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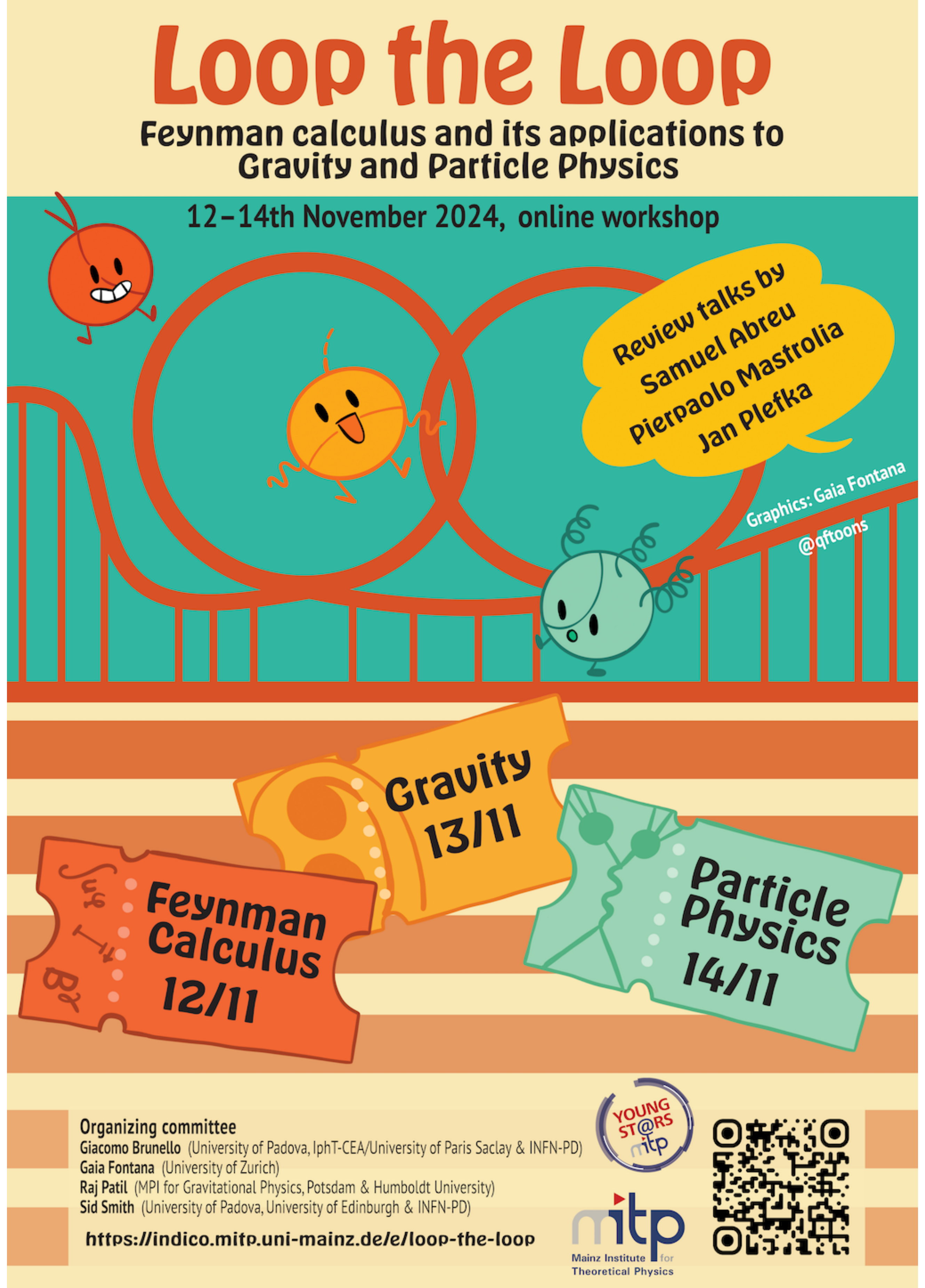
Twisted Cohomology & Canonical Differential Equations



Loop the Loop
Feynman calculus and its applications to Gravity and Particle Physics
12–14th November 2024, online workshop

Review talks by
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Jan Plefka

Graphics: Gaia Fontana
@qftoons

Feynman Calculus 12/11
Gravity 13/11
Particle Physics 14/11

Organizing committee
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https://indico.mitp.uni-mainz.de/e/loop-the-loop

YOUNG ST@RS mitp
mitp Mainz Institute for Theoretical Physics

Franziska Porkert

with Claude Duhr, Cathrin Semper & Sven Stawinski

arXiv: 2408.04904
arXiv: 2407.17175
+ work in progress (arXiv:2412.XXXX)

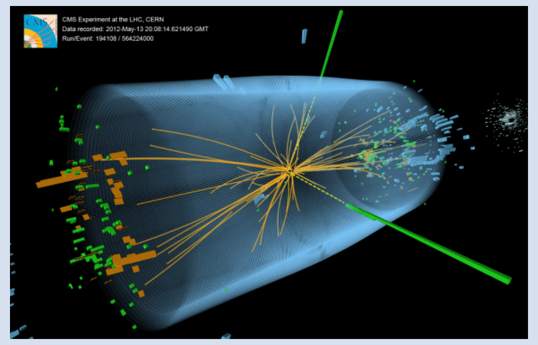
Loop - the - Loop Workshop, 12.11.2024

MOTIVATION

(canonical)
differential equation

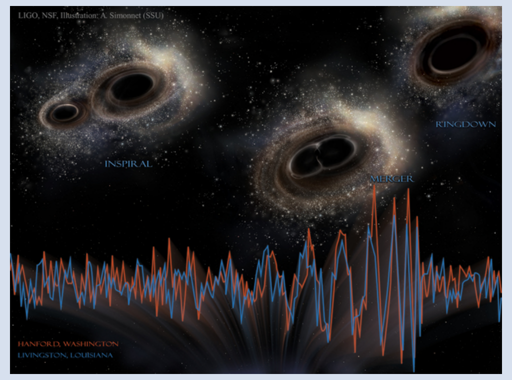
geometry

function space



CERN

FEYNMAN
INTEGRALS



Aurore Simones (Sonoma State U.)

Twisted
cohomology

CONTENTS

Feynman Integrals **Canonical Differential Equations**

Feynman Integrals **Twisted Cohomology**

Canonical Differential Equations **Twisted Cohomology**

Feynman Integrals

Canonical Differential Equations

FEYNMAN INTEGRALS FROM DIFFERENTIAL EQUATIONS

We want to compute a **Feynman integral family** analytically with ***differential equations***.

$D = d - 2\varepsilon$

$$I_{\boldsymbol{\nu}} \sim \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{\frac{D}{2}}} \right) \prod_{j=1}^{n_{\text{int}}} \frac{1}{D_i^{\nu_i}}$$

- Use IBPs to find a **basis of master integrals** for the integral family
- Set up a **differential equation** w.r.t the external (kinematic) parameters

  $\mathrm{d}\mathbf{I}(\mathbf{X}) = A(\mathbf{X}, \varepsilon)\mathbf{I}(\mathbf{X})$ with $\mathrm{d} = \sum \mathrm{d}X_i \partial_{X_i}$ where X_i are kinematic variables
- Find a **canonical differential equation** [Henn]

  $\mathbf{J}(\mathbf{X}) = \mathbf{U} \cdot \mathbf{I}(\mathbf{X})$ with $\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X})$ and $\varepsilon B(\boldsymbol{X}) = (\mathrm{d}\boldsymbol{U}) \cdot \boldsymbol{U}^{-1} + \boldsymbol{U} \cdot \boldsymbol{A}(\boldsymbol{X}, \varepsilon) \cdot \boldsymbol{U}^{-1}$

  and solve in terms of **iterated integrals.**

$$\mathbf{J}(\mathbf{X}) = \mathbb{P}\mathrm{exp}\left(\varepsilon \int_\gamma B \right) \cdot \mathbf{J}\left(\text{some point } \mathbf{X}^0\right) = \left(1 + \varepsilon \int_\gamma B + \varepsilon^2 \int_\gamma B \int_\gamma B + \dots \right) \cdot \mathbf{J}(\mathbf{X}^0)$$

FEYNMAN INTEGRALS FROM DIFFERENTIAL EQUATIONS

We want to compute a **Feynman integral family** analytically with ***differential equations***.

- Use IBPs to find a **basis of master integrals** for the integral family

How can this be improved? [See talks by Giulio, Vsevolod, Rourou]

Twisted cohomology

- Find a **canonical differential equation** and solve in terms of **iterated integrals.**

How can this be found systematically? [See talk by Sara]

What are these?

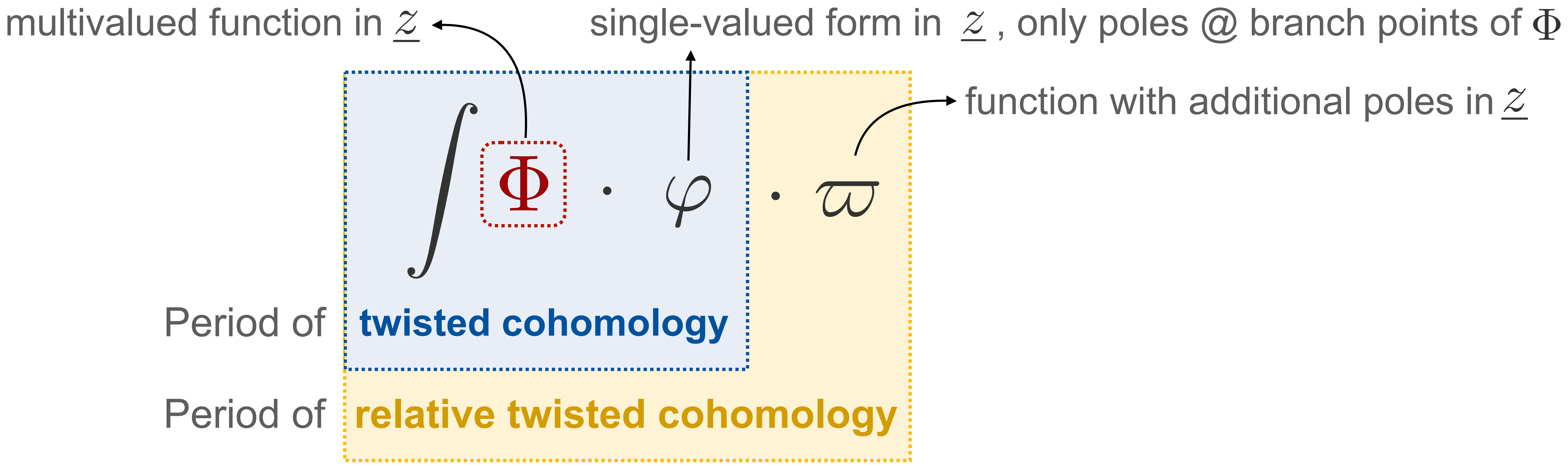
What can we learn from **Twisted cohomology** ?

See also: [Talk by Shounak] [Caron-Huot, Pokraka | Giroux, Pokraka]

Feynman Integrals

Twisted Cohomology

SHORT REVIEW: (RELATIVE) TWISTED COHOMOLOGY



$$\int \Phi \cdot \varphi \cdot \varpi$$

multivalued function in $\underline{z}$

single-valued form in $\underline{z}$, only poles @ branch points of Φ

function with additional poles in $\underline{z}$

Period of **twisted cohomology**

Period of **relative twisted cohomology**

$$\begin{tikzcd}
\textbf{cohomology}\text{ group with basis } \{\varphi_i\} \arrow[rr, leftrightarrow, "C_{ij} \sim \int \varphi_i \wedge \check{\varphi}_j"] \arrow[d, leftrightarrow, "P_{ij} = \int_{\gamma_i} \Phi \cdot \varphi_i"'] & & \textbf{dual cohomology}\text{ group with basis } \{\check{\varphi}_i\} \arrow[d, leftrightarrow, "\check{P}_{ij} = \int_{\check{\gamma}_i} \Phi^{-1} \cdot \check{\varphi}_i"] \\
\textbf{homology}\text{ group with basis } \{\gamma_i\} \arrow[rr, leftrightarrow, "H_{ij} \sim \text{weighted topological intersection of } \gamma_i \,\&\, \check{\gamma}_j"'] & & \textbf{dual homology}\text{ group with basis } \{\check{\gamma}_i\}
\end{tikzcd}$$

can choose:
$\check{\boldsymbol{P}}_\varepsilon = \boldsymbol{P}_{-\varepsilon}$

FEYNMAN INTEGRALS AS TWISTED RELATIVE PERIODS

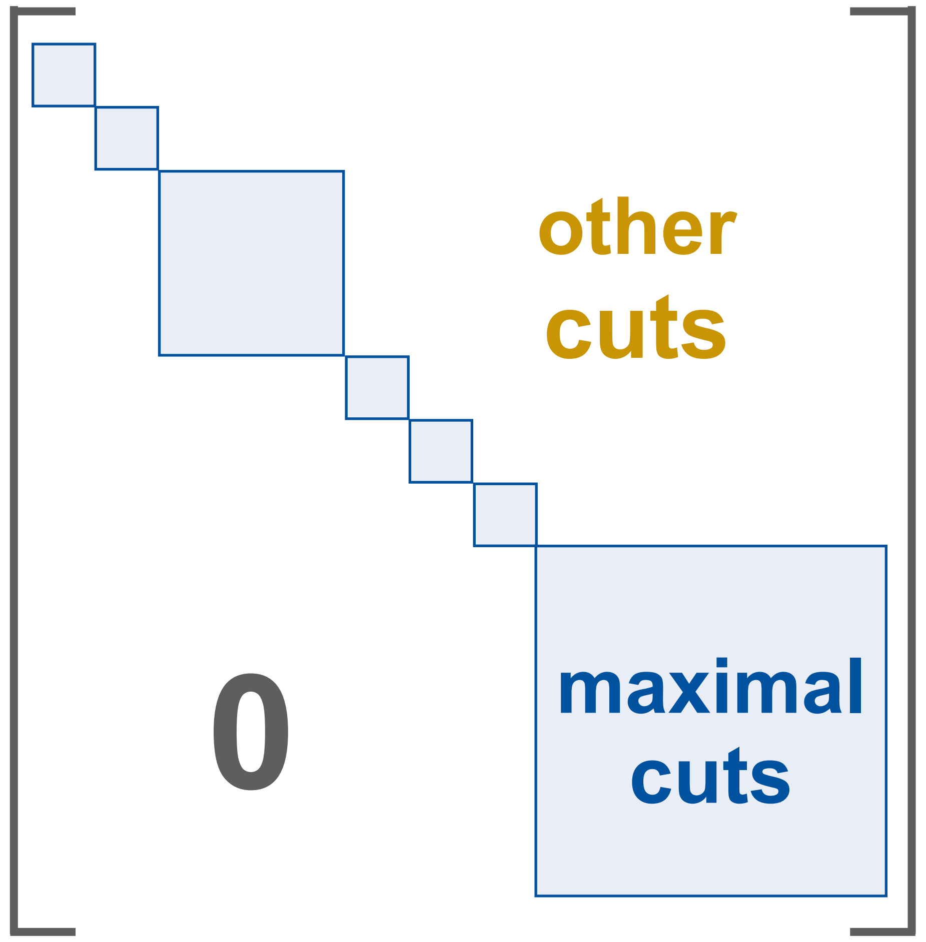
non-integer, contains ε

Baikov representation: $I_{\underline{\nu}}(\underline{X}) = \int_\Gamma \boxed{\mathcal{B}(\underline{z})^\mu} \cdot \mathrm{d}^n \underline{z} \; \prod_i z_i^{-\nu_i}$

Baikov polynomial

- Define relative twisted (co-)homology groups and their duals
  [Mastrolia, Mizera | Caron-Huot, Pokraka]
- Obtain period and intersection matrices $\boldsymbol{P}_\varepsilon, \check{\boldsymbol{P}}_\varepsilon, \boldsymbol{H}_\varepsilon, \boldsymbol{C}_\varepsilon$

$\boldsymbol{P}_\varepsilon \sim$



- Period matrix = Fundamental solution of DEQ: $\mathrm{d}\boldsymbol{P}_\varepsilon = \boldsymbol{A}(\boldsymbol{X},\varepsilon)\,\boldsymbol{P}_\varepsilon$

MAXIMAL CUTS AS TWISTED PERIODS

Maximal cut: $\sim \int \frac{\prod_{j=1}^{L} \mathrm{d}\ell_j^{D}}{\prod_{i=1}^{N} D_i^{\nu_i}} \bigg| \boxed{\frac{1}{D_i} \to \delta(D_i)}$

Baikov representation: $\boxed{I_{\underline{\nu}}(\underline{X}) = \int_{\Gamma} \boxed{\mathcal{B}(\underline{z})^{\mu}} \cdot \mathrm{d}^n \underline{z} \cdot} \boxed{\cancel{\prod_i z_i^{-\nu_i}}}$

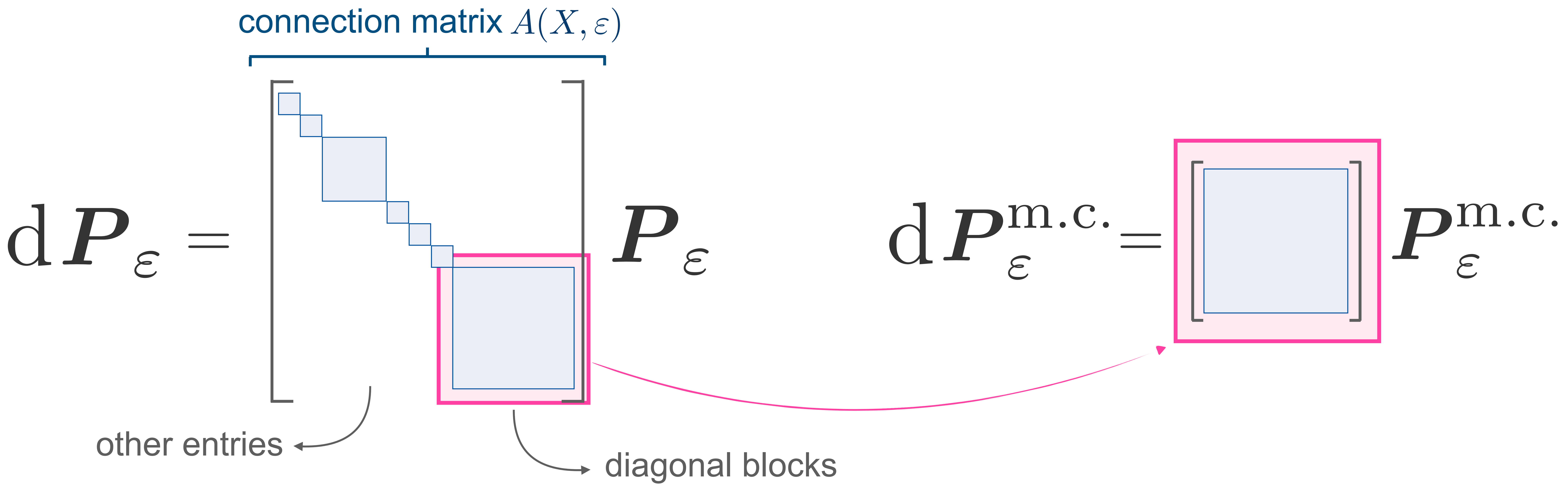
$z_1, \ldots, z_N \mapsto 0 \quad \mathrm{d}^n \underline{z} \mapsto \mathrm{d}^{n-N} \underline{z}$

- Define ~~relative~~ twisted (co-)homology groups and their duals
- Obtain period and intersection matrices $\boldsymbol{P}_\varepsilon, \check{\boldsymbol{P}}_\varepsilon, \boldsymbol{H}_\varepsilon, \boldsymbol{C}_\varepsilon$

$$\boldsymbol{P}_\varepsilon \sim \begin{bmatrix} \boxed{\textbf{maximal cuts}} \end{bmatrix}$$

DIFFERENTIAL EQUATIONS FOR PERIOD MATRICES

Maximal cut = Fundamental solution of the homogenous differential equation of the top sector



SELF-DUALITY AND BILINEAR RELATIONS FOR MAXIMAL CUTS

TWISTED RIEMANN BILINEAR RELATIONS

$$\boldsymbol{C} = \frac{1}{(2\pi i)^n} \boldsymbol{P} \left(\boldsymbol{H}^{-1}\right)^T \check{\boldsymbol{P}}^T$$

SELF - DUALITY FOR MAXIMAL CUTS

For maximal cuts, we can choose bases, s.t.

$$\check{\boldsymbol{P}}_\varepsilon = \boldsymbol{P}_{-\varepsilon} \Rightarrow \check{\boldsymbol{A}}(\boldsymbol{X}, \varepsilon) = \boldsymbol{A}(\boldsymbol{X}, -\varepsilon)$$

- See also:
  [Caron-Huot, Pokraka | Giroux, Pokraka | De, Pokraka]
- Self-duality for full Feynman integrals:
  [Pögel, Wang, Weinzierl, Wu, Xu] [Talk by Sebastian]

$$\boldsymbol{C} = \frac{1}{(2\pi i)^n} \boldsymbol{P}_\varepsilon \cdot \left(\boldsymbol{H}^{-1}\right)^T \boldsymbol{P}_{-\varepsilon}^T$$

Bilinear relations for maximal cuts

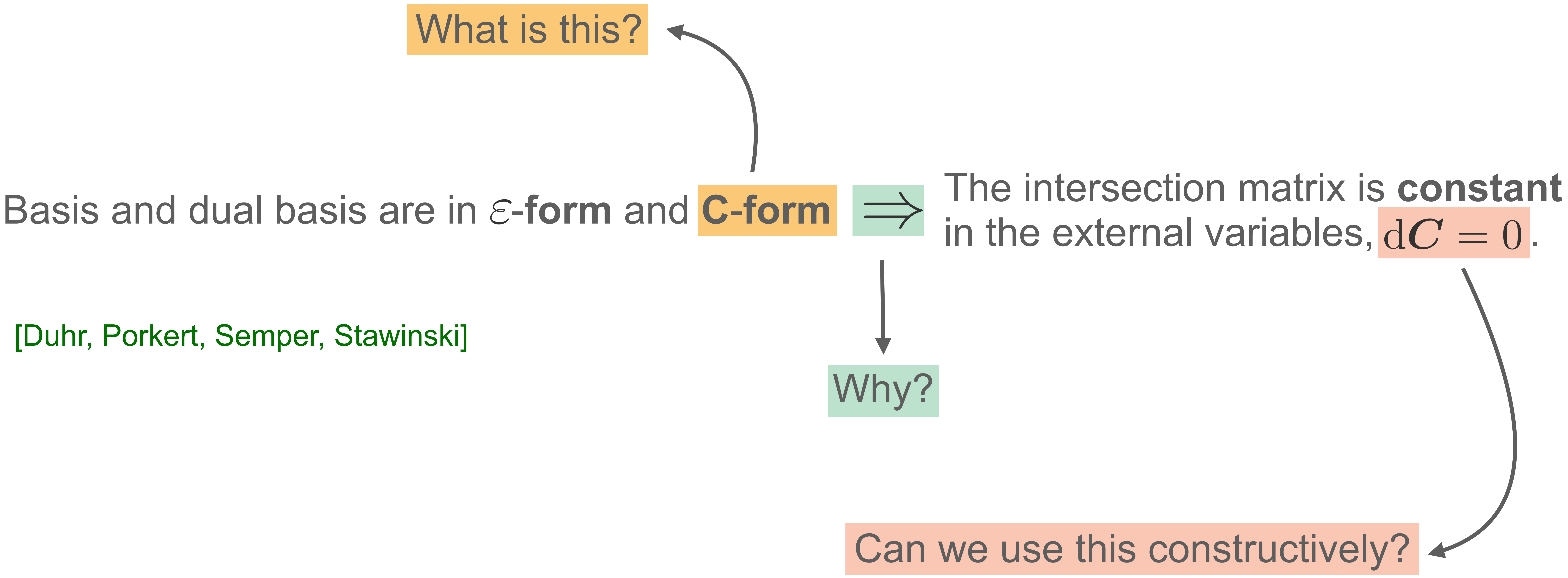
$\varepsilon = 0$

- For Calabi-Yau: Griffiths transversality
- For Hyperelliptic Curve: Legendre Relations

Canonical Differential Equations

Twisted Cohomology

CENTRAL THEOREM



What is this?

Basis and dual basis are in ε-**form** and **C-form** $\Longrightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}C = 0$.

[Duhr, Porkert, Semper, Stawinski]

Why?

Can we use this constructively?

Also observed for examples in [Caron-Huot, Pokraka | Giroux, Pokraka | De, Pokraka]

THE C - FORM

Short version: All known (to us) canonical DEQS for Feynman integrals are also in C-form!

THE C - FORM

Short version: All known (to us) canonical DEQS for Feynman integrals are also in C-form!

Long version:

$$\mathrm{d}\mathbf{J}(\mathbf{X}) = \varepsilon B(\mathbf{X})\mathbf{J}(\mathbf{X}) \quad \text{with} \quad B(\boldsymbol{X})_{ij} = \sum_{k=1}^{n} \mathrm{d}X_k f_{ijk}$$

$\mathcal{A}$ $=$ $\mathbb{K}$ - algebra of functions that contains all f_{ijk} and:

- Differentially closed ($f \in \mathcal{A} \Rightarrow \partial_{X_i} f \in \mathcal{A} \forall i$)
- Constants = $\mathbb{K}$ ($\partial_{X_i} f = 0 \, \forall i \Rightarrow f \in \mathbb{K}$)

$\mathbb{A}$ $=$ $\mathbb{K}$ - vector space of closed differential forms generated by the forms appearing in $B(\boldsymbol{X})$

$\mathcal{F}_{\mathbb{C}}$ $=$ $\mathrm{Frac}\,(\mathbb{C} \otimes_{\mathbb{K}} \mathcal{A})$

An ε- factorised differential equation is in C-form, if $\mathbb{A} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$.

[Duhr, Semper, Stawiński, FP]

Example:

$B(\boldsymbol{X})$ in dLog-form with

$$f_{ij} = \sum_r \frac{1}{a_{ijr} - X}$$

- $\mathcal{A}_{\mathrm{dLog}} =$ Rational functions in X with singularities at the a_{ijr}
- $\mathbb{A}_{\mathrm{dLog}} = \left\langle \frac{\mathrm{d}X}{a_{ijr} - X} \,\middle|\, \text{all } i, j, r \right\rangle$

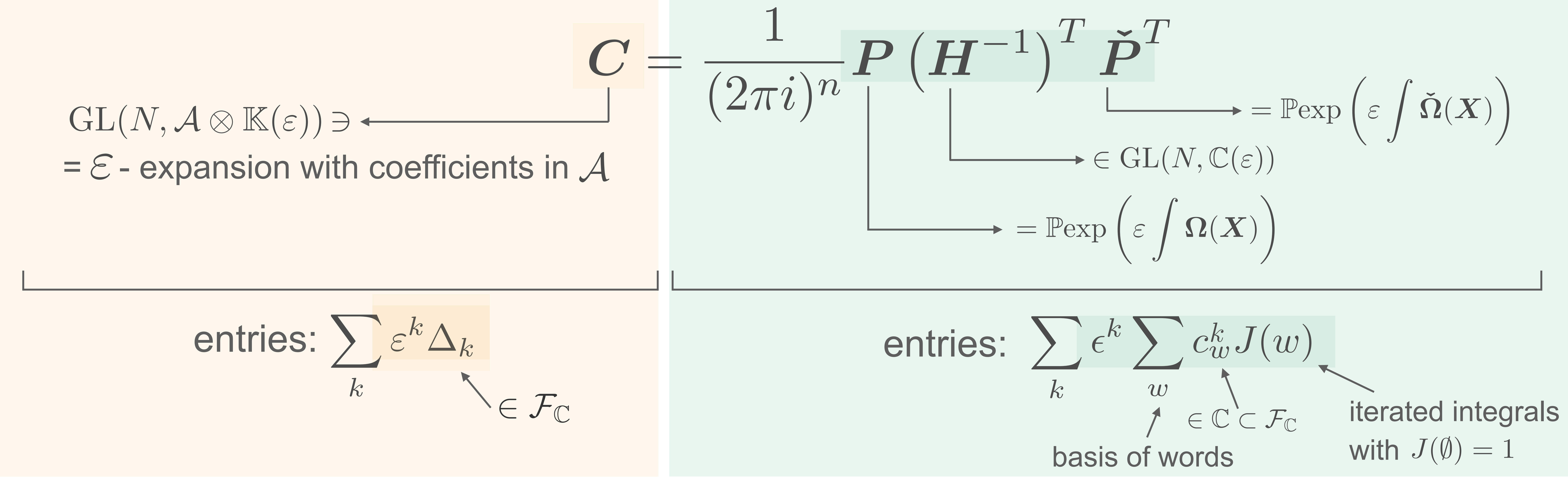
Elements of $\mathrm{d}\mathcal{F}_{\mathbb{C}}$:
no pole/ pole of order > 1

$$\Rightarrow \mathbb{A}_{\mathrm{dLog}} \cap \mathrm{d}\mathcal{F}_{\mathbb{C}} = \{0\}$$

WHY? PROOF!

Assumption: Period matrices $\boldsymbol{P}$ and $\check{\boldsymbol{P}}$ with differential equations in $\mathcal{E}$-form and C-form (same algebra)

Twisted Riemann bilinear relations:



$$\boldsymbol{C} = \frac{1}{(2\pi i)^n}\boldsymbol{P}\left(\boldsymbol{H}^{-1}\right)^T\check{\boldsymbol{P}}^T$$

$\mathrm{GL}(N,\mathcal{A}\otimes\mathbb{K}(\varepsilon))\ni$

= $\mathcal{E}$ - expansion with coefficients in $\mathcal{A}$

$= \mathbb{P}\mathrm{exp}\left(\varepsilon\int\check{\boldsymbol{\Omega}}(\boldsymbol{X})\right)$

$\in \mathrm{GL}(N,\mathbb{C}(\varepsilon))$

$= \mathbb{P}\mathrm{exp}\left(\varepsilon\int\boldsymbol{\Omega}(\boldsymbol{X})\right)$

entries: $\sum_k \varepsilon^k\Delta_k$

$\in\mathcal{F}_\mathbb{C}$

entries: $\sum_k \epsilon^k\sum_w c_w^k J(w)$

$\in\mathbb{C}\subset\mathcal{F}_\mathbb{C}$

basis of words

iterated integrals with $J(\emptyset)=1$

$$\Delta_k J(\emptyset) = \sum_w c_w^k J(w) \Rightarrow 0 = (c_\emptyset^k - \Delta_k)J(\emptyset) + \sum_{w\neq\emptyset} c_w^k J(w)$$

C-form $\Leftrightarrow$ $J(w)$ are linearly independent over $\mathcal{F}_\mathbb{C}$

[Deneufchatel, Duchamp, Hoang Ngoc Minh, Solomon]

$\Rightarrow c_w^k = 0$ for $w\neq\emptyset$ and $c_\emptyset\in\mathbb{C}$

$$\boldsymbol{C}\in\mathbb{C}(\varepsilon)\longrightarrow \mathrm{d}\boldsymbol{C}=0$$

EXAMPLE: HYPERELLIPTIC INTEGRAL FAMILY

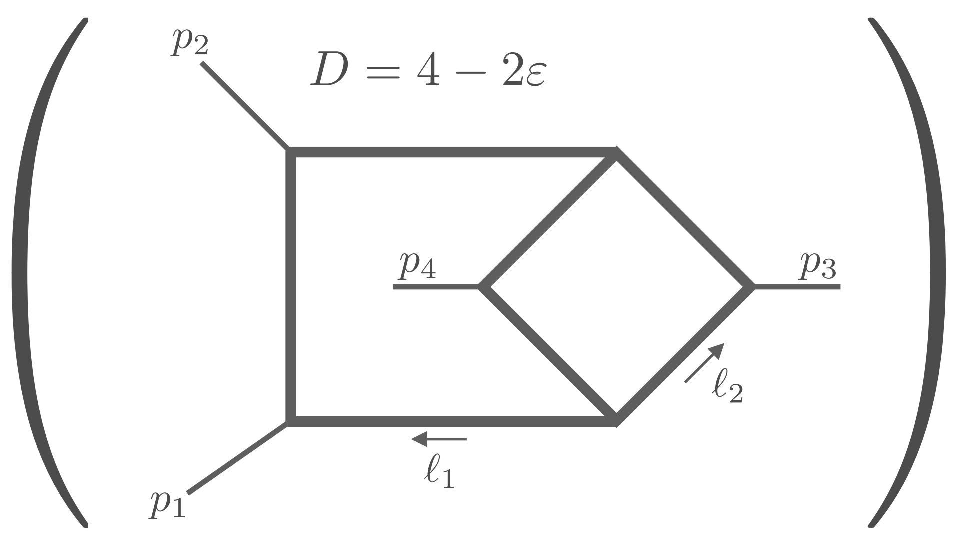
work in progress (arXiv:2412.XXXX)

$$L(\underline{X}, z) = \int (z - X_1)^{-\frac{1}{2}+a_1\varepsilon} \dots (z - X_6)^{-\frac{1}{2}+a_6\varepsilon} \mathrm{d}z$$

$$\Phi = \frac{1}{y} \prod_{i=1}^{6} (1 - \lambda_i^{-1} x)^{a_i \varepsilon} \,, \quad y^2 = (\lambda_1 - x) \dots (\lambda_6 - x)$$

Defines even hyperelliptic curve of genus 2

MC



- Choose basis of 5 master integrals $\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon)$ with $\mathrm{d}\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon) = \boldsymbol{A}(\boldsymbol{\lambda}, \varepsilon)\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon)$

- Rotate to a canonical basis $\boldsymbol{J}(\boldsymbol{\lambda}, \varepsilon) = \boldsymbol{U}(\boldsymbol{\lambda}, \varepsilon)\boldsymbol{I}(\boldsymbol{\lambda}, \varepsilon)$ with $\mathrm{d}\boldsymbol{J}(\boldsymbol{\lambda}, \varepsilon) = \varepsilon \boldsymbol{B}(\boldsymbol{\lambda})\boldsymbol{J}(\boldsymbol{\lambda}, \varepsilon)$

  Using the algorithm by [Görges, Nega, Tancredi, Wagner]

Last step: Ansatz for final rotation: $\boldsymbol{U}_{\text{fin}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & 0 & 0 \end{bmatrix}$

… and solve for unknowns ★ (8 coupled DEQs)

EXAMPLE: HYPERELLIPTIC INTEGRAL FAMILY

Basis and dual basis are in ε-**form** and **C-form** $\Rightarrow$ The intersection matrix is **constant** in the external variables, $\mathrm{d}C = 0$.

1. Compute intersection matrix C from basis & dual basis after final transformation:

   → Contains the 8 unknowns ★ of U_{fin}

2. Require all entries of C to be constant in parameters λ_i and solve for (some) ★ .

$$U_{\text{fin}} = \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ \bullet & \bullet & - & - & - \\ \bigstar & \bullet & \bigstar & - & - \\ \bullet & \bullet & \bigstar & - & - \end{bmatrix} \longrightarrow$$

All but three entries of the final transformation
(expressed in periods, branch points & the three remaining new functions)

$$C = \begin{bmatrix} - & - & - & - & \bullet \\ - & - & - & \bullet & - \\ - & - & \bullet & - & - \\ - & \bullet & - & - & - \\ \bullet & - & - & - & - \end{bmatrix} \longrightarrow$$

A constant skew-diagonal intersection

The requirement, that the **intersection matrix** is constant, can be used ***constructively***!

SUMMARY

- Twisted Riemann bilinear relations $\Rightarrow$ bilinear relations for maximal cuts
- Basis and dual basis in $\mathcal{E}$ - and C - form $\Rightarrow$ Constant intersection matrix
- Requiring constant intersection matrix can be used constructively

OUTLOOK

- Can we use constant intersection matrix for more examples (maximal cut & beyond)?
- Better understanding of the role of the C-form (more generally)

Thank you!