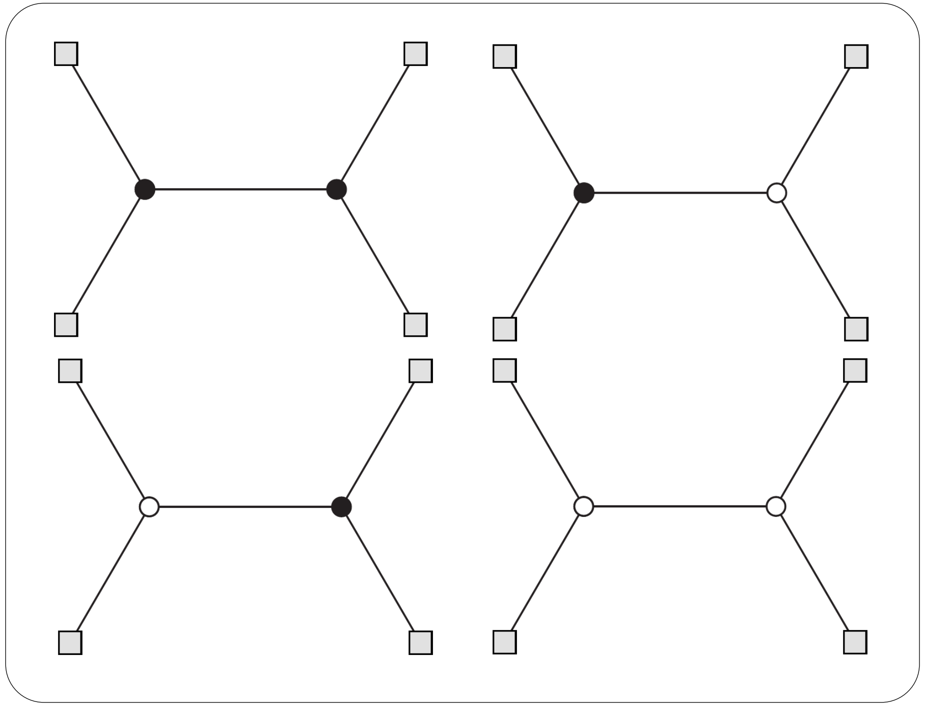
Systematically evaluate cosmological correlators by IBP & differential equations



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@Loop-the-Loop

Outline

- Generalize IBP to the function (Hankel) integrand case.
- Uniform formulas of iterative IBP and dlog DE of **arbitrary** (massive & time derivative interaction) tree-level cosmological correlators.

arxiv: 2401.00129

Jiaqi Chen, Bo Feng

- Hypergeometric Solutions of **arbitrary vertex integral family** (1 vertex with arbitrary legs) via power series expansion & dlog-form DE.
- Factorization and solutions of homogeneous parts of **arbitrary** tree-level cosmological correlators & example of solving non-homogeneous part.
- Discussion: inspiration of these techniques to **flat QFT**.

arxiv: 2411.03088

Jiaqi Chen, Bo Feng and Yi-Xiao Tao

Part 0 Background

Motivations of QFT in dS

- **De Sitter (dS) space is the simplest curved space** (together with AdS)

- **Cosmological Collider & e.t.c.**

- in-in formalism in cosmology:
  E. Calzetta and B.L. Hu, PhysRevD.35.495,   S. Weinberg, PhysRevD.72.043514
  Cosmology Collider:
  Xingang Chen, *Primordial Non-Gaussianities from Inflation Models*, 1002.1416
  Nima Arkani-Hamed & Juan Maldacena, *Cosmological Collider Physics*, 1503.08043

- **dS/CFT** duality
  Modave Lecture Notes on de Sitter Space & Holography 2306.10141

In-in formalism

refer to arXiv:1703.10166, Xingang Chen, Yi Wang, Zhong-Zhi Xianyu,
Schwinger-Keldysh Diagrammatics for Primordial Perturbations

The n-point correlation generated from the inflation of vacuum.

$$\langle Q(\tau)\rangle = \langle\Omega|\bar{F}(\tau,\tau_0)Q_I(\tau)F(\tau,\tau_0)|\Omega\rangle$$

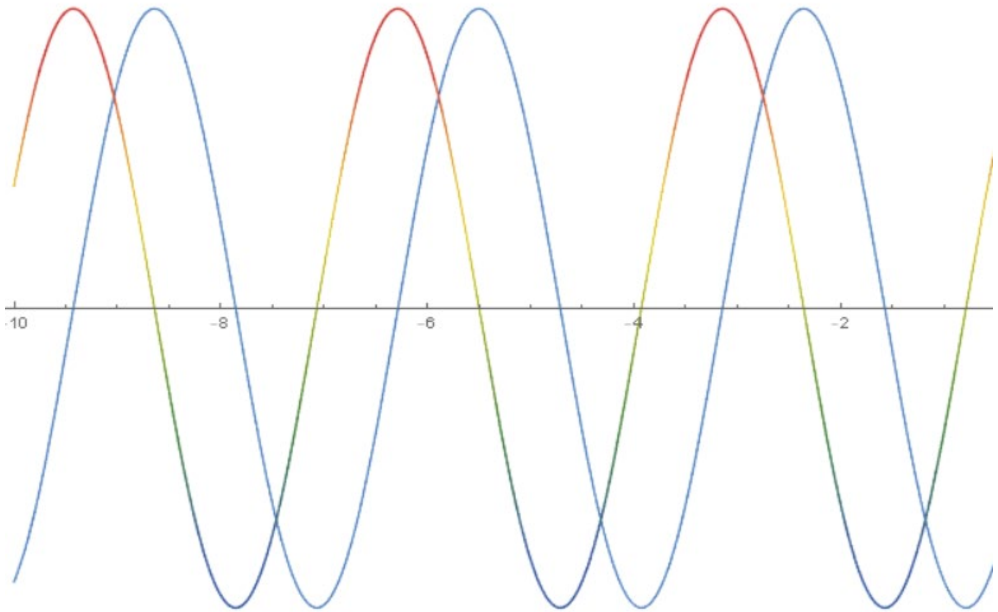
inflation → ... ← inflation

$$Q(\tau) \equiv \varphi^{A_1}(\tau,\mathbf{x}_1)\cdots\varphi^{A_N}(\tau,\mathbf{x}_N)$$

Metric of dS: $$\mathrm{d}s^2 = a^2(\tau)\big(-\mathrm{d}\tau^2+\mathrm{d}\boldsymbol{x}^2\big), \quad a(\tau) = 1/(-H\tau)$$

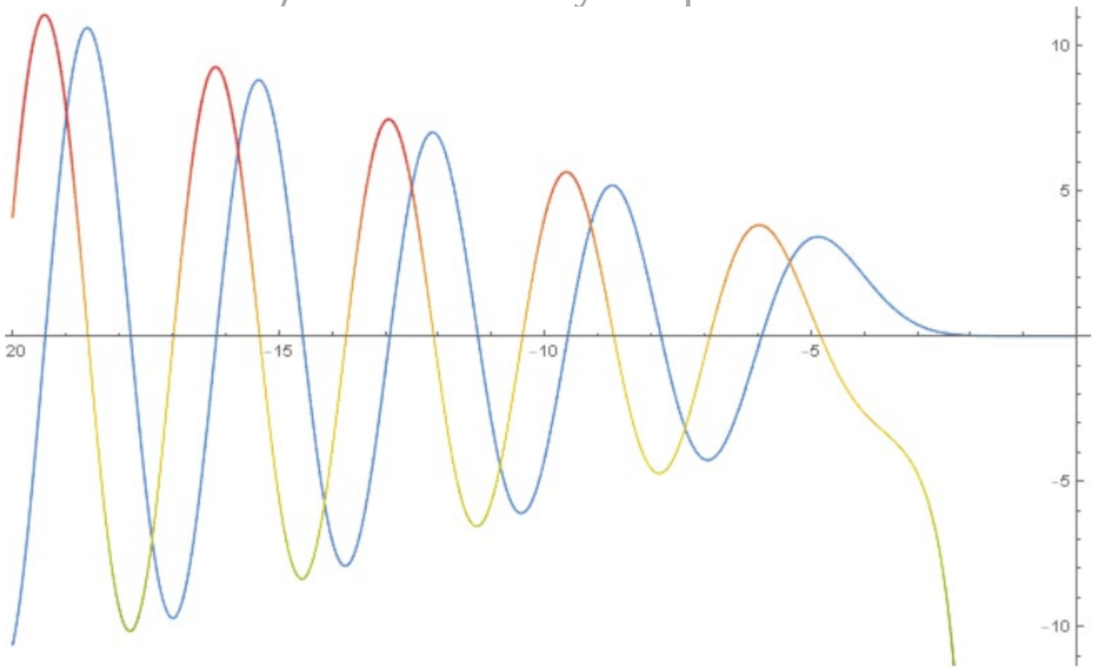
Flat vs dS - plane wave vs Hankel function

space mode： e^ikx



time mode： $(-\tau)^{3/2}\mathrm{H}^{(1)}_{\nu_a}(-k\tau)$

simplified when ν is half-odd integer
special cases: ν=3/2: massless,
ν=1/2: conformally coupled scalar.



The Hankel function results in leftover time integrals, including at tree level.

Feynman Rules from Schwinger-Keldysh Path Integral

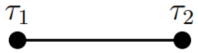
$$u_a(\tau, \mathbf{k}) = -\frac{\mathrm{i}\sqrt{\pi}}{2} e^{\mathrm{i}\pi(\nu/2+1/4)} H (-\tau)^{3/2} \mathrm{H}^{(1)}_{\nu_a}(-k\tau),$$

± for time ordered
& anti-time ordered

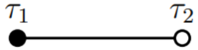
bulk propagator

$$G_>(k; \tau_1, \tau_2) \equiv u(\tau_1, k) u^*(\tau_2, k),$$

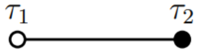
$$G_<(k; \tau_1, \tau_2) \equiv u^*(\tau_1, k) u(\tau_2, k).$$



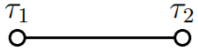
$$G_{++}(k; \tau_1, \tau_2) = G_>(k; \tau_1, \tau_2)\theta(\tau_1 - \tau_2) + G_<(k; \tau_1, \tau_2)\theta(\tau_2 - \tau_1),$$



$$G_{+-}(k; \tau_1, \tau_2) = G_<(k; \tau_1, \tau_2),$$



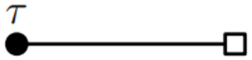
$$G_{-+}(k; \tau_1, \tau_2) = G_>(k; \tau_1, \tau_2),$$



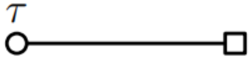
$$G_{--}(k; \tau_1, \tau_2) = G_<(k; \tau_1, \tau_2)\theta(\tau_1 - \tau_2) + G_>(k; \tau_1, \tau_2)\theta(\tau_2 - \tau_1),$$

bulk-to-boundary propagator

(massless)

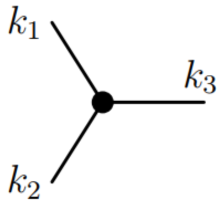


$$G_+(k; \tau) \equiv G_{+\pm}(k; \tau_1, 0) = \frac{H^2}{2k^3}(1 + ik\tau)e^{-ik\tau},$$



$$G_-(k; \tau) \equiv G_{-\pm}(k; \tau_1, 0) = \frac{H^2}{2k^3}(1 - ik\tau)e^{-ik\tau}.$$

Feynman Rules of the Typical Vertices

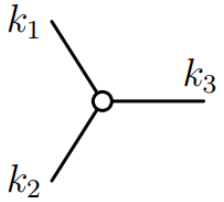


Scalar coupling

$$-\frac{\lambda}{24}a^4(\tau)\varphi^4 \to \mp i\lambda \int_{-\infty}^{0} \mathrm{d}\tau\, a^4(\tau)\cdots$$

Space-Derivative coupling

$$-\frac{\lambda}{6}a^2(\tau)\varphi(\partial_i\varphi)(\partial_i\varphi) \to \pm \frac{i\lambda}{3}(k_{12}+k_{23}+k_{13}) \int_{-\infty}^{0} \mathrm{d}\tau\, a^2(\tau)\cdots, \qquad k_{ij} \equiv \boldsymbol{k}_i \cdot \boldsymbol{k}_j$$

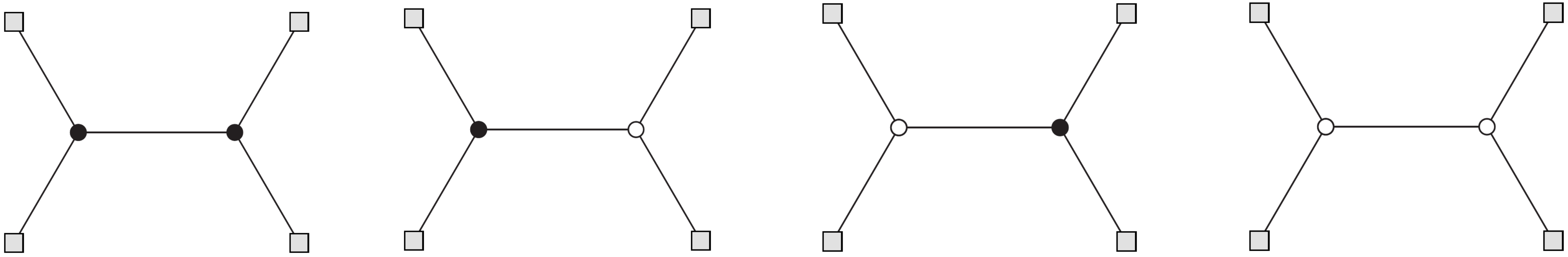


Time-Derivative coupling

$$-\frac{\lambda}{6}a^2(\tau)\varphi\varphi'^2 \to \mp \frac{i\lambda}{3} \int_{-\infty}^{0} \mathrm{d}\tau\, a^2(\tau) G_{+a_1}(k_1;\tau,\tau_1) \prod_{i=2,3} \boxed{[\partial_\tau G_{+a_i}(k_i;\tau,\tau_i)]}$$

$$+\ 2 \text{ permutations},$$

Example of **Diagrams:**



Current Methods for dS and Flat

Methods for dS

Cosmological Bootstrap
1811.00024, 1910.14051, 2005.04234, etc.

Mellin-Barnes (MB) Space
1906.12302, 1907.01143, 2109.02725, etc

Partial MB Transformation
2205.01692, 2208.13790, 2301.07047, etc

……

Methods for Flat

Conformal/Symbology Bootstrap

special cases (N=4 SYM,…) ⇒ powerful

general theory ⇒ not easy

1102.0062, 1612.08976, etc

Mellin-Barnes

Not easy for complicated cases

hep-ph/9905323, hep-ph/ 9909506, etc

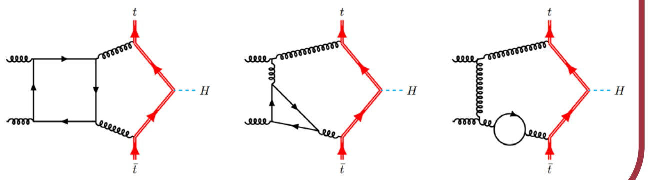
……

The Most Popular Methods:

Integration-By-Parts
&
Differential Equations

Many developed automatic packages.
(Kira, FIRE6, AMFlow, DiffEXP, etc)

Preferred for the **most complicated** multi-loop calculations e.g. 2312.08131.



Can we generalize?

Part 1

Generalize Integration-By-Parts

Traditional IBP (Integration-By-Parts)

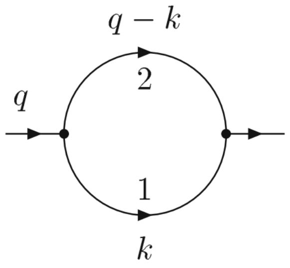
Polynomials
(monomials)

Integral family:

$$\int_{\mathcal{C}} \prod_i P_i(\boldsymbol{z})^{\alpha_i} \mathrm{d}\boldsymbol{z},$$

Example: flat space

$$\int \frac{\mathrm{d}^d k}{(k^2 - m^2)^{a_1}[(q-k)^2]^{a_2}}$$



Example: conformally-coupled scalar field in dS space

$$\mathcal{S} = \int \mathrm{d}^d x\, \mathrm{d}\eta \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \xi R \phi^2 - \sum_{k \geq 3} \frac{\lambda_k}{k!} \phi^k \right]$$

$$\int_0^\infty \mathrm{d}x_1 \wedge \mathrm{d}x_2 \, \frac{(x_1 x_2)^\varepsilon}{(x_1+x_2+X_1+X_2)(x_1+X_1+Y)(x_2+X_2+Y)}$$

ref: 2308.03753, 2312.05303, etc

IBP of Function Integrands

Integral family:

Polynomials — Functions

$$\int_{\mathcal{C}} \prod_i P_i(\boldsymbol{z})^{\alpha_i} \prod_j F_j(\boldsymbol{z})^{\beta_j} \mathrm{d}\boldsymbol{z}$$

Cosmological Integral Family:

$$\int_{\mathcal{C}} \mathrm{d}\boldsymbol{z} = \int_{-\infty}^{0} \mathrm{d}\tau_i \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{l}_i, \quad \boldsymbol{l}_i \text{ for loop momentum}$$

$$P_i = \tau_j, \quad \text{polynomial of loop momentum}$$

$$F_i = e^{ik\tau}, \quad \mathrm{H}_\nu^{(1,2)}(-k\tau), \quad \partial_\tau \mathrm{H}_\nu^{(1,2)}(-k\tau), \quad \theta(\tau_j - \tau_k)$$

The only term involves different τ_i

Example: IBP for dτ (dk is similar)

IBP:

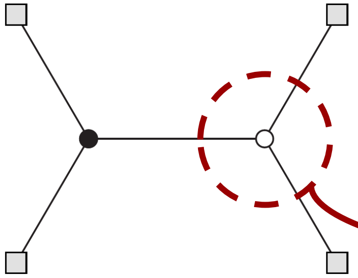
$$0 = \int_{\mathcal{C}} \mathrm{d}\left(\tau^{\nu_0} \mathrm{H}_\nu^{(1)}(-k\tau)\right) \Rightarrow 0 = \nu_0 \int_{\mathcal{C}} \tau^{\nu_0 - 1} \mathrm{H}_\nu^{(1)}(-k\tau) \mathrm{d}\tau + \int_{\mathcal{C}} \tau^{\nu_0} \partial_\tau \mathrm{H}_\nu^{(1)}(-k\tau) \mathrm{d}\tau$$

Second order derivative terms can be **reduced!**

$$\partial_\tau^2 \mathrm{H}_\nu^{(1,2)}(-k\tau) + \frac{1}{\tau} \partial_\tau \mathrm{H}_\nu^{(1,2)}(-k\tau) + \left(k^2 - \frac{\nu^2}{H^2 \tau^2}\right) \mathrm{H}_\nu^{(1,2)}(-k\tau) = 0\,,$$

Factorization of IBP of τ_i

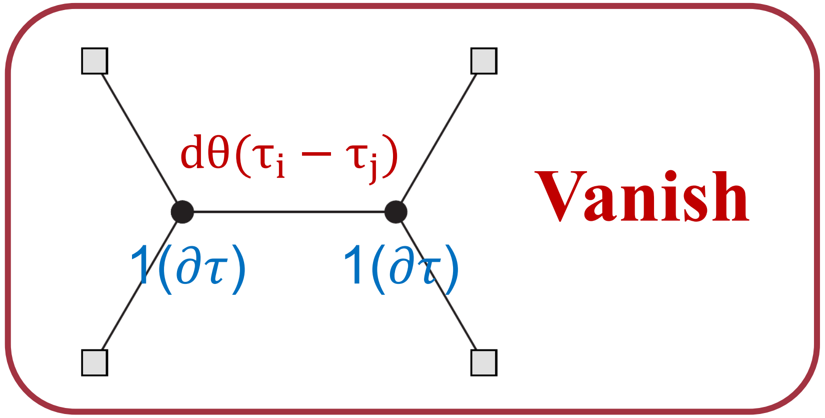
For $\mathbf{G}_{\pm\mp}$: no $d\theta(\tau_i - \tau_j)$, and integrals of different τ are automatically **factorized**.

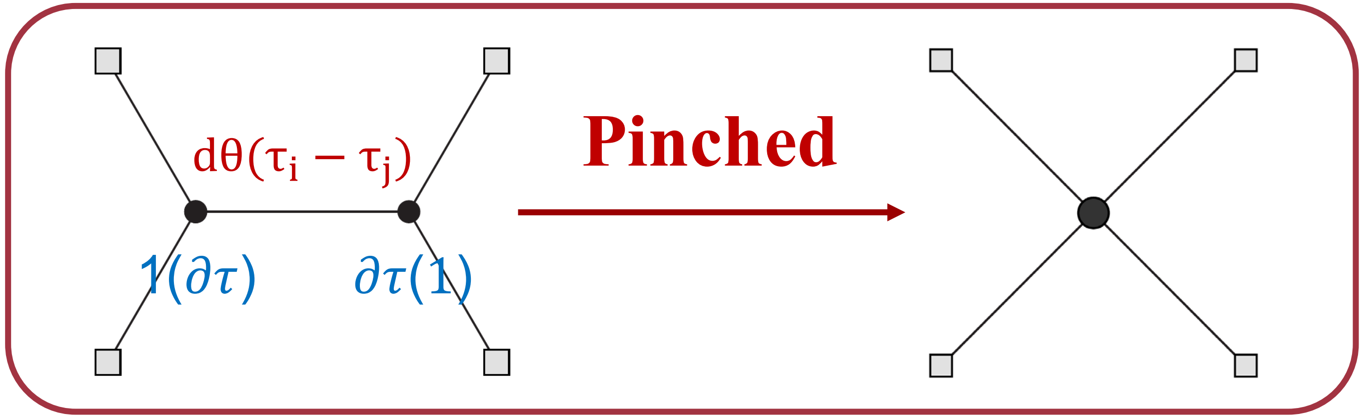


$$\int \left(\partial_{\tau_k}\hat{V}_k(\cdots;\tau_k)\mathrm{d}\tau_k\right)\times\prod_{j\neq k}\left(\hat{V}_j(\cdots;\tau_j)\mathrm{d}\tau_j\right) f(l_1,\cdots,l_L;k_1\cdots,k_E)\prod_i \mathrm{d}l_i$$

Reduced individually

For $\mathbf{G}_{\pm\pm}$: contribution of $d\theta(\tau_i - \tau_j)$ to IBP has two cases.





1 for constant vertex
$\partial\tau$ for time-derivative of the propagator

Subsector/
Remaining terms

Factorization of IBP

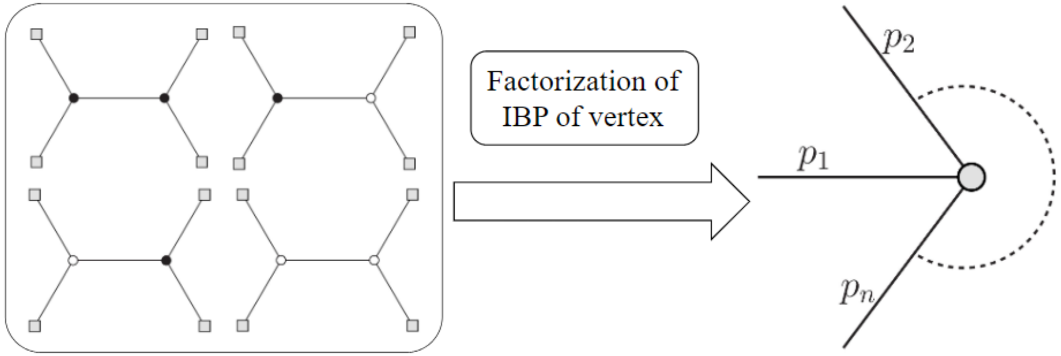
Subsector Remaining terms

IBP of τ:

$$\int \prod_i \mathrm{d}\tau_i \hat{V}_i \times \cdots = \left[\int \prod_i \left(\mathrm{d}\tau_i \sum_j c_j^{(i)} \underline{\hat{f}_j^{(i)}} \right) \times \cdots \right] + R,$$

MIs

IBP of **arbitrary tree-level** correlators (and **τ-integrals** in the **loop integral**) can be **factorized**



Reduction of **vertex integral family individually**

IBP of l & τ :

IBP relations of loop momentum → (Reduced via IBP of τ) → dτ-reduced IBP relations → (Solve) → Complete reduction

$$\int \mathrm{d}\left(\frac{1}{|l||l-p_1|\cdots} \boldsymbol{f} \right) = 0\,, \quad \boldsymbol{f} = \{f_1, f_2, \cdots\} \Longrightarrow \int \left(\mathrm{d}\frac{1}{|l||l-p_1|\cdots} \right) \boldsymbol{f} + \frac{1}{|l||l-p_1|\cdots} (\Omega.\boldsymbol{f}\mathrm{d}l) = 0\,, \quad \mathrm{d}\boldsymbol{f} = \Omega.\boldsymbol{f}\mathrm{d}l$$

Master integrals of τ-integral

DE of τ-integral

Part 2

IBP & DE of Vertex Integral Family

Vertex Integral Family & Construction of h-function

$$f^{(a_0)}_{\tilde{\boldsymbol{a}}} = \int_{-\infty}^{0} (-\tau)^{\nu_0+a_0} e^{ik_0\tau} \prod_{i=1}^{n} h(\nu_i, a_i; -k_i\tau)\mathrm{d}\tau \,,$$

$$a_0 \in \mathbb{Z}\,, \quad a_{i>1} = 0, 1; \quad \boldsymbol{a} = a_1, \cdots, a_n$$

$$h^{(1 \text{ or } 2)}(\nu, 0; -k\tau) \equiv (-k\tau)^{-\nu}\mathrm{H}^{(1)}_{\nu}(-k\tau) \ (\text{or } \mathrm{H}^{(2)}_{\nu^\star}) \propto \tau^{-\frac{3}{2}-\nu} u \ (\text{or } u^*),$$

$$h^{(1,2)}(\nu, 1; -k\tau) \equiv -\frac{1}{k}\partial_\tau h^{(1,2)}(\nu, 0; -k\tau) \,.$$

notation $\tilde{a}$:
base-2 → base-10

$$\boldsymbol{a} \equiv a_1, a_2, \cdots, a_n \,, \qquad a_i = 0, 1\,,$$

$$\tilde{\boldsymbol{a}} \equiv 1 + \sum_{i=1}^{n} a_i 2^{n-i}\,,$$

$$\mathrm{I}_{\tilde{\boldsymbol{a}}} \equiv \mathrm{I}_{\{\boldsymbol{a}\}} = \mathrm{I}_{\{a_1, a_2, \cdots\}}$$

For $k, \tau \in \mathbb{R}$ and $\nu = \pm\nu^\star$ u and u* satisfy same differential equation and thus IBP as well.

$$\partial_\tau^2 \mathrm{H}^{(1,2)}_{\nu}(-k\tau) + \frac{1}{\tau}\partial_\tau \mathrm{H}^{(1,2)}_{\nu}(-k\tau) + \left(k^2 - \frac{\nu^2}{H^2\tau^2}\right)\mathrm{H}^{(1,2)}_{\nu}(-k\tau) = 0\,,$$

$$\partial_{\tilde{\tau}}^2 h(\nu, 0; \tilde{\tau}) + \frac{1}{\tilde{\tau}}(2\nu + 1)\partial_{\tilde{\tau}} h(\nu, 0; \tilde{\tau}) + h(\nu, 0; \tilde{\tau}) = 0, \quad \tilde{\tau} = -k\tau$$

Example: 1-fold (1-massive-leg) Vertex Integral Family

Integral Family

$$V(\nu_0, a_1) = \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} h(\nu_1, a_1; -k_1\tau)\mathrm{d}\tau, \qquad a_i = 0, 1\,,$$

#MIs=2

Matrix Form IBP Relations

$$\boldsymbol{f}^{(a_0)} = \{V\left(\nu_0 + a_0, 0\right), V\left(\nu_0 + a_0, 1\right)\},$$

$$\left(M_1^{(1)} + \nu_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(-1)} + \left(M_0^{(1)} + ik_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(0)} = 0$$

$$M_1^{(j)} = -\frac{2\nu_j + 1}{2}\left(\mathbb{I}_2 - \sigma_3\right) = \begin{pmatrix} 0 & 0 \\ 0 & -2\nu_j - 1 \end{pmatrix}, \qquad M_0^{(j)} = -ik_j \sigma_2 = \begin{pmatrix} 0 & -k_j \\ k_j & 0 \end{pmatrix},$$

Pauli matrices

Example: 1-fold Vertex Integral Family Iterative reduction

Iterative reduction

$$\boldsymbol{f}^{(-1)} = A_-(\nu_0).\boldsymbol{f}^{(0)}, \quad A_+(\nu_0) \equiv (A_-(\nu_0+1))^{-1},$$

$$\boldsymbol{f}^{(-n)} = \left(\prod_{i=n-1}^{0} A_-(\nu_0 - i)\right).\boldsymbol{f}^{(0)}, \qquad A_-(\nu_0) = \begin{pmatrix} -\frac{ik_0}{\nu_0} & \frac{k_1}{\nu_0} \\ \frac{k_1}{-\nu_0+2\nu_1+1} & -\frac{ik_0}{\nu_0-2\nu_1-1} \end{pmatrix}$$

$$\boldsymbol{f}^{(n)} = \left(\prod_{i=n-1}^{0} A_+(\nu_0 + i)\right).\boldsymbol{f}^{(0)}, \qquad A_+(\nu_0) = \begin{pmatrix} \frac{ik_0(\nu_0+1)}{k_0^2-k_1^2} & \frac{k_1(\nu_0-2\nu_1)}{k_0^2-k_1^2} \\ \frac{k_1(\nu_0+1)}{k_1^2-k_0^2} & \frac{ik_0(\nu_0-2\nu_1)}{k_0^2-k_1^2} \end{pmatrix}$$

For $k_0 = 0$:

$$A_-(\nu_0) = \begin{pmatrix} 0 & \frac{k_1}{\nu_0} \\ \frac{k_1}{-\nu_0+2\nu_1+1} & 0 \end{pmatrix}, \qquad A_+(\nu_0) = \begin{pmatrix} 0 & -\frac{\nu_0-2\nu_1}{k_1} \\ \frac{\nu_0+1}{k_1} & 0 \end{pmatrix}$$

simplified

If we do not define h as introduced …

$$h(\nu_0, \boldsymbol{a}) = (-k\tau)^{\frac{1}{2}} \mathrm{H}_{\nu}^{(1,2)}$$

$$\partial_{\tilde{\tau}}^2 h(\mu, 0; \tilde{\tau}) + \left(1 + \frac{\mu^2}{\tilde{\tau}^2}\right) h(\mu, 0; \tilde{\tau}) = 0, \quad \tilde{\tau} = -k\tau, \quad \mu^2 = \frac{m^2}{H^2} - 2$$

$$M_2^{(1)} \boldsymbol{f}^{(-2)} + \nu_0 \mathbb{I}_2 . \boldsymbol{f}^{(-1)} + \left(M_0^{(1)} + i k_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(0)}$$

$$M_2^{(j)} = \begin{pmatrix} 0 & 0 \\ \frac{\mu_j^2}{k_j} & 0 \end{pmatrix}, \quad M_0^{(j)} = \begin{pmatrix} 0 & -k_j \\ k_j & 0 \end{pmatrix}.$$

Not easy to find the relation between $f^{(a)}$ and $f^{(a+1)}$

Example: 1-fold Vertex Integral Family d log-form DE

$$\partial_{k_0} \boldsymbol{f}^{(0)} = \begin{pmatrix} iV\left(\nu_0+1,0\right) \\ iV\left(\nu_0+1,1\right) \end{pmatrix} = iA_+(\nu_0+1)\boldsymbol{f}^{(0)}\,,$$

$$\partial_{k_1} \boldsymbol{f}^{(0)} = \begin{pmatrix} -V\left(\nu_0+1,1\right) \\ \frac{(-2\nu_1-1)V(\nu_0,1)}{k_1} + V\left(\nu_0+1,0\right) \end{pmatrix} = \left(\frac{1}{k_1}M_1^{(1)} - i\sigma_2.A_+(\nu_0+1)\right).\boldsymbol{f}^{(0)}.$$

dlog-form differential equations emerge automatically!!

$$\mathrm{d}\boldsymbol{f}^{(0)} = \mathrm{d}\Omega\boldsymbol{f}^{(0)}$$

$$\mathrm{d}\Omega = \sum_{i=0,1}\tilde{\Omega}_{k_i}\mathrm{d}k_i = C_1\mathrm{d}\log(k_1) + C_2\mathrm{d}\log[(k_0-k_1)(k_0+k_1)] + C_3\mathrm{d}\log\left(\frac{k_0+k_1}{k_0-k_1}\right)$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2\nu_1-1 \end{pmatrix},\quad C_2 = \begin{pmatrix} \frac{1}{2}\left(-\nu_0-1\right) & 0 \\ 0 & \frac{1}{2}\left(2\nu_1-\nu_0\right) \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 & -\frac{1}{2}i\left(\nu_0-2\nu_1\right) \\ \frac{1}{2}i\left(\nu_0+1\right) & 0 \end{pmatrix}.$$

n-fold Vertex Integral Family

Integral Family $V(\nu_0, a_1, \cdots, a_n) = \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} \prod_{i=1}^{n} h(\nu_i, a_i; -k_i\tau) \mathrm{d}\tau, \quad a_i = 0, 1\,.$

Master Integrals $f_{\boldsymbol{a}}^{(0)} = V(\nu_0, \boldsymbol{a}), \quad \boldsymbol{a} = a_1, \cdots, a_n, \quad \forall a_i =, 0, 1.$ $\#MIs = 2^n$

IBP Relations $(M_1)_{\boldsymbol{ba}} f_{\boldsymbol{a}}^{(-1)} + (M_0)_{\boldsymbol{ba}} f_{\boldsymbol{a}}^{(0)} = 0$

$$(M_1)_{\boldsymbol{ba}} = \sum_{j=1}^{n} \left[\left(M_1^{(j)}\right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + \nu_0 \delta_{\boldsymbol{ba}}$$

$$(M_0)_{\boldsymbol{ba}} = \sum_{j=1}^{n} \left[\left(M_0^{(j)}\right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + ik_0 \delta_{\boldsymbol{ba}}$$

$a_i = 0{,}1 \quad b_i = 0{,}1$

Tensor product of 2x2 matrices

Compare to n=1

$$\left(M_1^{(1)} + \nu_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(-1)} + \left(M_0^{(1)} + ik_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(0)} = 0$$

$$M_1^{(j)} = -\frac{2\nu_j + 1}{2} (\mathbb{I}_2 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & -2\nu_j - 1 \end{pmatrix}, \quad M_0^{(j)} = -ik_j \sigma_2 = \begin{pmatrix} 0 & -k_j \\ k_j & 0 \end{pmatrix},$$

n-fold Vertex Integral Family

IBP Relations $(M_1)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(-1)} + (M_0)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(0)} = 0$

$$M_1 = \sum_{j=1}^{n}\left(\nu_j + \frac{1}{2}\right)\Lambda_3^{(j)} + \left(\nu_0 - \frac{n}{2} - \sum_{i=1}^{n}\nu_i\right)\mathbb{I}_{2^n}$$

$$M_0 = -i\sum_{j=1}^{n} k_j \Lambda_2^{(j)} + ik_0\mathbb{I}_{2^n}$$

$$\left(\Lambda_k^{(j)}\right)_{\boldsymbol{ba}} \equiv (\sigma_k)_{b_j,a_j}\prod_{i\neq j}\delta_{b_i,a_i},\, k = 1,2,3$$

Pauli matrices

Tensor product of 2x2 matrices

Iterative reduction

$$A_-(\nu_0) = -M_1^{-1}.M_0\,, \qquad A_+(\nu_0 - 1) = -M_0^{-1}.M_1\,.$$

Non-diagonal, T_n for diagnolization

Universal formula of reduction of arbitrary Vertex Integral Family

$$A_-(\nu_0) = -M_1^{-1}.M_0\,,$$

$$A_+(\nu_0 - 1) = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\tilde{M}_1.\mathrm{T}_n = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\mathrm{T}_n.M_1\,,$$

No inverse of large matrix!

n-fold Vertex Integral Family d log-form DE

$$\partial_{k_0} \boldsymbol{f}^{(0)} = i\boldsymbol{f}^{(1)} = iA_+(\nu_0)\boldsymbol{f}^{(0)}\,,$$

$$\partial_{k_i} \boldsymbol{f}^{(0)} = \left(-\frac{1}{k_i}\frac{2\nu_i}{2}(\mathbb{I}_{2^n} - \Lambda_3^{(i)}) - i\Lambda_2^{(i)}.A_+(\nu_0)\right).\boldsymbol{f}^{(0)}\,,\ \text{for } i > 0\,.$$

Universal formula of d log-form DE of arbitrary n-fold Vertex Integral Family

$$\mathrm{d}\boldsymbol{f}^{(0)} = (\mathrm{d}\Omega).\boldsymbol{f}^{(0)} = \sum_{i=0}^{n} \Omega_{k_i}.\boldsymbol{f}^{(0)}\mathrm{d}k_i\,,$$

$$\Omega = \Omega_{ex} + i\mathrm{T}_n.\tilde{\Omega}_0.\mathrm{T}_n^{-1}.M_1(\nu_0+1)\,,$$

diagonal & only rely on ν_i

$$\left(\tilde{\Omega}_0\right)_{\boldsymbol{ba}} \equiv \begin{cases} -i\log\left[k_0 + \sum_i (2a_i - 1)k_i\right], & \boldsymbol{b} = \boldsymbol{a} \\ 0, & \boldsymbol{b} \neq \boldsymbol{a} \end{cases},$$

both diagonal

$$(\Omega_{ex})_{\boldsymbol{ba}} \equiv \begin{cases} -\sum_i a_i(2\nu_i+1)\log k_i, & \boldsymbol{b} = \boldsymbol{a} \\ 0, & \boldsymbol{b} \neq \boldsymbol{a} \end{cases},$$

Two structures: d log-form + tensor product of 2x2 matrices

IBP & dlog DE of Remaining Terms

Iterative Reduction:

$$\boldsymbol{f}^{(-1,0)}_{\boldsymbol{c},\boldsymbol{b}} = \left(A_{-;1}(\nu_0)\right)_{\boldsymbol{ca}} \boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} - \left(M^{-1}_{1;1}\right)_{\boldsymbol{ca}} \boldsymbol{R}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}},$$

$$\boldsymbol{f}^{(1,0)}_{\boldsymbol{c},\boldsymbol{b}} = \left(A_{+;1}(\nu_0)\right)_{\boldsymbol{ca}} \boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} - \left(\mathrm{T}^{-1}_{n_1}.\tilde{M}^{-1}_{0;1}.\mathrm{T}_{n_1}\right)_{\boldsymbol{ca}} \boldsymbol{R}^{(1,0)}_{\boldsymbol{a},\boldsymbol{b}}$$

$$\boldsymbol{R}^{(a_0,b_0)}_{\boldsymbol{a},\boldsymbol{b}} = \delta_{a_i,1-b_j}(-1)^{a_i+1}\frac{4i}{\pi}e^{\pi \mathrm{Im}[\nu]}(-k_{i;1})^{-2\nu_{i;1}-1} f^{(a_0+b_0-2\nu_{i;1}-1)}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}},$$

$$\boldsymbol{a}_{\hat{i}}, \boldsymbol{b}_{\hat{j}} = a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n_1}, b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_{n_2},$$

dlog DE:

$$\Omega_{(\boldsymbol{a},\boldsymbol{b})(\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}})} = -i\left(\mathrm{T}^{-1}_{n}.\tilde{\Omega}_{0;1}.\mathrm{T}_{n}\right)_{\boldsymbol{a}(\boldsymbol{c}_{\hat{i}};1-b_j)} \delta_{\boldsymbol{b}_{\hat{j}}\boldsymbol{d}_{\hat{j}}}(-1)^{b_j} - i\left(\mathrm{T}^{-1}_{n}.\tilde{\Omega}_{0;2}.\mathrm{T}_{n}\right)_{\boldsymbol{b}(\boldsymbol{d}_{\hat{i}};1-a_i)} \delta_{\boldsymbol{a}_{\hat{i}}\boldsymbol{c}_{\hat{i}}}(-1)^{a_i}$$

Selected MIs of the sub-sector: $\frac{4i}{\pi}e^{\pi \mathrm{Im}[\nu]}(-k_{i;1})^{-2\nu_{i;1}-1} f^{(-2\nu_{i;1})}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}$

Check details in

2401.00129

Part 3

Hypergeometric Solutions
of vertex integral family

1-fold Vertex Integral Family (1-massive-leg, 1 Hankel in integrand)

Integral Family:

$$\int_{-\infty}^{0} d\tau e^{ik_0\tau}(-\tau)^{\nu_0+a_0} h_{\nu_1}^{(2)}(a_1, -k_1\tau)\,.$$

For convenience, we consider $h^{(2)}$ and real ν_i here.

Master Integrals:

$$\mathrm{I}_1 = \int_{-\infty}^{0} d\tau e^{ik_0\tau}(-\tau)^{\nu_0} h_{\nu_1}^{(2)}(0, -k_1\tau)\,,$$

$$\mathrm{I}_2 = \int_{-\infty}^{0} d\tau e^{ik_0\tau}(-\tau)^{\nu_0} h_{\nu_1}^{(2)}(1, -k_1\tau)\,.$$

Differential equations:

$$d\mathrm{I}_{\tilde{a}} = (d\Omega_{\tilde{a}\tilde{b}})\,\mathrm{I}_{\tilde{b}}\,,$$

$$\Omega_{11} = -i(\nu_0+1)\left(-\frac{1}{2}i\log(k_0-k_1) - \frac{1}{2}i\log(k_0+k_1)\right),$$

$$\Omega_{12} = -i(\nu_0-2\nu_1)\left(\frac{1}{2}\log(k_0+k_1) - \frac{1}{2}\log(k_0-k_1)\right),$$

$$\Omega_{21} = -i(\nu_0+1)\left(\frac{1}{2}\log(k_0-k_1) - \frac{1}{2}\log(k_0+k_1)\right),$$

$$\Omega_{22} = -(2\nu_1+1)\log(k_1) - i(\nu_0-2\nu_1)\left(-\frac{1}{2}i\log(k_0-k_1) - \frac{1}{2}i\log(k_0+k_1)\right).$$

1-fold Vertex Integral Family around boundary $k_0 \to \infty$

$$x = \frac{1}{k_0}$$

$$\Omega_x = \begin{pmatrix} \frac{\nu_0+1}{x-k_1^2x^3} & \frac{ik_1(\nu_0-2\nu_1)}{k_1^2x^2-1} \\ -\frac{ik_1(\nu_0+1)}{k_1^2x^2-1} & \frac{\nu_0-2\nu_1}{x-k_1^2x^3} \end{pmatrix}$$

$$\Omega_x = \sum_{i=-1}^{\infty} \Omega_x^{(i)} x^i \,,$$

All order expansion of dlog DE is simple

$$\Omega_x^{(-1+2j)} = \begin{pmatrix} (\nu_0+1)\,k_1^{2j} & 0 \\ 0 & (\nu_0-2\nu_1)\,k_1^{2j} \end{pmatrix},$$

$$\Omega_x^{(0+2j)} = \begin{pmatrix} 0 & -i\,(\nu_0-2\nu_1)\,k_1^{1+2j} \\ i\,(\nu_0+1)\,k_1^{1+2j} & 0 \end{pmatrix},$$

1-fold Vertex Integral Family around boundary $k_0 \to \infty$

Ansatz of power series solutions:

$$f_i = x^\lambda \sum_{j=0}^{\infty} \mathrm{C}(i,j) x^j$$

λ solved by indicial equations

$$\partial_x \mathrm{C}(i,0) x^\lambda = \left(\Omega_x^{(-1)}\right)_{ij} x^{-1} \mathrm{C}(j,0) x^\lambda$$

$$\Rightarrow \lambda \begin{pmatrix} \mathrm{C}(1,0) \\ \mathrm{C}(2,0) \end{pmatrix} = \begin{pmatrix} \nu_0 + 1 & 0 \\ 0 & \nu_0 - 2\nu_1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{C}(1,0) \\ \mathrm{C}(2,0) \end{pmatrix}$$

solution 1: $\lambda = \nu_0 + 1\,, \quad \mathrm{C}(2,0) = 0\,,$

solution 2: $\lambda = \nu_0 - 2\nu_1\,, \quad \mathrm{C}(1,0) = 0\,.$

Equations at each order

$$(\lambda + j_0)\mathrm{C}(i, j_0) = \sum_{j=-1}^{j_0} \Omega_x^{(j)} \cdot \mathrm{C}(i, j_0 - j)\,.$$

2 general solutions X 2 MIs

1-fold Vertex Integral Family around boundary $k_0 \to \infty$

$$\mathrm{I}_{\tilde{a}} = \mathrm{C}^{[1]} f_{\tilde{a}}^{[1]} + \mathrm{C}^{[2]} f_{\tilde{a}}^{[2]}$$

Boundary Coefficients

General Solutions

$$f_1^{[1]}(1/x,k_1) = x^{\nu_0+1} \sum_{m=0}^{\infty} \frac{\left(\frac{\nu_0+1}{2}\right)_m \left(\frac{\nu_0+2}{2}\right)_m}{(\nu_1+1)_m} \frac{(k_1^2 x^2)^m}{m!},$$

$$f_2^{[1]}(1/x,k_1) = x^{\nu_0+1} i k_1 x \sum_{m=0}^{\infty} \frac{\left(\frac{\nu_0+1}{2}\right)_{m+1} \left(\frac{\nu_0+2}{2}\right)_m}{(\nu_1+1)_{m+1}} \frac{(k_1^2 x^2)^m}{m!},$$

$$f_1^{[2]}(1/x,k_1) = x^{\nu_0-2\nu_1}(-i k_1 x) \sum_{m=0}^{\infty} \frac{\left(\frac{\nu_0-2\nu_1}{2}\right)_{m+1} \left(\frac{\nu_0-2\nu_1+1}{2}\right)_m}{(-\nu_1)_{m+1}} \frac{(k_1^2 x^2)^m}{m!},$$

$$f_2^{[2]}(1/x,k_1) = x^{\nu_0-2\nu_1} \sum_{m=0}^{\infty} \frac{\left(\frac{\nu_0-2\nu_1}{2}\right)_m \left(\frac{\nu_0-2\nu_1+1}{2}\right)_m}{(-\nu_1)_m} \frac{(k_1^2 x^2)^m}{m!},$$

They can be expressed as the known hypergeometric function ${}_2\mathrm{F}_1$, e.g.

$$f_1^{[1]}(1/x,k_1) = x^{\nu_0+1} {}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1; k_1^2 x^2\right)$$

……

1-fold Vertex Integral Family around boundary $k_0 \to \infty$

$$\mathrm{I}_{\tilde{a}} = \mathrm{C}^{[1]} f^{[1]}_{\tilde{a}} + \mathrm{C}^{[2]} f^{[2]}_{\tilde{a}}$$

Boundary Coefficients

General Solutions

$$\int_{-\infty}^{0} \mathrm{d}\tau\, e^{ik_0\tau} (-\tau)^{\nu_0+a_0} h^{(2)}_{\nu_1}(a_1, -k_1\tau)$$

Exponential Suppression
(Wick Rotation)
$\tau \to 0$ contributes

$\tau \to 0$

$$\mathrm{C}^{[1]} = \mathrm{C}^{(k_0)}_1(\nu_1) \int_{-\infty}^{0} e^{ik_0\tau}(-k_0\tau)^{\nu_0} k_0 \mathrm{d}\tau = \mathrm{C}^{(k_0)}_1(\nu_1)(-i)^{\nu_0+1}\Gamma(\nu_0+1)\,,$$

$$\mathrm{C}^{[2]} = \frac{\mathrm{C}^{(k_0)}_2(\nu_1)}{k_1^{2\nu_1+1}} \int_{-\infty}^{0} e^{ik_0\tau}(-k_0\tau)^{\nu_0-2\nu_1-1} k_0 \mathrm{d}\tau = \frac{\mathrm{C}^{(k_0)}_2(\nu_1)}{k_1^{2\nu_1+1}}(-i)^{\nu_0-2\nu_1}\Gamma(\nu_0-2\nu_1)\,.$$

$$c[\nu] \equiv \frac{2^{-\nu}\Gamma(-\nu)}{i\pi}\,,\quad \mathrm{C}^{(k_0)}_1(\nu) = -e^{i\pi\nu}c[\nu]\,,\quad \mathrm{C}^{(k_0)}_2(\nu) = c[-\nu-1]\,.$$

Arbitrary n-fold Vertex Integral Family around boundary $k_0 \to \infty$

$$\mathrm{I}_{\tilde{\boldsymbol{a}}} = \sum_{\tilde{\boldsymbol{b}}=1}^{2^n} \mathrm{C}^{[\tilde{\boldsymbol{b}}]} f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]}$$

Through observation & conjecture (verified up to n=4)

General Solutions

$$f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]} = x^{\tilde{\mathrm{A}}} \frac{(\tilde{\mathrm{A}})_{|\boldsymbol{a}-\boldsymbol{b}|.\mathbf{1}}}{2^{|\boldsymbol{a}-\boldsymbol{b}|.\mathbf{1}}} \prod_{j=1}^{n} \left(\frac{(-1)^{b_j} i k_j x}{\tilde{\mathrm{B}}_j} \right)^{|a_j-b_j|} \times \tilde{\mathrm{F}}_4 (\mathrm{A}_1, \mathrm{A}_2;\ \mathrm{B}_1, \cdots, \mathrm{B}_n;\ k_1^2 x^2, \cdots, k_n^2 x^2)$$

Boundary Coefficients

For $h^{(2)},\ \nu_i \in R$

$$\mathrm{C}^{[\tilde{\boldsymbol{b}}]} = (-i)^{\nu_0+1} \Gamma(\tilde{\mathrm{A}}) \prod_{j=1}^{n} (-ik_i)^{-b_j(2\nu_j+1)} \mathrm{C}_{\tilde{b}_j}^{(k_0)}(\nu_j)$$

Notations

$$\tilde{\mathrm{A}} = \nu_0 + 1 - \boldsymbol{b}.(2\boldsymbol{\nu} + \mathbf{1}), \qquad \mathrm{A}_j = \frac{1}{2}\left(\tilde{\mathrm{A}} + |\boldsymbol{a} - \boldsymbol{b}|.\mathbf{1} - 1 + j\right),$$

$$\tilde{\mathrm{B}}_j = \nu_j + 1 - b_j(2\nu_j + 1), \qquad \mathrm{B}_j = \tilde{\mathrm{B}}_j + |a_j - b_j|,$$

$$\mathrm{C}^{[\tilde{\boldsymbol{b}}]} = (-i)^{\nu_0+1} \Gamma(\tilde{\mathrm{A}}) \prod_{j=1}^{n} (-ik_i)^{-b_j(2\nu_j+1)} \mathrm{C}_{\tilde{b}_j}^{(k_0)}(\nu_j),$$

$$\tilde{\mathrm{F}}_4(\mathrm{A}_1, \mathrm{A}_2; \mathbf{B}; \boldsymbol{z}) \equiv \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(\mathrm{A}_1)_{\boldsymbol{m}.\mathbf{1}} (\mathrm{A}_2)_{\boldsymbol{m}.\mathbf{1}}}{\prod_{i=1}^{n} (\mathrm{B}_i)_{m_i}} \prod_{i=1}^{n} \frac{z_i^{m_i}}{m_i!},$$

$$\boldsymbol{a} = a_1, a_2, \cdots, a_n, \qquad \boldsymbol{b} = b_1, b_2, \cdots, b_n,$$

$$\boldsymbol{\nu} = \nu_1, \nu_2, \cdots, \nu_n, \qquad \mathbf{1} = 1, 1, \cdots, 1,$$

$$|\boldsymbol{a} - \boldsymbol{b}| = |a_1 - b_1|, |a_2 - b_2|, \cdots, |a_n - b_n|.$$

Arbitrary n-fold Vertex Integral Family around boundary $k_0 \to \infty$ & $k_n \to \infty$

We give results of all these cases

General Solutions

- massless leg: $\boldsymbol{k} \to \infty$ in $\mathrm{e}^{-ik\tau}$
- massive leg: $\boldsymbol{k} \to \infty$ in $h_\nu(a, -k\tau)$

By our definition of $h^{(1,2)}$, IBP & DE & general solutions are the **same** for $h^{(1,2)}$ and $\nu_i \in \mathrm{Im}|\mathrm{R}$.

Boundary Coefficients

- massless leg: $\boldsymbol{k} \to \infty$ in $\mathrm{e}^{-ik\tau}$
- massive leg: $\boldsymbol{k} \to \infty$ in $h_\nu(a, -k\tau)$

- For $h^{(2)}$, $\nu_i \in \mathrm{R}$
- For $h^{(2)}, \nu_i \in \mathrm{Im}$
- For $h^{(1)}$, $\nu_i \in \mathrm{Im}|\mathrm{R}$

Check details of them and **analytic continuation** beyond the region of convergence by numerical DE in 2411.03088.

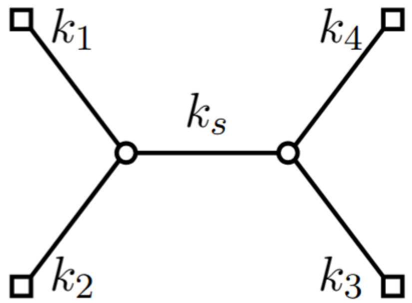
Part 04

Factorization of Homogeneous Solutions

dlog DE of 2-Vertex Integral

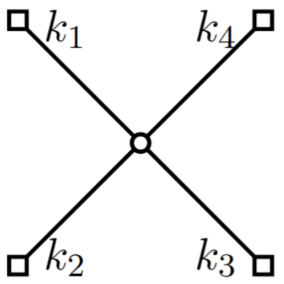
$$\mathrm{d}\mathbf{I} = \mathrm{d}\Omega\mathbf{I}$$

$$\Omega = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$$



(a) 2-vertex sector

4 MIs



(b) 1-vertex subsector

1 MI

$$k_{ij} = k_i + k_j$$

$$\mathbf{A} = \Omega_1(k_{12}, k_s) \otimes \mathbf{1}_{2\times 2} + \mathbf{1}_{2\times 2} \otimes \Omega_1(k_{34}, k_s),$$

$$\mathbf{C} = (-2\nu_0 + 2\nu_1 - 1)\log(k_{12} + k_{34}) + (-2\nu_1 - 1)\log(k_s)$$

$$\mathbf{R} = \begin{pmatrix} \frac{1}{2}i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\ \frac{1}{2}(\log(k_{12} - k_s) + \log(k_{12} + k_s) - \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\ \frac{1}{2}(-\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) + \log(k_{34} + k_s)) \\ \frac{1}{2}i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s)) \end{pmatrix}$$

Blow-up of DE: easy expansion of DE around arbitrary degenerate multivariate pole.

$$\mathbf{A} = \Omega_1(k_{12}, k_s) \otimes \mathbf{1}_{2\times 2} + \mathbf{1}_{2\times 2} \otimes \Omega_1(k_{34}, k_s),$$

$$\mathbf{C} = (-2\nu_0 + 2\nu_1 - 1)\log(k_{12} + k_{34}) + (-2\nu_1 - 1)\log(k_s)$$

$$\mathbf{R} = \begin{pmatrix} \frac{1}{2}i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\ \frac{1}{2}(\log(k_{12} - k_s) + \log(k_{12} + k_s) - \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\ \frac{1}{2}(-\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) + \log(k_{34} + k_s)) \\ \frac{1}{2}i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s)) \end{pmatrix}$$

$$\mathrm{I}_{\{a,b\}} \equiv \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 (-\tau_1)^{\nu_0} e^{ik_{12}\tau_1} (-\tau_2)^{\nu_0} e^{ik_{34}\tau_2} h(\nu_1, a, -k_s\tau_1)\theta^{(1,1)}_{1,2} h(\nu_1, b, -k_s\tau_2)$$

Exponential suppression boundary $(k_{12}, k_{34}) \to (\infty, \infty)$ is a degenerate pole

blow-up: $x = k_{34}/k_{12}$, $y = 1/k_{34}$ ⇨ pole: $(x, y) = (0, 0)$; region: $k_{12} \gg k_{34} \gg 1$

Indicial equations of 2-Vertex Integral Family

Power series expansion ansatz

$$x^\lambda y^\mu \sum_{j,k=0}^{\infty} \mathrm{C}(i,j,k) x^j y^k$$

$C(5,0,0) = 0 \rightarrow f_5 = 0$

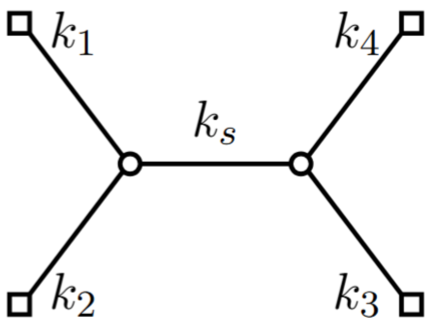
four **homogeneous** solutions

$$\{\mathrm{C}(i \neq 1,0,0) = 0, \lambda = \nu_0 + 1, \mu = 2\nu_0 + 2\},$$
$$\{\mathrm{C}(i \neq 2,0,0) = 0, \lambda = \nu_0 + 1, \mu = 1 + 2\nu_0 - 2\nu_1\},$$
$$\{\mathrm{C}(i \neq 3,0,0) = 0, \lambda = \nu_0 - 2\nu_1, \mu = 1 + 2\nu_0 - 2\nu_1\},$$
$$\{\mathrm{C}(i \neq 4,0,0) = 0, \lambda = \nu_0 - 2\nu_1, \mu = 2\nu_0 - 4\nu_1\},$$

One **non-homogeneous** solution

$$\Big\{\mathrm{C}(2,0,0) = \frac{\mathrm{C}(5,0,0)}{2\nu_1 - \nu_0}, \mathrm{C}(3,0,0) = \frac{\mathrm{C}(5,0,0)}{\nu_0 + 1},$$
$$\mathrm{C}(1,0,0) = \mathrm{C}(4,0,0) = 0, \lambda = \mu = 2\nu_0 - 2\nu_1 + 1\Big\}$$

Factorization of homogeneous general solutions



(a) 2-vertex sector

$$\Omega = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$$

4 homogeneous general solutions

$$\mathrm{I} = \sum_{i=1}^{5} \mathrm{C}^{[i]} \boldsymbol{f}^{[i]}$$

Factorized IBP→ factorized DE

$$\mathbf{A} = \Omega_1(k_{12}, k_s) \otimes \mathbf{1}_{2\times 2} + \mathbf{1}_{2\times 2} \otimes \Omega_1(k_{34}, k_s)$$

→ factorized homogeneous general solution

$$f_{\tilde{\boldsymbol{b}}}^{[\tilde{\boldsymbol{a}}]} = V_{\tilde{b}_1}^{[\tilde{a}_1]}(xy) V_{\tilde{b}_2}^{[\tilde{a}_2]}(y)\,, \quad f_5^{[\tilde{\boldsymbol{a}}]} = \mathbf{0}$$

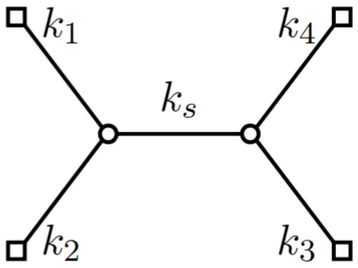
$$V_j^{[i]}(x) = f_j^{[i]}(1/x, k_s)$$

2 general solutions of **1-fold vertex integral family** we solved

Factorization of homogeneous boundary coefficients

region of blow-up
$k_{12} \gg k_{34} \gg 1$

region for integrand: $-1 \ll \tau_2 \ll \tau_1 < 0$



(a) 2-vertex sector

$$\mathrm{I}_{\{a,b\}} = -\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 (i\tau_1)^{\nu_0} e^{\tau_1/(xy)} (i\tau_2)^{\nu_0} e^{\tau_2/y}$$
$$\times \left[h^{(1)}(\nu_1, a, ik_s\tau_1)\theta(\tau_1 - \tau_2)h^{(2)}(\nu_1, b, ik_s\tau_2) + h^{(2)}(\nu_1, a, ik_s\tau_1)\theta(\tau_2 - \tau_1)h^{(1)}(\nu_1, b, ik_s\tau_2) \right]$$

$\theta(\tau_1 - \tau_2) = 1$, $\theta(\tau_2 - \tau_1) = 0$

the integrand for getting boundary coefficients

Factorized! $\left[\int \mathrm{d}\tau_1 (-\tau_1)^{\nu_0} e^{ik_{12}\tau_1} h^{(1)}(\nu_1, a, -k_s\tau_1)\right] \left[\int \mathrm{d}\tau_2 (-\tau_2)^{\nu_0} e^{ik_{34}\tau_2} h^{(2)}(\nu_1, b, -k_s\tau_2)\right]$

expand　　expand

Homogeneous boundary coefficients
$=\prod$ boundary coefficients of the vertex integral family

$$\mathrm{C}^{[1]} = -e^{-i\pi\nu_0}\Gamma(\nu_0+1)^2 \mathrm{C}_1^{*(k_0)}(\nu_1)\mathrm{C}_1^{(k_0)}(\nu_1)$$
$$\mathrm{C}^{[2]} = -ie^{-i\pi(\nu_0-\nu_1)}k_s^{-2\nu_1-1}\Gamma(\nu_0+1)\Gamma(\nu_0-2\nu_1)\mathrm{C}_1^{*(k_0)}(\nu_1)\mathrm{C}_2^{(k_0)}(\nu_1)$$
$$\mathrm{C}^{[3]} = -ie^{-i\pi(\nu_0-\nu_1)}k_s^{-2\nu_1-1}\Gamma(\nu_0+1)\Gamma(\nu_0-2\nu_1)\mathrm{C}_2^{*(k_0)}(\nu_1)\mathrm{C}_1^{(k_0)}(\nu_1)$$
$$\mathrm{C}^{[4]} = (k_s^2)^{-2\nu_1-1} e^{-i\pi(\nu_0-2\nu_1)}\Gamma(\nu_0-2\nu_1)^2 \mathrm{C}_2^{*(k_0)}(\nu_1)\mathrm{C}_2^{(k_0)}(\nu_1).$$

Factorization of homogeneous solution

Factorization of IBP & DE

⇩

Factorization of homogeneous general solutions

blow-up of k_i

⇩

order of τ_i

⇩ $\theta(\tau_i - \tau_j) = 0,1$

Factorization of homogeneous boundary coefficients

⏟

Factorization of homogeneous (special) solutions

→

Hypergeometric solutions of **arbitrary** vertex integral family by **power series expansion** of **dlog-form DE**

massless leg $\boldsymbol{k} \to \infty$ in $e^{-ik\tau}$

massive leg $\boldsymbol{k} \to \infty$ in $h_\nu(a, -k\tau)$

analytic continuation to any region via NDE

→

Homogeneous solutions of **arbitrary** tree-level cosmological correlator

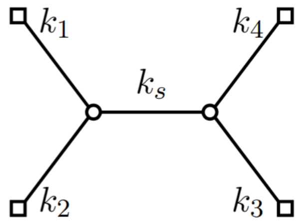
$= \prod$ solutions of vertex integral family

Example of non-homogeneous solution

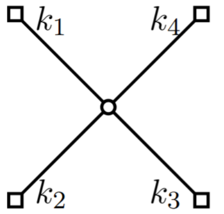
$$
\boldsymbol{f}^{[5]} = x^{2\nu_0-2\nu_1+1} y^{2\nu_0-2\nu_1+1} \times \begin{pmatrix}
\sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 xy\right)^{n+1} (2\nu_0-2\nu_1+1)_{m+2n+1} (4ix^n y^n)}{4m! k_s \left(\frac{\nu_0+1}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1+1)\right)_{n+1}} \\
\sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 x^2 y^2\right)^{n} (2\nu_0-2\nu_1+1)_{m+2n}}{2m! \left(\frac{\nu_0+2}{2}+\frac{m}{2}\right)_{n} \left(\frac{1}{2}(m+\nu_0-2\nu_1)\right)_{n+1}} \\
\sum_{m,n=0}^{\infty} \frac{(-x)^m (2\nu_0-2\nu_1+1)_{m+2n} \left(\frac{1}{4}k_s^2 x^2 y^2\right)^{n}}{2m! \left(\frac{\nu_0+1}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1+1)\right)_{n}} \\
\sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 xy\right)^{n+1} (2\nu_0-2\nu_1+1)_{m+2n+1} (4ix^n y^n)}{4m! k_s \left(\frac{\nu_0+2}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1)\right)_{n+1}} \\
(1+x)^{-1-2\nu_0+2\nu_1}
\end{pmatrix}
$$

⇩

$$
\begin{pmatrix}
\sum_{m=0}^{\infty} \frac{-ik_s xy(-x)^m (2\nu_0-2\nu_1+1)_{m+1}}{m!(\nu_0+m+1)(\nu_0-2\nu_1+m+1)} {}_3\mathrm{F}_2\left(\nu_0-\nu_1+\frac{m+2}{2}, \nu_0-\nu_1+\frac{m+3}{2}, 1; \frac{\nu_0+m+3}{2}, \frac{\nu_0-2\nu_1+m+3}{2}; k_s^2 x^2 y^2\right) \\
\sum_{m=0}^{\infty} \frac{-(-x)^m (2\nu_0-2\nu_1+1)_{m}}{m!(\nu_0-2\nu_1+m)} {}_3\mathrm{F}_2\left(\nu_0-\nu_1+\frac{m+1}{2}, \nu_0-\nu_1+\frac{m+2}{2}, 1; \frac{\nu_0+m+2}{2}, \frac{\nu_0-2\nu_1+m+2}{2}; k_s^2 x^2 y^2\right) \\
\sum_{m=0}^{\infty} \frac{(-x)^m (2\nu_0-2\nu_1+1)_{m}}{m!(\nu_0+m+1)} {}_3\mathrm{F}_2\left(\nu_0-\nu_1+\frac{m+1}{2}, \nu_0-\nu_1+\frac{m+2}{2}, 1; \frac{\nu_0+m+3}{2}, \frac{\nu_0-2\nu_1+m+1}{2}; k_s^2 x^2 y^2\right) \\
\sum_{m=0}^{\infty} \frac{-ik_s xy(-x)^m (2\nu_0-2\nu_1+1)_{m+1}}{m!(\nu_0+m+2)(\nu_0-2\nu_1+m)} {}_3\mathrm{F}_2\left(\nu_0-\nu_1+\frac{m+2}{2}, \nu_0-\nu_1+\frac{m+3}{2}, 1; \frac{\nu_0+m+4}{2}, \frac{\nu_0-2\nu_1+m+2}{2}; k_s^2 x^2 y^2\right) \\
(1+x)^{-1-2\nu_0+2\nu_1}
\end{pmatrix}
$$



(a) 2-vertex sector



(b) 1-vertex subsector

Part 5

Discussion:
Techniques & Flat QFT

d log-form DE: towards analytic evaluation beyond MPL?

MPL cases

elliptic & beyond cases

Canonical Differential Equations
arxiv: 1304.1806
ϵ-form
d log-form
MPL GPL

ϵ-form
eMPL & e.t.c.

comparison:
algebraic & simpler
easier to get

d log-form
Multivariate Hypergeometric Functions (**MHF**)

properties of MHF

~~tensor product of 2x2 matrices~~
getting MHF may not be easy

define new special functions via **d log DE**

All-order expansion of dlog DE is simple

$$\Omega_0 = A_i \frac{1}{c_0 - c_i} = \sum_i \frac{A_i}{c_0} \sum_{j=0}^{\infty} \frac{c_i^j}{c_0^j}$$

Fast evaluation

Properties of singularities

Analytic Continuation

Relations between power series solutions around different singular or regular points

……

Notes: power series expansion & blow-up of DE

When do we need the "generalized"?

power series expansion
$r^{\nu}(1+r+\cdots)$

vs

generalized power series expansion
arxiv: 1907.13234

$$\vec{f}^{(i)}_{\text{sing}}(t) = \sum_{j_1 \in S_i} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{N_i} \vec{c}^{\,(i,j_1,j_2,j_3)} (t-\tau)^{w_{j_1}+j_2} \log(t-\tau)^{j_3}$$

$r^{-1+\epsilon}$ expand

$$\frac{1}{r} + \frac{\mathrm{Log}[r]\,\epsilon}{r} + \frac{\mathrm{Log}[r]^2\,\epsilon^2}{2\,r} + O[\epsilon]^3$$

Blow-up of DE vs Method of Region

one blow-up transformation for one of the regions around the selected pole
→ boundary condition at a regular point
→ analytic continuation to all region by NDE

Sum over the contributions of all regions

I'm not saying it's better (at this moment, I don't know), but worth a try.

Part 05 Summary

Summary

- We generalize IBP & DE to integrals with function (Hankel) integrand case.
- Factorization of IBP → Factorization of DE → Factorization of homogeneous part solutions
- Construction of the h function → Formulas of iterative IBP & dlog DE (including remaining/subsector terms)
- Solutions of arbitrary vertex integral family $\xrightarrow{\text{+factorization}}$ homogeneous part solutions of arbitrary tree level cosmological correlators
- Techniques: solving dlog DE by power series expansion and give MHF solutions & blow-up of DE.

Thank you for listening

Backup

Details of IBP of θ-function

For G+- and G-+, no dθ(τi-τj), IBP of τ automatically factorized.

$$\int \left(\partial_{\tau_k}\hat{V}_k(\cdots;\tau_k)\mathrm{d}\tau_k\right) \times \prod_{j\neq k}\left(\hat{V}_j(\cdots;\tau_j)\mathrm{d}\tau_j\right) f(l_1,\cdots,l_L;k_1\cdots,k_E)\prod_i \mathrm{d}l_i$$

For G++ and G--, contributions from dθ(τi-τj) to IBP have two cases.

$$\left[h^{(a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,0,-k\tau_j)+h^{(3-a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,0,-k\tau_j)\right]\times\cdots$$

$$\left[h^{(a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,0,-k\tau_j)+h^{(3-a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,0,-k\tau_j)\right]\times\cdots$$

$$\left[h^{(a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,1,-k\tau_j)+h^{(3-a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,1,-k\tau_j)\right]\times\cdots$$

$$\left[h^{(a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,1,-k\tau_j)+h^{(3-a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,1,-k\tau_j)\right]\times\cdots$$

$$\theta_{ij}\equiv\theta(\tau_i-\tau_j)\,,\quad a=1,2\,.$$

Details of IBP of θ-function

Vanish

$$\int \mathrm{d}\tau_i \mathrm{d}\tau_j \left[h^{(a)}(\nu,0,-k\tau_i)h^{(3-a)}(\nu,0,-k\tau_j) - h^{(3-a)}(\nu,0,-k\tau_i)h^{(a)}(\nu,0,-k\tau_j)\right]\delta(\tau_i-\tau_j)\times\cdots$$
$$= \int \mathrm{d}\tau_i \left[h^{(a)}(\nu,0,-k\tau_i)h^{(3-a)}(\nu,0,-k\tau_i) - h^{(3-a)}(\nu,0,-k\tau_i)h^{(a)}(\nu,0,-k\tau_i)\right]\times\cdots = \boxed{0}$$
$$\int \mathrm{d}\tau_i \mathrm{d}\tau_j \left[h^{(a)}(\nu,1,-k\tau_i)h^{(3-a)}(\nu,1,-k\tau_j) - h^{(3-a)}(\nu,1,-k\tau_i)h^{(a)}(\nu,1,-k\tau_j)\right]\delta(\tau_i-\tau_j)\times\cdots$$
$$= \int \mathrm{d}\tau_i \left[h^{(a)}(\nu,1,-k\tau_i)h^{(3-a)}(\nu,1,-k\tau_i) - h^{(3-a)}(\nu,1,-k\tau_i)h^{(a)}(\nu,1,-k\tau_i)\right]\times\cdots = \boxed{0}.$$

Pinched

$$-(-1)^a \int \mathrm{d}\tau_i \left[F(-k\tau_i)\right]\times\cdots = +\int \mathrm{d}\tau_i\, C_\nu \frac{4i}{\pi}(-k\tau_i)^{-2\nu-1}\times\cdots$$
$$+(-1)^a \int \mathrm{d}\tau_i \left[F(-k\tau_i)\right]\times\cdots = -\int \mathrm{d}\tau_i\, C_\nu \frac{4i}{\pi}(-k\tau_i)^{-2\nu-1}\times\cdots$$
$$F(-k\tau_i) = h^{(1)}(\nu,1,-k\tau_i)h^{(2)}(\nu,0,-k\tau_i) - h^{(2)}(\nu,1,-k\tau_i)h^{(1)}(\nu,0,-k\tau_i)$$

$$F(-k\tau_i) = C_\nu \frac{4i}{\pi}(-k\tau_i)^{-2\nu-1},$$
$$C_\nu = \begin{cases} 1 & \text{for real } \nu \\ e^{-i\pi\nu} & \text{for imaginary } \nu \end{cases}$$

Construction of h-function & IBP and DE of ki

For IBP of τ

$$\partial_\tau h(\nu, 0, -k\tau) = -kh(\nu, 1, -k\tau)$$

$$\partial_\tau h(\nu, 1, -k\tau) = -k\left[\frac{1}{k\tau}(2\nu+1)h(\nu, 1; -k\tau) - h(\nu, 0; -k\tau)\right]$$

For IBP and DE of k

$$\partial_k h(\nu, 0, -k\tau) = -\tau h(\nu, 1, -k\tau)$$

$$\partial_k h(\nu, 1, -k\tau) = -\tau\left[\frac{1}{k\tau}(2\nu+1)h(\nu, 1; -k\tau) - h(\nu, 0; -k\tau)\right].$$

n-fold Vertex Integral Family

Integral Family

$$V(\nu_0, a_1, \cdots, a_n) = \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} \prod_{i=1}^{n} h(\nu_i, a_i; -k_i\tau) \mathrm{d}\tau, \qquad a_i = 0, 1\,.$$

Master Integrals

$$f_{\boldsymbol{a}}^{(0)} = V(\nu_0, \boldsymbol{a}), \quad \boldsymbol{a} = a_1, \cdots, a_n, \quad \forall a_i =, 0, 1. \qquad \#MIs = 2^n$$

IBP Relations

$$(M_1)_{\boldsymbol{ba}} f_{\boldsymbol{a}}^{(-1)} + (M_0)_{\boldsymbol{ba}} f_{\boldsymbol{a}}^{(0)} = 0$$

$$(M_1)_{\boldsymbol{ba}} = \sum_{j=1}^{n} \left[\left(M_1^{(j)}\right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + \nu_0 \delta_{\boldsymbol{ba}}$$

$$(M_0)_{\boldsymbol{ba}} = \sum_{j=1}^{n} \left[\left(M_0^{(j)}\right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + ik_0 \delta_{\boldsymbol{ba}}$$

$a_i = 0{,}1 \quad b_i = 0{,}1$

Tensor product of 2x2 matrices

Compare to n=1

$$\left(M_1^{(1)} + \nu_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(-1)} + \left(M_0^{(1)} + ik_0 \mathbb{I}_2\right) . \boldsymbol{f}^{(0)} = 0$$

$$M_1^{(j)} = -\frac{2\nu_j + 1}{2} (\mathbb{I}_2 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & -2\nu_j - 1 \end{pmatrix}, \qquad M_0^{(j)} = -ik_j \sigma_2 = \begin{pmatrix} 0 & -k_j \\ k_j & 0 \end{pmatrix},$$

n-fold Vertex Integral Family

IBP Relations

$$(M_1)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(-1)} + (M_0)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(0)} = 0$$

$$M_1 = \sum_{j=1}^{n} \left(\nu_j + \frac{1}{2}\right) \Lambda_3^{(j)} + \left(\nu_0 - \frac{n}{2} - \sum_{i=1}^{n} \nu_i\right) \mathbb{I}_{2^n}$$

$$M_0 = -i \sum_{j=1}^{n} k_j \Lambda_2^{(j)} + i k_0 \mathbb{I}_{2^n}$$

$$\left(\Lambda_k^{(j)}\right)_{\boldsymbol{ba}} \equiv (\sigma_k)_{b_j, a_j} \prod_{i \neq j} \delta_{b_i, a_i}, \; k = 1, 2, 3$$

Pauli matrices

n=2 matrix form

$$M_1 = M_1^{(1)} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes M_1^{(2)} + \nu_0 \mathbb{I}_2 \otimes \mathbb{I}_2$$

$$M_0 = M_0^{(1)} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes M_0^{(2)} + i k_0 \mathbb{I}_2 \otimes \mathbb{I}_2 .$$

$$\left(M_1 \middle| M_0\right) . \boldsymbol{f}^{\mathsf{T}} = \left(\begin{array}{cccc|cccc} \nu_0 & 0 & 0 & 0 & i k_0 & -k_2 & -k_1 & 0 \\ 0 & \nu_0 - 2\nu_2 - 1 & 0 & 0 & k_2 & i k_0 & 0 & -k_1 \\ 0 & 0 & \nu_0 - 2\nu_1 - 1 & 0 & k_1 & 0 & i k_0 & -k_2 \\ 0 & 0 & 0 & \nu_0 - 2\nu_1 - 2\nu_2 - 2 & 0 & k_1 & k_2 & i k_0 \end{array}\right) . \boldsymbol{f}^{\mathsf{T}} = 0 ,$$

$$\boldsymbol{f} = \{\boldsymbol{f}^{(-1)}, \boldsymbol{f}^{(0)}\}, \qquad \boldsymbol{f}^{(i)} = \{f_{0,0}^{(i)}, f_{0,1}^{(i)}, f_{1,0}^{(i)}, f_{1,1}^{(i)}\} ,$$

n-fold Vertex Integral Family Iterative Reduction

$$\left(\begin{array}{cccc|cccc} \nu_0 & 0 & 0 & 0 & ik_0 & -k_2 & -k_1 & 0 \\ 0 & \nu_0-2\nu_2-1 & 0 & 0 & k_2 & ik_0 & 0 & -k_1 \\ 0 & 0 & \nu_0-2\nu_1-1 & 0 & k_1 & 0 & ik_0 & -k_2 \\ 0 & 0 & 0 & \nu_0-2\nu_1-2\nu_2-2 & 0 & k_1 & k_2 & ik_0 \end{array}\right)$$

M_1 M_0

$$A_-(\nu_0) = -M_1^{-1}.M_0\,, \qquad A_+(\nu_0-1) = -M_0^{-1}.M_1\,.$$

Inverse of diagonal matrix

Inverse of this matrix is NOT easy to compute for large n, although you may not meet such case.

Diagonalization of M_0 can help.

$$\tilde{h}(\nu_i, a_i; -k\tau) = \sum_{b_i=0,1} \mathrm{T}_{a_i b_i} h(\nu_i, b_i; -k\tau)$$

$$\mathrm{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad \mathrm{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$\mathrm{T}.\sigma_2.\mathrm{T}^{-1} = \sigma_3\,, \quad \mathrm{T}.\sigma_3.\mathrm{T}^{-1} = -\sigma_2\,,$$

$$\tilde{\boldsymbol{f}}^{(a_0)} = \mathrm{T}_n.\boldsymbol{f}^{(a_0)}\,, \qquad (\mathrm{T}_n)_{\boldsymbol{ba}} = \prod_{i=1}^{n} T_{b_i a_i}\,, \qquad \mathrm{T}_n.\Lambda_2^{(j)}.\mathrm{T}_n^{-1} = \Lambda_3^{(j)}\,, \qquad \mathrm{T}_n.\Lambda_3^{(j)}.\mathrm{T}_n^{-1} = -\Lambda_2^{(j)}\,.$$

n-fold Vertex Integral Family Iterative Reduction

After diagonalization of M_0

$$\tilde{M}_1^{(j)} = -\frac{2\nu_i+1}{2}\left(\mathbb{I}_2+\sigma_2\right) , \qquad \tilde{M}_0^{(j)} = -ik_j\sigma_3 ,$$

$$\tilde{M}_1 = -\sum_{j=1}^{n}\left(\nu_j+\frac{1}{2}\right)\boxed{\Lambda_2^{(j)}} + \left(\nu_0 - \frac{n}{2} - \sum_{i=1}^{n}\nu_i\right)\mathbb{I}_{2^n},$$

$$\tilde{M}_0 = -\sum_{j=1}^{n} ik_j\boxed{\Lambda_3^{(j)}} + ik_0\mathbb{I}_{2^n}.$$

Example: n=2 matrix form IBP relations after diagonalization

$$\left(\tilde{M}_1\middle|\tilde{M}_0\right).\tilde{\boldsymbol{f}}^{\mathsf{T}} = 0 , \qquad \tilde{\boldsymbol{f}} = \{\tilde{\boldsymbol{f}}^{(-1)}, \tilde{\boldsymbol{f}}^{(0)}\} ,$$

$$\tilde{M}_1 = \begin{pmatrix} \nu_0-\nu_1-\nu_2-1 & \frac{1}{2}i\left(2\nu_2+1\right) & \frac{1}{2}i\left(2\nu_1+1\right) & 0 \\ -\frac{1}{2}i\left(2\nu_2+1\right) & \nu_0-\nu_1-\nu_2-1 & 0 & \frac{1}{2}i\left(2\nu_1+1\right) \\ -\frac{1}{2}i\left(2\nu_1+1\right) & 0 & \nu_0-\nu_1-\nu_2-1 & \frac{1}{2}i\left(2\nu_2+1\right) \\ 0 & -\frac{1}{2}i\left(2\nu_1+1\right) & -\frac{1}{2}i\left(2\nu_2+1\right) & \nu_0-\nu_1-\nu_2-1 \end{pmatrix}$$

$$\tilde{M}_0 = i\begin{pmatrix} k_0-k_1-k_2 & 0 & 0 & 0 \\ 0 & k_0-k_1+k_2 & 0 & 0 \\ 0 & 0 & k_0+k_1-k_2 & 0 \\ 0 & 0 & 0 & k_0+k_1+k_2 \end{pmatrix}$$

Universal formula of reduction of arbitrary n-Hankel Vertex Integral Family

$$A_-(\nu_0) = -M_1^{-1}.M_0 ,$$

$$A_+(\nu_0-1) = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\tilde{M}_1.\mathrm{T}_n = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\mathrm{T}_n.M_1 ,$$

No inverse of large matrix!