Random tensor models for quantum gravity

(a look at tensor models with a mixed $\mathrm{U}(N)$ and $\mathrm{O}(D)$ symmetry)

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Introducing random tensor models...

Random tensor models and tensor field theories

- Consider a field theory defined by a complex field $\phi : G^d \to \mathbb{C}$, where G is a compact Lie group admitting Peter-Weyl decomposition.
- The Fourier transform of ϕ yields an order-d complex tensor[1] $\phi_{\mathbf{P}}$, with $\mathbf{P} = (p_1, p_2, \dots, p_d)$ a multi-index, where $p_1, p_2, \dots, p_d \in \mathbb{Z}$.
- $\bar{\phi}_{\mathbf{P}}$ denotes its complex conjugate.

The partition function is $$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\; e^{-(S^{\text{kinetic}}[\bar{\phi},\phi] + S^{\text{interaction}}[\bar{\phi},\phi])},$$

where the action $S[\bar{\phi}, \phi]$ is given by convolutions of tensors, e.g.,

$$S^{\text{kinetic}}[\bar{\phi},\phi] \;=\; \sum_{\mathbf{P},\mathbf{P}'} \bar{\phi}_{\mathbf{P}}\, K(\mathbf{P};\mathbf{P}')\, \phi_{\mathbf{P}'} =: \mathrm{Tr}_2(\bar{\phi} \cdot K \cdot \phi)\,,$$

where $\mathrm{Tr}_{2n_{\mathcal{B}}}$ represents sums over all indices p_s of $\mathbf{P}$ on $n_{\mathcal{B}}$ tensors ϕ and $\bar{\phi}$.

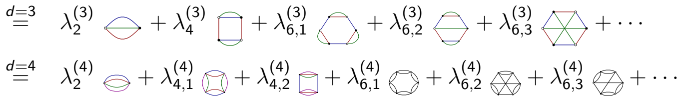
We are studying the space of tensors $\phi = \phi_{p_1,\dots,p_d}$, $p_i \in \{1, \cdots, N\}$, equipped with the measure
$$d\mu(\bar{\phi},\phi) = d\nu_K(\bar{\phi},\phi)\; e^{-S^{\text{interaction}}[\bar{\phi},\phi]} \qquad \text{and} \qquad \mathcal{Z} = \int d\mu(\bar{\phi},\phi)\,,$$
where $d\nu_K(\bar{\phi},\phi)$ is a Gaussian measure with covariance K^{-1}.

[1] Considering $\phi_{\mathbf{P}}$ as a tensor is a slight abuse because the modes p_i range up to infinity. We cut off at N, then $\phi_{\mathbf{P}}$ is a tensor.

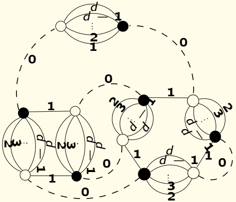
Random tensor models and tensor field theories

$$S^{\text{kinetic}}[\phi,\bar\phi] = \mathrm{Tr}_2(\bar\phi\cdot K\cdot\phi)$$

$$S^{\text{interaction}}[\phi,\bar\phi] = \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\,\mathrm{Tr}_{2n_{\mathcal{B}}}(\bar\phi^{n_{\mathcal{B}}}\cdot\mathcal{V}_{\mathcal{B}}\cdot\phi^{n_{\mathcal{B}}})$$



After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs, e.g.,



Remark

- *If $K(\mathbf{P};\mathbf{P}') = \delta_{\mathbf{P};\mathbf{P}'}$ (trivial delta function), then this model is a statistical model i.e., a tensor model.*
- *Otherwise, if the propagator is nontrivial, e.g., $K(\mathbf{P};\mathbf{P}') = \delta_{\mathbf{P};\mathbf{P}'}\mathbf{P}^{2b}$, this is a QFT, i.e., tensor field theories (generalisation of Grosse-Wulkenhaar model).*

Random tensor models and tensor field theories

- After Wick contraction, random tensor models (with d indices) generates $(d+1)$-edge-colored Feynman graphs.
- $(d+1)$-edge-colored graphs (also, called graph encoding manifolds (GEM)) are dual to simplicial triangulations of piecewise linear (PL) d-dimensional pseudo-manifolds.
  [Pezzana 1974; Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986]
- In other words, tensor models generate discrete (pseudo-)manifolds, and the path integral formulation provides us a way to sum over all of them.

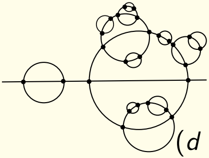
Relevant for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$.

- Encouragingly, the lower dimensional counterpart ($d = 2$), matrix models yield the Brownian sphere at criticality and are rigorously proven to be equivalent to 2-dimensional Liouville quantum gravity.
  [Le Gall, Miermont 2011; Miller, Sheffield 2015]

Promising for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$!

Tensor models

Melons dominate in the large N (size of tensors) limit.
[Gurau Rivasseau 2011]



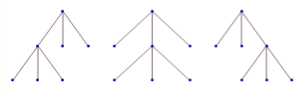
$(d = 3)$

The melonic 2-point function admits the following expansion

$$G_{\mathrm{melonic}}(t) = \sum_{n=0}^{\infty} t^n \, FC_n^{(d+1)} , \qquad FC_n^{(d+1)} = \frac{1}{(d+1)n+1} \binom{(d+1)n+1}{n} .$$

Fuss-Catalan numbers $FC_n^{(d+1)}$ ($d = 1$ is Catalan) correspond to

- the numbers of planar $(d+1)$-ary trees with n vertices and with $dn+1$ leaves.
- the numbers of non-crossing partitions of the set $\{1, 2, \cdots, dn\}$ that contain only subsets of size d.
- etc.



$(d = 2, \ n = 2)$

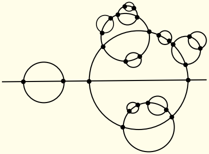
Tensor models

At criticality, melons are branched polymers.
(with Hausdorff dimension 2 and spectral dimension 4/3)
(converge to the *continuous random tree* [Aldous (1991)].)

...a strong universality statement regardless of the order (d) of tensors.

[Bonzom, Gurau, Riello, Rivasseau, "*Critical behavior of colored tensor models in the large N limit*," Nucl. Phys. B 853, 174 (2011)]

[Gurau, Ryan, "*Melons are branched polymers*," Annales Henri Poincare 15, no. 11, 2085 (2014).]



Corresponding to a branched polymer (tree-like) geometric structure in the continuum, melons are undesirable in the (random geometric) quantum gravity context.

$\sim$

Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$\mathcal{Z}_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\; e^{-S_{\text{gravity}}} \quad \rightarrow \quad \sum_{\text{random triangulations}} e^{-S_{\text{discretised gravity}}}$$

$d = 2$: matrix models

- Large N expansion of partition function is controlled by a topological invariant (genus g),

$$\mathcal{Z} = \sum_{g} N^{2-2g} Z_g, \quad \text{where } \; Z_g \sim |\lambda - \lambda_c|^{(2-\gamma)(2-2g)/2}\, f_g\,.$$

  (for the sphere, $Z_{g=0} \sim |\lambda - \lambda_c|^{2-\gamma}$, where $\gamma = -1/2$ (Liouville quantum gravity, "Brownian map"))

- A clear relation between topology and the critical behavior in the double scaling limit ($N \rightarrow \infty$, $\lambda \rightarrow \lambda_c$ while κ being constant),

$$\mathcal{Z} = \sum_{g} \kappa^{2g-2} f_g, \quad \text{where } \; \kappa^{-1} = N|\lambda - \lambda_c|^{(2-\gamma)/2}\,.$$

Random geometrical path integral formulations

Partition function of quantum gravity may be given by

$$\mathcal{Z}_{\text{gravity}} \sim \sum_{\text{top}} \int \mathcal{D}\mathbf{g}\, e^{-S_{\text{gravity}}} \quad \to \quad \sum_{\text{random triangulations}} e^{-S_{\text{discretised gravity}}}$$

$d > 2$: tensor models

- Large N limit is controlled by Gurau degree ω, having strong influence from combinatorics and encoding geometrical information via the number of bicolor cycles (faces), $F(\mathcal{G})$,

$$\mathcal{Z} = \sum_{\omega} N^{d-\frac{2}{(d-1)!}\omega} Z_{\omega}, \quad \text{where } \omega(\mathcal{G}) = \frac{(d-1)!}{2}\Big(\frac{d(d-1)}{2}p(\mathcal{G}) + d - F(\mathcal{G})\Big)$$
$$= \sum_{\text{jackets}, \mathcal{J}(\mathcal{G})} g_{\mathcal{J}(\mathcal{G})} \geq 0\,.$$

- Then, $\omega(\mathcal{G}) = 0$ (subclass of the sphere, "melons") dominate. For melons, $Z_{\omega=0} \sim |\lambda - \lambda_c|^{2-\gamma}$, where $\gamma = 1/2$ (“branched polymer").
- A systematic analysis of subleading contributions ($\omega(\mathcal{G}) > 0$) turns out to be hard. (how to go beyond melons ?)

$\sim$

Outline

- Random tensor models
  - with $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ symmetry
    - ⋆ critical limits of double and triple scalings
    - ⋆ classification of the Feynman graphs
  - with $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ symmetry
    - ⋆ counting of invariants

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor (multi-matrix, multi-orientable) model

multi-scaling limits
and
classification of its Feynman graphs

(with Dario Benedetti, Sylvain Carrozza, Guillaume Valette)
[Ann. Inst. H. Poincare D Comb. Phys. Interact. 9 (2022) 2]

(with Remi Avohou, Matthias Vancraeynest)
[arXiv:2310.13789]

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model

$$\begin{array}{rcl} \mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D) \text{ tensor} & = & N \times N \times D \text{ tensor of order-3} \\ & = & \text{a collection of } D \text{ matrices of size } N \times N \\ & (\sim & \text{affinity to matrix models}) \end{array}$$

Therefore, denote this tensor as a vector of complex matrices of size $N \times N$,

$$X_\mu \in M_N(\mathbb{C}), \qquad 1 \leq \mu \leq D,$$

and in index notation

$$(X_\mu)_{ab}, \quad 1 \leq a, b \leq N\,,$$

with a mixed symmetry $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$: $\quad X_\mu \to X'_\mu = O_{\mu\mu'} U_{(L)} X_{\mu'} U^\dagger_{(R)}$, where $O \in \mathrm{O}(D)$, $U_{(L)}, U_{(R)} \in \mathrm{U}(N)$ (independent unitary matrices in distinct $\mathrm{U}(N)$s).

$\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model admits an expansion in **two parameters** owing to the presence of N and D (g and ℓ) and yields a more **refined classification** of Feynman graphs generated by the model.

$\rightarrow$ possibility of finding a new universality class.

Some background

- A tensor model with $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ symmetry shares similar properties with the holographic SYK model with complex fermions at large N, which can be potentially useful in understanding black holes.

  [Klebanov, Tarnopolsky, Phys. Rev. D 95 (2017) no.4, 046004]

- D matrices of size $N \times N$ can be interpreted as transverse excitations of strings, $(X_\mu)_{ab}$, where a, b is associated with $\mathrm{U}(N)$ and the transverse directions μ to the branes is with $\mathrm{O}(D)$.

  String theory interprets the large D limit of the $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model as the limit of large spacetime dimension in general relativity, where important features of classical black holes may be kept.

  [Emparan, Suzuki, Tanabe, JHEP, 06, (2013), 009.]

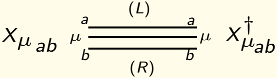
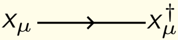
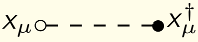
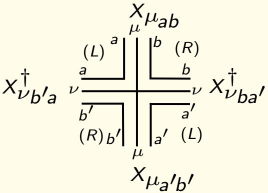
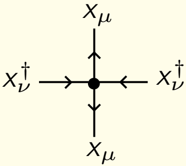
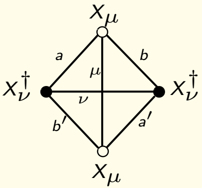
Our model

Free energy

$$\mathcal{F}(\lambda) = \log \int [dX] e^{-S[X,X^\dagger]}\,, \quad \text{where} \quad [dX] = \prod_{\mu,a,b} d\mathrm{Re}(X_\mu)_{ab}\, d\mathrm{Im}(X_\mu)_{ab}\,,$$

$$\text{and} \quad S[X,X^\dagger] = N\,D\Big(\sum_{\mu=1}^{D} \mathrm{Tr}[X_\mu X_\mu^\dagger] - \frac{\lambda}{2}\sqrt{D}\sum_{\mu,\nu=1}^{D} \mathrm{Tr}[X_\mu X_\nu^\dagger X_\mu X_\nu^\dagger]\Big)\,.$$

Perturbative expansion in λ admits a graphical representation in terms of Feynman graphs.

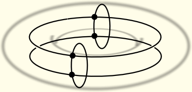
	stranded rep.	Feynman graph rep.	colored rep.
propagator			
interaction			

Free energy

→ $\mathcal{F}(\lambda) = \sum_{g\in\mathbb{N}} N^{2-2g} \sum_{\ell\in\mathbb{N}} D^{1+g-\ell/2} \mathcal{F}_{g,\ell}(\lambda)$ [Ferrari 2017]

- genus $g \equiv g_{LR}$ $\left(\mathrm{U}(N)^{\otimes 2}\text{ ribbon graph}\right)$
- grade ℓ: $\frac{\ell}{2} = g_{OR} + g_{OL} = 1 + g + \frac{v}{2} - \varphi$,

  where φ is the number of $\mathrm{O}(D)$ faces/loops,
  v is the number of vertices.

e.g., $\ell = 0$, $g = 1$ graph 

→ If $D = N$, $\mathcal{F}(\lambda) = \sum_{\omega\in\frac{\mathbb{N}}{2}} N^{3-\omega} \mathcal{F}_\omega(\lambda)$, with $\omega = g + \frac{\ell}{2}$ (Gurau degree).

→ reorganise further, and $\mathcal{F}(\lambda) = D^2 \sum_{g\in\mathbb{N}} \left(\frac{N}{\sqrt{D}}\right)^{2-2g} \sum_{\ell\in\mathbb{N}} D^{-\frac{\ell}{2}} \mathcal{F}_{g,\ell}$,

where $\frac{N}{\sqrt{D}} =: M$ (double scaling parameter).

Double scaling limit

→ reorganise further, and $\mathcal{F}(\lambda) = D^2 \sum_{g\in\mathbb{N}} \left(\frac{N}{\sqrt{D}}\right)^{2-2g} \sum_{\ell\in\mathbb{N}} D^{-\frac{\ell}{2}} \mathcal{F}_{g,\ell}$,

where $\frac{N}{\sqrt{D}} =: M$ (double scaling parameter).

Take the double scaling limit to collect only $\ell = 0$;

$$\lim_{\substack{N\to\infty \\ D\to\infty \\ M=\frac{N}{\sqrt{D}} \text{ finite}}} \frac{1}{D^2}\mathcal{F}(\lambda) = \sum_{g\in\mathbb{N}} M^{2-2g} \mathcal{F}_{g,\ell=0} \equiv \mathcal{F}^{(0)}(M,\lambda)$$

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2]
  - characterisation of $\ell = 0$ graphs, any g.
  - critical behavior of $\mathcal{F}_{g,\ell=0}$
  - triple scaling limit

Triple scaling limit

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433]
  - characterisation of $\ell = 0$ graphs, any g.
  - critical behavior of $\mathcal{F}_{g,\ell=0}$
  - triple scaling limit

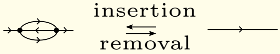
$$\left.\begin{array}{l} \lambda \to \lambda_c \\ M = \frac{N}{\sqrt{D}} \to \infty \\ \text{keep } \kappa^{-1} := M(\lambda - \lambda_c)^{2/b} \text{ fixed} \end{array}\right\}$$

$$\lim_{\substack{M \to \infty \\ \lambda \to \lambda_c \\ \kappa^{-1} := M(\lambda - \lambda_c)^{2/b} \text{ fixed}}} \frac{1}{M^2(\lambda - \lambda_c)^{2-a}} \mathcal{F}^{(0)}(M, \lambda) = \sum_{g \in \mathbb{N}} \kappa^{2g} f_g \quad \text{can be resummed.}$$

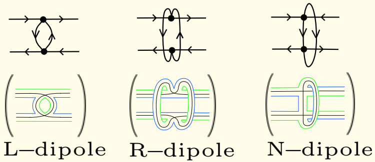
- [Avohou, Toriumi, Vancraeynest, [arXiv:2310.13789[math-ph]]] (motivated by topological recursion)
  - characterisation of $\ell = 1, 2, (3)$ graphs, for any g.
  - characterisation of any ℓ, $g = 0$ graphs with one $\mathrm{O}(D)$ face ($\varphi = 1$).

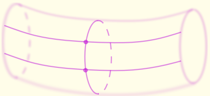
key graph-theoretical objects

-Melon:

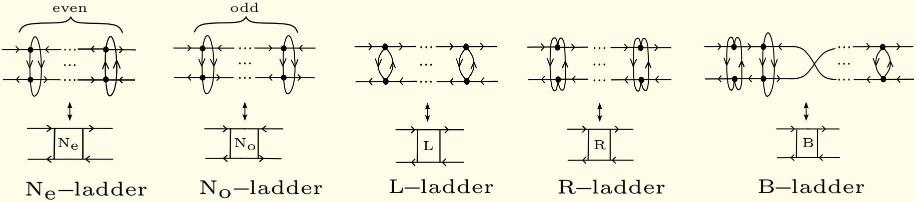


-Dipoles:




N—dipole

-Ladders:



-Scheme: equivalent classes of Feynman graphs, up to insertion/deletion of infinite families of subgraphs, leaving ℓ and g invariant. [Gurau Shaeffer 2013; Fusy Tanasa 2015]

central tools and ideas [Avohou, Toriumi, Vancraeynest, [arXiv:2310.13789]]

Lemma

- *Each Feynman graph with $\ell = 1$ and $g = 0$ has at least one $\mathrm{O}(D)$-loop with length 2.*
- *For each Feynman graph with $\ell = 0, 1, 2, 3$, with any $g \geq 1$, it always contains at least one $\mathrm{O}(D)$-loop of length 2.*

Corollary

For any Feynman graph (and its corresponding scheme) of $\ell = 0, 1, 2, 3$ with $g \geq 1$, there exists an N-dipole.

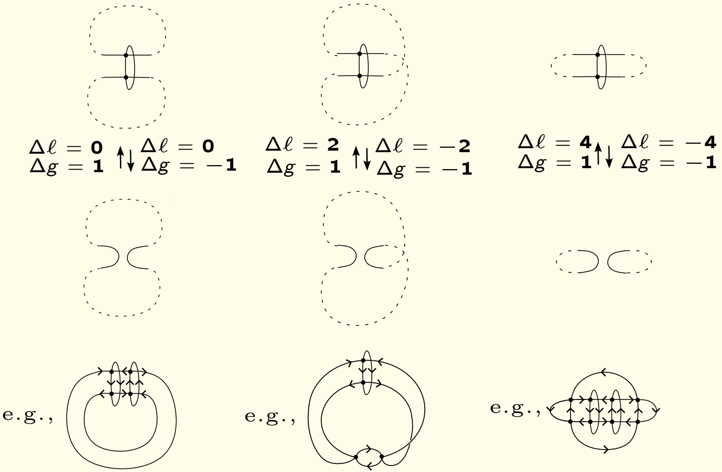
$\rightarrow$ allows **recursive construction** of $\ell = 0, 1, 2, 3$ order by order in g.
i.e., can construct higher genus graphs from lower genus graphs.

Theorem

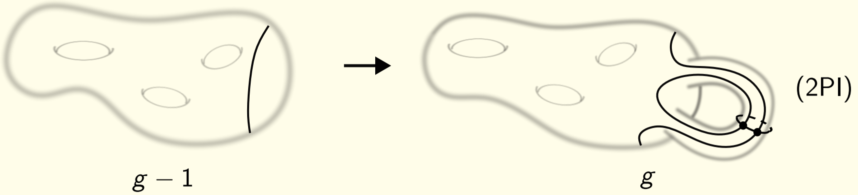
Any $\ell = 0$ resp. $1, 2, 3$ scheme of genus g can be reconstructed from $\ell = 0$, resp. $1, 2, 3$ schemes of genus $g' < g$.

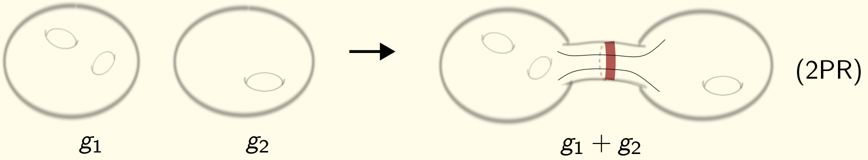
key tools for recursive construction of graphs/schemes

- insertion and removal of non-separating N-dipoles



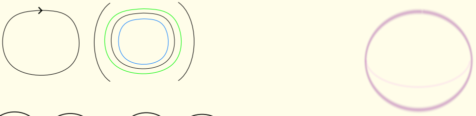
topological picture



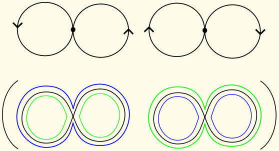


2PI $g = 0$ schemes [Avohou, Toriumi, Vancraeynest, [arXiv:2310.13789]]

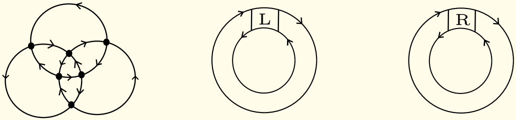
- $\ell = 0$, $g = 0$



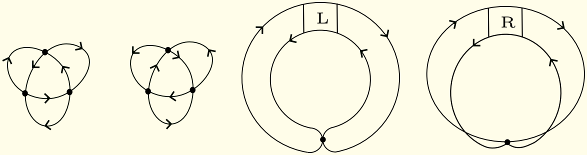
- $\ell = 1$, $g = 0$



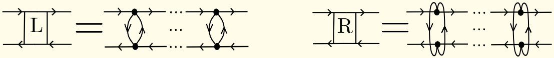
- $\ell = 2$, $g = 0$



- $\ell = 3$, $g = 0$



where



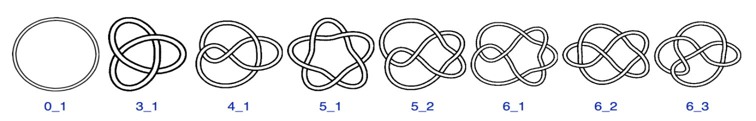
higher ℓ graphs (planar $g = 0$)

Q. What about higher ℓ graphs?

A. Use 1-1 correspondence between 4-regular planar diagrams and alternating knot diagrams.

Theorem ([Avohou, Toriumi, Vancraeynest, [arXiv: 2310.13789[math-ph]]])

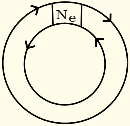
Each 2PI (except the infinity graph) 4-regular planar graph with $\varphi = 1$ **(one $\mathrm{O}(D)$-face/loop)** *and any ℓ, ignoring the orientation assignment on the edges, is in one-to-one correspondance with reduced alternating knot diagrams with ℓ crossings which are*
1) projections of the prime knots as listed in the Rolfsen knot table, or
2) obtained after performing the Tait flyping moves.
Furthermore, we can correspond each 2PR 4-regular planar graphs with $\varphi = 1$ and any ℓ to an alternating knot diagram obtained by performing a connected sum or a Reidemeister move I on the reduced alternating knot diagrams referred above.

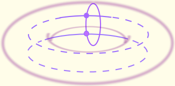


etc.

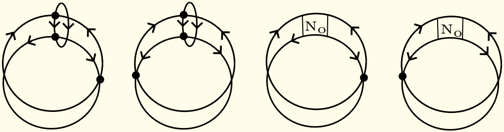
(higher g schemes) 2PI $g = 1$ schemes

- $\ell = 0$, $g = 1$

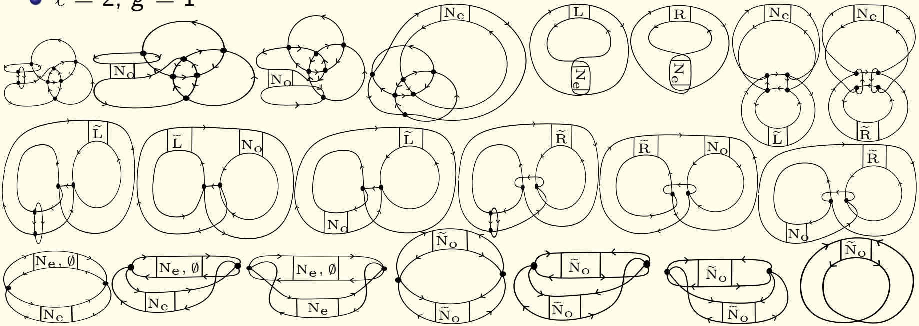




- $\ell = 1$, $g = 1$



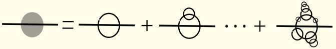
- $\ell = 2$, $g = 1$



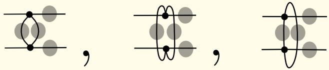
critical behavior of the double scaling limit

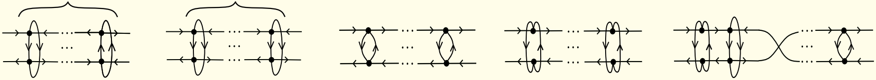
To resum $\displaystyle\sum_{\substack{\ell=0 \text{ schemes}, S \\ \text{(finite sum)}}} \mathcal{F}_S$ is still too hard (double scaling limit).

Focus on the subclass of schemes (“**dominant schemes**") that contribute to the dominant singularity, arising from resumming of melons and consequently ladders.



$$= G_{\text{melon}}(\lambda) \sim \frac{1}{3}\left(4 - \sqrt{\frac{8}{3}}\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}\right), \quad \lambda_c = (3^3/4^4)^{1/2}$$

$\rightarrow$  $\leftrightarrow \quad \lambda^2 G_{\text{melon}}{}^4(\lambda) =: u$

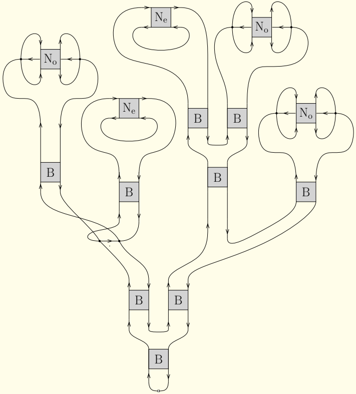


N_e–ladder	N_o–ladder	L–ladder	R–ladder	B–ladder
$\frac{u^2}{1-u^2}$	$\frac{u^3}{1-u^2}$	$\frac{u^2}{1-u}$	$\frac{u^2}{1-u}$	$\frac{6u^2}{(1-3u)(1-u)}$

critical behavior of double scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2])

Dominant schemes *of $\ell = 0$, genus g have $2g - 1$ B-ladder vertices, and in one-to-one correspondence with (decorated)* **plane binary trees** *with g leaves.*



A dominant scheme of genus $g = 5$. It has the structure of a rooted binary tree with: $g = 5$ leaves, $g - 1 = 4$ inner vertices, and $2g - 1 = 9$ edges.

triple scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433])

Dominant schemes *of* $\ell = 0$*, genus* g *have* $2g-1$ *B-ladder vertices, and in one-to-one correspondence with (decorated)* **plane binary trees** *with* g *leaves.*

- can count with $(g-1)^{\text{th}}$ Catalan numbers. $C_g = \frac{1}{2g-1}\binom{2g-1}{g-1}$.
- critical behavior is known for B-ladders arising from melonic resummation

$$\mathcal{F}_{g,\ell=0} \sim C_g \left(\frac{5}{48}\right)^g \left(\sqrt{1-\frac{\lambda^2}{\lambda_c^2}}\right)^{1-2g} .$$

- can resum over genera

$$\rightarrow \sum_g \mathcal{F}_{g,\ell=0} \sim \frac{1}{2}\left(1-\sqrt{1-4\frac{5}{48}\kappa^2}\right), \quad \text{where} \quad \kappa = \left(M\sqrt{1-\frac{\lambda^2}{\lambda_c^2}}\right)^{-1}.$$

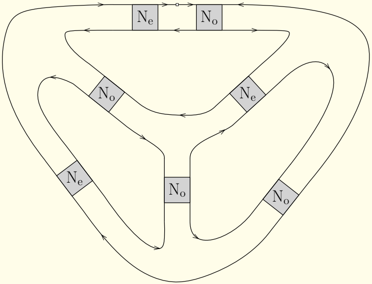
$$\langle g \rangle = \frac{1}{2}\kappa\partial_\kappa \ln\left(\sum_g \mathcal{F}_{g,\ell=0}\right) = \frac{1}{2\sqrt{1-\frac{\kappa^2}{\kappa_c^2}}} \rightarrow \text{diverges}, \quad \kappa_c = \sqrt{\frac{12}{5}}.$$

Large random trees (2PR) dominate, representing surfaces with large g.

critical behavior of double scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2])

2PI dominant schemes *of $\ell = 0$, genus g have $3g - 2$ N-ladder vertices, and are in one-to-one correspondence with (decorated) cubic, and bridgeless* **planar maps**.



A 2PI-dominant scheme of $\ell = 0$, genus $g = 3$. It has $3g - 2 = 7$ N-vertices.

triple scaling limit

Theorem ([Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2, 367-433])

2PI dominant schemes *of* $\ell = 0$*, genus* g *have* $3g - 2$ *N-ladder vertices, and are in one-to-one correspondence with (decorated) cubic, and bridgeless* **planar maps***.*

- can count with A000309 integer sequence in OEIS.
- critical behavior is known coming from N-ladders arising from melonic resummation.
- can resum over genera

$$\sum_g \mathcal{F}_g \sim \left(1 - \frac{\kappa^2}{\kappa_c^2}\right)^{3/2}, \quad \kappa^{-1} = M(1-\lambda)^{3/2}, \quad \kappa_c = \sqrt{\frac{2^5}{3^3}}$$

$$\langle g \rangle \sim \left(1 - \frac{\kappa^2}{\kappa_c^2}\right)^{1/2} \kappa \quad = \quad \text{finite.}$$

Conclusions so far, for $\mathrm{U}(N)^{\otimes 2} \otimes \mathrm{O}(D)$ tensor model

We studied

- "double" scaling limit ($\to$ organisation in ℓ)
  - $N \to \infty$ (size of matrices),
  - $D \to \infty$ (number of matrices),
  - keeping $\frac{N}{\sqrt{D}} =: M$ fixed and finite.
- "triple" scaling limit
  - $M \to \infty$ (i.e., $N \gg D$),
  - $\lambda \to \lambda_c$,
  - keeping $M(\lambda_c - \lambda)^\alpha$ fixed and finite.

and obtained

- [Benedetti, Carrozza, Toriumi, Valette, AIHPD 9 (2022) 2]
  - characterisation of $\ell = 0$ graphs, any g.
  - at criticality (continuum limit),
    - trees (2PR).
    - 2D quantum gravity (2PI).
- [Avohou, Toriumi, Vancraeynest, [arXiv: 2310.13789[math-ph]]]
  - characterisation of $\ell = 1, 2, (3)$ graphs, for any g.
  - characterisation of any ℓ, $g = 0$ graphs with one $\mathrm{O}(D)$ face ($\varphi = 1$).

$\to$ topological recursion?

$\sim$

Enumeration of $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

(with Rémi Cocou Avohou, Joseph Ben Geloun)

Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404 [hep-th]]

Summary of the main results

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

We enumerated $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants using group theoretic formulas.

- Our enumerations unveiled a wide array of novel integer sequences that have not been previously known.
- For a general order (r, q), the counting can be interpreted as the partition function of a topological quantum field theory (TQFT) with the symmetric group as the gauge group. We identified the 2-complex pertaining to the enumeration of the invariants, which in turn defines the TQFT, and establish a correspondence with countings associated with covers of diverse topologies, in general with branched points.
- At order $(r, q) = (1, 1)$, the numbers of invariants corresponds to the numbers of certain cicular words with pattern avoidance, offering insights into enumerative combinatorics and potentially to linguistics.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

Consider

- A tensor T transforms under the action of the fundamental representation of the Lie group $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$.

$$T_{a_1,a_2,\dots,a_r,b_1,b_2,\dots,b_q} \to U^{(1)}_{a_1 c_1} U^{(2)}_{a_2 c_2} \dots U^{(r)}_{a_r c_r} O^{(1)}_{b_1 d_1} O^{(2)}_{b_2 d_2} \dots O^{(q)}_{b_q d_q} T_{c_1,c_2,\dots,c_r,d_1,d_2,\dots,d_q} \,.$$

- A $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$ invariant (UO-invariant) is constructed by contractions of complex tensors of order $r+q$ (of a given number, n, of tensors T and the same number of complex conjugate $\bar{T}$.)

  $\to$ Therefore, UO invariants are tensor model invariants/bubbles .

- An UO-invariant is algebraically denoted

$$\mathrm{Tr}_{K_n}(T,\bar{T}) = \sum_{a^i_k, b^i_k, a'^i_k, b'^i_k} K_n(\{a^i_k, b^i_k\}; \{a'^i_k, b'^i_k\}) \prod_{i=1}^{n} T_{a^i_1,a^i_2,\dots,a^i_r,b^i_1,b^i_2,\dots,b^i_q} \bar{T}_{a'^i_1,a'^i_2,\dots,a'^i_r,b'^i_1,b'^i_2,\dots,b'^i_q} \,.$$

  K_n is a kernel composed of a product of Kronecker delta functions that match the indices of n copies of T's and those of n copies of $\bar{T}$'s. A given tensor contraction dictates the pattern of an edge-colored graph, which can, in turn, be used to label the invariant.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

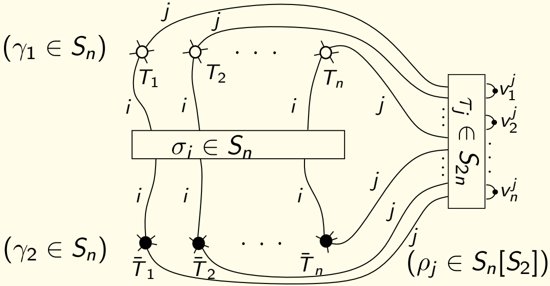


Diagram of contraction of n tensors T and n tensors $\bar{T}$. For a given color $i = 1, 2, \ldots, r$, σ_i represents the contraction in the unitary sector and, for any color $j = 1, 2, \ldots, q$, τ_j represents the contraction in the orthogonal sector.

Consider $(r, q) = (3, 3)$. An UO-invariant is defined by a $(3 + 3)$-tuple of permutations $(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ from the product space $(S_n)^{\times 3} \times (S_{2n})^{\times 3}$.

We will remove the vertex labeling (two configurations are equivalent if their resulting unlabeled graphs coincide), which introduces more permutations $\gamma_1, \gamma_2 \in S_n$, and $\varrho_1, \varrho_2, \varrho_3 \in S_n[S_2]$ the so-called wreath product subgroup of S_{2n}.

The equivalence relation is

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \quad \sim \quad (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3)$$

Counting UO tensor invariants

Idea:

We work with the equivalence relation to count the graphs, i.e., tensor invariants

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \sim (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3).$$

- $G \times X \to X$.
- Recall: orbit of an element x in X: the set of elements in X to which x can be moved by the elements of G. $G \cdot x = \{g \cdot x : g \in G\}$.
- a point ($\in X$) on an orbit $\to$ another point on the orbit.
- number of equivalent classes of graphs $=$ number of orbits
- Burnside's lemma
  $\sharp\mathrm{orb} = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \{x \in X : gx = x\}$.

Therefore the counting of UO invariants of order (r, q) is

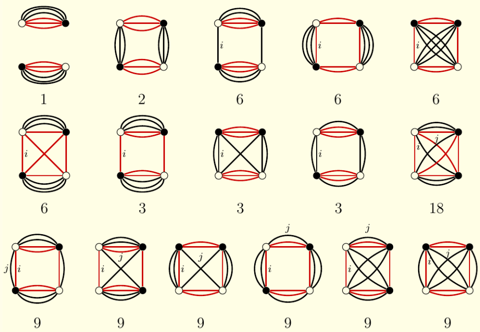
$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q} \sum_{\gamma_1,\gamma_2\in S_n}\ \sum_{\varrho_1,\ldots,\varrho_q\in S_n[S_2]}\ \sum_{\substack{\sigma_1,\ldots,\sigma_r\in S_n\\ \tau_1,\ldots,\tau_q\in S_{2n}}} \left[\prod_{i=1}^{r}\delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\right]\left[\prod_{i=1}^{q}\delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\right].$$

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

example: $\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

$\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants are enumerated in the increasing number of tensors: $1, 108, 20385, 27911497, 101270263373, 808737763302769, \ldots$



UO-invariant graphs at order $(r, q) = (3, 3)$ with 4 tensors ($n = 2$). The integer below each graph enumerates various possibilities based on index colors, summing to 108 for all configurations. Black edges are in the U-sector, and red are in the O-sector.

TQFT (lattice gauge theories)

- On a **cellular complex** X, we can define a partition function for a finite **group** G by assigning a group element g_e to each edge and to each **plaquette** P a weight $w_P\left(\prod_{e\in P} g_e\right)$. The partition function of this **lattice gauge theory** is

$$Z[X;G] = \frac{1}{|G|^V}\sum_{g_e}\prod_P w_P\Big(\prod_{e\in P} g_e\Big)\,,$$

with V the number of vertices in the cell decomposition.

- The theory is **topological** because it is invariant under refinement of the cellular decomposition.
- When $G = S_n$ (symmetric group or permutation group), it has applications to QFT combinatorics. [Ben Geloun, Ramgoolam, Ann. Inst. H. Poincare Comb. Phys. Interact. 1 (2014) 1]
- The partition function for a topological space X counts equivalence classes of homomorphisms from $\pi_1(X)$ to S_n, i.e., counts equivalence classes of covering spaces of X of degree n counted with a certain weight.

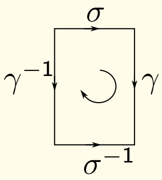
an example of permutation TQFT

e.g., Consider the torus realised as a rectangle.

- The partition function of this lattice gauge theory is given by

$$Z(T^2; S_n) = \frac{1}{n!} \sum_{\sigma,\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1})\,.$$

- $\delta(\gamma\sigma\gamma^{-1}\sigma^{-1})$ or $\gamma\sigma\gamma^{-1}\sigma^{-1} = \text{id}$ is represented by the torus and γ and σ are the generators of the fundamental group of the torus.



- $Z(T^2; S_n)$ counts n-fold covers of the torus.

permutation TQFT for UO tensor invariants

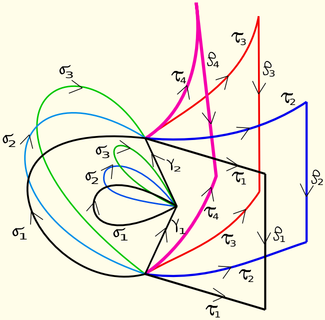
[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

Recall the counting of UO invariants of order (r,q)

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q}$$

$$\sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\varrho_1,\ldots,\varrho_q \in S_n[S_2]} \sum_{\substack{\sigma_1,\ldots,\sigma_r \in S_n \\ \tau_1,\ldots,\tau_q \in S_{2n}}} \left[\prod_{i=1}^{r} \delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\right]\left[\prod_{i=1}^{q} \delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\right].$$

TQFT reformulates our enumeration.



2-cellular complex associated with the $TQFT_2$ of $Z_{(3,4)}$ made of 3+4 cylinders sharing boundaries.

permutation TQFT for UO tensor invariants

The counting of UO invariants of order $(r \geq 2, q)$ can be massaged:

$$Z_{(r\geq 2,q)}(n) = \frac{1}{n!}\sum_{\gamma\in S_n} Z^q_{n;\gamma} \sum_{\sigma_0,\sigma_2,\sigma_3\ldots,\sigma_r\in S_n} \Big[\prod_{i=2}^{r} \delta(\gamma^{-1}\sigma_i\gamma\sigma_i^{-1})\Big]\delta(\gamma^{-1}\sigma_0\gamma\sigma_0^{-1})\delta(\sigma_0\prod_{i=2}^{r}\sigma_i)\,,$$

$$\text{with} \quad Z^q_{n;\gamma} = \frac{1}{(n!)[n!(2!)^n]^q} \sum_{\substack{\sigma_1\in S_n \\ \tau_1,\ldots,\tau_q\in S_{2n} \\ \tau_i^{-1}\sigma_1\gamma^{-1}\sigma_1^{-1}\gamma\tau_i\in S_n[S_2]}} 1\,.$$

We are counting equivalence classes of r permutations σ_i under the conjugation $\sigma_i \sim \gamma\sigma_i\gamma^{-1}$, and the group generated by r generators subject to one relation by the last constraint $\sigma_0\prod_{i=2}^{r}\sigma_i = \mathrm{id}$, i.e., the fundamental group of the 2-sphere with r-punctures.

Therefore, $Z_{(r\geq 2,q)}(n)$ counts the covers of the r-punctured sphere with each cover weighted by $Z^q_{n;\gamma}$, i.e., enumerates $Z^q_{n;\gamma}$-weighted equivalence classes of branched covers (with r branched points of degree n) of the sphere.

On the other hand, $Z_{(r=1,q)}(n)$ counts the number of covers of the sphere with $(q+3)$-punctures.

Consequences and Outlook

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

- We added more correspondence between the enumeration of tensor invariants and 2-dimensional permutation TQFT.
- The sequences of numbers corresponding to our enumerations [2] [3] are new and unknown before in OEIS (Online Encyclopedia of Integer Sequences).
- So far, regardless of whether the invariants are unitary [Ben Geloun, Ramgoolam 2013], orthogonal [Avohou, Ben Geloun, Dub 2019], or UO symmetric, we consistently find a correspondence with (branched) covers of the sphere (possibly with punctures).

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[2] except purely U case ($r, q = 0$) was reported before [Ben Geloun, Ramgoolam 2013] and also ($r = 2, q = 1$) case was reported in [Bulycheva, Klebanov, Milekhin Tarnopolsky 2017].

[3] Remark that our formulation cannot be reduced to purely O case which was studied before [Avohou, Ben Geloun,Dub 2022].

Take home messages

The counting of tensor invariants, in addition to their essential role in the pertubative analysis of tensor models in theoretical physics, reveals unexpected connections between combinatorics, algebra, and topology.

What is intriguing is the connection between tensor models and branched covers of the 2-sphere suggests that two dimensional holomorphic maps know about higher dimensional combinatorial topology.

the end