# Symanzik Effective Field Theory

Provides a description of discretization effects of the underlying lattice QFT. [Refs 1-5, 6, 7]

→ functional form for continuum extrapolation of lattice data

→ procedure for building improved actions

Describe lattice $\mathcal{L}$ by a local effective $\mathcal{L}$ (LE$\mathcal{L}$) which which is written in terms of continuum fields: $\mathcal{L}_{Sym}$
ie.

$$\mathcal{L}_{lat} \doteq \mathcal{L}_{Sym}$$

where $\doteq$ means equality of on-shell quantities (matrix elements) each evaluated in the theories on the LHS & RHS

The EFT construction on the RHS takes advantage of the usual separation of scales, where short-distance effects are expressed in the coefficients and long-distance effects in the operators.

The LHS corresponds to the lattice QFT used for the numerical simulations, while the RHS is an expansion in terms of continuum fields and matrix elements of physical continuum states. For QCD, we have:

$$\mathcal{L}_{Sym} = \mathcal{L}_{QCD} + \mathcal{L}_{I}$$

with

$$\mathcal{L}_{QCD} = \frac{1}{2g^2} \, tr\, [G_{\mu\nu}{}^2] + \sum_f \bar{\psi}_f (\not{D} + m_f) \psi_f$$

where $g^2, m$ are the renormalized couplings or masses that depend on the bare parameters of the lattice theory:

$$g^2 = g^2(g_0^2; m_0 a, c_i; \mu a) \qquad m = m_0 Z_m (g_0^2; m_0 a, c_i; \mu a)$$

and the $c_i$ are adjustable parameters (coefficients) of higher dimensional terms ($d > 4$) in $\mathcal{L}_{lat}$.

The 2nd term on RHS describes discretization effects in $\mathcal{L}_{lat}$ in terms of higher dimensional operators ($\dim \mathcal{O}_n > 4$)

$$\mathcal{L}_I = \sum_n a^{\dim \mathcal{O}_n - 4} \cdot k_n \cdot \mathcal{O}_n^R(\mu)$$

with

$$k_n = k_n(g, ma, c_i; \mu a)$$

where matrix-elements of the $\mathcal{O}_n$ capture long-distance effects and scale with the typical momenta $\Lambda$ of the participating particles:

$$\langle \mathcal{O}_n \rangle \sim \Lambda^{\dim \mathcal{O}_n - 4}$$

hence describe discretization effects of $(a\Lambda)^{\dim \mathcal{O}_n - 4}$

If $a$ is small enough so that $a\Lambda \ll 1$, we can treat effects from $\mathcal{L}_I$ as perturbations, so that we can use the physical states of $\mathcal{L}_{QCD}$.

- It is enough to consider only <u>on-shell</u> quantities $\Rightarrow$ fewer operators in $\mathcal{L}_I$
  [ Refs 8, 9 ]

- Operators that don't affect on-shell quantities are <u>redundant</u>.

- matrix elements as functional integrals: redundant operators are obtained from field redefinitions (a.k.a. spectrum-conserving transformations)

- can also use EoMs

<u>Examples</u>:

(1) <u>Wilson gauge action</u>: start gauge-link definition

$$U_\mu(x) \equiv P \exp\left( -i \int_x^{x+a\hat{\mu}} A_\mu(y)\, dy \right)$$

plaquette

$$P_{\mu\nu} \equiv \frac{1}{3} \operatorname{Re} \operatorname{Tr}\left[ U_\mu(x)\, U_\nu(x+a\hat{\mu})\, U_\mu^\dagger(x+a\hat{\nu})\, U_\nu^\dagger(x) \right]$$

and the action $S_g^W = \sum_x \mathcal{L}_g^W(x)$ with

$$\mathcal{L}_g^W = \beta \sum_{\mu>\nu} \left[ 1 - P_{\mu\nu}(x) \right] \qquad \beta = \frac{6}{g_0^2}$$

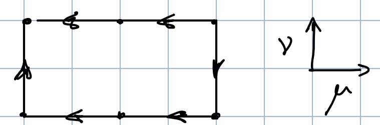
Exercise I:

(a) Show that in the classical continuum limit

$$\mathcal{L}_g^W \longrightarrow \frac{1}{2g_0^2} \operatorname{Tr}[G_{\mu\nu}^2] + \frac{a^2}{24} \operatorname{Tr}[G_{\mu\nu}(D_\mu^2 + D_\nu^2) G_{\mu\nu}] + \dots$$

(b) Show that the leading discretization effects can be removed by adding rectangular Wilson loops to the action, e.g.

$$R_{\mu\nu} = \frac{1}{3} \operatorname{Re} \operatorname{Tr}$$



Expand this operator in the small $a$ limit and show that the improved action

$$\mathcal{L}_g^{imp} = -\beta \sum_{\mu>\nu} \left[ \frac{5}{3} P_{\mu\nu} - \frac{1}{12}(R_{\mu\nu} + R_{\nu\mu}) \right]$$

yields:

$$\mathcal{L}_g^{imp} \longrightarrow \frac{1}{2g_0^2} \operatorname{Tr}[G_{\mu\nu}^2] + \mathcal{O}(a^4)$$

This is called Symanzik improvement.
This analysis can be extended to include 1-loop corrections,
see: M. Lüscher + P. Weisz in Ref. 10.

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(2) Wilson fermion action

$$\mathcal{L}_f^W = m_0 \bar{\psi}\psi + \bar{\psi}\gamma_\mu D_\mu^{lat}\psi - \frac{ar}{2}\bar{\psi}\Delta^2\psi$$

with

$$D_\mu^{lat}\psi(x) = \frac{1}{2a}\left[ U_\mu(x)\psi(x+a\hat{\mu}) - U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) \right]$$

and

$$\bar{\psi}\Delta^2\psi = \frac{1}{a^2}\bar{\psi}\left[ U_\mu(x)\psi(x+a\hat{\mu}) + U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu}) - 2\psi(x) \right]$$

where the usual choice is $r=1$. The operator $\bar{\psi}\Delta^2\psi$ is dimension 5 and therefore carries an explicit factor of $a$. Hence it doesn't affect the classical continuum limit of the action; it however gives rise to discretization effects that are linear in $a$. This term is called the Wilson term and was added to the action to remove the doublers.

Exercise II:

Starting with the naive fermion action (with $m_0 = 0$)

$$\mathcal{L}_f^{naive} = \bar{\psi} \gamma_\mu D_\mu^{lat} \psi$$

show that the free-field quark propagator in momentum space takes the form:

$$\tilde{D}^{naive}(p) = \frac{i}{a} \gamma_\mu \sin(p_\mu a)$$

(omitting color indices)

which means that the naive action contains 15 unwanted quark states.

Then show that the free Wilson propagator takes the form

$$\tilde{D}^W(p) = \frac{i}{a} \gamma_\mu \sin(p_\mu a) + \frac{1}{a} \sum_\mu [1 - \cos(p_\mu a)]$$

which removes the poles at $p_\mu = \pi/a$, by giving the doublers a mass term $\sim 2/a$.

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We now consider the Symanzik LE $\mathcal{L}$ for the Wilson fermion action:

$$\mathcal{L}_{Sym}^W = m_f \bar{\psi}_f \not{D} \psi_f + \mathcal{L}_I^W$$

We already know that $\mathcal{L}_I^W$ starts at dimension 5. There are two possible operators (see S+W):

$$\mathcal{O}_5 = i \bar{\psi}_f \sigma_{\mu\nu} G_{\mu\nu} \psi_f \qquad \mathcal{O}_5' = \bar{\psi}_f \not{D}^2 \psi_f$$

The two operators are related: $\mathcal{O}_5' = \bar{\psi}_f D^2 \psi_f - \frac{1}{2} \mathcal{O}_5$
and $\mathcal{O}_5'$ can be generated by a field redefinition of the form:

$$\psi_f \to e^{\epsilon a \not{D}} \psi_f \qquad \bar{\psi}_f \to \bar{\psi}_f e^{\bar{\epsilon} a \not{D}}$$

[see: El-Khadra, Kronfeld, Mackenzie, Ref. 11]

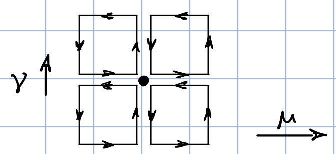
This means that $\mathcal{O}_5'$ is redundant and the Symanzik LE $\mathcal{L}$ for the Wilson fermion action takes the form

$$\mathcal{L}_I^W = a k_5^W \mathcal{O}_5 + \sum_{\dim \mathcal{O}_n > 5} a^{\dim \mathcal{O}_n - 4} k_n^W \mathcal{O}_n$$

We can now use this analysis to improve the Wilson action, by adding a discretized version of $\mathcal{O}_5$ with a coefficient adjusted so that the resulting Symanzik LE $\mathcal{L}$ has $k_5 = 0$:

$$\mathcal{L}_f^{SW} = \mathcal{L}_f^{W} + \frac{i}{4} c_{SW} \bar{\psi} \sigma_{\mu\nu} G_{\mu\nu}^{lat} \psi$$

where $G_{\mu\nu}^{lat}$ is constructed from a "clover-leaf" of plaquettes:




$- (\mu \leftrightarrow \nu)$

To determine the coefficient $c_{SW}$ for tree-level improvement, a perturbative treatment using on-shell matrix elements of quark states is sufficient. For non-perturbative improvement, the improvement conditions must be formulated in terms of hadronic external states.
We note that once the improvement coefficients are determined to a given order, all on-shell quantities are automatically improved to that order.
However, if we want to compute processes involving external currents, such as weak matrix elements, a separate matching calculation is needed to obtain improved currents.

For the tree-level $\mathcal{O}(a)$ improvement of the Wilson action, we could, for example compute forward scattering matrix elements of the gauge-current. It turns out that when $m_0 a \ll 1$, the discretization effects in the external state spinors start at $\mathcal{O}(a^2)$ [see El-Khadra, Kronfeld, Mackenzie].

In short, we can obtain the tree-level $\mathcal{O}(a)$ improvement condition by simply considering the quark-gluon vertex.

<u>Exercise III:</u>

(a) Derive the quark-gluon vertex for the naive fermion action and show that:

$$\Gamma_\mu^{naive}(p, p') = -i g_0 t^a \gamma_\mu \cos[\tfrac{1}{2}(p+p')_\mu a]$$

(b) Same for the Wilson action to show that:

$$\Gamma_\mu^{W}(p, p') = -i g_0 t^a \gamma_\mu \left\{ \cos[\tfrac{1}{2}(p+p')_\mu a] - i \sin[\tfrac{1}{2}(p+p')_\mu a] \right\}$$

(c) Same for the improved Wilson (SW) action):

$$\Gamma_\mu^{SW}(p, p') = \Gamma_\mu^{W}(p, p') - i g_0 t^a \tfrac{1}{2} c_{SW} \sigma_{\mu\nu} \cos[\tfrac{1}{2}(p+p')_\mu a] \sin[\tfrac{1}{2}(p+p')_\nu a]$$

Use the Gordon Identity (sandwiching the vertex between quark spinors) to see that with $c_{SW} = 1$ the $\mathcal{O}(a)$ terms cancel.

Now consider external currents:

For example axial vector current: $A_\mu^{lat} = \bar{\psi}_2 \gamma^\mu \gamma^5 \psi_1$

It can be described in Symanzik EFT by

$$A_\mu^{lat} \doteq \bar{Z}_A^{-1} A_\mu + a K_A \partial_\mu \bar{\psi}_2 \gamma^5 \psi_1 + \dots$$

where $A_\mu = \bar{\psi}_2 \gamma^\mu \gamma^5 \psi_1$ is the (renormalized) continuum current.

Then we have

$$\begin{aligned} \langle H_2 | \bar{Z}_A A_\mu^{lat} | H_1 \rangle_{lat} &= \langle H_2 | A_\mu | H_1 \rangle_{cont} \\ &+ a \bar{Z}_A K_A \partial_\mu \langle H_2 | \bar{\psi} \gamma^5 \psi | H_1 \rangle_{cont} \\ &+ a K_5 \int d^4 y \langle H_2 | T A_\mu \mathcal{O}_5(y) | H_1 \rangle_{cont} \\ &+ \mathcal{O}(a^2) \end{aligned}$$

As before, if $K_A \neq 0$, add correction operators to the lattice current: $A_\mu^{lat} = \bar{\psi}_2 \gamma^\mu \gamma^5 \psi_1 + a c_A \partial_\mu^{lat} \bar{\psi}_2 \gamma^5 \psi_1$

and adjust $c_A$ so that $K_A = 0$. At tree-level one finds that $c_A = 0$.

As always, $\bar{Z}_A$, $K_A$, $K_5$ are functions of $(g^2, ma, c_i\text{'s}; \mu a)$

So far we have assumed that $ma \ll 1$, so we can expand the coefficients in powers of $ma$. For example:

$$\bar{Z}_A(ma) = Z_A \left[ 1 + b_A \cdot \tfrac{1}{2} (m_{f_1} a + m_{f_2} a) + \mathcal{O}(ma)^2 \right]$$

At tree-level $Z_A = 1$ and $b_A = 1$.

In the traditional presentation of Symanzik EFT (Refs. 1-5, 8-10) one expands in $ma$ from the start, and the coefficients are obtained explicitly at $ma=0$.

For example, for the axial current, the dim 4 operators in the Symanzik EFT are

$$m A_\mu \,, \quad \partial_\mu \bar{\psi}_2 \gamma^5 \psi_1$$

which would yield

$$A_\mu^{lat} \doteq Z_A^{-1} \left(1 - b_A' a (m_{f_1} + m_{f_2}) + \dots\right) A_\mu$$
$$+ a\, k_A' \, \partial_\mu \bar{\psi}_2 \gamma^5 \psi_1$$

with the same results as before.

For heavy quarks, $m \gg \Lambda_{QCD}$, however, as long as $ma \ll 1$, the Symanzik EFT still works. However, discretization effects will be dominated by $(ma)^n$ terms. Keeping $ma \ll 1$ is feasible for charm quarks, but not for b quarks (on currently available gauge field ensembles).

## Heavy Quarks: $ma \not\ll 1$

We now consider the case of heavy quarks with $ma \not\ll 1$. We will see that we need to modify the Symanzik EFT, because contributions which scale as $(ma)^n$ are no longer small.

First, to explicitly see what happens, an example:

Exercise IV:

Consider the quark propagator as a function of Euclidean time $t$ and 3-momentum $\vec{p}$:

$$\langle \psi(\vec{p}', t') \bar{\psi}(\vec{p}, t) \rangle = (2\pi)^3 \, \delta^3(p - p') \, C(\vec{p}, t' - t)$$

$$\text{where} \qquad \psi(\vec{p}, t) = a^3 \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \psi(\vec{x}, t)$$

The quark propagator can be obtained from $\tilde{D}(p)$:

$$C(\vec{p},t) = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi}\, e^{ip_0 t}\, \tilde{D}^{-1}(\vec{p}, p_0)$$

It is instructive to evaluate $C(\vec{p},t)$ for different actions, for example to see how the doublers appear for the naive action and are removed for the Wilson action with $r=1$.

Definitions: $\quad S_\mu = \frac{1}{a}\sin(p_\mu a) \qquad \hat{p}_\mu = \frac{2}{a}\sin(p_\mu a/2)$

$$\hat{p}^2 = \frac{4}{a^2}\sum_\mu \sin^2(p_\mu a/2) \qquad \sum_\mu [1-\cos(p_\mu a)] = \frac{1}{2} a\, \hat{p}^2$$

Wilson propagator ($r=1$):

$$\tilde{D}_W(p) = \frac{i}{a}\gamma_\mu \sin(p_\mu a) + \frac{1}{a}\sum_\mu [1-\cos(p_\mu a)] + m_0$$

$$= i\not{S} + m_0 + \frac{1}{2} a\, \hat{p}^2$$

$$\tilde{D}_W(\vec{p},p_0) = \frac{i}{a}\gamma^0 \sin(p_0 a) + m_0 + \frac{1}{a}[1-\cos(p_0 a)] + i\vec{\gamma}\cdot\vec{S} + \frac{1}{2} a\, \hat{p}_i^2$$

$$C_W(\vec{p},t) = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi}\, e^{ip_0 t}\, \tilde{D}_W^{-1}(\vec{p}, p_0) \qquad (a=1)$$

$$= \int_{-\pi}^{\pi} \frac{dp_0}{2\pi} \frac{e^{ip_0 t}\left[m_0 + \frac{1}{2}\hat{p}_i^2 + 1 - \cos p_0 a - i\gamma^0 \sin p_0 - i\vec{\gamma}\cdot\vec{S}\right]}{\sin^2 p_0 a + \vec{S}^2 + (m_0 + 1 - \cos p_0 a + \frac{1}{2}\hat{p}_i^2)^2}$$

$t>0$: let $z = e^{ip_0}$ $\quad t<0$: $z = e^{-ip_0}$

$\Rightarrow z = e^{ip_0\, \mathrm{sgn}(t)}$

$$\mu_0 = m_0 + \frac{1}{2}\hat{p}_i^2$$

$$\longrightarrow \oint \frac{dz}{2\pi i} \frac{z^{|t|}}{z} \frac{\frac{1}{2}(z^2+1) + \frac{1}{2}\mathrm{sgn}(t)(z^2-1)\gamma_0 - z(1+\mu_0 - i\vec{\gamma}\cdot\vec{S})}{-z + (z^2+1)(1+\mu_0) + z[\vec{S}^2 + (1+\mu_0)^2]}$$

poles at $\quad z = e^{\pm E}$

$$\Rightarrow C_W(\vec{p},t) = Z_2 \cdot e^{-E|t|} \times \frac{\gamma_0\, \mathrm{sgn}(t) \sinh E + m_0 + 1 - \cosh E - i\vec{\gamma}\cdot\vec{S} + \frac{1}{2}\hat{p}_i^2}{2\sinh E}$$

with $$\cosh E = 1 + \frac{[m_0 + \frac{1}{2}\hat{p}_i^2]^2 + \vec{S}^2}{2[1 + m_0 + \frac{1}{2}\hat{p}_i^2]}$$

$$Z_2 = \frac{1}{1 + m_0 + \frac{1}{2}\hat{p}_i^2}$$

Now expand in powers of $\vec{p}$ – note $a=1$, so $\vec{S}^2 = \vec{S}^2 a^2$, $\hat{p}_i^2 = \hat{p}_i^2 a^2, \ldots$

$$\Rightarrow E^2 = m_1^2 + \frac{m_1}{m_2}\vec{p}^2 + \mathcal{O}(\vec{p}^4)$$

where $m_1, m_2$ are the rest and kinetic masses, respectively:

$$m_1 \equiv E(\vec{p}=0) \qquad \frac{1}{m_2} \equiv \left.\frac{\partial^2 E}{\partial p_i^2}\right|_{\vec{p}=0}$$

and we have: $$m_1 = \ln(1+m_0)$$

$$\frac{1}{m_2} = \frac{2}{m_0(2+m_0)} + \frac{1}{1+m_0}$$

Clearly, if $m_0 \neq 0$ $m_1 \neq m_2$. Expand the ratio $m_1/m_2$ in powers of $m_0$:

$$\frac{m_1}{m_2} = 1 - \frac{2}{3}m_1^2 + \ldots \xrightarrow[m_0 a \to 0]{} 1$$

While the difference is an $\mathcal{O}(a^2)$ effect, it is not small when $m_0 a \sim 1$.

Similarly, for the residue: $$Z_2(\vec{p}) = \frac{1}{1+m_0} + \mathcal{O}(\vec{p}^2)$$

Note $Z_2$ is a function of $\vec{p}$ not $\vec{p}^2$, because at finite lattice spacing Euclidean $SO(4)$ invariance is broken.

Define: $$Z_2 \equiv Z_2(\vec{p}=0) = \frac{1}{1+m_0} = e^{-m_1}$$

and the properly normalized field is $Z_2^{-1/2}\psi(x) = e^{m_1/2}\psi(x)$

One main take-away is that the energy-momentum relation is affected by $a m_0 \sim 1$ so that
$$E^2 \neq m^2 + \vec{p}^2 + \ldots$$

Recall that for $m_0 a \sim 1$ we have $m_0 \gg |\vec{p}| \sim \Lambda$, so the non-relativistic expansion of $\cosh E$ yields:

$$E = m_1 + \frac{\vec{p}^{\,2}}{2 m_2} \qquad \text{(same } m_1, m_2 \text{ as before)}$$

Note that the mass term ($m_1$) is conventionally omitted from NR or HQ eff. Lagrangians. Hence, if we take the HQ EFT view of QCD, $m_1 \neq m_2$ is no longer a source of discretization error in the lattice theory, simply tune $m_0$ so that $m_2 = m$.

This is a consequence of the Wilson action having the same HQ limit as QCD.

The Symanzik EFT relies on the terms in $\mathcal{L}_I$ being small. When $m_0 a \sim 1$, $\mathcal{L}_I$ contains "large" $O(1)$ terms. Generically, $\mathcal{L}_I$ contains terms

$$\mathcal{L}_I = \dots + \sum_{n=3} a^n k_n \sum_{\mu=0}^{3} \bar{\psi} (\gamma_\mu D_\mu)^n \psi$$

Using the EoM

$$(-\gamma_0 D_0)\psi = (\vec{\gamma}\cdot\vec{D} + m)\psi$$

we see that $a \bar{\psi}\gamma_0 D_0 \psi \sim am$, hence not small. Repeated application of the EoM yields

$$a^n (\vec{\gamma}\cdot\vec{D} + m)^n$$

Terms like $[D_0, D_i] \sim F_{0i}$ stay "small". Expanding

$$a^n (\vec{\gamma}\cdot\vec{D} + m)^n \sim (ma)^{n-\ell} a^\ell \bar{\psi}(\vec{\gamma}\cdot\vec{D})^\ell \psi$$

$\ell = 0 \longrightarrow$ modifies coefficient of $\bar{\psi}\psi$ term in $\mathcal{L}_{QCD}$

$\ell = 1 \longrightarrow$ modifies coefficient of $\vec{\gamma}\cdot\vec{D}$ term in $\mathcal{L}_{QCD}$

and the Symanzik LE$\mathcal{L}$ takes the form

$$\mathcal{L}_{sym} = \underbrace{\bar{\psi}\left[m_1 + \gamma_0 D_0 + \sqrt{\tfrac{m_1}{m_2}}\,\vec{\gamma}\cdot\vec{D}\right]\psi}_{\neq \mathcal{L}_{QCD}!} + \mathcal{L}_I'$$

While the terms in $\mathcal{L}_I'$, containing higher order operators describing the remaining discretization effects are now "small" (no time derivatives left) with coefficients $k_n$ that are bounded functions of $ma$, the leading term no longer corresponds to (the fermionic part of) QCD.

Solutions:

(1) Relativistic heavy quarks (I)

Start with the improved Wilson action and add an asymmetry (time-space) parameters, allowing time- and space-like operators to have different coefficients:

$$\mathcal{L}_f^{FNAL} = m_0 \bar\psi\psi + \bar\psi \gamma_0 D_0^{lat}\psi - \frac{a}{2}\bar\psi\Delta_0^2\psi$$
$$+ \zeta \bar\psi \vec\gamma\cdot\vec D^{lat}\psi - \frac{a r_s}{2}\zeta\bar\psi\vec\Delta^2\psi$$
$$- \frac{i}{2} a c_B \zeta \bar\psi \vec\Sigma\cdot\vec B^{lat}\psi - \frac{1}{2} a c_E \zeta \bar\psi \vec\alpha\cdot\vec E^{lat}\psi$$

where the asymmetry parameter is tuned (as a function of $m_0$) so that $m_1 = m_2$. This can be done at any order in PT or non-perturbatively.

Example: Repeat Exercise IV with $\mathcal{L}_f^{FNAL}$:

$$\zeta = \left[\left(\frac{r_s m_0(2+m_0)}{4(1+m_0)}\right)^2 + \frac{m_0(2+m_0)}{2\ln(1+m_0)}\right]^{1/2} - \frac{r_s m_0(2+m_0)}{4(1+m_0)}$$

yields $m_1 = m_2$ (at tree-level) i.e. $E^2 = m^2 + \vec p^{\,2} + O(p^4)$

To consider improvement at $O(a^2)$ and beyond construct improvement operators without $(\gamma^0 D_0^{lat})^n$ terms – they can always be removed by using the EoM, see Ref. 13.

In summary, this solution yields

$$\mathcal{L}_{sym} = \mathcal{L}_{QCD} + \mathcal{L}_I$$

where $\mathcal{L}_I$ contains no "large $ma$" terms, with coefficients

$K_n = K_n(g^2, ma, \ldots)$ which are bounded functions of $ma$.

This approach works for all values of $ma$ (i.e. is not limited to $ma < 1$.

(2) Relativistic Heavy Quarks II

Start with the improved Wilson action, no asymmetry term, and construct the Symanzik EFT using continuum HQET (or NRQCD) fields with short-distance coefficients which depend on the HQET and lattice parameters:

$$\mathcal{L}_{Sym} = \dots - \bar{h}(D_0 + m_1)h + \sum_i C_i^{lat}\, O_i$$

with $C_i^{lat} = C_i^{lat}(g^2, m_Q; m_2 a, c_j; \mu/m_Q)$

where cont QCD is then also matched to continuum HQET:

$$\mathcal{L}_{QCD} = \dots + \mathcal{L}^{cont}_{HQET}$$

which differs from the HQET terms in $\mathcal{L}_{Sym}$ above only in the

$$C_i^{cont}(g^2; m_Q; \mu/m_Q)$$

The $c_j$ are couplings (coefficients) of the lattice action which are then adjusted so that

$$C_i^{lat}(g^2, m_2; m_2 a, c_j; \mu/m_2) - C_i^{cont}(g^2; m_2; \mu/m_2) = 0$$

An important piece of the construction is identifying $m_2$ as the physical mass of the heavy quark, i.e.

$$m_2 = m_Q$$

This ensures that the kinetic term in $\mathcal{L}^{lat}_{HQET}$, $D^2/2m$ is correctly normalized, while the rest mass $m_1$ is ignored, since it has no effect on mass splittings in the spectrum or on matrix elements.

In summary, heavy quark discretization errors reside in the mismatch of the coefficients $C_i^{lat} - C_i^{cont} \neq 0$.

The $O_i$ in the HQET $\mathcal{L}$ can be organized dimension, i.e

$$\mathcal{L}_{HQET} = -\bar{h}(D_4 + m_1)h + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots \mathcal{L}^{(s)}$$

where the $O_i$ have dimension $4+s$.

The first two terms are

$$\mathcal{L}^{(1)} = C_2 \cdot \mathcal{O}_2 + C_B \cdot \mathcal{O}_B$$

with $\mathcal{O}_2 = \bar{h}\, \vec{D}^2 h \qquad \mathcal{O}_B = \bar{h}\, \vec{\sigma} \cdot \vec{B}\, h$

and $C_2^{cont} = \frac{1}{2m_Q} = C_B^{cont}$

It is straightforward to show that if

$$m_Q = m_2 \qquad C_2^{lat} = C_2^{cont}$$

$$c_{sw} = 1 \qquad C_B^{lat} = C_B^{cont}$$

(3) Lattice HQET and Lattice NRQCD – see Refs 6,7.

(4) Improved light-quark action:

Example: HISQ action: tree-level $(ma)^2$ removed

Requires field ensembles with fine lattice spacings

$$a \lesssim 1/m_b$$

see slides.

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Pedagogical introduction to Symanzik Improvement:


G. P. Lepage https://arxiv.org/pdf/hep-lat/0506036