Syzygy Method of Integration by Parts

NeatIBP

Zihao Wu, Janko Boehm, RM, Hefeng Xu, Yang Zhang, Comput.Phys.Commun. 295 (2024) 108999

Syzygy IBP for Feynman-like integrals ($E^n C$)

work with: Jianyu Gong, Jingwen Lin, Kai Yan, Yang Zhang, 24xx.xxxx

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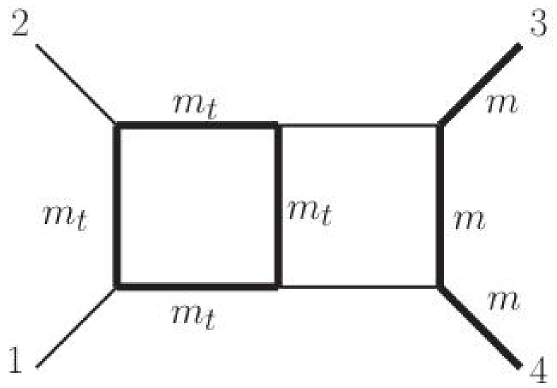
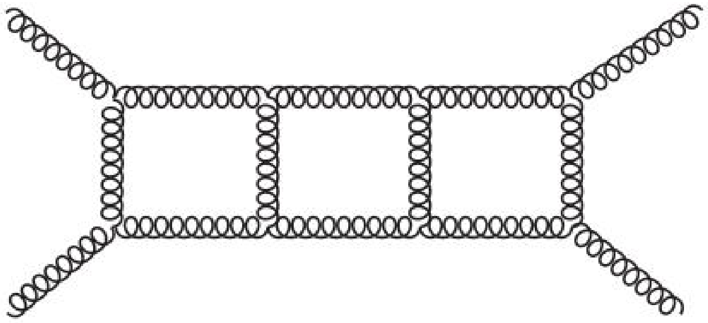
Loop-the-Loop **12/11/2024**

Feynman integral evaluation





Multi-loop Feynman integrals are the hardcore objects for a perturbative QFT computation

- Important for high-energy phenomenology
- Theoretically important

Integration by Parts (IBP)

[Analytic Tool for Feynman Integrals] *Smirnov*

For the integrals $F(a) = \int \frac{d^d k}{(k^2 - m^2)^a}$

The IBP identity $\int d^d k \frac{\partial}{\partial k_\mu} \cdot k_\mu \frac{1}{(k^2 - m^2)^a} = 0$

$$(d - 2a)F(a) - 2am^2 F(a+1) = 0$$

IBP relation system $\xrightarrow{\text{reduction}}$ (Irreducible integrals) Master integrals

Traditional IBP

$$0 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\frac{\partial v^\mu}{\partial l_k^\mu} - v^\mu \sum_{i=1}^n \frac{\partial D_i}{\partial l_k^\mu} \frac{\alpha_k}{D_i}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

target integrals

```
{G[1, 1, 1, 1, 1, 1, 1, 1, 0, -4, 0], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -3, -1],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, -2, -2], G[1, 1, 1, 1, 1, 1, 1, 1, 0, -1, -3],
 G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -4], G[1, 1, -2, 1, 1, 1, 1, 1, 0, 0, 0], ...
```

redundent integrals G[1, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0], G[1, 1, 2, 1, 1, 1, 1, 1, 0, -1, 0] ...

master integrals

```
G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, -1, 0], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -2],
G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, -1], G[1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0],...
```

redundent IBPs Time consuming! Memory consuming!

IBP syzygy method

$$0 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\frac{\partial v^\mu}{\partial l_k^\mu} - v^\mu \sum_{i=1}^n \frac{\partial D_i}{\partial l_k^\mu} \frac{\alpha_k}{D_i}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

redundent integrals

IBP operator

$$O_{IBP} = \sum_{i=1}^{L} \frac{\partial}{\partial l_k^\mu} (v_k^\mu \cdot\ )$$

avoid increasing propagators' degree

Syzygy equation

$$\sum_{k=1}^{L} v_k^\mu \frac{\partial D_i}{\partial l_k^\mu} = g_i D_i \qquad i \in \{j \,|\, \alpha_j > 0\}$$

Janusz Gluza, Krzysztof Kajda, David A. Kosower: Phys.Rev.D 83 (2011) 045012

Contents

- NeatIBP
- an IBP package to get a shorter IBP system based on syzygy method
- recent devolopment

- Feynman-like integral
- 3-point energy correlator
- 4-point energy correlator

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NeatIBP in Baikov representation

$$I[\alpha_1, \cdots, \alpha_n] = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

↓ Baikov representation

$$I[\alpha_1, \cdots, \alpha_n] = C \int \mathrm{d}z_1 \cdots \mathrm{d}z_n P(z)^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

IBP operator $$O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \;) \qquad a_i \in Q(\vec{s}_{ij})[z_1, \cdots, z_n]$$

IBP relation $$\left(O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \;) \right) I[\alpha_1, \cdots, \alpha_n] = 0$$

Syzygy IBP in Baikov represetation

$$0=\int \mathrm{d}z_1\cdots \mathrm{d}z_n \sum_{i=1}^{n}\frac{\partial}{\partial z_i}\left(a_i(z)P^{\alpha}\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}\right)$$

$$\alpha=\frac{D-L-E-1}{2}$$

$$=\int \mathrm{d}z_1\cdots \mathrm{d}z_n\left(\sum_{i=1}^{n}\frac{\partial a_i}{\partial z_i}P^{\alpha}+\sum_{i=1}^{n}\alpha a_i\frac{\partial P}{\partial z_i}P^{\alpha-1}-P^{\alpha}\sum_{i=1}^{n}\alpha_i\frac{a_i}{z_i}\right)\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

$$\left(\sum_{i=1}^{n}a_i(z)\frac{\partial P}{\partial z_i}\right)+b(z)P=0 \qquad a_i(z)=b_i(z)z_i \quad \text{for } i\in\{j|\alpha_j>0\}$$

Relate integrals in the same dimention

Avoid incearsing the degree of propagators

$$0=\int \mathrm{d}z_1\cdots \mathrm{d}z_n\left(\sum_{i=1}^{n}\left(\frac{\partial a_i}{\partial z_i}-\alpha_i b_i\right)+\alpha b\right)P^{\alpha}\frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

Syzygy equation

$$O_{IBP} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (a_i \cdot \ ) \qquad a_i \in Q(\vec{s}_{ij})[z_1, \cdots, z_n]$$

① $$\left(\sum_{i=1}^{n} a_i(z) \frac{\partial P}{\partial z_i} \right) + b(z) P = 0$$

$$M_1 =< f_1, f_2, \cdots >$$

② $$a_i(z) = b_i(z) z_i \ \text{ for } i \in \{ j | \alpha_j > 0 \}$$

$$M_2 =< g_1, g_2, \cdots >$$

Module Intersection $$\begin{pmatrix} a_i \\ b \end{pmatrix} \in M_1 \cap M_2$$

SINGULAR

IBP linear system

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \left(\sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial z_i} - \alpha_i b_i \right) + \alpha b \right) P^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

$$\begin{pmatrix} a_i \\ b \end{pmatrix} \in M_1 \cap M_2$$

$$\vec{\alpha} \to (1, \cdots, 1, -2, -3)$$
$$\vec{\alpha} \to (1, \cdots, 1, 0, -5)$$
$$\vdots$$

Seeding

#IBP relations = #O_{IBP} × #Seeding

NeatIBP workflow

config.txt kinematics.txt targetIntegrals.txt

Baikov representation

```
Start initialization steps.
L=2 E=4
Parameters {t1, t2, w1, w2, yy}
Baikov variables: var={z[1], z[2], z[3], z[4], z[5], z[6], z[7], z[8], z[9], z[10], z[11]}
```

Module intersection

```
Module intersections solved. Time Used: 6 second(s). Memory used: 81 MB.
VectorList Length: 269
VectorList ByteCount: 58466712
```

Generate IBP relations

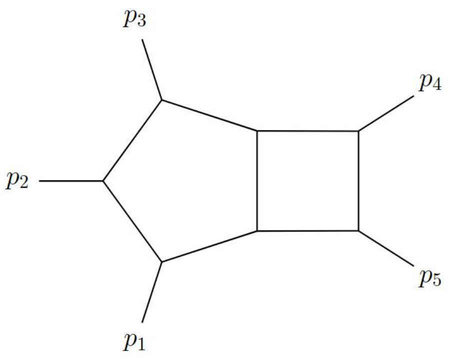
```
Generating Formal IBPs...
    269 Formal IBPs generated. Time Used: 7 second(s). Memory used: 104 MB.
Removing FIBPs for lower sectors seeding with DenominatorTypes...
    269 Formal IBPs are generated; 144 Formal IBPs are used. 125 Formal IBPs are removed.
    IBPs for lower sectors removed. Time Used: 1 second(s). Memory used: 29 MB.
Generating numerical FIBPs...
    nFIBPs generated. Time Used: 1 second(s). Memory used: 53 MB.
Generating nFIBPFunctions...
    nFIBPFunctions generated. Time Used: 1 second(s). Memory used: 36 MB.
```

Select independent IBPs

```
    Independent IBPs selected. Time Used: 0 second(s). Memory used: 1 MB.
    50 IBPs are selected with 56 integrals in current sector.
Removing the unneeded IBPs...
```

IBP_all.txt MI_all.txt OrderedIntegrals.txt

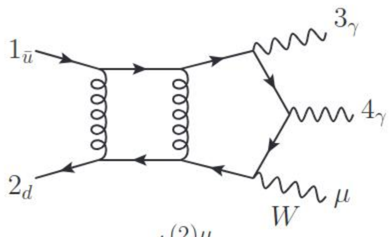
NeatIBP examples



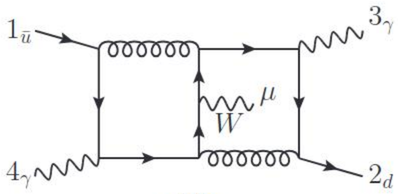
#Target integrals: 2483

#IBP (FIRE6): 11207942
#IBP (NeatIBP): 14120

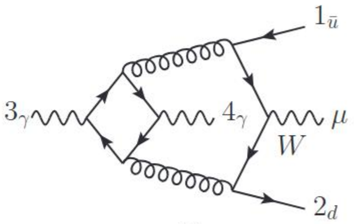
2 or 3 orders of magnitude simplification than Laporta algorithm



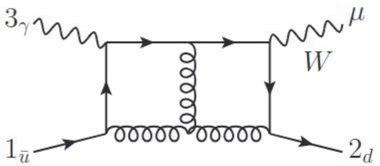
$A^{(2)\mu}_{5,uu}$



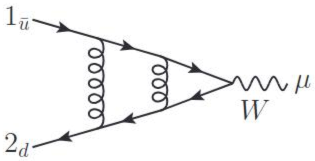
$A^{(2)\mu}_{5,ud}$



$A^{(2)\mu}_{5,q}$



$A^{(2)\mu}_{4,u}$



$A^{(2)\mu}_{3}$

$W\gamma\gamma$ production

S. Badger, H. B. Hartanto, Z. Wu, Y. Zhang and S. Zoia, Two-loop amplitudes for $O(\alpha_s^2)$ corrections to $W\gamma\gamma$ production at the LHC, 2409.08146

Planar diagram: NeatIBP+ Finiteflow
8 times faster than Finiteflow itself

non-Planar diagram:
NeatIBP+ Finiteflow works
Finiteflow itself does not provide the result

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Recent devolopment

- Spanning cut IBP

  Splitting the hard problem into simpler pieces.

- NeatIBP + Kira interface

  Providing automated IBP reduction.

- Maximal cut syzygy generator

  Making the seeding process computation cheaper.

Stay tuned RM, Johann Usovitsch, Zihao Wu, Yingxuan Xu, Yang Zhang, 24xx.xxxx

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Feynman-like integrals

Where are Fynman-like integrals?

- observables or cross section level: energy correlator, cosmological correlator...

$$\frac{5x_1x_2x_3x_4\left(2\zeta_{12}x_2x_1+2\zeta_{13}x_3x_1+2\zeta_{14}x_4x_1+2\zeta_{23}x_2x_3+2\zeta_{34}x_3x_4-2x_1-x_2-2x_3-x_4+1\right)}{\left(\zeta_{12}x_2+\zeta_{13}x_3+\zeta_{14}x_4-1\right)\left(\zeta_{14}x_1+\zeta_{24}x_2+\zeta_{34}x_3-1\right)\left(\zeta_{13}x_1x_3+\zeta_{23}x_2x_3+\zeta_{34}x_4x_3+\zeta_{14}x_1x_4+\zeta_{24}x_2x_4-x_3-x_4\right)\cdots}$$

Why energy correlator?

- Phenomenology: energy correlators can be used as jet observables
- Computation: energy correlators is perhaps the simplest infared safe observable
- Theory: observables probing the spatial correlation among flow operators.

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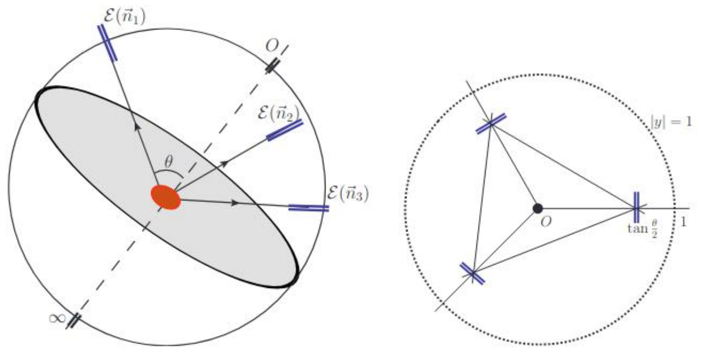
Three point energy correlator



energy parameters: $x_i = \frac{2\, q \cdot p_i}{q^2}, \quad i = 1, \cdots, n$

angle parameters: $\zeta_{ij} = \frac{q^2\, p_i \cdot p_j}{2\, q \cdot p_i\, q \cdot p_j}, \qquad i, j = 1, \cdots, n$

$$E^3 C(\vec{\zeta}_{ij})|_{LO} \sim \int_0^1 dx_1 dx_2 dx_3 (x_1 \cdots x_3)^2 \delta(1 - Q_3) |F_4^{(0)}|^2_{sym}$$

$$\frac{4 x_1 x_2 x_3}{\zeta_{13} \left(\zeta_{12} x_1 + \zeta_{23} x_3 - 1\right) \left(\zeta_{12} x_1 x_2 + \zeta_{13} x_1 x_3 + \zeta_{23} x_2 x_3\right) \left(\zeta_{12} x_2 x_1 + \zeta_{13} x_3 x_1 + \zeta_{23} x_2 x_3 - x_1 - x_2\right)}$$

Kai Yan and Xiaoyuan Zhang. Three-Point Energy Correlator in N=4 Supersymmetric Yang-Mills Theory. *Phys. Rev. Lett.*, 129(2):021602, 2022.

Power counting for finite integrals

$$D_1 = -1 + x_3,\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_3 = -1 + x_1\zeta_{12} + x_3\zeta_{23},$$
$$D_\delta = 1 - x_1 - x_2 - x_3 + x_1x_2\zeta_{12} + x_2x_3\zeta_{23} + x_1x_3\zeta_{13}$$

$$\mathrm{Int}[1,1,-1,1] = \int dx_1dx_2dx_3 \frac{D_3\,\delta(D_\delta)}{D_1D_2}$$

$$\text{region: } \{x_1 \to c,\ x_2 \to c,\ x_3 \to 1-2c\}|_{c\to 0}$$

$$dx_1dx_2dx_3\delta(D_\delta) \xrightarrow[\text{counting}]{\text{power}} 2$$

$$\{D_1,\ D_2,\ D_3\} \xrightarrow[\text{counting}]{\text{power}} \{1,0,0\}$$

```
{Int[-2, 0, 0, 1], Int[-2, 0, 1, 1], Int[-2, 1, -1, 1], Int[-2, 1, 0, 1],
 Int[-2, 1, 1, 1], Int[-2, 1, 2, 1], Int[-2, 2, -1, 1], Int[-2, 2, 0, 1],
 Int[-2, 2, 1, 1], Int[-1, -1, 0, 1], Int[-1, 0, -1, 1], Int[-1, 0, 0, 1],
 Int[-1, 1, -2, 1], Int[-1, 1, -1, 1], Int[-1, 1, 2, 1], Int[-1, 2, -2, 1],
 Int[-1, 2, -1, 1], Int[-1, 2, 0, 1], Int[-1, 2, 1, 1], Int[0, -1, -1, 1],
 Int[0, -1, 0, 1], Int[0, 0, -2, 1], Int[0, 0, -1, 1], Int[0, 1, -3, 1],
 Int[0, 1, -2, 1], Int[0, 1, 2, 1], Int[0, 2, -3, 1], Int[0, 2, -2, 1],
 Int[0, 2, -1, 1], Int[0, 2, 0, 1], Int[0, 2, 1, 1], Int[1, -2, -1, 1],
 Int[1, -1, -2, 1], Int[1, -1, -1, 1], Int[1, 0, -3, 1], Int[1, 0, -2, 1],
 Int[1, 1, -3, 1], Int[1, 1, -2, 1], Int[1, 2, -4, 1], Int[1, 2, -3, 1],
 Int[1, 2, -2, 1], Int[1, 2, -1, 1], Int[1, 2, 0, 1], Int[-1, 0, 1, 1],
 Int[-1, 1, 0, 1], Int[0, 0, 0, 1], Int[0, 0, 1, 1], Int[0, 1, -1, 1],
 Int[0, 1, 0, 1], Int[1, -1, 0, 1], Int[1, 0, -1, 1], Int[-1, 1, 1, 1],
 Int[0, 1, 1, 1], Int[1, 0, 0, 1], Int[1, 1, -1, 1], Int[1, 1, 0, 1]}
```

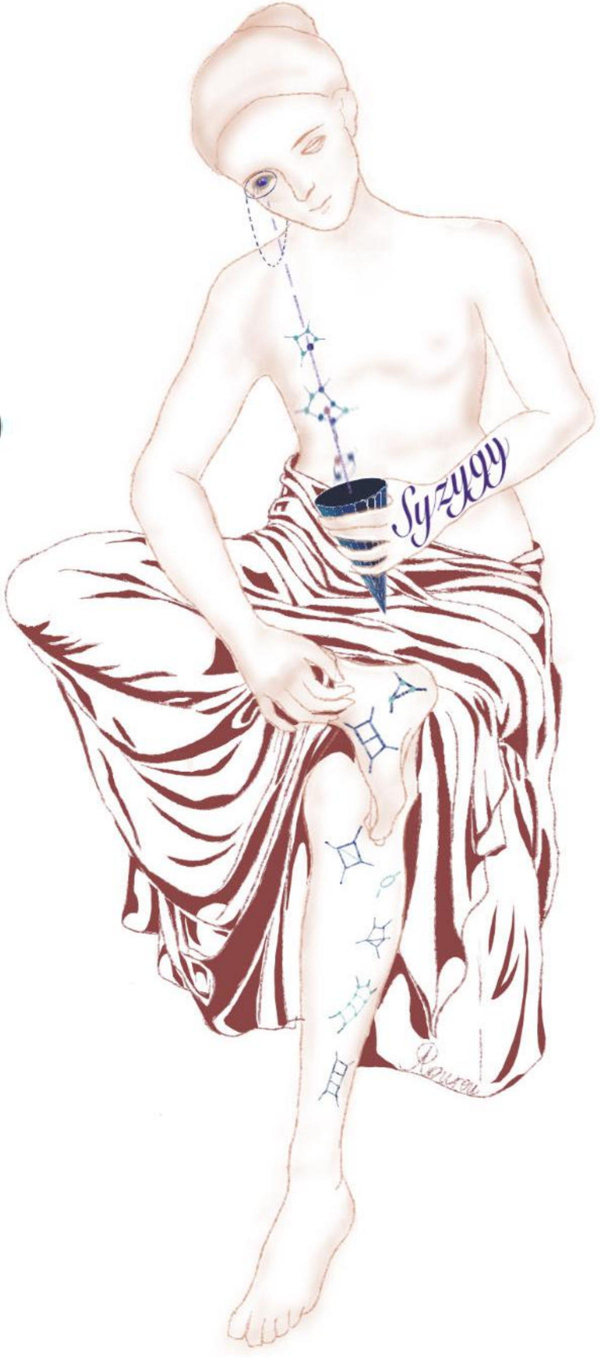
Syzygy method for finite integrals

$$\text{Int}[1,1,-1,1] = \int dx_1 dx_2 dx_3 \frac{D_3\,\delta(D_\delta)}{D_1 D_2} = \int dx_1 dx_2 dx_3 \frac{D_3}{D_1 D_2 D_\delta}\Big|_{cut(D_\delta)}$$

IBP Operator $$O_{IBP} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (a_i \ \cdot\ )$$

syzygy equation $$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \quad j \in \{1, \delta\}$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3].$$



Boundary IBP

$$\left(O_{IBP} = \sum_{i=1}^{3} \frac{\partial}{\partial z_i} (a_i \cdot\ )\right) I[\alpha_1, \alpha_2, \alpha_3] = \textbf{Boundary term}$$

subfamily 1 $\quad D_1 = -1 + x_1\zeta_{12},\ D_2 = -1 + x_1\zeta_{13} + x_2\zeta_{23},\ D_\delta = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,;$

subfamily 2 $\quad D_1 = -1 + x_3\zeta_{13},\ D_2 = -1 + x_1\zeta_{13},\ D_\delta = 1 - x_1 - x_3 + x_1x_3\zeta_{13}\,;$

subfamily 3 $\quad D_1 = -1 + x_3\zeta_{23},\ D_2 = -1 + x_2\zeta_{23},\ D_\delta = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,.$

Master Integrals

$$\{\mathrm{Int}[1,1,-1,1],\ \mathrm{Int}[-1,1,1,1],\ \mathrm{Int}[1,1,0,1],\ \mathrm{Int}[0,1,1,1],\ \mathrm{Int}[1,0,0,1],$$
$$\mathrm{Int}_2[1,\{0,0,1\}],\ \mathrm{Int}_2[2,\{0,1,1\}],\ \mathrm{Int}_2[2,\{0,0,1\}],\ \mathrm{Int}_2[3,\{0,0,1\}],\ 1\}$$

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E^3C propagators

$$\frac{s_{134}}{q^2} \propto -1 + x_2,\ \frac{s_{124}}{q^2} \propto -1 + x_3,\ \frac{s_{123}}{q^2} \propto -1 + x_1 + x_2 + x_3,$$
$$\frac{s_{34}}{q^2} \propto -1 + x_1\zeta_{13} + x_2\zeta_{23},\ \frac{s_{24}}{q^2} \propto -1 + x_1\zeta_{12} + x_3\zeta_{23},$$
$$\delta(D_\delta) = 1 - x_1 - x_2 - x_3 + x_1x_2\zeta_{12} + x_2x_3\zeta_{23} + x_1x_3\zeta_{13}$$

E^4C propagators

$$\frac{s_{45}}{x_4} = 1 - x_1\zeta_{14} - x_2\zeta_{24} - x_3\zeta_{34}\,, \quad \frac{s_{15}}{x_1} = 1 - x_2\zeta_{12} - x_3\zeta_{13} - x_4\zeta_{14}\,,$$
$$s_{2345} = 1 - x_1\,, \quad s_{1345} = 1 - x_2\,, \quad s_{1235} = 1 - x_4\,, \quad s_{1234} = 1 - x_1 - x_2 - x_3 - x_4\,,$$
$$s_{345} = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,, \quad s_{145} = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,, \quad s_{125} = 1 - x_3 - x_4 + x_3x_4\zeta_{34}\,,$$
$$s_{123} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}\,, \quad s_{234} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}\,.$$
$$\delta(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_1x_4\zeta_{14} + x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}).$$

Power counting

$$\frac{s_{45}}{x_4} = 1 - x_1\zeta_{14} - x_2\zeta_{24} - x_3\zeta_{34}\,, \quad \frac{s_{15}}{x_1} = 1 - x_2\zeta_{12} - x_3\zeta_{13} - x_4\zeta_{14}\,,$$

$$s_{2345} = 1 - x_1\,, \quad s_{1345} = 1 - x_2\,, \quad s_{1235} = 1 - x_4\,, \quad s_{1234} = 1 - x_1 - x_2 - x_3 - x_4\,,$$

$$s_{345} = 1 - x_1 - x_2 + x_1x_2\zeta_{12}\,, \quad s_{145} = 1 - x_2 - x_3 + x_2x_3\zeta_{23}\,, \quad s_{125} = 1 - x_3 - x_4 + x_3x_4\zeta_{34}\,,$$

$$s_{123} = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23}\,, \quad s_{234} = x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}\,.$$

cadidate divergent regions

$$\{x_1 \to c\,, \quad x_2 \to c\,, \quad x_3 \to c\,, \quad x_4 \to 1 + (-3 + \zeta_{14} + \zeta_{24} + \zeta_{34})c\}$$

$$\{x_1 \to c\,, \quad x_2 \to c\,, \quad x_3 \to 1 + (-3 + \zeta_{13} + \zeta_{23} + \zeta_{34})c\,, \quad x_4 \to c\}$$

$$\{x_1 \to c\,, \quad x_2 \to 1 + (-3 + \zeta_{12} + \zeta_{23} + \zeta_{24})c\,, \quad x_3 \to c\,, \quad x_4 \to c\}$$

$$\{x_1 \to 1 + (-3 + \zeta_{12} + \zeta_{13} + \zeta_{14})c\,, \quad x_2 \to c\,, \quad x_3 \to c\,, \quad x_4 \to c\}$$

$$\text{region 1:} \;\; \deg(1) = \{0,0,1,0,0,1,1,0,0,1,2\}$$

$$\text{region 2:} \;\; \deg(2) = \{0,0,0,1,0,1,1,1,0,1,1\}$$

$$\text{region 3:} \;\; \deg(3) = \{0,0,0,0,0,1,0,1,1,1,1\}$$

$$\text{region 4:} \;\; \deg(4) = \{0,0,0,0,1,1,0,0,1,2,1\}$$

$$dx_1 dx_2 dx_3 dx_4 \delta(D_\delta) \xrightarrow[\text{counting}]{\text{power}} 3$$

Example family

family: $D_1 = 1 - x_1\zeta_{14} - x_2\zeta_{24} - x_3\zeta_{34},\ D_2 = 1 - x_4,\ D_3 = x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_2x_3\zeta_{23},$
$D_\delta = 1 - x_1 - x_2 - x_3 - x_4 + x_1x_2\zeta_{12} + x_1x_3\zeta_{13} + x_1x_4\zeta_{14} + x_2x_3\zeta_{23} + x_2x_4\zeta_{24} + x_3x_4\zeta_{34}$

target integrals:

```
{Int[{0, 0, 0, 1}, 1], Int[{0, 0, 1, 1}, 1], Int[{0, 0, 1, 1}, x[1]], Int[{0, 0, 1, 1}, x[2]],
 Int[{0, 0, 1, 1}, x[3]], Int[{0, 0, 1, 1}, x[4]], Int[{0, 0, 1, 1}, x[3] x[4]],
 Int[{0, 1, 1, 1}, x[2]], Int[{0, 1, 1, 1}, x[3]], Int[{1, 0, 0, 1}, 1], Int[{1, 1, 0, 1}, 1],
 Int[{1, 1, 0, 1}, x[2]], Int[{1, 1, 0, 1}, x[3]], Int[{1, 1, 1, 1}, x[2]],
 Int[{1, 1, 1, 1}, x[3]], Int[{1, 1, 1, 1}, x[2] x[3]], Int[{1, 1, 1, 1}, x[3]^2],
 Int[{2, 0, 0, 1}, 1], Int[{2, 0, 0, 1}, x[2]], Int[{2, 0, 0, 1}, x[3]],
 Int[{2, 1, 0, 1}, 1], Int[{2, 1, 0, 1}, x[2]], Int[{2, 1, 0, 1}, x[1] x[2]],
 Int[{2, 1, 0, 1}, x[3]], Int[{2, 1, 0, 1}, x[1] x[3]], Int[{2, 1, 0, 1}, x[2] x[3]],
 Int[{2, 1, 0, 1}, x[3]^2], Int[{2, 1, 1, 1}, x[1]], Int[{2, 1, 1, 1}, x[1]^2],
 Int[{2, 1, 1, 1}, x[2]], Int[{2, 1, 1, 1}, x[1] x[2]], Int[{2, 1, 1, 1}, x[2]^2],
 Int[{2, 1, 1, 1}, x[3]], Int[{2, 1, 1, 1}, x[1] x[3]], Int[{2, 1, 1, 1}, x[2] x[3]],
 Int[{2, 1, 1, 1}, x[3]^2], Int[{2, 1, 1, 1}, x[2] x[3]^2], Int[{2, 1, 1, 1}, x[3]^3]} ...
```

Syzygy IBP

$$\sum_{i=1}^{4} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \quad j \in \{2, 3, \delta\}$$

$$a_i,\ b_j \in Q(\vec{z})[x_1, x_2, x_3, x_4]$$

Integrand reduction:

$$I[\{n_1, n_2, \ldots, n_i, \ldots\}, N] = \sum_{m \in \text{monomial in } D_i} c_i I[\{n_1, n_2, \ldots, n_i + 1, \ldots\}, mN]$$

kill integrals with high degree numerator, such as

$$\mathrm{Int}[\{2,1,1,1\}, x_3^3],\ \mathrm{Int}[\{2,1,1,1\}, x_2 x_3^2],\ \mathrm{Int}[\{2,1,0,1\}, x_2 x_3],\ \mathrm{Int}[\{1,1,1,1\}, x_2 x_3], \cdots$$

Master integrals

syzygy IBP : 10 irreducible integrals

```
{Int[{0, 0, 1, 1}, 1], Int[{1, 1, 0, 1}, 1], Int[{1, 1, 1, 1}, x[1]], Int[{1, 1, 1, 1}, x[2]],
 Int[{2, 1, 0, 1}, x[1] x[2]], Int[{2, 1, 1, 1}, x[1]], Int[{2, 1, 1, 1}, x[1]^2],
 Int[{2, 1, 1, 1}, x[1] x[2]], Int[{2, 1, 1, 1}, x[2]^2], Int[{2, 1, 1, 1}, x[1] x[3]]}
```

syzygy IBP + integrand reduction: 4 irreducible integrals

```
{Int[{0, 0, 1, 1}, 1], Int[{1, 1, 0, 1}, 1], Int[{1, 1, 1, 1}, x[1]], Int[{1, 1, 1, 1}, x[2]]}
```

Summary

- Syzygy IBP method

  Shorten IBP system efficiently

  Generator IBP system for Feynman-like integrals

- NeatIBP: An IBP package for generator a small-size IBP system
- recent devolopment: spanning cut, NeatIBP + Kira interface, simplify syzygy generators

- Syzygy IBP for particular integrals: n-point energy correlator (finite integrals)

Thank you

n-point energy correlator

energy flow operator $\mathcal{E} = \int_0^\infty \mathrm{dt} \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\vec{n})$

$$E^n C(\theta_{ij}) = \int \prod_{i=1}^{n} d\Omega_{\vec{n}_i} \prod_{i\neq j} \delta(\vec{n}_i \cdot \vec{n}_j - \cos(\theta_{ij})) \frac{\int d^4x e^{iqx} \langle 0|\mathcal{O}^+(x)\mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\mathcal{O}(0)|0\rangle}{(q^0)^n \int d^4x e^{iqx} \langle 0|\mathcal{O}^+(x)\mathcal{O}(0)|0\rangle}$$

create the final state X

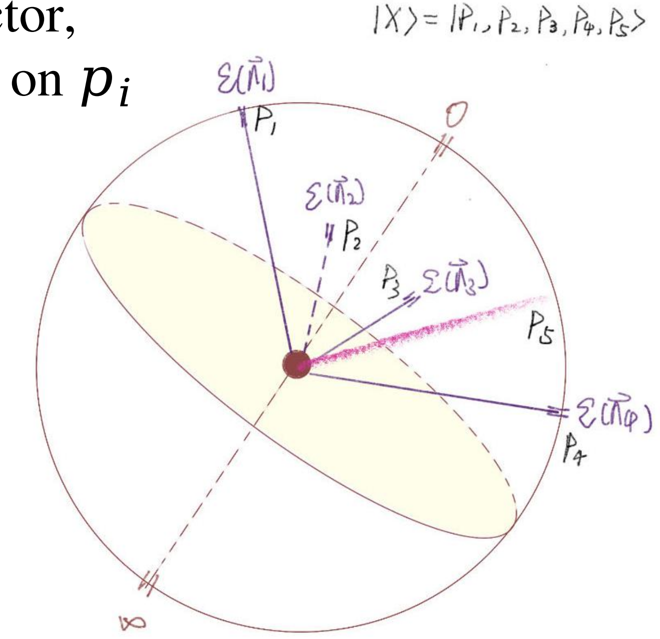
$$\Downarrow \quad \int d^4x\, e^{iqx} \langle X|\mathcal{O}(x)|0\rangle \equiv (2\pi)^4 \delta^4(q - q_X) F_X$$

form factor, depends on p_i

$$\sim \sum_{(n_1,\cdots,n_n)\in X} \int d\Pi_X \left(\prod_{i=1}^{n} \delta^2(\vec{n}_i - \hat{p}_{n_i}) \frac{E_i}{q^0} \right) |F_X|^2$$

where $d\Pi_X$ is onshell phase-space of the finial state.

$$\frac{2\left(2q^4 (p_1+p_2+p_4+p_5)\cdot(p_2+p_3+p_4+p_5) + \cdots - q^4 (p_2+p_3+p_4+p_5)\cdot(p_1+p_2+p_3+p_4+p_5) + q^6\right)}{(p_3+p_4)^2 (p_1+p_5)^2 (p_1+p_4+p_5)^2 (p_1+p_2+p_4+p_5)^2 (p_3+p_4+p_5)^2 (p_2+p_3+p_4+p_5)^2}$$



Kai Yan and Xiaoyuan Zhang. Three-Point Energy Correlator in N=4 Supersymmetric Yang-Mills Theory. *Phys. Rev. Lett.*, 129(2):021602, 2022.

Verify finite integrals by power counting

$$I(x_0, y_0) = \int_0^{x_0} dx \int_0^{y_0} dy \frac{1}{x+y} \qquad \{x \to c, y \to c\} \quad \text{when } c \to 0$$

$$= (x_0 + y_0) \log (x_0 + y_0) - x_0 \log (x_0) - y_0 \log (y_0)$$

$$cPower[I(x_0, y_0)] = cPower[\int_0^{x_0} dcx \int_0^{y_0} dcy \frac{1}{cx + cy}] = 1 > 0$$

syzygy equation

$$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \quad j \in \{1, \delta\}$$

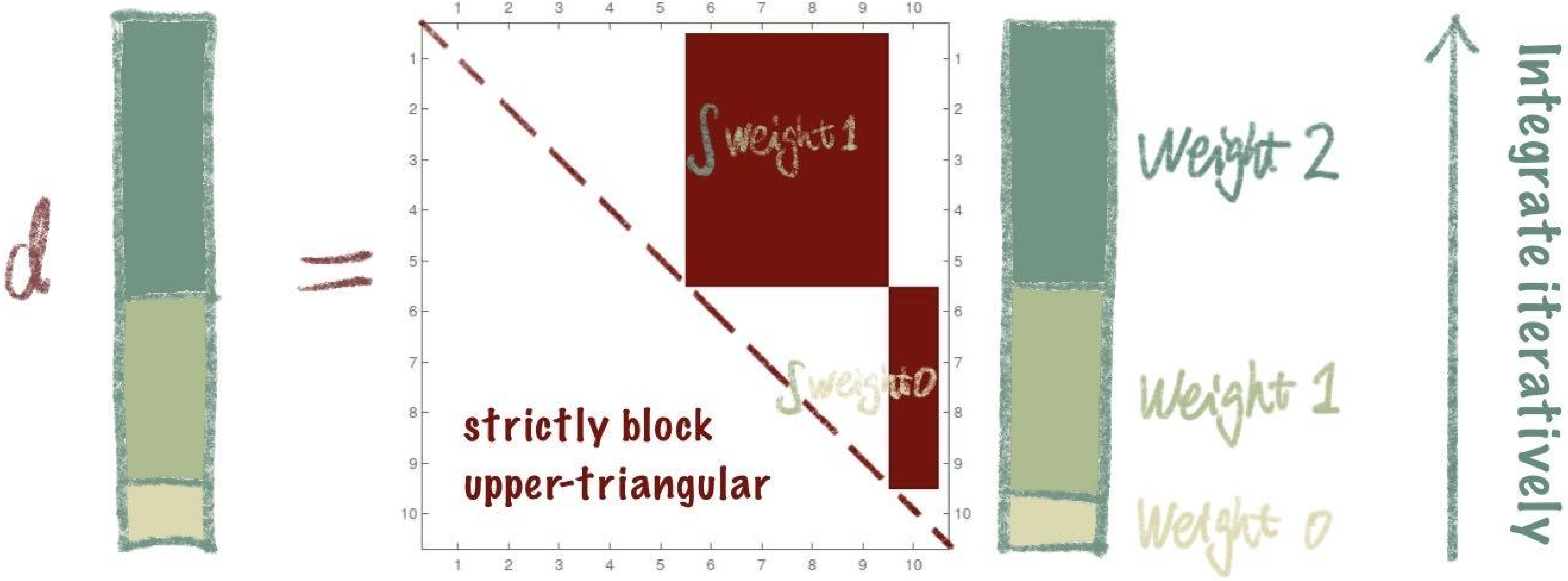
$$a_i,\, b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3].$$

Lift differential equation

$$O_{\partial\zeta_{**}} = \frac{\partial}{\partial\zeta_{**}} + O_{IBP}$$

$$\frac{\partial}{\partial\zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \quad j \in \{1, \delta\}$$

$$a_i,\, b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$



d
=
∫ weight 1
∫ weight 0
strictly block
upper-triangular
Weight 2
Weight 1
Weight 0
Integrate iteratively