Exercises for Lattice determinations of α_s

MITP School "Frontiers and challenges in Lattice Gauge Theory"
Jul 20 2025 – Aug 2 2025 Alberto ramos <alberto.ramos@ific.uv.es>

Problem 1 *Show that the first two coefficients of the β-function are in fact universal.*

1. *Assume that one coupling g_s^2 has a β_s-function with asymptotic expansion*
$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_\mathrm{s}(\mu) = \beta_\mathrm{s}(g_\mathrm{s}) \overset{g_\mathrm{s}\to 0}{\sim} -g_\mathrm{s}^3(b_0 + b_1 g_\mathrm{s}^2 + \dots)$$
2. *Now assume that another coupling $g_{\mathrm{s}'}(\mu)$ is perturbatively related with $g_\mathrm{s}(\mu)$ via the relation*
$$g_{\mathrm{s}'}^2(\mu) \overset{g_\mathrm{s}\to 0}{\sim} g_\mathrm{s}^2(\mu) + c_{ss'}g_\mathrm{s}^4(\mu) + \dots .$$
3. *Show that the $\beta_{\mathrm{s}'}$ function has also the same asymptotic expansion*
$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_{\mathrm{s}'}(\mu) = \beta_{\mathrm{s}'}(g_{\mathrm{s}'}) \overset{g_{\mathrm{s}'}\to 0}{\sim} -g_{\mathrm{s}'}^3(b_0 + b_1 g_{\mathrm{s}'}^2 + \dots)$$

Problem 2 *From the exact formula of the Λ_s parameter associated with the coupling $g_\mathrm{s}(\mu)$*
$$\frac{\Lambda_\mathrm{s}}{\mu} = \left[b_0 g^2\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0 g^2}} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{1}{\beta_\mathrm{s}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

1. *Show that the integral is finite. i.e. that the integral behaves as $\overset{x\to 0}{\sim}\mathcal{O}(x)$ given the asymptotic expansion*
$$\beta_\mathrm{s}(g_\mathrm{s}) \overset{g_\mathrm{s}\to 0}{\sim} -g_\mathrm{s}^3(b_0 + b_1 g_\mathrm{s}^2 + \dots)$$
2. *Show that if another coupling $g_{\mathrm{s}'}(\mu)$ is perturbatively related with $g_\mathrm{s}(\mu)$ via the relation*
$$g_{\mathrm{s}'}^2(\mu) \overset{g_\mathrm{s}\to 0}{\sim} g_\mathrm{s}^2(\mu) + c_{ss'}g_\mathrm{s}^4(\mu) + \dots .$$
the ratio of Λ-parameters is exactly given by
$$\frac{\Lambda_{\mathrm{s}'}}{\Lambda_\mathrm{s}} = \exp\left(\frac{-c_{\mathrm{ss}'}}{2b_0}\right) .$$

A future extraction of α_s

Problem 3 *Using Gradient Flow techniques, one can define not only the standard t_0 scale, but the general t_x scale, defined by the condition*
$$t_\mathrm{x}^2\langle E(r_\mathrm{x})\rangle = \mathrm{x} \tag{1}$$
In a close future Alice has managed to do an impressive calculation: The ratio
$$\frac{t_0}{t_{0.088}} = \lim_{a\to 0}\frac{t_0/a^2}{t_{0.088}/a^2} = 16.97(23)\,. \tag{2}$$
(Note that $t_{0.088}$ is a very short distance scale. Alice has simulated lattice spacings $a \approx 0.01, 0.013, 0.017, 0.024$, using lattices as large as $400^3\times 800$. This was only possible thanks to several improvements in algorithms [Kanwar lectures]).

- *Using (current $N_f = 3$) value $\sqrt{t_0} = \sqrt{t_{0.3}} = 0.14474(57)$ fm, determine $t_{0.088}$ in physical units.*

- *Determine*

$$\alpha_{\rm GF}(\mu_{0.088}) = \frac{4\pi}{3} t_{0.088}^2 \langle E(t_{0.088})\rangle \,, \qquad \left(\mu_{\rm x} = \frac{1}{\sqrt{8t_{\rm x}}}\right) \tag{3}$$

- *Using the known perturbative relation*

$$\alpha_{\rm GF}(\mu) \overset{\alpha\to 0}{\sim} \alpha_{\overline{\rm MS}}(\mu) + 1.122\alpha_{\overline{\rm MS}}^2(\mu) - 1.174\alpha_{\overline{\rm MS}}^3(\mu) + \dots \tag{4}$$

  determine $\alpha_{\overline{\rm MS}}(\mu_{\rm x})$

- *Determine using the known $N_f = 3$ coefficients of the $\beta_{\overline{\rm MS}}$ function (see below), the value of $\Lambda^{(3)}_{\overline{\rm MS}}$*

- *As an estimate of the PT error, determine the contribution of the last known perturbative order in Eq. (3) to $\Lambda^{(3)}_{\overline{\rm MS}}$.*

If you can do it, propagate the errors of t_0 and $t_0/t_0.088$ into Λ. Compare it with your estimate of the PT errors.

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b0 = 0.056993165798815
b1 = 0.0025664955636710844
b2 = 0.00016349868861254387
b3 = 1.9442896826846894e-5
b4 = 1.3280696495205311e-6
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Problem 4 *Using Gradient Flow techniques we have determined in finite volume the step scaling function. The continuum values are given in table 1.*

1. *Parametrize the step scaling function, and fit the data to the parametrization.*
2. *With your parametrization, start a recursion at $g^2(\mu_0) = 11.31$, and determine the values of $g^2(2^k\mu)$ for $k = 1, \dots, 4$.*
3. *You are covering a factor $2^4 = 16$ in scale. Compare the cost (assuming a cost $\propto (L/a)^6$) to the numerical exercise of the previous exercise.*

u_i	$\sigma_i = \sigma(u_i)$	
6.5489	14.005(175)	14.184(197)
5.8673	11.464(123)	11.654(146)
5.3013	9.371(79)	9.468(89)
4.4901	7.139(47)	7.181(51)
3.8643	5.622(28)	5.641(30)
3.2029	4.354(19)	4.367(21)
2.7359	3.541(14)	3.550(15)
2.3900	2.991(10)	2.996(10)
2.1257	2.575(9)	2.578(9)

Table 1: Step scaling function in the continuum. Each value has been obtained from a set of three pairs of simulations ($8 \to 16; 12 \to 24; 16 \to 32$) to control the continuum limit. Columns two and three are two different models for the continuum extrapolation.