Recent Advancements in Intersection Theory

Giulio Crisanti

Based on

[2401.01897]— G. Brunello, V. Chestnov, G. Crisanti, H. Frellesvig, M. Mandal, P. Mastrolia

[2405.18178]— G. Crisanti, S. Smith

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Part 1 — Introduction and Motivation

Master Integrals and IBPs

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Modern multiloop calculations often require the computation of very large sets of integrals

Can be cut down by orders of magnitude by reducing to master integrals

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Integration by Parts Identities (IBPs)

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0$$

Requires solving a very large linear system

Systematised by Laporta's Algorithm

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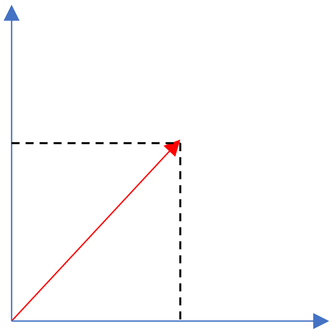
Arbitrary vector → **Vector Decomposition** → Basis vectors



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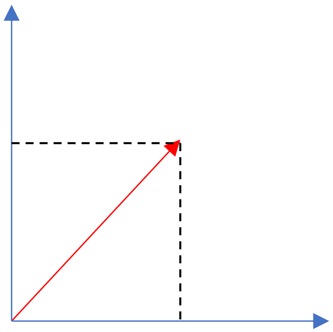

A given family of Feynman Integrals forms a (finite dimensional) vector space

$$J = \sum_{i=1}^{n} c_i e_i$$

Feynman Integrals → J

Master Feynman Integrals → e_i

IBP Coefficients → c_i

[Smirnov, Petukhov, 2011]
[Frellesvig, Gasparotto, Mandal, Mastrolia, Matiazzi, Mizera 2019]

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We can endow this vector space with an inner product $\langle - | - \rangle$, known as the intersection number. This inner product allows us to compute IBP decomposition coefficients. [Mastrolia, Mizera, 2018]

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$$c_i = \sum_{j=1}^{n} \langle J | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

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"Master Decomposition Formula"

To define intersection numbers properly we must switch representation of Feynman integrals.

Twisted Feynman Integrals

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Super quick Baikov review:

Momentum Representation

$$I_{\alpha_1 \cdots \alpha_m} \sim \int \left(\prod_i d^d k_i \right) \frac{1}{D_1^{\alpha_1} \cdots D_m^{\alpha_m}}$$

Baikov Representation

$$I_{\alpha_1 \cdots \alpha_m} \sim \int \left(\prod_i dz_i \right) \frac{b(z_1, \cdots, z_m)^\gamma}{z_1^{\alpha_1} \cdots z_m^{\alpha_m}}$$

[Baikov, 1996]

In Baikov the integration variables are the propagators of the Feynman Integral

The function $b(z_1, \cdots, z_m)$ is a multivariate polynomial that is unique for each family

$\mathcal{B}$ in Seva's talk

γ is a non integer parameter that depends on d

$b^\gamma = u$ is a multivalued function called the "twist"

Integrals of this kind are called "twisted integrals" — central objects of study for intersection theory

Twisted Cohomology Groups

Twisted integrals can be thought of as pairings between differential forms and contours

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H^n_ω The twisted cohomology group: Closed modulo exact forms

$\nu = \dim\ H^n_{\pm\omega} = \{\#\text{ of solutions to }\omega = 0\}$ is the number of *Master Integrals*

[Lee, Pomeransky, 2013]

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Matiazzi, Mastrolia, Mizera 2021]

Intersection Numbers

The intersection number is a pairing between two elements of H^n_ω

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$$\langle \varphi_L | \varphi_R \rangle = \int \iota(\varphi_L) \wedge \varphi_R$$

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For more than one variable things are more complicated, and there are several strategies one can adopt.

[Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia, 2022]

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Matiazzi, Mastrolia, Mizera 2021]

Part 2 — Recent Advancements in Intersection Theory

Relative Cohomology (1/3)

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$\rho \to 0$ at the end of the calculation

$$c_i = \lim_{\rho \to 0} \left(\sum_{i=1}^{n} \langle J | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \right)$$

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diverge!

finite!

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In practice this means we can use a new type of differential form, known as the "delta form"

$$\langle \varphi_L | \delta_z \rangle \equiv \mathrm{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi_L \right)$$

[Matsumoto,2018]
[Caron-Huot,Pokraka,2021]
[Caron-Huot,Pokraka,2022]
[Duhr,Porkert et al, 2024]

δ forms act as residue operators, "integrating out" variables from intersection numbers

(See Franziska's talk!)

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1-loop tadpole: $I_a = \int \frac{d^d k}{(k^2-1)^a} \propto \int_C \frac{(z-1)^{\gamma(d)}}{z^a} dz$ $\quad I_2 = c\, I_1 \longrightarrow \left\langle \frac{1}{z^2} \right| = c \left\langle \frac{1}{z} \right|$

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Okay, but why do these delta forms even work?

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Diverge, but how?

It turns out that intersection numbers diverge in a very systematic way

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{\rho} \, \mathrm{Res}_{z=0} \left(u(z) \, \varphi_L(z) \right) \times \mathrm{Res}_{z=0} \left(\varphi_R(z) / u(z) \right) + \mathcal{O}(\rho^0)$$

any 1-forms forms

[Brunello, Crisanti, Chestnov, Frellesvig, Mandal, Mastrolia 2023]

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$$u(0) \neq 0$$
$$u_\rho(0) = 0$$

$$c_i = \lim_{\rho \to 0} \left(\sum_{i=1}^{n} \langle J | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \right)$$

Diverge, but how?

It turns out that intersection numbers diverge in a very systematic way

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{\rho} \mathrm{Res}_{z=0} \left(u(z) \, \varphi_L(z) \right) \times \mathrm{Res}_{z=0} \left(\varphi_R(z) / u(z) \right) + \mathcal{O}(\rho^0)$$

any 1-forms forms

[Brunello, Crisanti, Chestnov, Frellesvig, Mandal, Mastrolia 2023]

Regulators Again (1/3)

Okay, but why do these delta forms even work?

Relative cohomology is deeply related to analytic regularisation

Can be interpreted as a careful analysis of what survives the limit as $\rho \to 0$

Regulated twist

Unregulated twist

$$u_\rho(z) = z^\rho \, u(z)$$

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only part that contributes to c_i

any 1-forms forms

[Brunello, Crisanti, Chestnov, Frellesvig, Mandal, Mastrolia 2023]

Regulators Again (2/3)

$$c_i = \lim_{\rho \to 0} \left(\sum_{i=1}^{n} \langle J | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji} \right)$$

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

$$=$$

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Regulators Again (2/3)

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[Fontana, Peraro 2023]

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Regulators Again (2/3)

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[Fontana, Peraro 2023]

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Regulators Again (2/3)

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Regulators Again (2/3)

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[Fontana, Peraro 2023]

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The delta-form secretly is a specific case of the formula above for $\varphi_R = 1/z$

$$\lim_{\rho\to 0}\left\langle\varphi_L\middle|\frac{\rho}{z}\right\rangle = \mathrm{Res}_{z=0}\left(u(z)\,\varphi_L(z)\right)\times\mathrm{Res}_{z=0}\left(1/(z\,u(z))\right) = \frac{\mathrm{Res}_{z=0}\left(u(z)\,\varphi_L(z)\right)}{u(0)} \equiv \langle\varphi_L|\delta(z)\rangle$$

[Brunello, Crisanti, Chestnov, Frellesvig, Mandal, Mastrolia 2023]

Regulators Again (3/3)

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Consider the 1-loop tadpole $I_a = \int \frac{d^d k}{(k^2-1)^a} \propto \int_C \frac{(z-1)^{\gamma(d)}}{z^a} dz \quad u_\rho = z^\rho (z-1)^\gamma$

Regulators Again (3/3)

Università degli Studi di Padova

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$$\left\langle \frac{1}{z^2} \middle| \frac{1}{z} \right\rangle = \frac{\gamma}{\rho(\rho-1)}$$

$$\left\langle \frac{1}{z} \middle| \frac{1}{z} \right\rangle = \frac{\gamma}{\rho(\rho+\gamma)}$$

$$c = \lim_{\rho \to 0} \left(\frac{\rho(\rho+\gamma)}{\rho(\rho-1)} \right) = -\gamma$$

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Regulators Again (3/3)

Università degli Studi di Padova

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$$\lim_{\rho\to 0} \left\langle \frac{1}{z^2} \middle| \frac{\rho}{z} \right\rangle = -\gamma \quad \leftarrow \quad \mathrm{Res}_{z=0}\left(\frac{(z-1)^\gamma}{z^2} \right) \times \mathrm{Res}_{z=0}\left(\frac{(z-1)^{-\gamma}}{z} \right) = -\gamma$$

$$\lim_{\rho\to 0} \left\langle \frac{1}{z} \middle| \frac{\rho}{z} \right\rangle = 1 \quad \leftarrow \quad \mathrm{Res}_{z=0}\left(\frac{(z-1)^\gamma}{z} \right) \times \mathrm{Res}_{z=0}\left(\frac{(z-1)^{-\gamma}}{z} \right) = 1$$

Example Computation (2/2)

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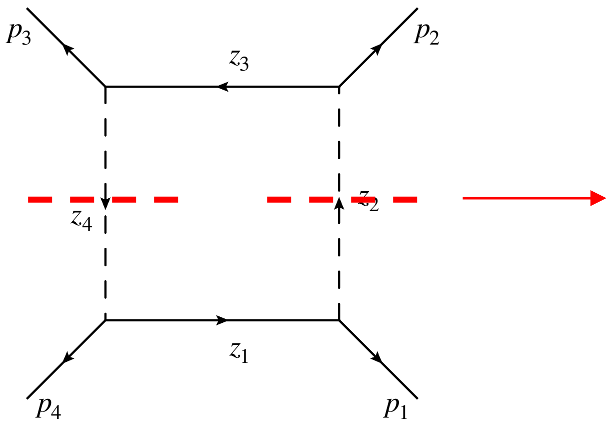
Consider a one loop box on the double cut



$$\langle \varphi_L | = \left\langle \frac{1}{z_1 z_3^2} \right|$$

$$b = 4m^2 \left(st + (z_1 - z_3)^2 \right) - s^2 t + 2st (z_1 + z_3) + 4s z_1 z_3 - t (z_1 - z_3)^2$$

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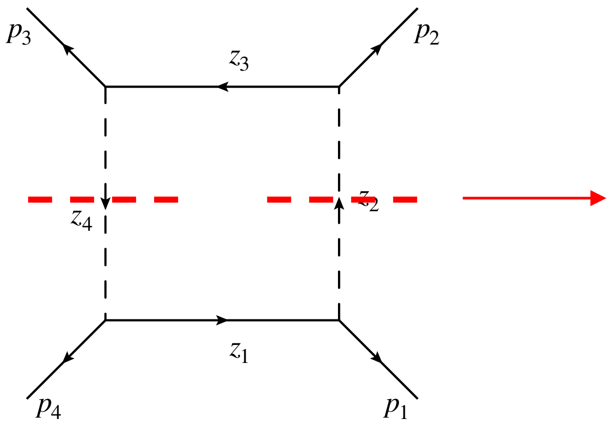

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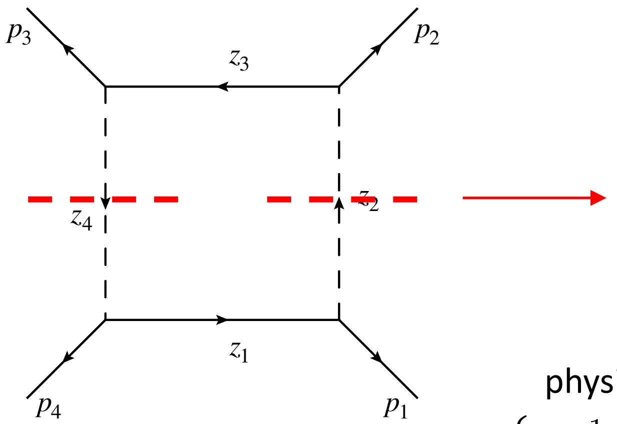

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physical basis

$$\tilde{e} = \left\{1\,,\ \frac{1}{z_1}\,,\ \frac{1}{z_3}\,,\ \frac{1}{z_1 z_3}\right\}$$

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Università degli Studi di Padova

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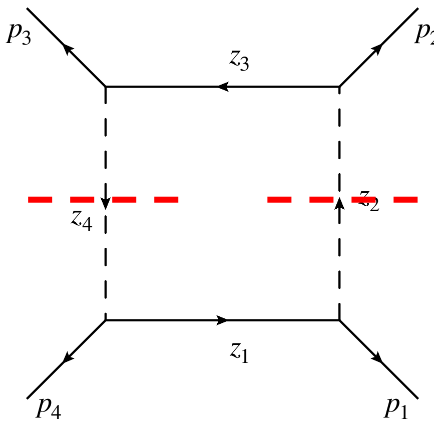

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dual basis

$$\tilde{h} = \{1, \delta_1, \delta_3, \delta_{13}\}$$

$$\delta_{ab} = \delta(a)\delta(b)$$

Example Computation (2/2)

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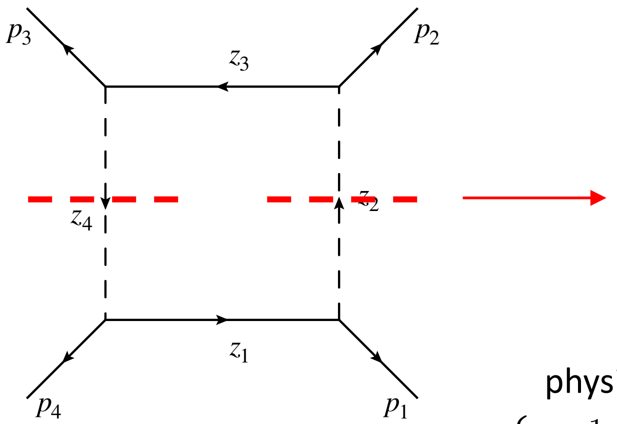

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$$\tilde{h} = \{1, \delta_1, \delta_3, \delta_{13}\} \qquad \delta_{ab} = \delta(a)\delta(b)$$

$$\tilde{\mathbf{C}} = \langle \tilde{e} | \tilde{h} \rangle = \begin{pmatrix} -\frac{st^2\left(-4m^2+s+t\right)}{4(d-7)(d-3)} & 0 & 0 & 0 \\ \frac{st^2\left((44-8d)m^2+(d-6)t\right)\left(-4m^2+s+t\right)}{2(d-7)(d-6)(d-4)(t-4m^2)^2} & -\frac{4(d-5)m^2 st\left(4m^2-s-t\right)}{(d-6)(d-4)(t-4m^2)^2} & 0 & 0 \\ \frac{st^2\left((44-8d)m^2+(d-6)t\right)\left(-4m^2+s+t\right)}{2(d-7)(d-6)(d-4)(t-4m^2)^2} & 0 & -\frac{4(d-5)m^2 st\left(4m^2-s-t\right)}{(d-6)(d-4)(t-4m^2)^2} & 0 \\ \frac{st^2}{(d-7)(d-6)(4m^2-t)} & \frac{st}{(d-6)(t-4m^2)} & \frac{st}{(d-6)(t-4m^2)} & 1 \end{pmatrix}$$

Recent Applications

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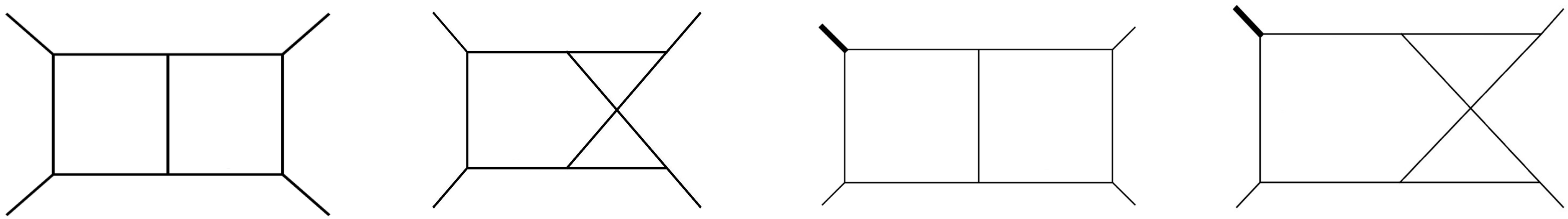
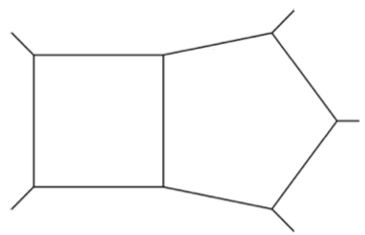
Up to four point non-planar Feynman Integrals at two loops:



[Brunello, Crisanti, Chestnov, Frellesvig, Mandal, Mastrolia 2023]

Pentabox:



[Brunello, Chestnov, Mastrolia 2024]

Fourier Integrals in QCD and Gravity:

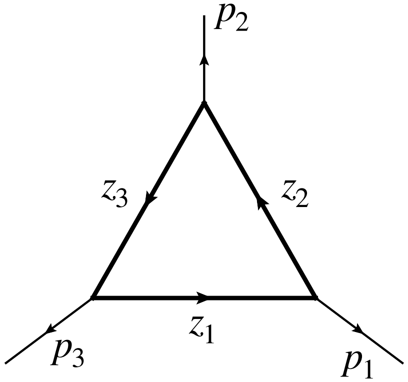
$$\mathcal{I}_\alpha = \int_{\mathcal{M}} d^d q \frac{\delta(u_1 \cdot q)\delta(u_2 \cdot (q-k))e^{-iq\cdot b}}{[q^2 - i\varepsilon]^\alpha} \qquad G^{ij} = \int d^d q_1 d^d q_2 \frac{N_I^{ij}(q_1, q_2)e^{i(q_1\cdot x_1 + q_2 \cdot x_2)}}{(q_1+q_2)^2(q_1^2\tau + q_2^2)}$$

[Brunello, Crisanti, Giroux, Smith, Mastrolia 2023]

Hidden Structures (1/3)

Delta forms allow for the computation of much simpler intersection numbers for Feynman Integrals

Can help us shed light on properties of Feynman integral bases previously not known



$$e = \left\{ \frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_1 z_2 z_3} \right\}$$

$$h = \left\{ \frac{\delta_1}{b^2}, \frac{\delta_2}{b^2}, \frac{\delta_3}{b^2}, \frac{\delta_{12}}{b}, \frac{\delta_{13}}{b}, \frac{\delta_{23}}{b}, \delta_{123} \right\}$$

$$\mathbf{C} = \begin{pmatrix} \frac{4}{3(d-2)^2 m_2^4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3(d-2)^2 m_2^4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3(d-2)^2 m_2^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{(d-3)m_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(d-3)m_2^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(d-3)m_2^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

[Crisanti, Smith, 2024]

Hidden Structures (2/3)

By studying the elements on the diagonal of the C-matrix unexpected patterns seem to emerge

We conjecture a new closed formula for computing intersection numbers for quadratic twists

$$\langle b(\mathbf{z})^p | b(\mathbf{z})^q \rangle = f_n(p, q; \gamma) \times \frac{\det\left(\mathbf{H}(b_h)\right)^{n+p+q}}{\det\left(\mathbf{H}(b)\right)^{n+p+q+1}} \qquad u = b^\gamma$$

Factorised prefactor

Homogenised polynomial

Hessian determinants are multivariate discriminants of quadratic polynomials

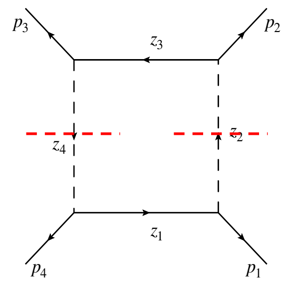
First formula of its kind beyond dLog forms

[Crisanti, Smith, 2024]

Hidden Structures (3/3)

Example usage:

$$b = 4m^2\left(st + (z_1 - z_3)^2\right) - s^2 t + 2st\,(z_1 + z_3) + 4s z_1 z_3 - t\,(z_1 - z_3)^2$$
$$u = b^{(d-5)/2}$$



Evaluate: $\left\langle 1 \middle| \frac{1}{b^2} \right\rangle$

The various components evaluate to

$$f_2\left(0, -2; \frac{d-5}{2}\right) = \frac{4}{(d-3)^2}$$
$$\det\left(\mathbf{H}(b)\right) = -16 s t^2 (-4m^2 + s + t)$$
$$\det\left(\mathbf{H}(b_h)\right) = -32 s^2 t^4 (-4m^2 + s + t)^2$$

And thus

$$\left\langle 1 \middle| \frac{1}{b^2} \right\rangle = f_2\left(0, -2; \frac{d-5}{2}\right) \times \frac{1}{\det\left(\mathbf{H}(b)\right)} = -\frac{1}{4(d-3)^2 s t^2\left(-4m^2 + s + t\right)}$$

No intersection number computation strategy needed!

Conclusions

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- The (twisted cohomology) intersection number is an inner product between elements of a cohomology group (and its dual).

- Through the usage of parametric representations such as Baikov Feynman integrals can be cast as twisted period integrals. These twists must be regulated however for ordinary intersection numbers to apply.

- Relative cohomology intersection numbers use a different formalism to allow for the removal of regulators from intersection computations. Through careful analysis, we can show how the relative intersection number can be thought of as a limit of ordinary intersection numbers.

- Relative cohomology intersection numbers seem to be closer to the “true nature” of Feynman integrals — many patterns seem to emerge when studying their properties.

Thank you for listening!

Relative Cohomology Extra

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Regulated twist

Unregulated twist

$$u_\rho(z) = z^\rho\, u(z)$$

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi \varphi_R) \qquad \nabla_p \psi \equiv \left(d + \frac{du}{u} \right) \psi = \varphi_L$$

Formal solution

$$\psi_p(z) = \frac{u_\rho(z)}{(e^{2\pi i \alpha_p} - 1)} \int_{C_p(z)} \frac{\varphi_L(t)}{u_\rho(t)} \quad \xrightarrow{p=0} \quad \psi_0(z) = \frac{z^\rho\, u(z)}{(e^{-2\pi i \rho} - 1)} \int_{C_0(z)} t^{-\rho} \frac{\varphi_R(t)}{u(t)}$$

$$\psi_0(z) = -\frac{u(z)}{2\pi i \rho} \int_{C_0(z)} \frac{\varphi_R(t)}{u(t)} + \mathcal{O}(\rho^0) = -\frac{u(z)}{2\pi i \rho} \oint_\epsilon \frac{\varphi_R(t)}{u(t)} + \mathcal{O}(\rho^0)$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i \rho} \mathrm{Res}_{z=0} \left(u(z)\, \varphi_L(z) \oint_\epsilon \frac{\varphi_R(t)}{u(t)} \right) + \mathcal{O}(\rho^0) = \frac{1}{\rho} \mathrm{Res}_{z=0} \left(u(z)\, \varphi_L(z) \right) \times \mathrm{Res}_{z=0} \left(\varphi_R(z)/u(z) \right)$$