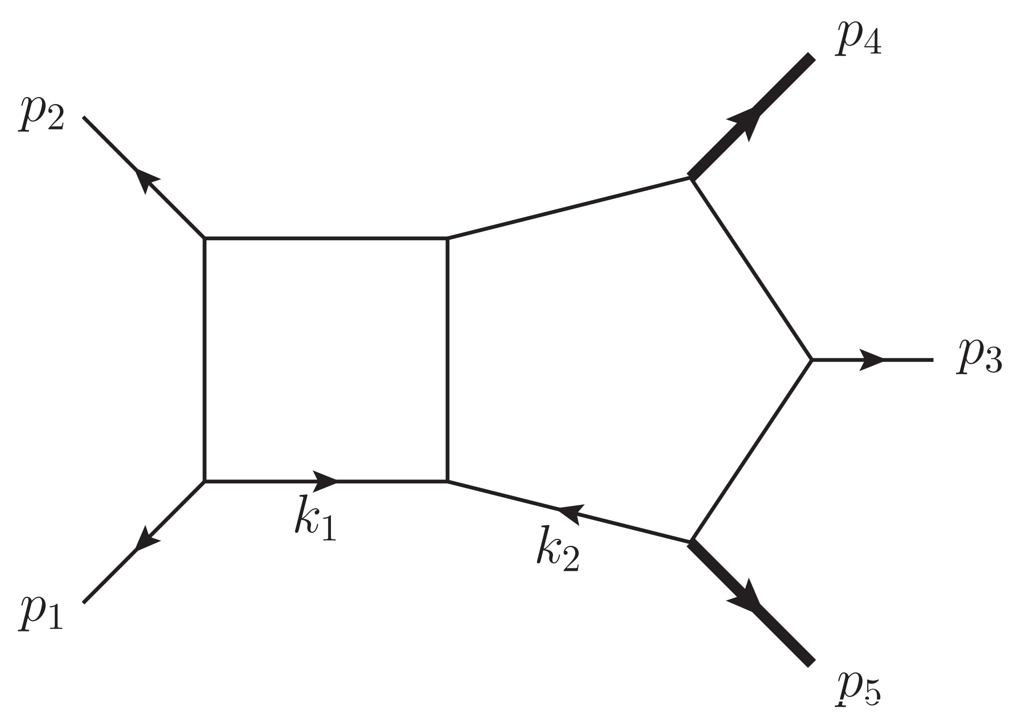
Two-loop five-point two-mass Feynman integrals

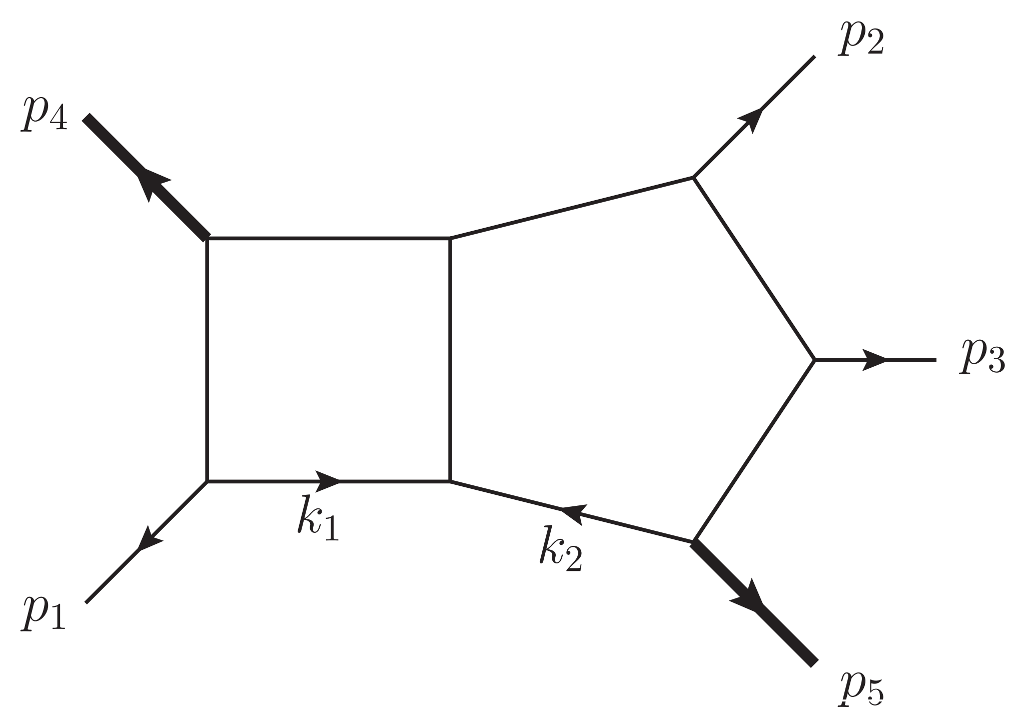
Simone Zoia

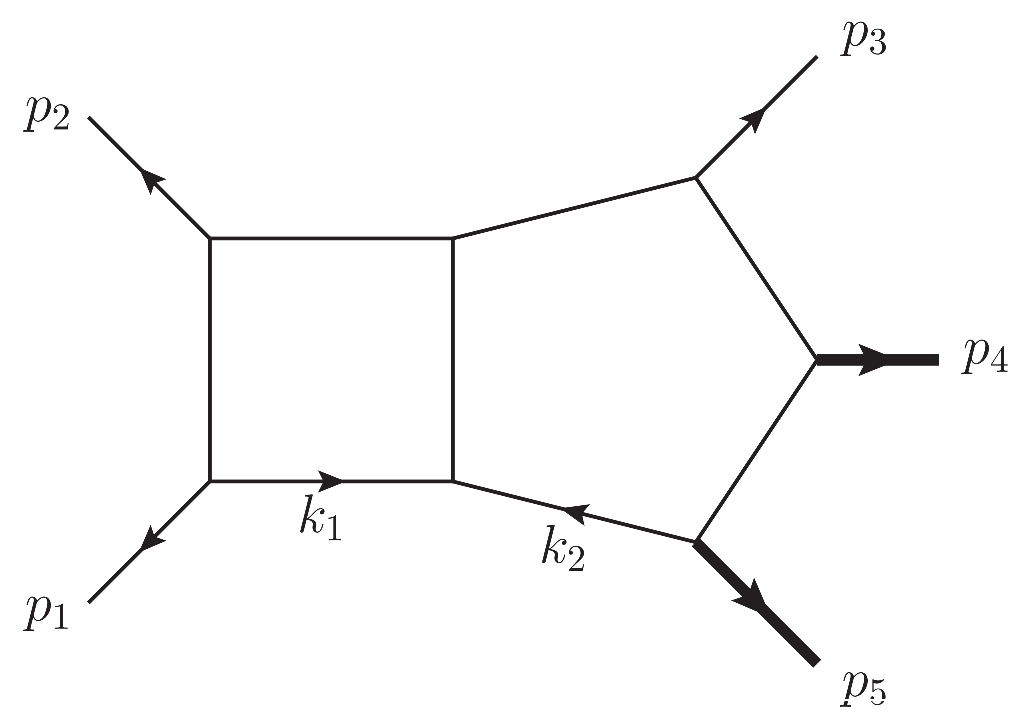
Loop-the-Loop, 12th Nov 2024

Swiss National Science Foundation

Universität Zürich UZH

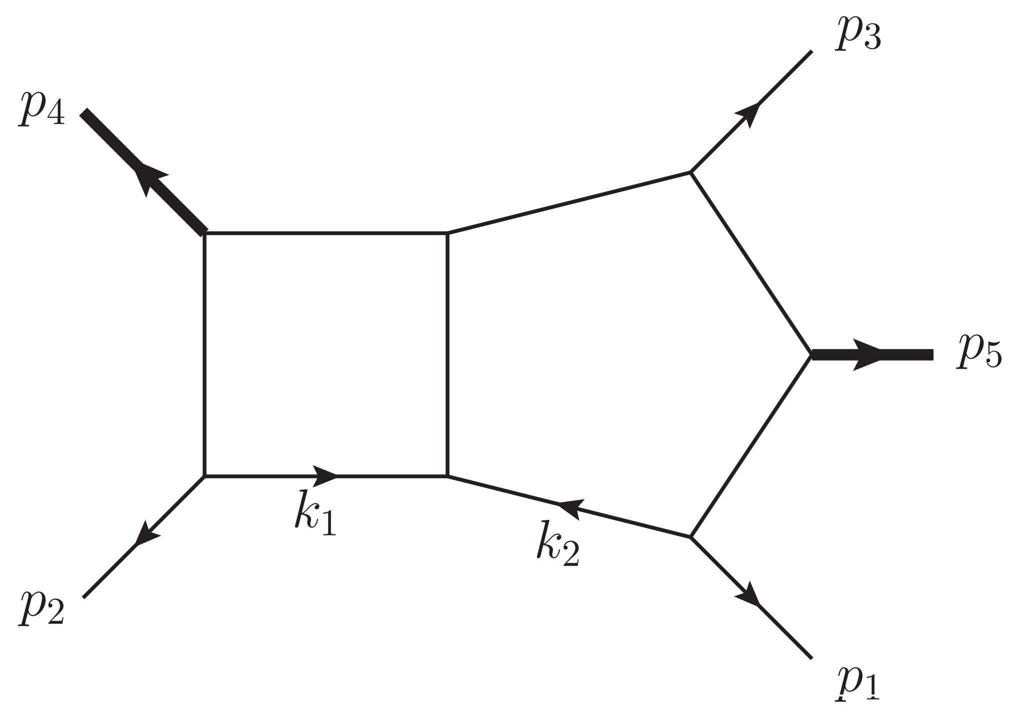


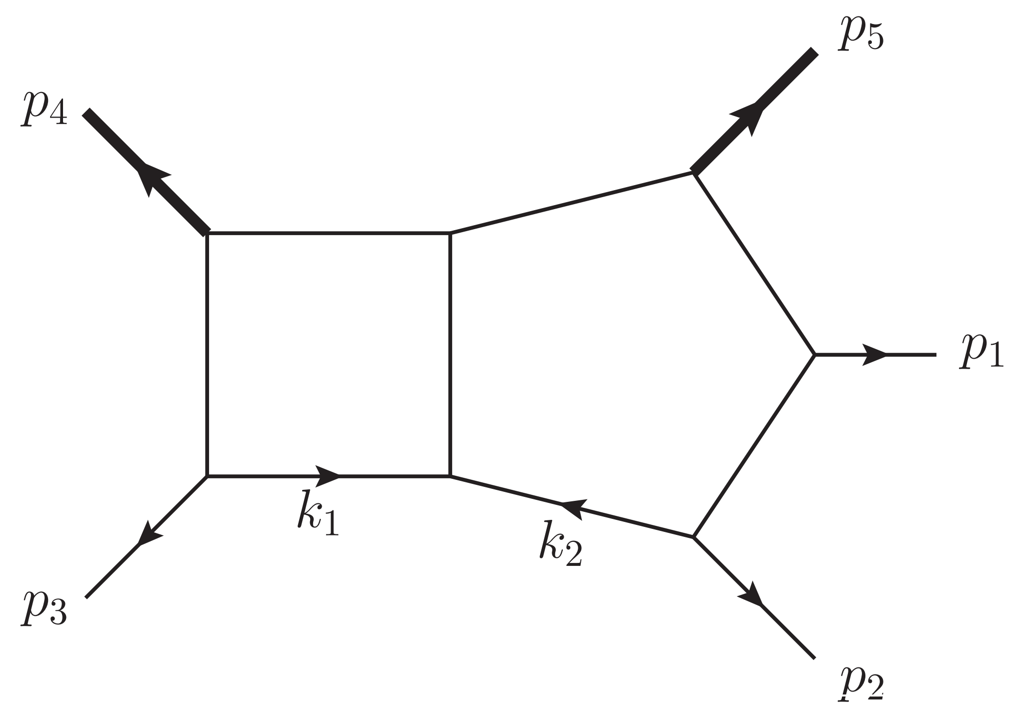


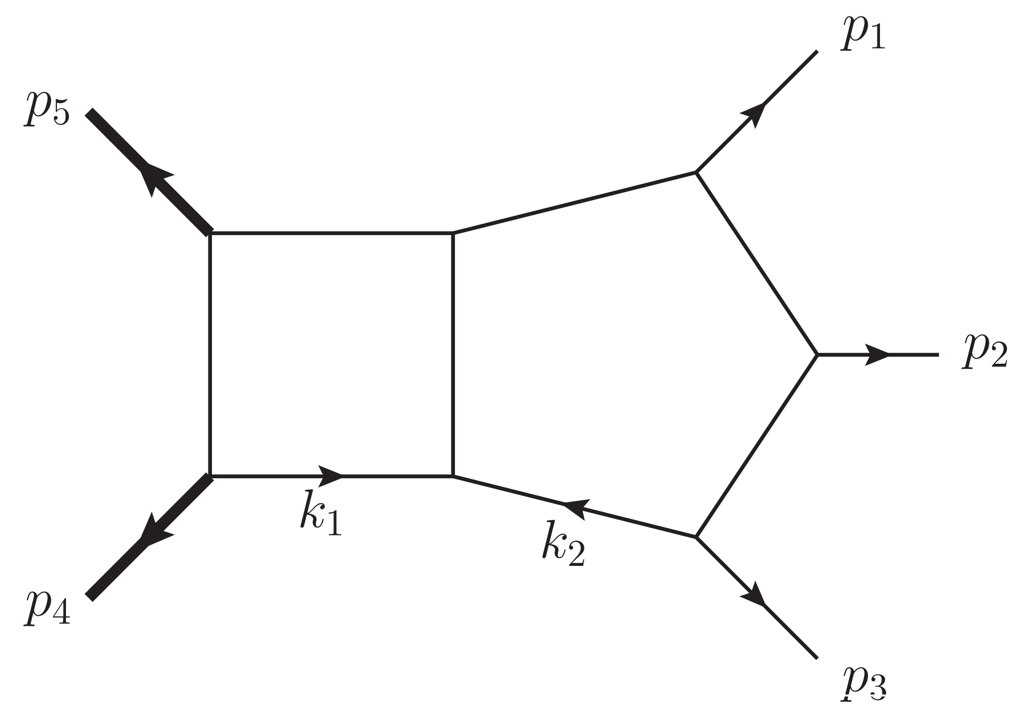


Based on arXiv:2408.05201, *JHEP* 10 (2024) 167

with **Samuel Abreu**, **Dmitry Chicherin** & **Vasily Sotnikov**







Jiang, Liu, Xu, Yang 2024

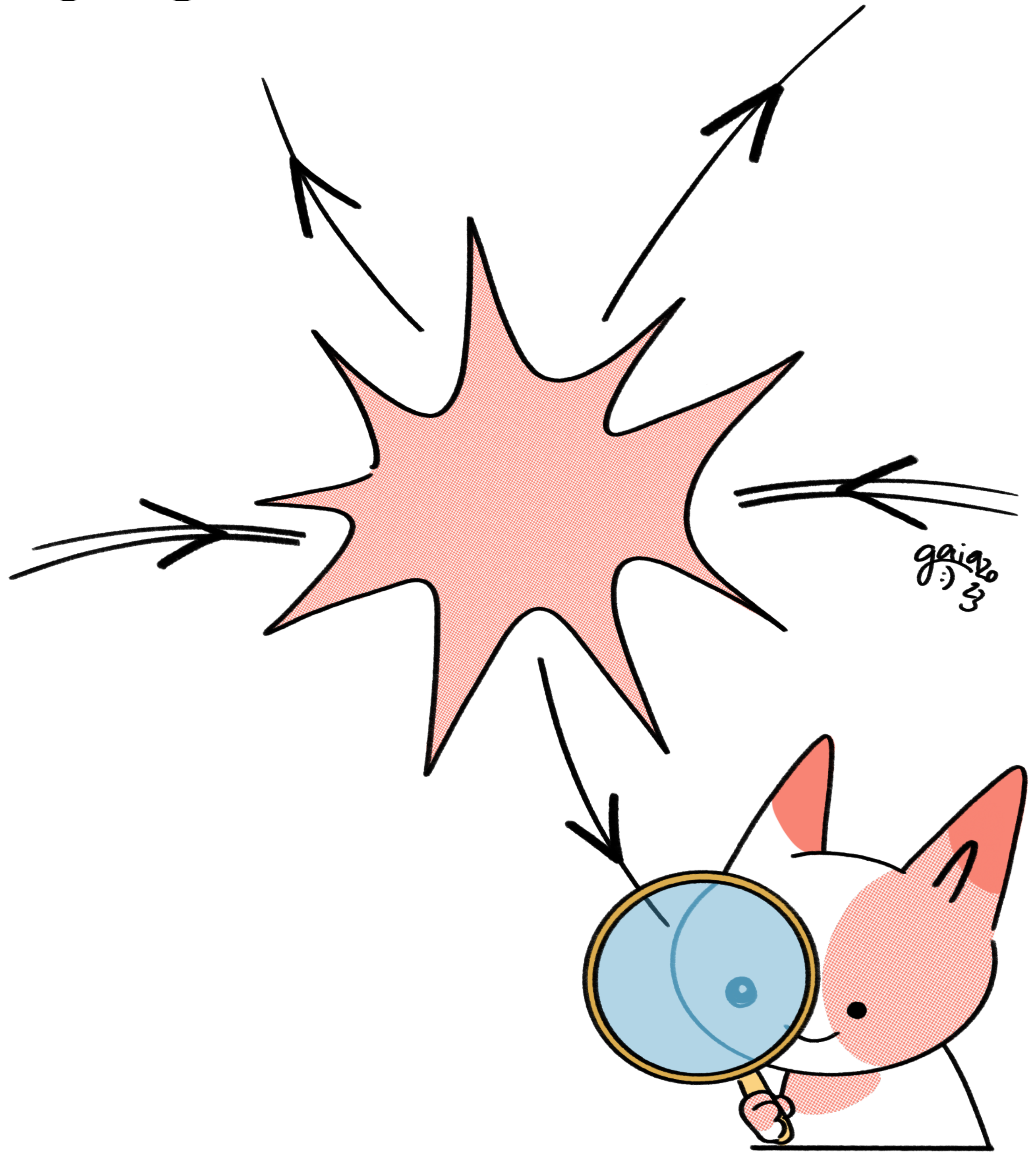
Urgent demand for NNLO QCD for LHC physics

D. Canko

Current frontier: $2 \to 3$ processes @ NNLO, $2 \to 2$ @ N3LO

Bottleneck: 2-loop 5-particle scattering amplitudes

Hot topic! See talks by **C. Brancaccio**, **M. Vicini**



Analytical data to search for new properties

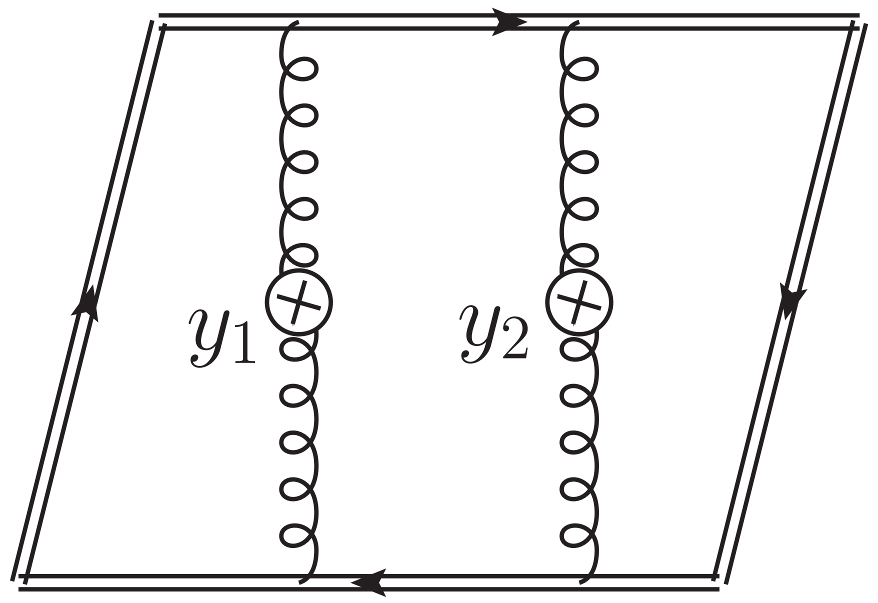
Special functions, (dual) conformal symmetry, positivity, cluster algebras, intersection theory…

P. Mastrolia

Why two external massive legs?

- 0-mass and 1-mass already done… natural next step
- Two heavy vector bosons + jet/photon at hadron colliders @ NNLO QCD
- Two heavy vector bosons at hadron colliders @ N3LO QCD
- Double Lagrangian insertion in four-cusp Wilson loop in planar *N*=4 sYM theory

Single Lagrangian insertion revealed intriguing hidden properties (positivity, conformal symmetry…) *Chicherin, Henn 2022*



Bonus result of our paper 🎁

Good news: we're getting better at this

Multi-scale integrals → **method of differential equations**

Going from 1-mass to 2-mass was much easier than from 0 to 1

- We have **better tools**. In this work we used:
  - NeatIBP *Wu, Boehm, Ma, Xu, Zhang '23*
  - AMFlow *Liu, Ma, Wang '18; Liu, Ma '22*
  - Baikovletter *Jiang, Liu, Xu, Yang '24*
  - FiniteFlow *Peraro '19*

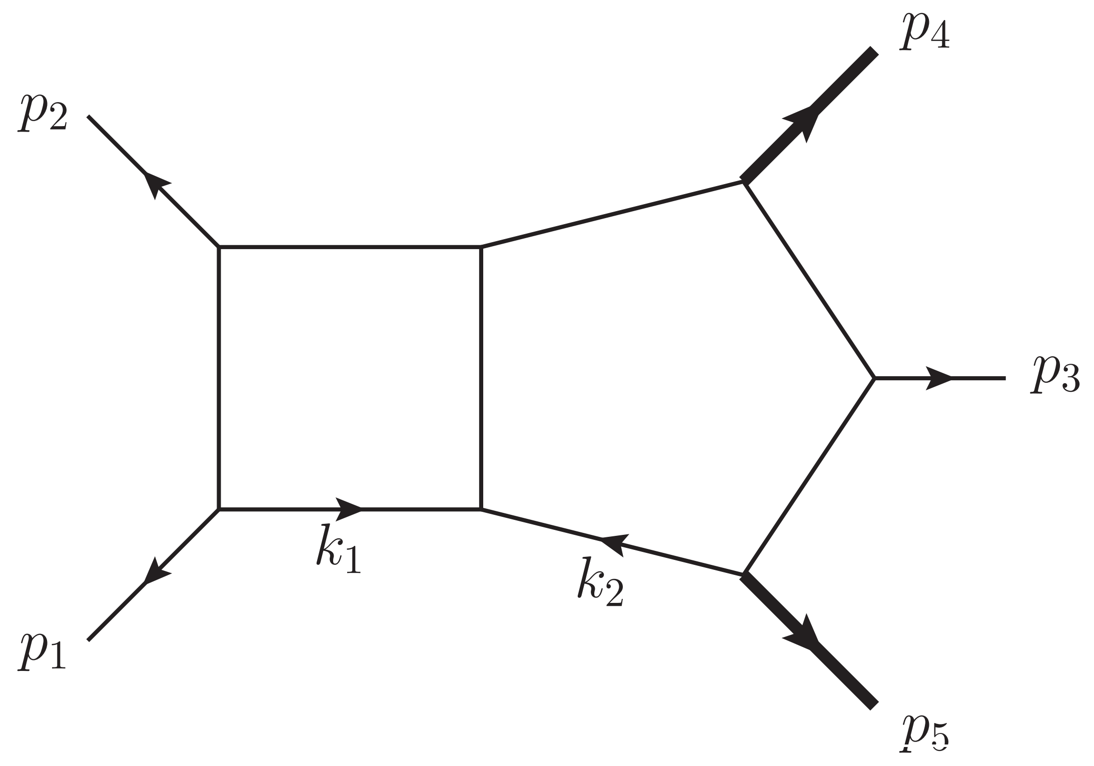
- **Finite-field arithmetic** to overcome the algebraic complexity

  von Manteuffel, Schabinger '15; Peraro '16

- We see **patterns** emerging

Integral families

Scalar Feynman integrals with the same propagator structure = **integral family**



$$\mathrm{I}_{\vec{a}}(X;\epsilon) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{D/2}} \frac{\rho_9^{-a_9}\rho_{10}^{-a_{10}}\rho_{11}^{-a_{11}}}{\rho_1^{a_1}\dots\rho_8^{a_8}}$$

$$\left\{\mathrm{I}_{\vec{a}}(X;\epsilon)\,|\,\forall\,\vec{a}\in\mathbb{Z}^{11}\right\}$$

$$\rho_1 = k_1^2$$
$$\rho_2 = (k_1+p_1)^2$$
$$\rho_3 = (k_1+p_1+p_2)^2$$
$$\dots$$

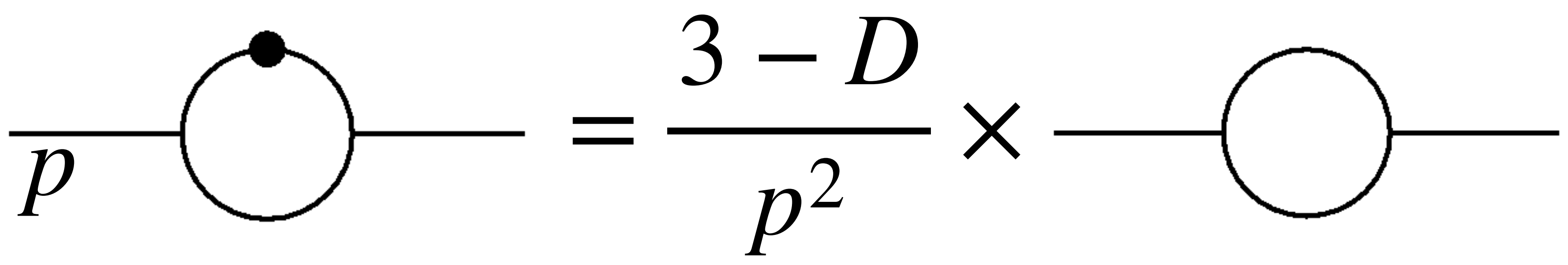
Dimensional regularisation: $D = 4-2\epsilon$

7 kinematic variables: $X = \left(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_4, s_5\right)$

$$s_{ij} = (p_i+p_j)^2$$
$$s_i = p_i^2$$

Integral bases

Identities among the $\mathrm{I}_{\vec{a}}$'s

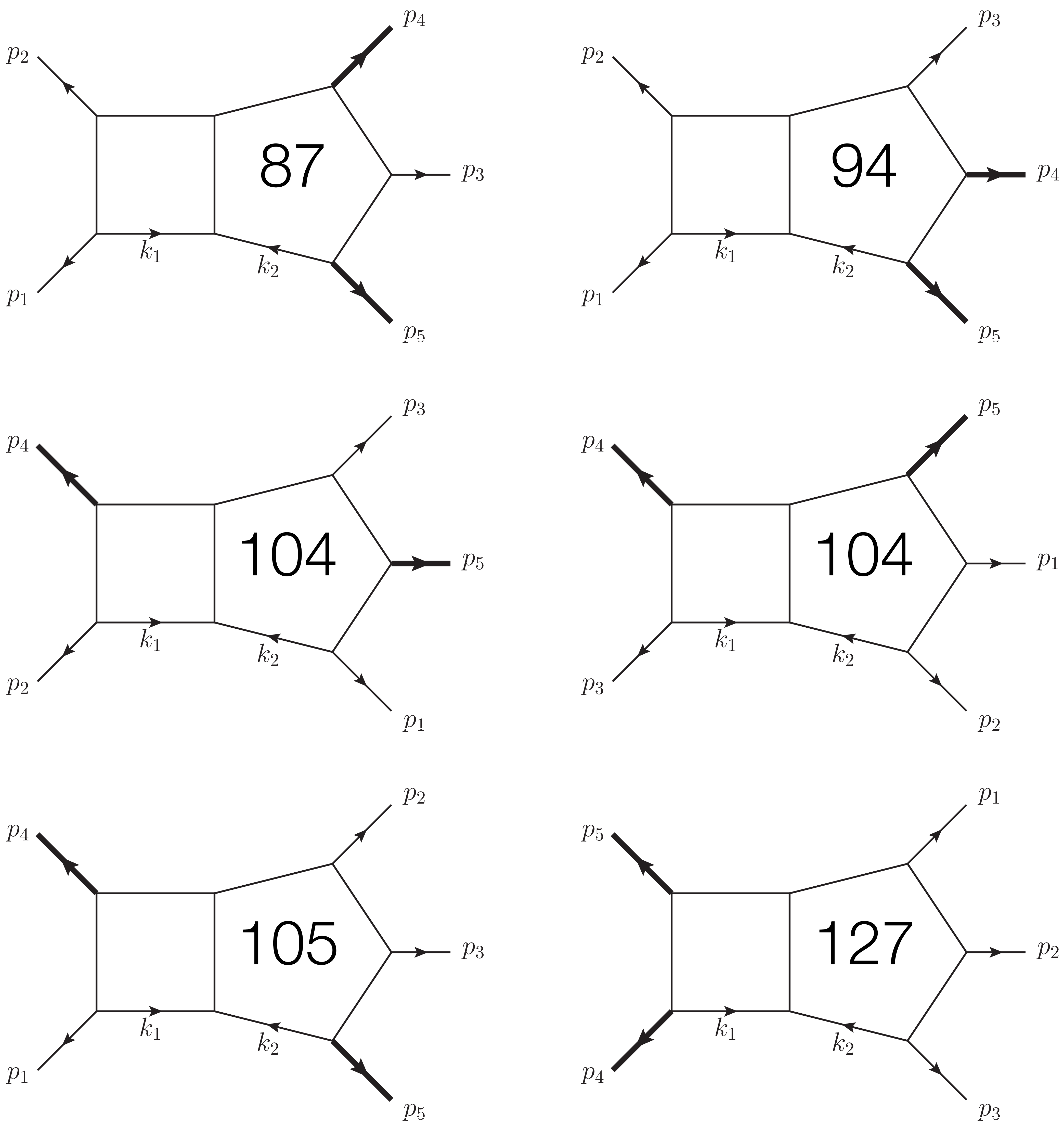


e.g. Integration-By-Parts relations

Chetyrkin, Tkachov '81; Laporta 2000

Finite-dimensional basis:

master integrals $\vec{\mathrm{F}}(X;\epsilon)$



Integrating by differentiating

Barucchi, Ponzano '73; Kotikov '91; Bern, Dixon, Kosower '94; Gehrmann, Remiddi 2000

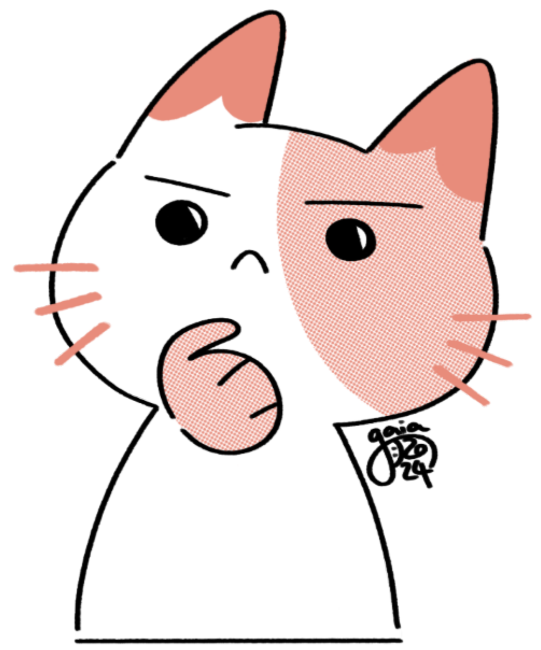
Integral families by construction closed under differentiation

$$\frac{\partial}{\partial s_{12}} \overrightarrow{\mathrm{F}}(X;\epsilon) = \sum_{\vec{a}} c_{\vec{a}}(X;\epsilon)\ \mathrm{I}_{\vec{a}}(X;\epsilon)$$

$$= A_{s_{12}}(X;\epsilon) \cdot \overrightarrow{\mathrm{F}}(X;\epsilon)$$

IBP reduction

System of 1st order linear PDEs for the MIs $\overrightarrow{\mathrm{F}}$



How do we solve it? Is there a "natural" basis?

Canonical form

Henn 2013

$$\mathrm{d}\,\overrightarrow{\mathrm{F}} = \frac{\partial \overrightarrow{\mathrm{F}}}{\partial s_{12}} \mathrm{d}s_{12} + \ldots + \frac{\partial \overrightarrow{\mathrm{F}}}{\partial s_5} \mathrm{d}s_5$$

$$\mathrm{d}\,\overrightarrow{\mathrm{F}}(X;\epsilon) = \epsilon\, \mathrm{d}\tilde{A}(X) \cdot \overrightarrow{\mathrm{F}}(X;\epsilon)$$

- Factorisation of ϵ makes ϵ-expansion of the solution easy
- In the best understood cases, the connection matrix $\tilde{A}(X)$ takes the form

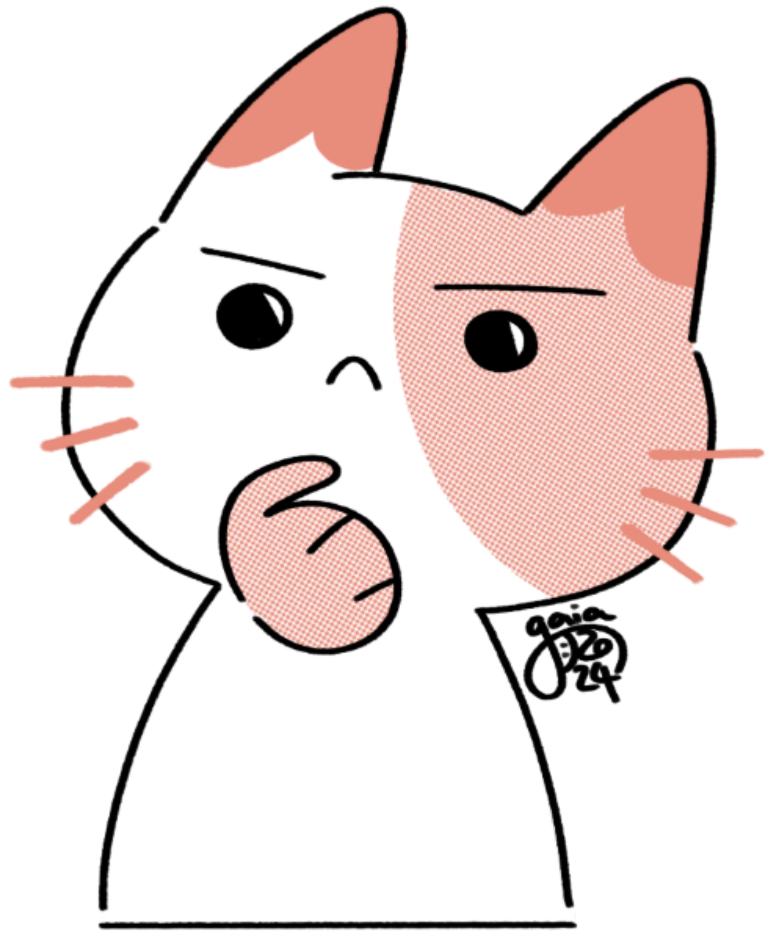
$$\tilde{A}(X) = \sum_i A_i \log W_i(X)$$

$W_i(X)$ = **letters** = singularities of the solution

We know how to proceed from here!

How do we construct a canonical basis?

Is the IBP reduction a problem?



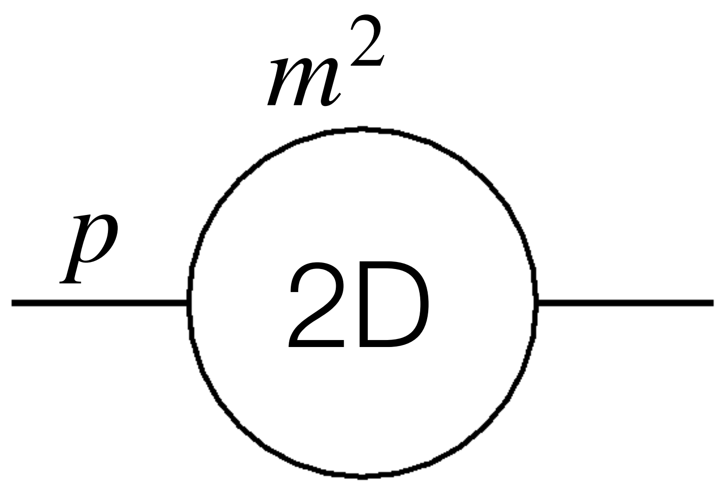
How to construct a canonical basis?

A lot of progress, but still no general algorithm

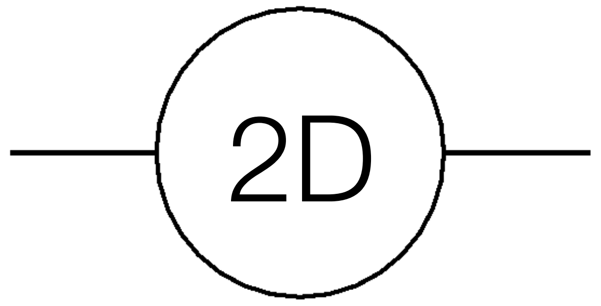
General approach: study **leading singularities** *Arkani-Hamed, Bourjaily, Cachazo, Trnka 2012*

Parameterise loop integrand and take residues (≡ partial fraction) until all integration variables are localised

Construct integrals s.t.: at most simple poles + constant leading singularities



$$\propto \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{p^2\,(\alpha_1\alpha_2 - x)\big[(1+\alpha_1)(1+\alpha_2) - x\big]} \to \frac{\mathrm{d}\alpha_1}{p^2\,(\alpha_1^2 + \alpha_1 + x)} \to \pm\frac{1}{p^2\sqrt{1-4x}}$$

→ $p^2\sqrt{1-4x}\times$  ✓

$$x = \frac{m^2}{p^2}$$

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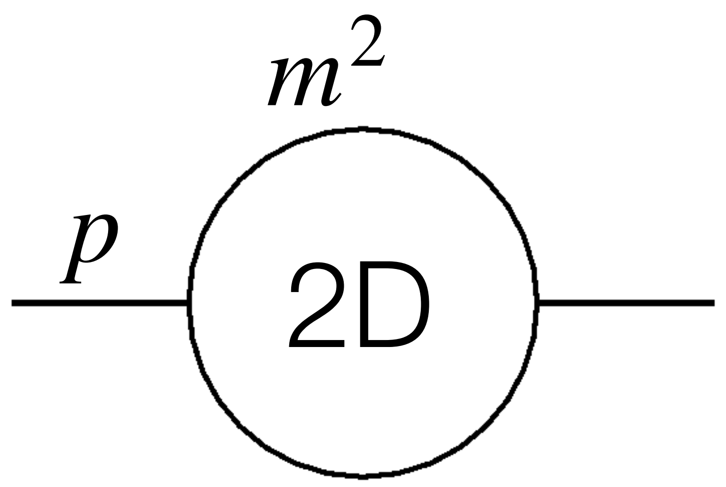

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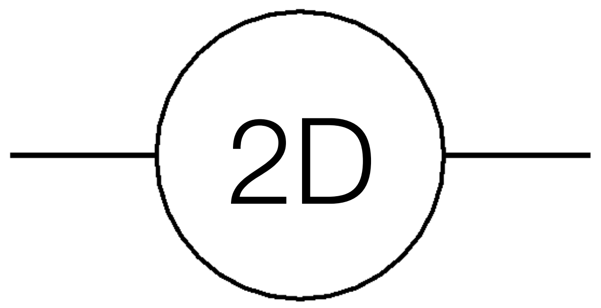
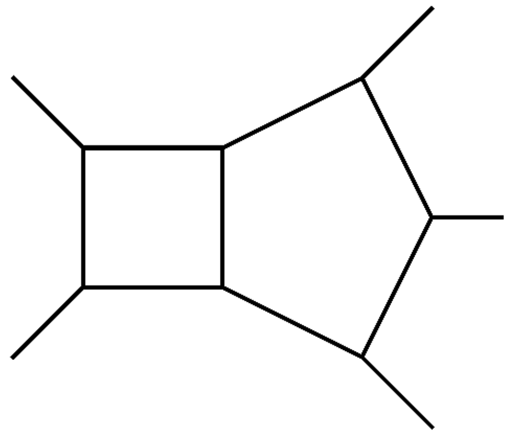
→ $p^2\sqrt{1-4x}\times$ 

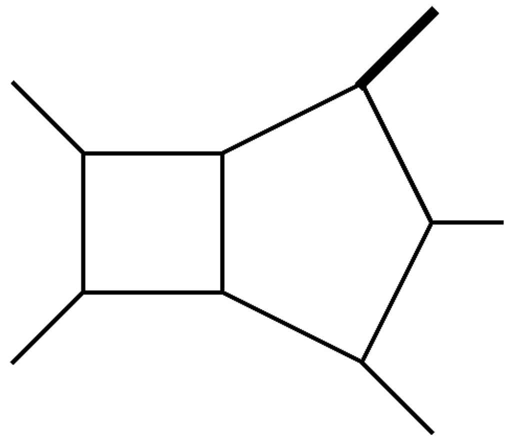
✓

This is how square roots enter the game!

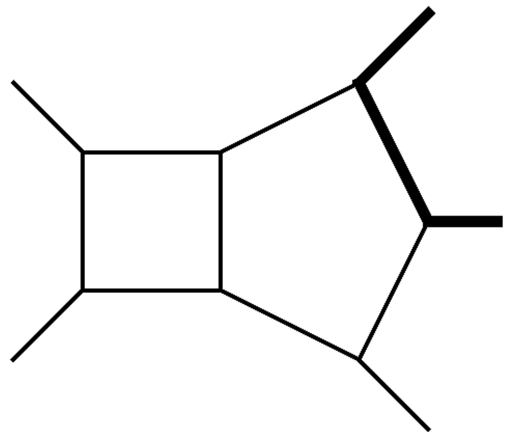
Study emerging patterns



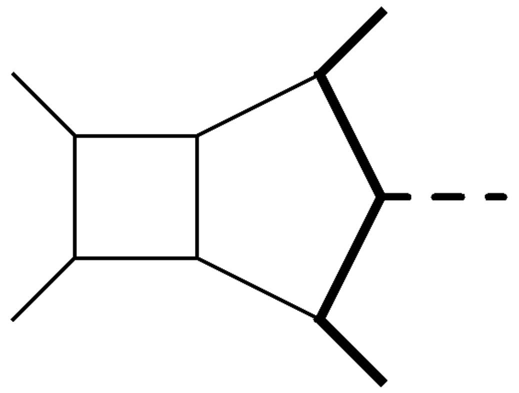
Gehrmann, Henn, Lo Presti '15; Papadopoulos, Tommasini, Wever '15



Abreu, Ita, Moriello, Page, Tschernow, Zeng '20



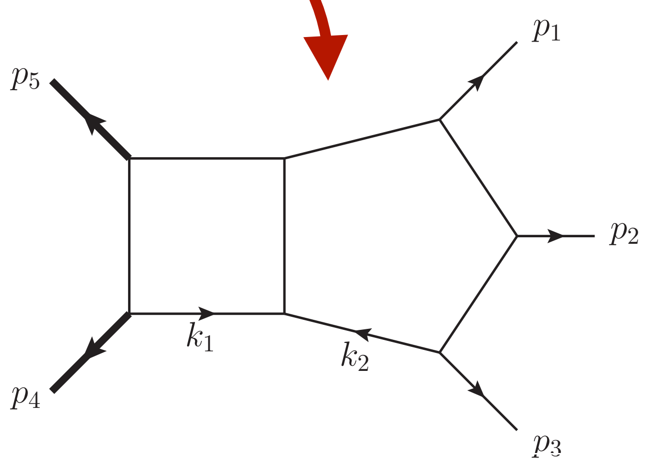
*Badger, Becchetti, Chaubey, Marzucca '23; Badger, Becchetti, Giraudo, **SZ** '24*



Febres Cordero, Figueiredo, Kraus, Page, Reina '24

Top sector has 3 MIs for all but PBzzz:

1. $\sqrt{\Delta_5}\,(p_i+p_j)^2\,\mu_{12}$
2. $\sqrt{\Delta_5}\,(p_i+p_j)^2\,\mu_{22}$
3. $C_1\,(k_2-p_i)^2 + C_2\,k_2^2$



Also in *Jiang, Liu, Xu, Yang 2024*

4th MI for PBzzz:

4. $s_{45}\sqrt{\Delta_5}\sqrt{\lambda(s_4,s_5,s_{45})}\,\left(\ldots\mu_{12}\ldots+\mu_{22}+\ldots\right)$

$$\mu_{ij} = k_i^{[-2\epsilon]}\cdot k_j^{[-2\epsilon]} \propto G\big(\{k_i,p_1,\ldots,p_4\},\{k_j,p_1,\ldots,p_4\}\big)$$

Complicated if not recognised!

A LOT of trial and error

Often need to try many candidates → cheap tests are crucial

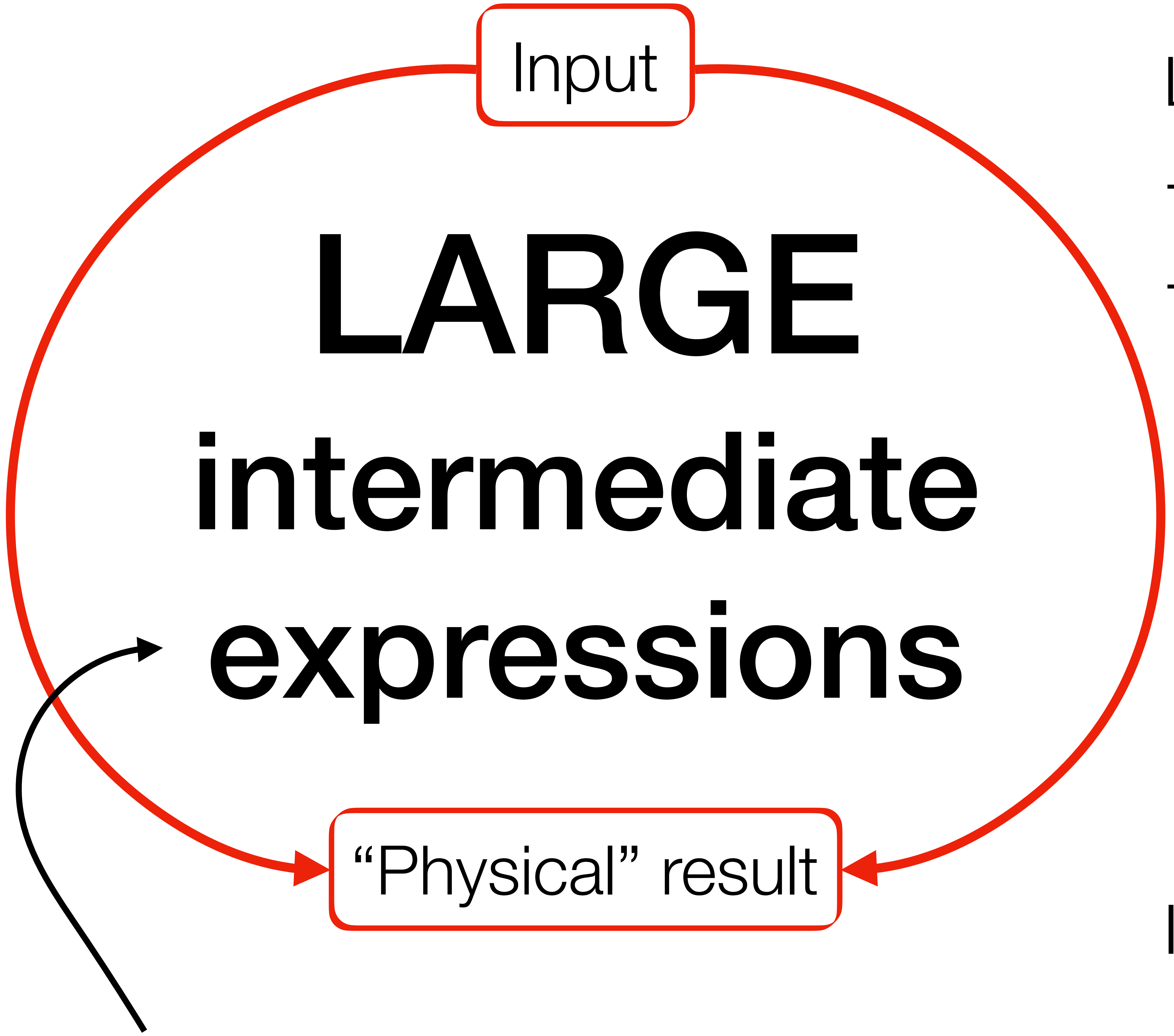
- Start from the **maximal cuts**, sector by sector

$$\mathrm{d}\vec{F} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \cdot \vec{F}$$

- Work on **univariate slices**
  - Vary only ϵ and fix kinematics
    → check ϵ-degrees + spurious singularities mixing ϵ and kinematics
  - Vary kinematic variables w.r.t. single parameter: $X = A + t\,B$
    → Polynomial degrees of entries, singularities, simple poles…

Go fully analytic only when the basis is already "good"

IBP reduction hindered by algebraic complexity



e.g. the “IBP tables”

Laporta algorithm: *Laporta 2000*

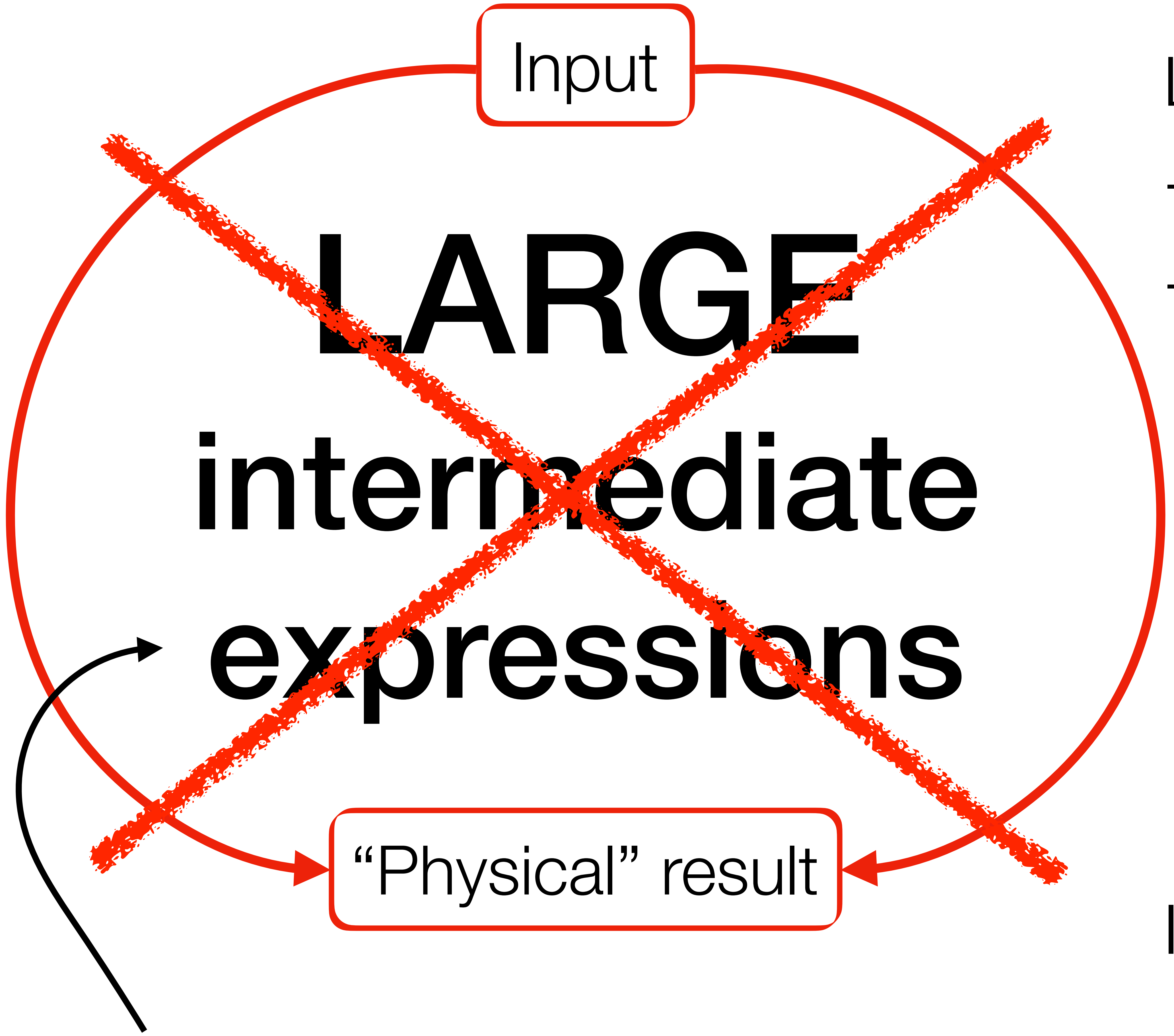
- generate large linear system of equations
- solve with special ordering of the variables

$$I_i(X;\epsilon) = \sum_j C_{ij}(X;\epsilon)\, \mathrm{F}_j(X;\epsilon)$$

needed integrals — master integrals

Intermediate expression swell

IBP reduction hindered by algebraic complexity



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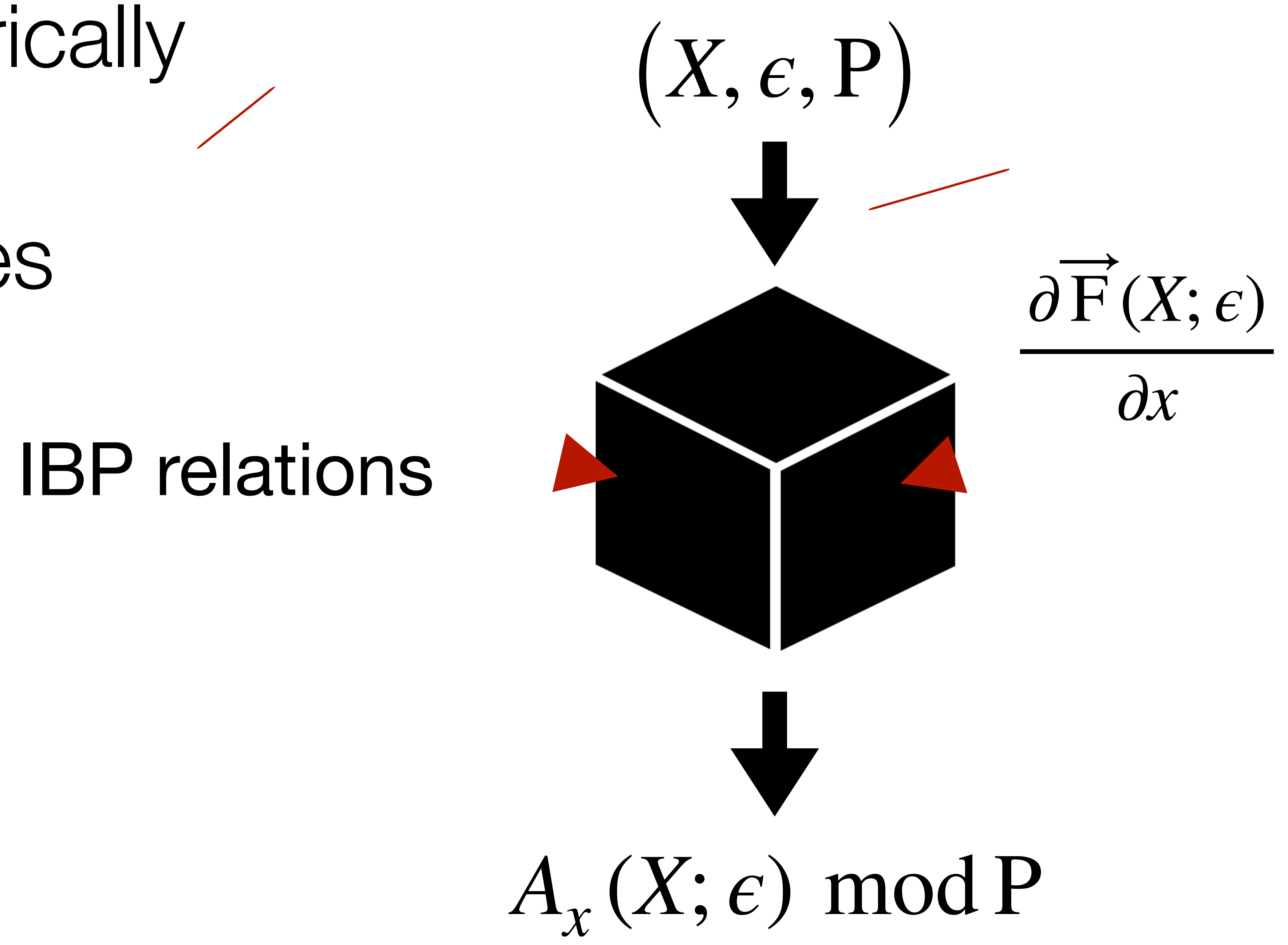
needed integrals master integrals

Intermediate expression swell

Expression swell bypassed through finite fields

von Manteuffel, Schabinger 2015; Peraro 2016

- Evaluate rational functions at numerical points (X, ϵ) modulo prime → finite field
- Perform all intermediate rational operations numerically
- Reconstruct the final result from numerical samples



Public implementations for IBP reduction:
- FIRE *Smirnov, Chukharev 2019*
- Kira *Klappert, Lange, Maierhöfer, Usovitsch 2020*

Mathematica/C++ framework **FiniteFlow** *Peraro 2019*

Functional reconstruction

$A_x(X;\epsilon)$ reconstructed in $\dfrac{\#\ \text{points} \times \text{eval. time}}{\#\ \text{CPUs}}$

$f(x, y) = 1 + x + y$ → 5 pts.

$f(x, y) = \dfrac{(1 + x + y)^{10}}{1 - x}$ → 70 pts.

Functional reconstruction

$A_x\,(X;\epsilon)$ reconstructed in $\dfrac{\#\ \text{points}\ \times\ \text{eval. time}}{\#\ \text{CPUs}}$ ⌛

$f(x,y) = 1 + x + y \quad \to$ 5 pts.

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- Choose a good basis
- Smarter algorithms
- Make ansätze

If we knew the letters before-hand,

we would only need to fit…

$$A_x(X) = \sum_i A_i \frac{\partial \log W_i(X)}{\partial x}$$

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Improve the IBP system

NeatIBP *Wu, Boehm, Ma, Xu, Zhang '23*

based on syzygy techniques *Gluza, Kajda, Kosower 2011; Ita 2016; Larsen, Zhang 2016*

→ Rourou Ma

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💰

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Alphabet can be determined without knowing the DEs

- Leading singularities
- Package Baikovletter *Jiang, Liu, Xu, Yang 2024*
- Landau equations (*Fevola, Mizera, Telen 2024; Caron-Huot, Correia, Giroux 2024*), coactions, cluster algebras, intersection theory, Schubert problem…

talk by M. Giroux

If all else fails… Rational letters = denominators of the DEs: $\mathrm{d}\log W(x) = \dfrac{\mathrm{d}W(x)}{W(x)}$

letters = # linearly independent entries of $\sum_x A_x(X)\,\mathrm{d}x \to$ computable numerically!

Systematic construction of algebraic letters

Algebraic letters inherit notion of “**charge**” from the master integrals

$$\vec{g} = \begin{pmatrix} \sqrt{\Delta}\, I_1 \\ I_2 \end{pmatrix}$$

$$g_1\Big|_{\sqrt{\Delta}\to-\sqrt{\Delta}} = -\, g_1 \;\rightarrow\; \text{odd}$$

$$g_2\Big|_{\sqrt{\Delta}\to-\sqrt{\Delta}} = g_2 \;\rightarrow\; \text{even}$$

$$\mathrm{d}\vec{g} = \begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \cdot \vec{g}$$

Letters with different charges do not talk to each other → split the problem!

Algebraic letters are singular where (some of) the rational letters vanish

$$\mathrm{d}\log\left(\frac{P+\sqrt{\Delta}}{P-\sqrt{\Delta}}\right) = \frac{\cdots}{\sqrt{\Delta}\,(P^2-\Delta)} \;\longrightarrow\; P^2 - \Delta \propto \prod_{\text{rational}} W_i^{k_i}$$

Manifestly **odd**

Ansatz

Heller, von Manteuffel, Schabinger 2019; ***SZ*** *2020*

Planar 2-loop 5-point 2-mass alphabet

(Covering all permutations of external legs)

570 independent letters,

215 rational letters,

44 square roots

236 letters odd w.r.t. 1 square root

119 letters odd w.r.t. 2 square roots

Most of the letters obtained with **Baikovletter**

$$\frac{P + Q\sqrt{\Delta_1}\sqrt{\Delta_2}}{P - Q\sqrt{\Delta_1}\sqrt{\Delta_2}}$$

Planar 2-loop 5-pt. alphabets:

0-mass: **31** letters, **1** square root

Gehrmann, Henn, Lo Presti '15; Chicherin, Henn, Mitev '17

1-mass: **156** letters, **4** square roots

*Abreu, Ita, Moriello, Page, Tschernow, Zeng '20; Chicherin, Sotnikov, **SZ** '21*

Numerical evaluation through series expansions

Numerical evaluation in Euclidean and physical region with **AMFlow** *Liu, Ma, Wang ’18; Liu, Ma ’22* + FiniteFlow *Peraro ’19* & LiteRed *Lee ’12* for the IBPs → boundary values ✔

Evolve solution to other phase-space points with **DiffExp** *Hidding 2020*

Generalised power-series solutions with finite convergence radius

$$\sum_{j_1\geq 0}\sum_{j_2=0}^{N_i} c_i^{j_1,j_2}(t-t_i)^{\frac{j_1}{2}}\log^{j_2}(t-t_i)$$

Moriello ’19; Hidding ’20; Armadillo et al. ’22; Liu, Ma ‘22

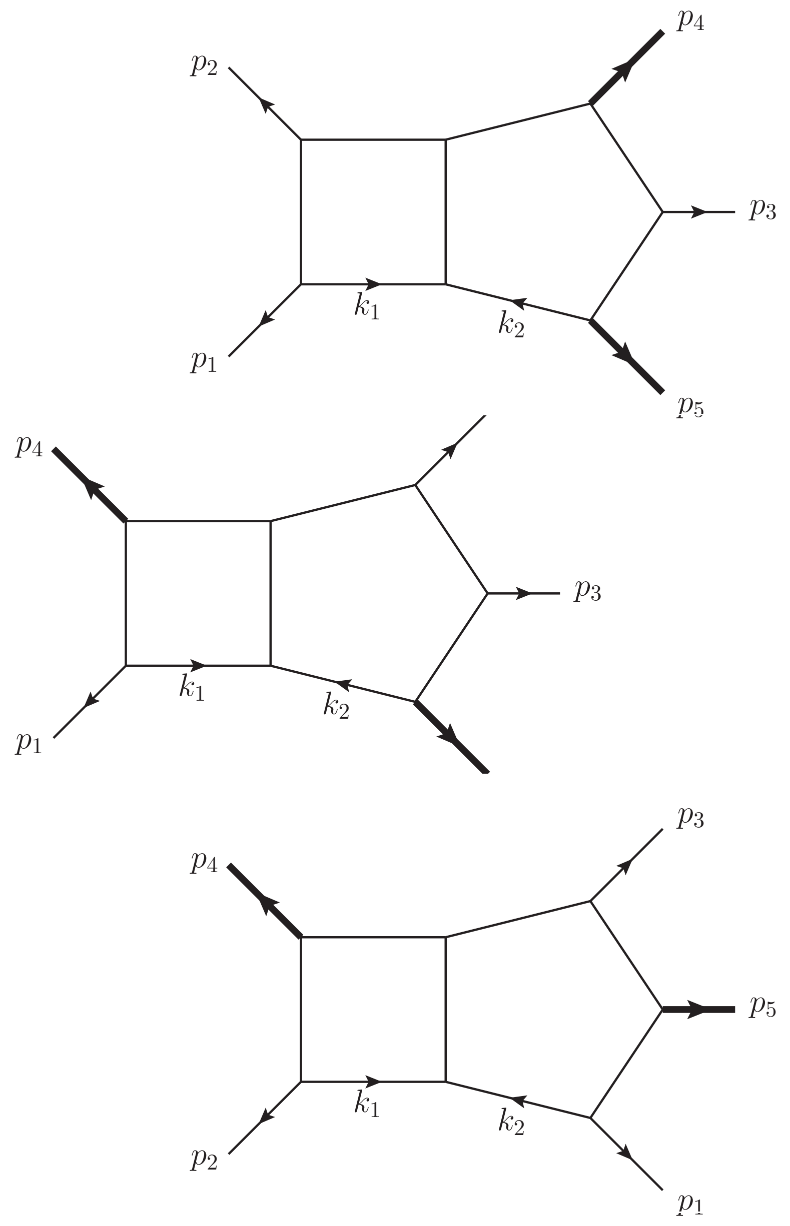
T. Armadillo’s talk

Solution with method of **pentagon functions** underway

*Gehrmann, Henn, Lo Presti ’15; Chicherin, Sotnikov ’20; + **SZ** ’21; + Abreu, Page, Tschernow ‘23*

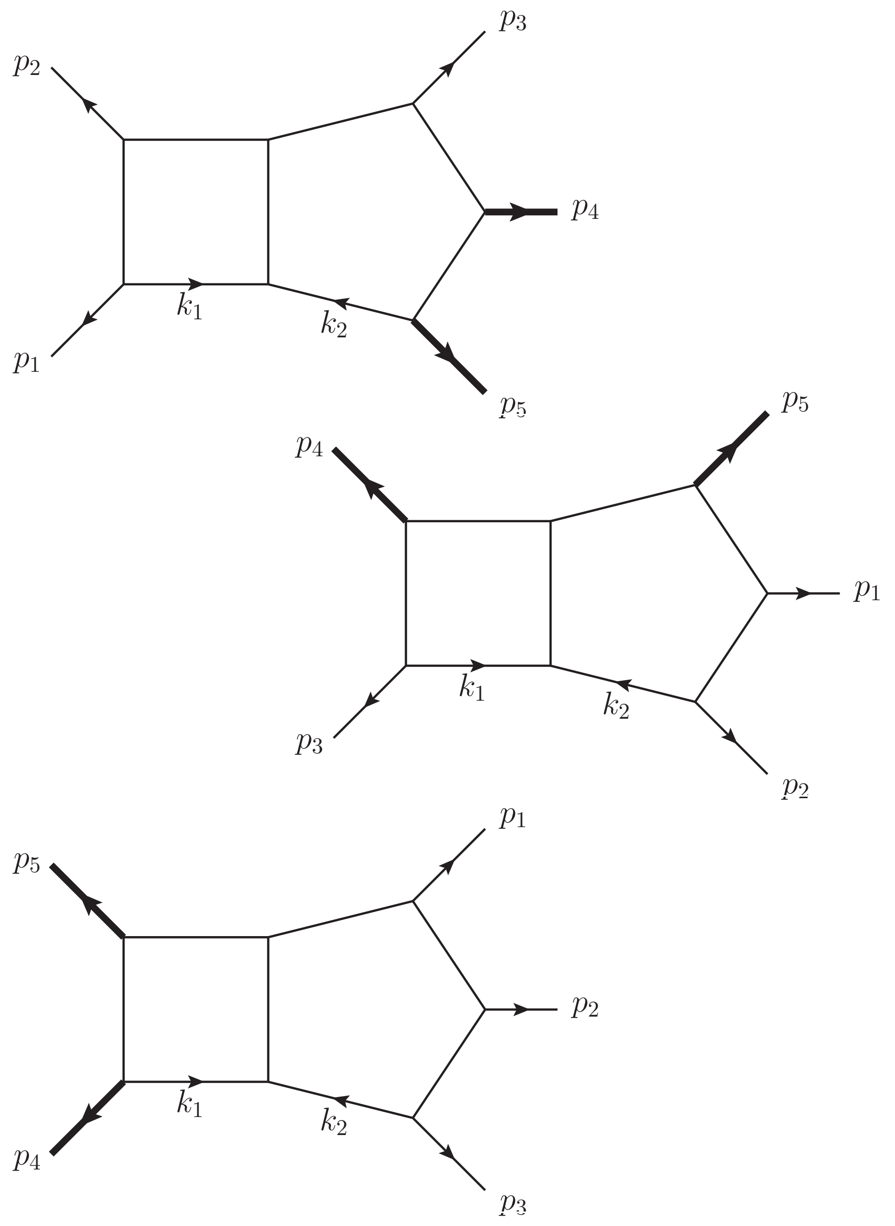
*Abreu, Chicherin, Sotnikov, **SZ** work in progress…*

Summary & outlook



Canonical differential equations for all planar 2-loop 5-point 2-mass Feynman integrals

Phenomenology + formal studies

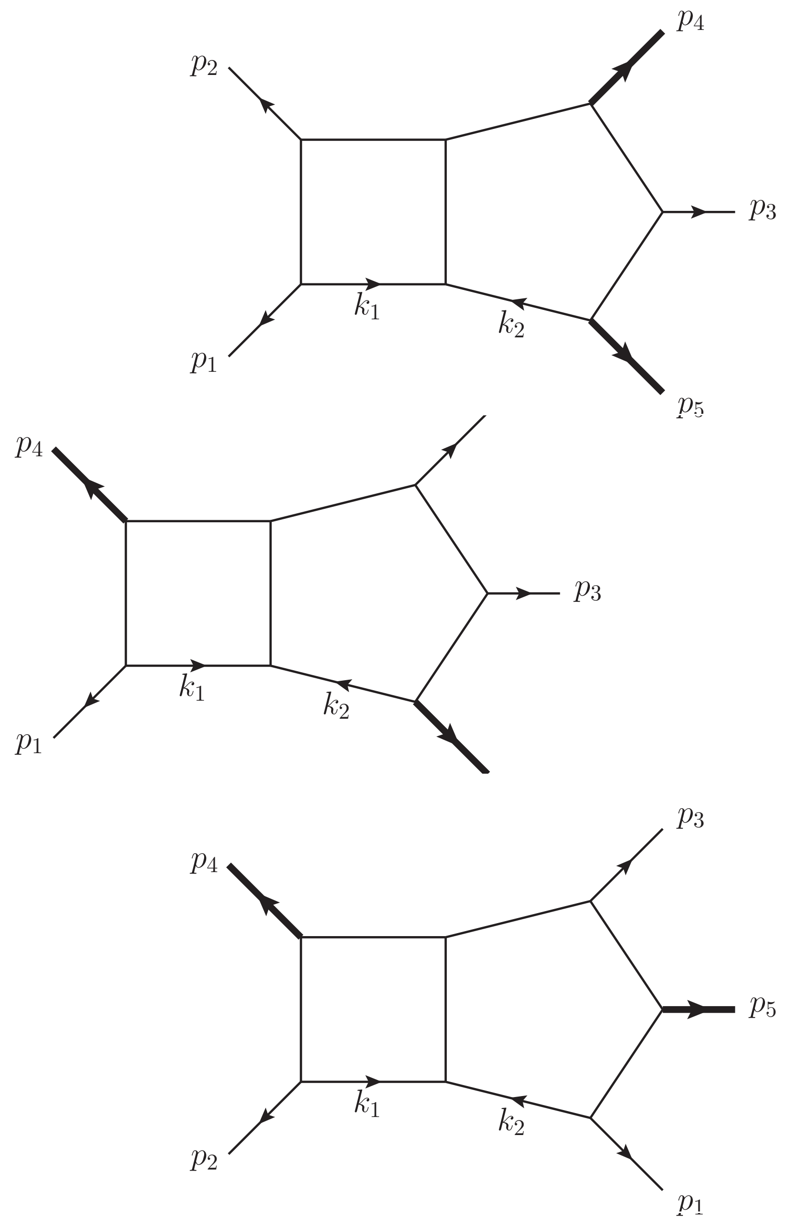


Building upon new tools and techniques resulting from effort of entire community

More challenges ahead: more legs, more loops, more masses…

Colomba Brancaccio's talk

Summary & outlook



Canonical differential equations for all planar 2-loop 5-point 2-mass Feynman integrals

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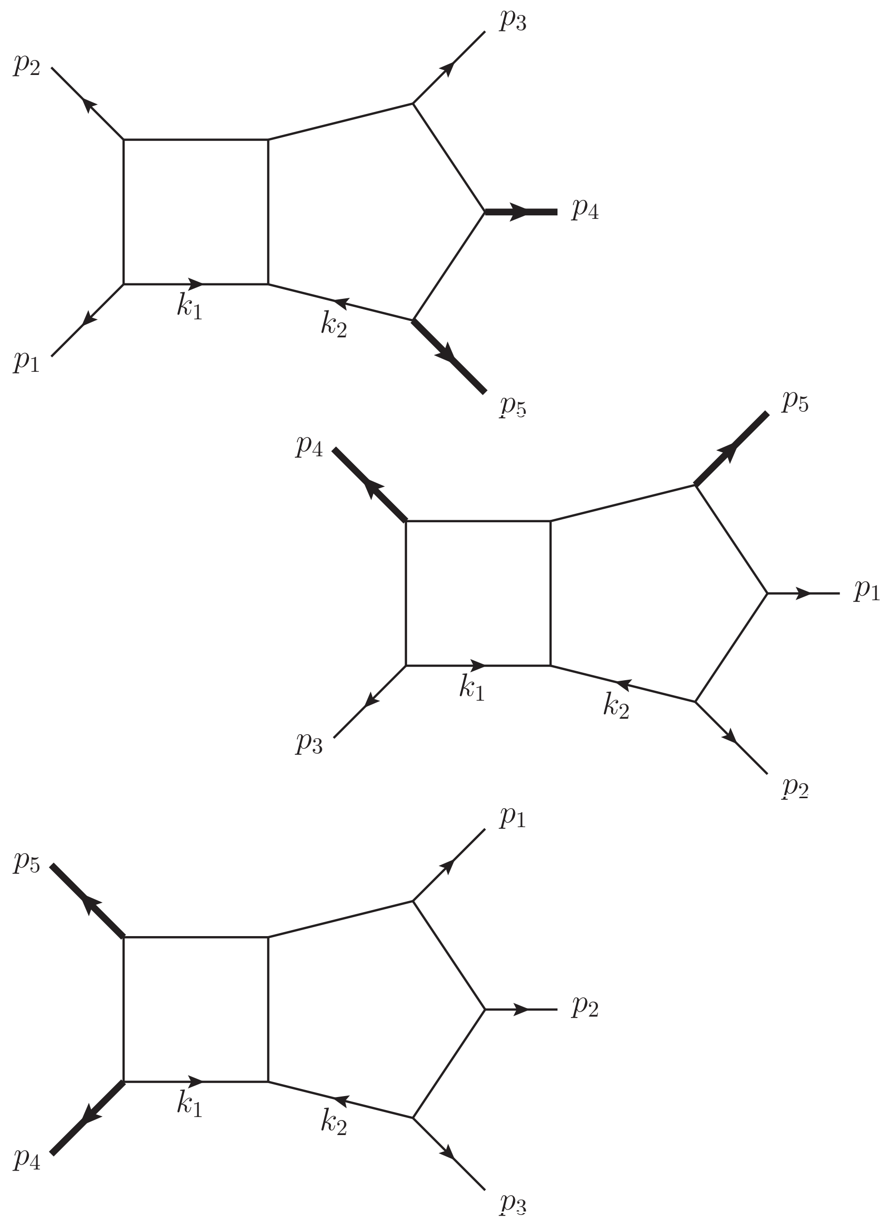

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Thank you!