Intersecting companion matrices for Feynman integrals

Seva Chestnov

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

Theory and Phenomenology of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

Istituto Nazionale di Fisica Nucleare

Based on joined work with:
Giacomo Brunello, Giulio Crisanti, Hjalte Frellesvig,
Federico Gasparotto, Manoj Mandal, Pierpaolo Mastrolia
+ Andrzej Pokraka

[2401.01897] [2408.16668] + ongoing

Loop the Loop
November 12, 2024

Introduction

Feynman twisted period integrals

- Amplitude at fixed loop order = sum of families. Each family = sum of master integrals

$$\mathcal{A} = \sum_i c_i \, \mathcal{I}_i$$

- Master decomposition formula (abusing notation) [Mastrolia Mizera '18] [FGLMMMM '19]

$$\langle \mathcal{A} | = \sum_{ij} \langle \mathcal{A} \,|\, \mathcal{I}_j^\vee \rangle \left(C^{-1}\right)_{ji} \langle \mathcal{I}_i |$$

of a vector $\langle \mathcal{A}|$ into a basis $\langle \mathcal{I}_i|$ using metric

$$C_{ij} := \langle \mathcal{I}_i \,|\, \mathcal{I}_j^\vee \rangle$$

- The scalar product $\langle \bullet \,|\, \bullet \rangle$ is called the **intersection number** [Cho '95 Matsumoto]
- Decomposition agrees with traditional IBP-based methods [Tkachov '81] [Chetyrkin Tkachov '81] [Laporta '00]
- Also used to study differential equations! [Duhr, Porkert '24 Semper, Stawinski] [Duhr, Porkert '24 Semper, Stawinski] [Franziska's talk next]

Baikov representation

Parametric representation of Feynman integrals: [Baikov '96] [Frellesvig '17 Papadopoulos]

$$\mathcal{I}(x) = \int \mathrm{d}^d k \, \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}} \Longrightarrow \mathcal{I}(x) = \int_{C_R} (\mathcal{B}(z;\, x))^\gamma \, \frac{\mathrm{d}^n z}{z_1^{a_1} \dots z_n^{a_n}} = \int_{C_R} u \, \varphi$$

Features:

1. Holomorphic measure $\mathrm{d}^n z := \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n$
2. Baikov polynomial $\mathcal{B}(z;\, x) \in \mathbb{C}[z, x]$ with $x = \{p_i \cdot p_j, m^2, \dots\}$
3. $\gamma = (d - \texttt{\#loops} - \texttt{\#legs} - 1)/2 \in \mathbb{C} \Longrightarrow \mathcal{B}^\gamma$ is multivalued!
4. Integration contour $C_R \subset \mathbb{C}^n$ such that $\mathcal{B}(\partial C_R) = 0$
5. Compute it with Hjalte's `BaikovPackage.m` [repo @]

Twisted cohomology

- Twist, connection, and covariant derivative [Mastrolia Mizera '18]

$$u := \prod_i (\mathcal{B}_i(z))^{\gamma_i}\,, \ \omega := \mathrm{d}\log(u)\,, \text{ and } \nabla\psi := \mathrm{d}\psi + \omega\,\psi$$

- Intersection number integral

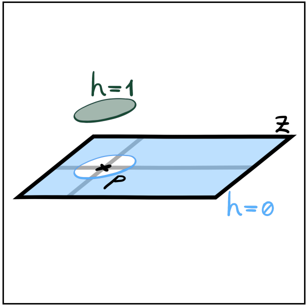
$$\langle \varphi \,|\, \varphi^\vee \rangle = \int_{\mathbb{C}} \mathrm{reg}(\varphi) \wedge \varphi^\vee$$

- Regularization in subtraction scheme . . . [Cho '95 Matsumoto]

$$\mathrm{reg}(\varphi) = \varphi - \sum_p \nabla(h_p\,\psi_p)\,, \quad \nabla\psi_p = \varphi \quad \text{near } z = p$$

- . . . localizes on singularities of ω = *zeroes* of $\mathcal{B}(z)$

$$\langle \varphi \,|\, \varphi^\vee \rangle = \sum_p \mathrm{Res}_{z=p}\Big(\psi_p\,\varphi^\vee\Big)$$



Polynomial reduction and $p(z)$-adic expansion

- Ansatz for $p(z)$-adic expansion and the β-deformation [Fontana '22] [Fontana Peraro '23]

$$\psi = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \left(\mathcal{B}(z)\right)^n z^i \, \psi_{ni} \quad \rightsquigarrow \quad \psi(\beta, z) = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \beta^n z^i \, \psi_{ni}$$

- Differential equation with polynomial reduction

$$(\mathcal{B}'(z)\,\partial_\beta + \partial_z + \omega)\psi - \varphi = 0 \quad \text{modulo} \quad \mathcal{B}(z) - \beta = 0$$

Polynomial reduction and $p(z)$-adic expansion

- Ansatz for $p(z)$-adic expansion and the β-deformation [Fontana '22] [Fontana Peraro '23]

$$\psi = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg \mathcal{B}-1} \big(\mathcal{B}(z)\big)^n \, z^i \, \psi_{ni} \quad \rightsquigarrow \quad \psi(\beta, z) = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg \mathcal{B}-1} \beta^n \, z^i \, \psi_{ni}$$

- Differential equation with polynomial reduction

$$(\mathcal{B}'(z)\,\partial_\beta + \partial_z + \omega)\psi - \varphi = 0 \quad \text{modulo} \quad \mathcal{B}(z) - \beta = 0$$

- Sum over poles via the global residue $\mathrm{Res}_{\langle\mathcal{B}\rangle}$ [Weinzierl '20]

$$\langle \varphi \,|\, \varphi^\vee \rangle = \sum_p \mathrm{Res}_{z=p}\Big(\psi_p \, \varphi^\vee\Big)$$

Polynomial reduction and $p(z)$-adic expansion

- Ansatz for $p(z)$-adic expansion and the β-deformation [Fontana '22] [Fontana Peraro '23]

$$\psi = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \big(\mathcal{B}(z)\big)^n \, z^i \, \psi_{ni} \quad \rightsquigarrow \quad \psi(\beta, z) = \sum_{n\in\mathbb{Z}} \sum_{i=0}^{\deg\mathcal{B}-1} \beta^n \, z^i \, \psi_{ni}$$

- Differential equation with polynomial reduction

$$\big(\mathcal{B}'(z)\,\partial_\beta + \partial_z + \omega\big)\psi - \varphi = 0 \quad \text{modulo} \quad \mathcal{B}(z) - \beta = 0$$

- Sum over poles via the global residue $\mathrm{Res}_{\langle\mathcal{B}\rangle}$ [Weinzierl '20]

$$\langle \varphi \mid \varphi^\vee \rangle = \mathrm{Res}_{\langle\mathcal{B}\rangle}\big(\psi\,\varphi^\vee\big) = c_{-1,\,\deg\mathcal{B}-1}\,, \quad \psi\,\varphi^\vee = \sum_{ni} \big(\mathcal{B}(z)\big)^n \, z^i \, c_{ni}$$

- Ansatz issues: many variables and expressions $\Rightarrow$ replace everything with matrices!

Intersection numbers with tensors

Running example

- Consider a family of integrals with quadratic twist

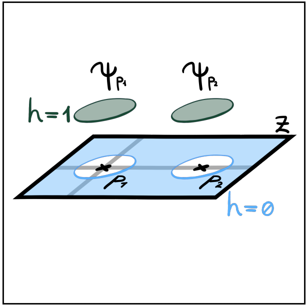
$$\mathcal{I} = \int u\,\varphi\,,\; u = z^{\rho}\,\mathcal{B}(z)^{\gamma}\,,\; \varphi = \frac{\mathrm{d}z}{z^{n}\,\mathcal{B}(z)^{m}}\,,\; \mathcal{B}(z) := b_0 + b_1\,z + z^2\,,\; \omega = \frac{\rho}{z} + \gamma\,\frac{\mathcal{B}'}{\mathcal{B}}$$

- And let's compute

$$\langle \varphi \mid \varphi^{\vee} \rangle = \langle 1/\mathcal{B} \mid 1/\mathcal{B}^2 \rangle$$

$$\langle \varphi \mid \varphi^{\vee} \rangle = \mathrm{Res}_{z=p_1}\Big(\psi_{p_1}\,\varphi^{\vee}\Big) + \mathrm{Res}_{z=p_2}\Big(\psi_{p_2}\,\varphi^{\vee}\Big)$$

$\Big(\partial_z + \omega\Big)\psi_p = \varphi$ near roots p_1 and p_2 of $\mathcal{B}(z)$



Running example

- Consider a family of integrals with quadratic twist

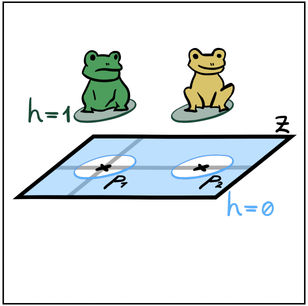
$$\mathcal{I} = \int u\, \varphi\,,\; u = z^{\rho}\, \mathcal{B}(z)^{\gamma}\,,\; \varphi = \frac{\mathrm{d}z}{z^{n}\, \mathcal{B}(z)^{m}}\,,\; \mathcal{B}(z) := b_0 + b_1\, z + z^2\,,\; \omega = \frac{\rho}{z} + \gamma\, \frac{\mathcal{B}'}{\mathcal{B}}$$

- And let's compute

$$\langle \varphi \,|\, \varphi^{\vee} \rangle = \langle 1/\mathcal{B} \,|\, 1/\mathcal{B}^2 \rangle$$

$\langle \varphi \,|\, \varphi^{\vee} \rangle = \mathrm{Res}_{z=p_1}\Big($  $\varphi^{\vee}\Big) + \mathrm{Res}_{z=p_2}\Big($  $\varphi^{\vee}\Big)$

$\left(\partial_z + \omega\right) \Big[$   $\Big] = \varphi$ near roots p_1 and p_2 of $\mathcal{B}(z)$

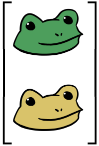
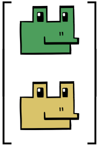


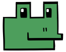
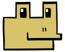
Changes of variables and gauges

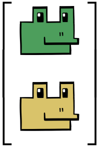
- Want to work simultaneously near the roots $\implies$ new variable β

$$\mathcal{B}(z) - \beta = 0$$

-  and  correspond to $z^{\pm}(\beta) = -b_1/2 \pm \sqrt{b_1^2 - 4b_0 + 4\beta}/2$
- How to remove square roots from differential equation? Gauge transform!

 $= \begin{bmatrix} 1 & z^{+}(\beta) \\ 1 & z^{-}(\beta) \end{bmatrix} \cdot$ 

-  and  correspond to irreducible monomials 1 and z

$\psi(z, \beta) = \psi_0(\beta) + z\,\psi_1(\beta) \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix} = \begin{bmatrix} 1 & z \end{bmatrix} \cdot$ 

Companion matrix dictionary

- Basic building blocks from $\mathcal{B}(z) - \beta = 0 \implies$ rule $z^2 \mapsto -b_0 + \beta - b_1\, z$

$$\begin{bmatrix} z & z^2 \end{bmatrix} = \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix} =: \begin{bmatrix} 1 & z \end{bmatrix} \cdot Q_z$$

Companion matrix dictionary

- Basic building blocks from $\mathcal{B}(z) - \beta = 0 \Longrightarrow$ rule $z^2 \mapsto -b_0 + \beta - b_1\, z$

$$Q_z = \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix}, \quad Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Companion matrix dictionary

- Basic building blocks from $\mathcal{B}(z) - \beta = 0 \Longrightarrow$ rule $z^2 \mapsto -b_0 + \beta - b_1\, z$

$$Q_z = \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix}, \quad Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Algebra works: division is matrix inversion

$$Q_{1/z} \equiv Q_z^{-1} = \frac{1}{-b_0 + \beta} \begin{bmatrix} b_1 & -b_0 + \beta \\ 1 & 0 \end{bmatrix}$$

$$Q_{\mathcal{B}'} \equiv b_1\, \mathbb{1} + 2\, Q_z = \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}, \qquad \mathcal{B}' = b_1 + 2\, z$$

Companion matrix dictionary

- Basic building blocks from $\mathcal{B}(z) - \beta = 0 \implies$ rule $z^2 \mapsto -b_0 + \beta - b_1\, z$

$$Q_z = \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix}, \quad Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Algebra works: division is matrix inversion

$$Q_{1/z} \equiv Q_z^{-1} = \frac{1}{-b_0 + \beta} \begin{bmatrix} b_1 & -b_0 + \beta \\ 1 & 0 \end{bmatrix}$$

$$Q_{\mathcal{B}'} = \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}$$

$$Q_{\mathcal{B}'}^{-1} = \frac{1}{\Delta + 4\beta} \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}, \qquad \Delta := b_1^2 - 4b_0$$

Companion matrix dictionary

- Basic building blocks from $\mathcal{B}(z) - \beta = 0 \implies$ rule $z^2 \mapsto -b_0 + \beta - b_1\, z$

$$Q_z = \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix}, \quad Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Algebra works: division is matrix inversion

$$Q_{1/z} \equiv Q_z^{-1} = \frac{1}{-b_0 + \beta} \begin{bmatrix} b_1 & -b_0 + \beta \\ 1 & 0 \end{bmatrix}$$

$$Q_{\mathcal{B}'} = \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}$$

$$Q_{\mathcal{B}'}^{-1} = \frac{1}{\Delta + 4\beta} \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}, \qquad \Delta := b_1^2 - 4b_0$$

- System of differential equations becomes rational!

$$\left(\partial_\beta - \frac{\gamma}{\beta} - \rho\, Q_{\mathcal{B}'}^{-1} \cdot Q_{1/z} + Q_{\mathcal{B}'}^{-1} \cdot Q_{\partial_z}\right) \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix} - Q_{\mathcal{B}'}^{-1} \cdot \begin{bmatrix} \beta^{-1} \\ 0 \end{bmatrix} = 0$$

Series expansion

- Only need $\beta \to 0$ behavior $\Longrightarrow$ Laurent expand

$$\psi(z,\beta)\Big|_{\beta\to 0} = \sum_{a=0}^{1}\sum_{n\in\mathbb{Z}} z^a\,\psi_{an}\,\beta^n \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} \cdots\cdots & \psi_{0\,-1} & \psi_{00} & \psi_{01} & \cdots\cdots \\ \cdots\cdots & \psi_{1\,-1} & \psi_{10} & \psi_{11} & \cdots\cdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \beta^{-1} \\ 1 \\ \beta \\ \vdots \end{bmatrix}$$

Series expansion

- Only need $\beta \to 0$ behavior $\implies$ Laurent expand and cut

$$\psi(z,\beta)\Big|_{\beta\to 0} = \sum_{a=0}^{1}\sum_{n\geq 0} z^a\,\psi_{an}\,\beta^n \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} 0 & \psi_{00} & \psi_{01} \\ 0 & \psi_{10} & \psi_{11} \end{bmatrix} \cdot \begin{bmatrix} \beta^{-1} \\ 1 \\ \beta \end{bmatrix}$$

Series expansion

- Only need $\beta \to 0$ behavior $\Longrightarrow$ Laurent expand and cut

$$\psi(z,\beta)\Big|_{\beta\to 0} = \sum_{a=0}^{1}\sum_{n\geq 0} z^a\,\psi_{an}\,\beta^n \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} 0 & \psi_{00} & \psi_{01} \\ 0 & \psi_{10} & \psi_{11} \end{bmatrix} \cdot \begin{bmatrix} \beta^{-1} \\ 1 \\ \beta \end{bmatrix}$$

- Corresponding blocks of differential equation

$$\left(\partial_\beta - \frac{\gamma}{\beta} + \frac{-\rho}{\Delta}\begin{bmatrix} \frac{2b_0 - b_1^2}{b_0} & b_1 \\ -\frac{b_1}{b_0} & 2 \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} 0 & b_1 \\ 0 & 2 \end{bmatrix} + \dots\right) \cdot \left(0 + \begin{bmatrix} \psi_{00} \\ \psi_{10} \end{bmatrix} + \begin{bmatrix} \psi_{01} \\ \psi_{11} \end{bmatrix}\beta\right)$$

$$-\left(\frac{1}{\beta\,\Delta} + \frac{-4}{\Delta^2} + \dots\right)\begin{bmatrix} b_1 \\ 2 \end{bmatrix} = 0$$

Series expansion

- Only need $\beta \to 0$ behavior $\implies$ Laurent expand and cut

$$\psi(z,\beta)\Big|_{\beta\to 0} = \sum_{a=0}^{1}\sum_{n\geq 0} z^a\,\psi_{an}\,\beta^n \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} 0 & \psi_{00} & \psi_{01} \\ 0 & \psi_{10} & \psi_{11} \end{bmatrix} \cdot \begin{bmatrix} \beta^{-1} \\ 1 \\ \beta \end{bmatrix}$$

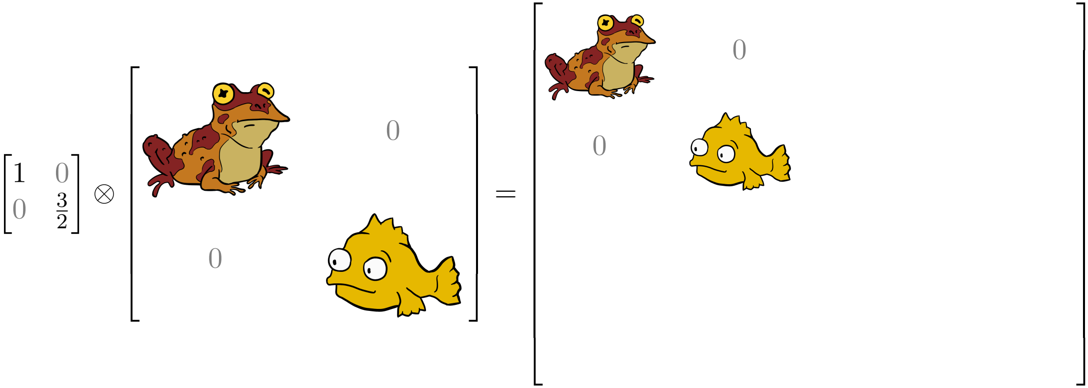
- Corresponding blocks of differential equation

$$\left(\partial_\beta - \frac{\gamma}{\beta} + \frac{-\rho}{\Delta}\begin{bmatrix} \frac{2b_0-b_1^2}{b_0} & b_1 \\ -\frac{b_1}{b_0} & 2 \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} 0 & b_1 \\ 0 & 2 \end{bmatrix} + \ldots\right) \cdot \left(0 + \begin{bmatrix} \psi_{00} \\ \psi_{10} \end{bmatrix} + \begin{bmatrix} \psi_{01} \\ \psi_{11} \end{bmatrix} \beta\right)$$

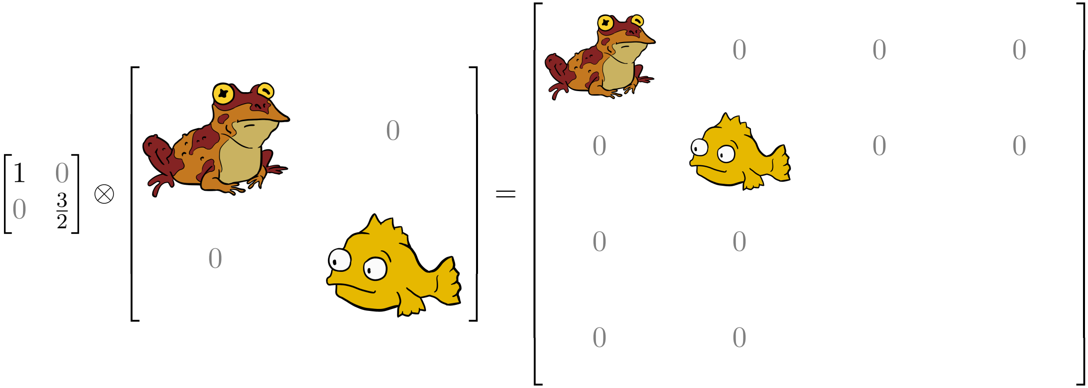
$$-\left(\frac{1}{\beta\,\Delta} + \frac{-4}{\Delta^2} + \ldots\right)\begin{bmatrix} b_1 \\ 2 \end{bmatrix} = 0$$

- Convert expressions into tensors!

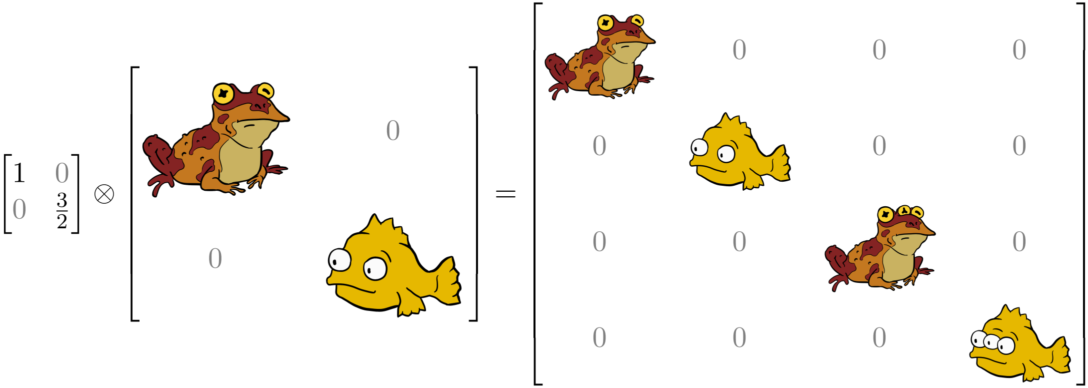
A quick reminder about tensor product



A quick reminder about tensor product



A quick reminder about tensor product



Tensor algebra for series expansion

- Dictionary for ∂_β and β^{-1} terms

$$L_{\partial_\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad L_{\beta^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

- Leading term of differential equation

$$\partial_\beta \otimes \mathbb{1} - \frac{\gamma}{\beta} \otimes \mathbb{1} \rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -\gamma & 0 \\ 0 & -\gamma \end{bmatrix}$$

Full system of differential equations

$$
\begin{bmatrix}
* & * & -\gamma & 0 & 0 & 0 \\
* & * & 0 & -\gamma & 0 & 0 \\
* & * & \frac{-\rho}{\Delta}\frac{2b_0-b_1^2}{b_0} & \frac{1-\rho}{\Delta}b_1 & -\gamma+1 & 0 \\
* & * & \frac{\rho}{\Delta}\frac{b_1}{b_0} & \frac{2(1-\rho)}{\Delta} & 0 & -\gamma+1 \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{bmatrix}
\cdot
\begin{bmatrix}
0 \\ 0 \\ \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11}
\end{bmatrix}
-
\begin{bmatrix}
\frac{b_1}{\Delta} \\ \frac{2}{\Delta} \\ \frac{-4b_1}{\Delta^2} \\ \frac{-8}{\Delta^2} \\ * \\ *
\end{bmatrix}
= 0 \,,
$$

Global residue as covector

$$\langle \varphi \mid \varphi^\vee \rangle = -\operatorname{Res}_{\langle \mathcal{B} \rangle}(\varphi\, \psi)$$

$$= -\left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{c} 0 \\ 0 \\ \hline \psi_{00} \\ \psi_{10} \\ \hline \psi_{01} \\ \psi_{11} \end{array}\right]$$

Global residue as covector

$$\langle \varphi \mid \varphi^\vee \rangle = -\operatorname{Res}_{\langle \mathcal{B} \rangle}(\varphi\,\psi)$$

$$= -\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c} 0 \\ 0 \\ \hline \psi_{00} \\ \psi_{10} \\ \hline \psi_{01} \\ \psi_{11} \end{array}\right]$$

Putting all pieces together

$$\left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 1 \\ \hline 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 \\ 0 & \frac{-\rho}{\Delta}\frac{2b_0-b_1^2}{b_0} & \frac{1-\rho}{\Delta}b_1 & -\gamma+1 & 0 \\ 0 & \frac{\rho}{\Delta}\frac{b_1}{b_0} & \frac{2(1-\rho)}{\Delta} & 0 & -\gamma+1 \end{array}\right] \cdot \left[\begin{array}{c} \langle \varphi \mid \varphi^\vee \rangle \\ \hline \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline \frac{b_1}{\Delta} \\ \frac{2}{\Delta} \\ \frac{-4b_1}{\Delta^2} \\ \frac{-8}{\Delta^2} \end{array}\right]$$

$$\langle \varphi \mid \varphi^\vee \rangle = -\psi_{11} = \frac{8}{\Delta^2(\gamma-1)} + \frac{-4b_0 + \Delta\rho}{b_0\Delta^2(\gamma-1)\gamma}$$

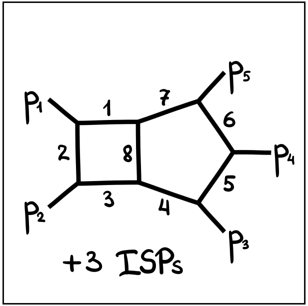
In agreement with the awesome analytic formula of [Crisanti Smith '24] [Giulio's talk later]

Application to 2-loop 5-point reduction

- Result of recent efforts [2401.01897] [2408.16668] [repo @]
- Numerical decomposition into 62 masters (no symmetry)

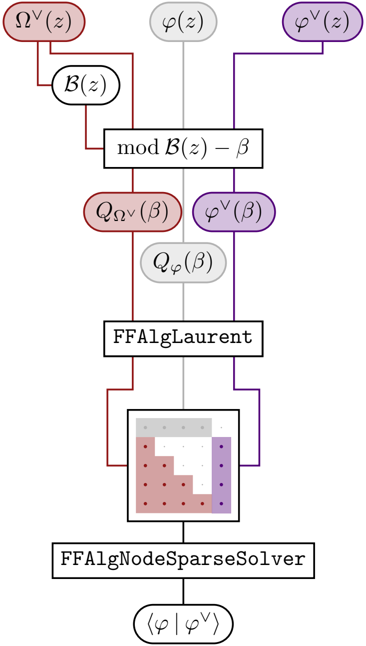
$$\mathcal{I} = \int u \, \frac{z_9^2}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8} \, \mathrm{d}z \equiv \mathcal{I}_{11111111-200}$$

- 11-variate intersections using spanning cuts
- Automatic intermediate (candidate) bases generation `GetBasis.m`
- Extensive usage of tensors in FiniteFlow [Peraro '19, FiniteFlow]



rank	$\times 10^4$ sec
5	2.04
10	4.19
15	6.97
20	10.82

The FiniteFlow data graph



Conclusion

Conclusion

- **Twisted cohomology** describes multivariate integrals of multivalued functions, which are commonly encountered in theoretical physics.
- **Intersection number** is a scalar product on the space of twisted periods, allowing for the direct decomposition of Feynman integrals via the master decomposition formula.
- Today we explored **tensor structures** of intersection numbers that arise from the companion matrix representation.
- This new formulation enabled the full numerical reduction to master integrals of the massless 5-point 2-loop Feynman integral family.