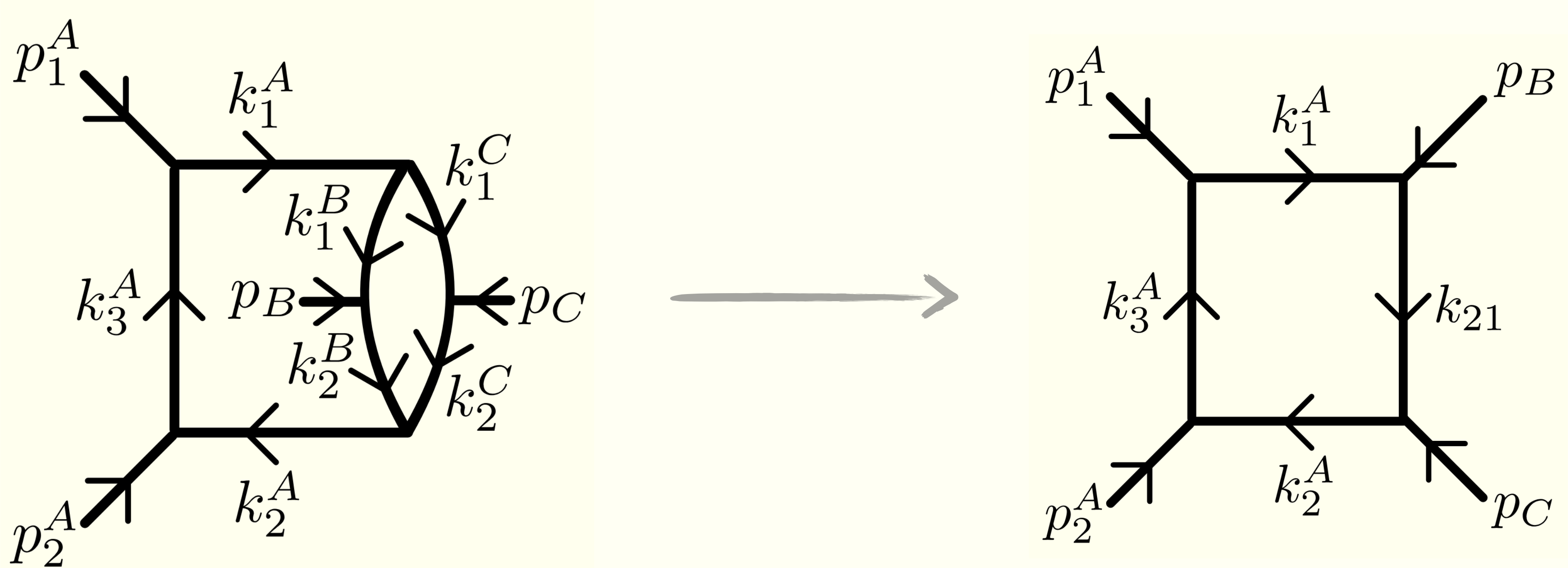
Systematic way to include higher-cut subgraphs into the recursion without knowing *a priori* their singularities ?

We now know,** how to deal with higher cuts*



Systematic way to find if a singularity is physical or not ?

Strong clues that we can also recurse in α-parameter space



Effective (recursive) α:

$$\alpha_{ij} = \frac{\alpha_i^B \alpha_j^C}{\alpha_1^B + \alpha_2^B + \alpha_1^C + \alpha_2^C}$$

Thank you!

Dirac on his way to cut (actual) trees

Extra slides

TYPES OF SOLUTIONS

Leading or subleading singularities

When all or a subset of propagators are set on-shell

[Bjorken, Landau, Nakanishi (1954)]

Second- or mixed-type singularities

When all or a subset of loop momenta diverge ($\ell_i \to \infty$)

[Cutkosky (1960), Fairlie, Landshoff, Nuttall, Polkinghorne (1962)]

[Drummond (1963), Boyling (1967)]

Beyond the standard classification singularities

When a subset of loop momenta diverge ($\ell_i \to \infty$) at different rates

[Berghoff, Panzer (2022), Fevola, Mizera, Telen (2023)]

Higher-cuts diagrams

Examples of (sub)graphs whose singularities cannot be resolved *systematically* by the two-particle cut recursion (may need to use, e.g., `PLD.jl`)

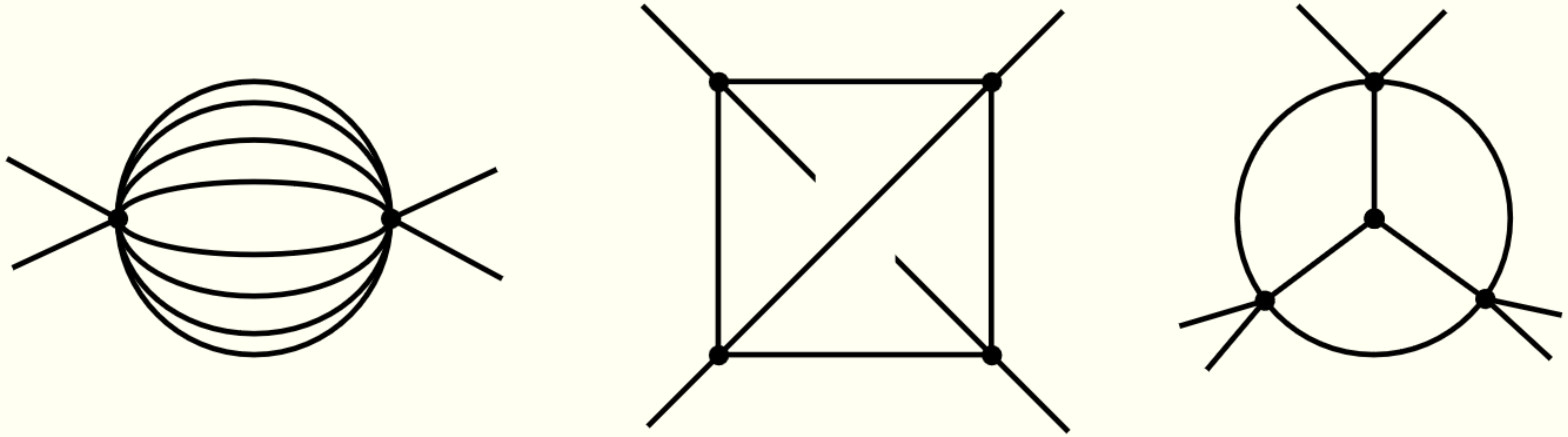


Figure 3. Examples of diagrams with *no* two-particle cuts splitting the graph in two disjoint subgraphs.

Recursively finding singularities

But wait! `PLD.jl` flags another leading singularity :

```
##################################
# Component 2
##################################

D[2] = M[3]^2 - 2*M[3]*M[4] - 2*M[3]*s + M[4]^2 - 2*M[4]*s + s^2
Ï‡[2] = 16
weights[2] = [[-1, -1, -1, -1], [0, 0, 0, 0]]
computed_with[2] = ["PLD_num", "HyperInt"]
```

Where is it in our approach ?

The singularity depends solely on *external* invariants

$$\begin{vmatrix} s & \frac{m_2^2-m_1^2+s}{2} & \frac{M_4^2-M_3^2-s}{2} \\ \frac{m_2^2-m_1^2+s}{2} & m_2^2 & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} \\ \frac{M_4^2-M_3^2-s}{2} & \frac{(m_4\pm m_3)^2-m_2^2-M_3^2}{2} & M_3^2 \end{vmatrix} = 0$$

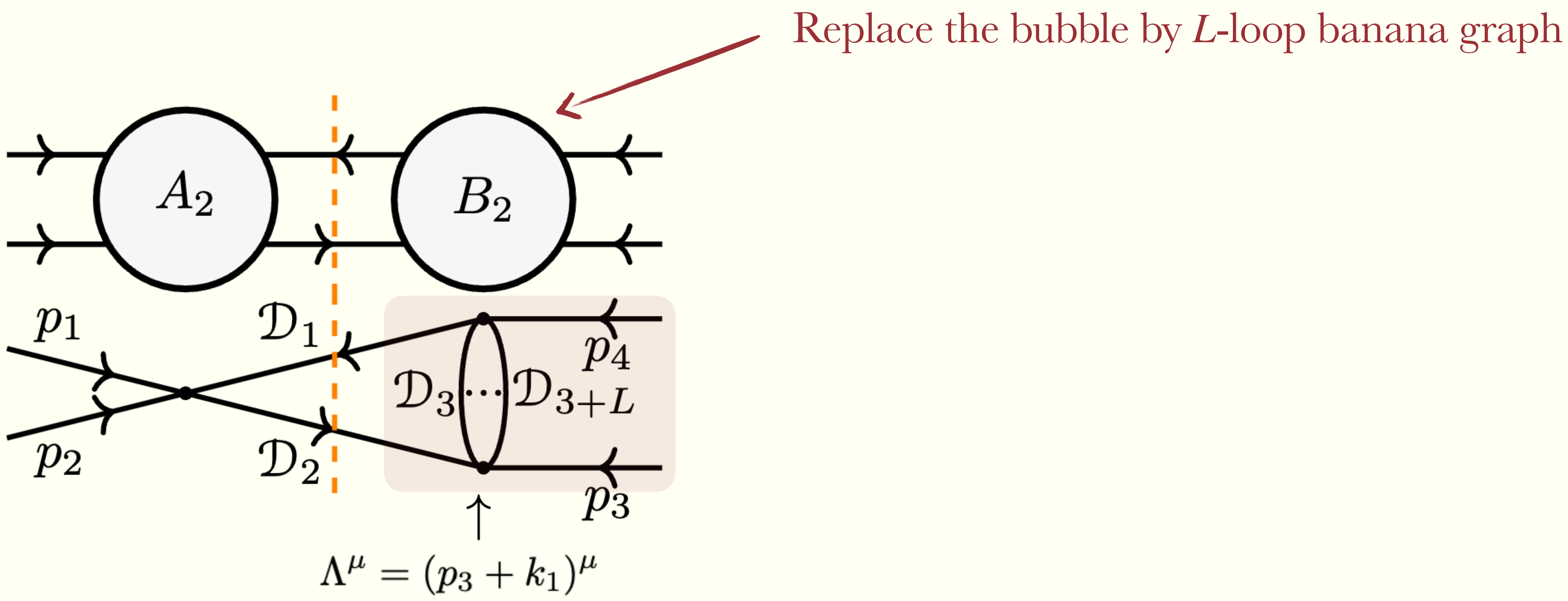
It is the expected (from C_{bub}) *collinear divergence* between p_{12} and p_3

(supported even on the maximal cut)

L-LOOP RESULTS

Some times, this method makes it easy to make L-loop statements

$$\begin{bmatrix} p_{12}^2 & p_{12}\cdot k_1 & p_{12}\cdot p_3 \\ p_{12}\cdot k_1 & k_1^2 & k_1\cdot p_3 \\ p_{12}\cdot p_3 & k_1\cdot p_3 & p_3^2 \end{bmatrix}$$

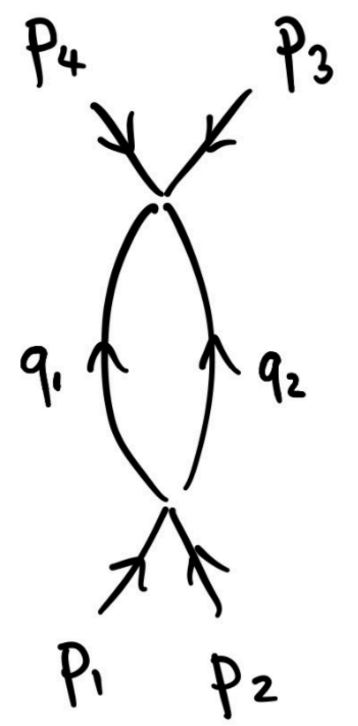


Although the banana subgraph does not have a two-particle cut,

we can still find the parachute singularities because the analytic structure of the banana is known *beforehand*

$$k_1 \cdot p_3 = \frac{1}{2}\left[(m_3 \pm m_4 \pm \ldots \pm m_{3+L})^2 - m_2^2 - M_3^2\right]$$

Bubble diagram



$$s = (p_1 + p_2)^2$$

momentum conservation

$$p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu = -p_3^\mu - p_4^\mu$$

locality

$$\alpha_1 q_1^\mu = \alpha_2 q_2^\mu$$

on-shellness

$$q_1^2 - m_1^2 = 0$$

$$q_2^2 - m_2^2 = 0$$

$$q_1^\mu = \ell^\mu, \qquad q_2^\mu = p_1^\mu + p_2^\mu - \ell^\mu$$

$$\ell^\mu = (p_1^\mu + p_2^\mu)\frac{\alpha_2}{\alpha_1+\alpha_2}$$

$$s\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2 - m_1^2 = 0, \qquad s\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 - m_2^2 = 0$$

Lorentz invariant

The solutions are

$$(\alpha_1 : \alpha_2) = \left(\frac{1}{m_1} : \pm\frac{1}{m_2}\right) \qquad s = (m_1 \pm m_2)^2$$

Projective invariance in Schwinger parameters and kinematic variables separately

$+$ normal threshold

$-$ pseudo-normal threshold

[Slides from Sebastian Mizera]