

i) A general algorithmic strategy

- ii) Time evolution in the Schwinger model
 - In purely fermionic formulation
 - In fermion-boson formulation
- iii) Outlining the differences between Abelian and non-Abelian algorithms

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SIMULATION STRATEGIES: FEW QUESTIONS

Q1: What is the best breakdown of Hamiltonian term to H_i terms such that:

- i) each term can be simulated with the least resources,
- ii) the number of terms to be simulated is minimized,
- iii) the Trotter error is minimized,
- iv) as many symmetries as possible are retained?

We may not be able to simultaneously satisfy all these conditions so we need to seek a balance.

Q2: How to simulate each e^{-itH_i} ? This amounts to:

- i) finding the unitary transformation that diagonalizes e^{-itH_i} in the computational basis, i.e., $e^{-itH_i} = \mathcal{U}_i e^{-it\mathcal{D}_i} \mathcal{U}_i^{\dagger}$.
- ii) circuitizing the unitary transformation $\mathcal{U}_{i'}$
- iii) circuitizing the diagonal form $e^{-it\mathcal{D}_i}$.

If e^{-itH_i} is already diagonal, steps i) and ii) are not needed.

Q3: What quantum resources should we minimize given those choices in the previous Qs?

- i) In the near-term scenario,
 - the hardware systems are small so the **less ancillary qubits** the better,
 - single-qubit gates are almost free but two-qubit gates (CNOT) are of low fidelity.
- ii) In the far-term scenario,
 - we likely do not have qubit-resource constraints,
 - compilation of all Clifford gates (including CNOT) is less costly but non-Clifford (T gates) have high fault-tolerant implementation cost.

Q4: Given all these consideration, which Hamiltonian formulation and basis states of the theory are most suitable? We may need to consider formulations that:

- i) give the desired continuum physics faster with the least resources,
- ii) have the least encoding overhead,
- ii) have less complex terms,
- iii) respect more symmetries by construction.

We are not considering state preparation and measurements here, but those often enter our considerations of what the most suitable formulation is given the observable of interest.

RESOURCE ANALYSIS

$$\left\| V_p(T) - e^{-iTH} \right\| \leq \varepsilon$$

Given the accuracy ε on the time evolution operator, how many ancilla qubits and costly gates are needed for simulating a Hamiltonian with given parameters for time T using the p^{th} -order product formula?

For a LGT Hamiltonian, these are volume, lattice spacing, couplings, masses, and truncation scale of the bosonic fields.

The errors that accumulate to add up to the total error ϵ include:



ii) function-evaluation approximation error,

iii) gate-synthesis error,

iv) measurement error, and

v) theoretical errors (finite-volume, discretization, truncation, etc.).

The p^{th} -order product formula requires $\boldsymbol{\ell}$

$$O\left(\frac{T^{(p+1)/p}}{\epsilon^{1/p}}\right)$$
Ti

Trotter steps. Near-optimal

algorithms based on completely different digitization strategies achieve ${\it O}$

(IMPROVED) THEORY OF PRODUCT FORMULAS

Consider the Hamiltonian

 $H = \sum_{i=1}^{\Gamma} H_i$

First-order product formula

$$V_1(t) = e^{-itH_1}e^{-itH_2}\cdots e^{-itH_{\Gamma}}$$

is bounded by:

$$|V_1(t) - e^{-itH}|| \le \frac{t^2}{2} \sum_{i=1}^{\Gamma} \left\| \left[\sum_{j=i+1}^{\Gamma} H_j, H_i \right] \right\|$$

Second-order formula

$$V_2(t) = (e^{-itH_{\Gamma}/2} \cdots e^{-itH_2/2} e^{-itH_1/2})(e^{-itH_1/2} e^{-itH_2/2} \cdots e^{-itH_{\Gamma}/2})$$

is bounded by:

$$\|V_2(t) - e^{-itH}\| \le \frac{t^3}{12} \sum_{i=1}^{\Gamma} \left\| \left[\sum_{k=i+1}^{\Gamma} H_k, \left[\sum_{j=i+1}^{\Gamma} H_j, H_i \right] \right] \right\| + \frac{t^3}{24} \sum_{i=1}^{\Gamma} \left\| \left[H_i, \left[H_i, \sum_{j=i+1}^{\Gamma} H_j \right] \right] \right\|$$

A general bound also exist, see: Childs, Su, Tran, Wiebe, Zhu, Phys. Rev. X 11, 011020 (2021).

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(RESCALED HENCE DIMENSIONLESS) LATTICE SCHWINGER MODEL HAMILTONIAN



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LET'S START SIMPLE: THE **FULLY FERMIONIC** REPRESENTATION WITH **FIRTS-ORDER** PRODUCT FORMULA.

$$H = x \sum_{n=0}^{N-2} \left(\sigma_n^+ \sigma_{n+1}^- + \text{h.c.} \right) + \frac{\mu}{2} \sum_{n=0}^{N-1} (-1)^{n+1} \sigma_n^z + \sum_{n=0}^{N-2} \left\{ \epsilon_0 + \frac{1}{2} \sum_{m=0}^n \left[\sigma_m^z - (-1)^m \right] \right\}^2$$
$$= H^x + H^{ZZ} + H^Z \quad \text{or} \quad H^{(XX)} + H^{(YY)} + H^{ZZ} + H^Z$$

Two time orderings, one that respects the global charge conservation:

$$V_1(\delta t) = e^{-i\delta t \,\hat{H}^Z} e^{-i\delta t \,\hat{H}^{ZZ}} \prod_{k=1}^{(N/2)-1} e^{-i\delta t \,\hat{H}_{2k,2k+1}^x} \prod_{k=1}^{N/2} e^{-i\delta t \,\hat{H}_{2k-1,2k}^x}$$

and one that breaks it!

$$V_1'(\delta t) = e^{-i\delta t \,\hat{H}^Z} e^{-i\delta t \,\hat{H}^{ZZ}} \prod_{k=1}^{N-1} e^{-i\delta t \,\hat{H}_{k,k+1}^{(YY)}} \prod_{k=1}^{N-1} e^{-i\delta t \,\hat{H}_{k,k+1}^{(XX)}}$$

and many more!



What is the global conserved charge in the Schwinger-model Hamiltonian? Why is one of the schemes in the previous slide conserves the global charge and the other does not?

- $R_{ij}^{zz}(\theta) \equiv e^{-i\theta\sigma_i^z\sigma_j^z/2}$ can be implemented either directly (like in trapped ions) or by two CNOTs and one single-qubit rotation since $e^{-i\theta\sigma_i^z\sigma_j^z/2} = \text{CNOT}_{ij}R_i^z(\theta) \text{CNOT}_{ij}$.
- $e^{-i\theta\sigma_i^x\sigma_j^x/2}$ and $e^{-i\theta\sigma_i^y\sigma_j^y/2}$ can be implemented similarly by rotating to the eigenstates of σ^z .
- $e^{-i\theta\sigma_i^z/2}$ is already an elementary gate and can be applied directly.



Martinez et al, Nature 534, 516 EP (2016).





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NOW WHAT ABOUT **FERMIONIC-BOSONIC** REPRESENTATION WITH THE **SECOND-ORDER** PRODUCT FORMULA?

$$H = x \sum_{n} \left[\sigma_n^+ U_n \sigma_{n-1}^- + \text{h.c.} \right] + \frac{\mu}{2} \sum_{n} (-1)^{n+1} \sigma_n^z + \sum_{n} E_n^2$$

One can split the terms in the Hamiltonian as:

Shaw, Lougovski, Stryker, Wiebe, Quantum 4, 306 (2020).

$$H = \sum_{x} (T_{x} + D_{x}), \quad \text{with} \quad D_{x} \coloneqq D_{x}^{(M)} + D_{x}^{(E)}.$$

$$T_{x} \coloneqq x \left(\frac{1}{4}(U_{x} + U_{x}^{\dagger})(X_{x}X_{x+1} + Y_{x}Y_{x+1}) + \frac{i}{4}(U_{x} - U_{x}^{\dagger})(X_{x}Y_{x+1} - Y_{x}X_{x+1})\right)$$

$$D_{x}^{(M)} \coloneqq \frac{\mu}{2}(-1)^{x}Z_{x} \quad \text{and} \quad D_{x}^{(E)} \coloneqq E_{x}^{2}$$

and do the following ordering of the terms

$$V_{2}(t) = \prod_{x} \left(\prod_{k \in \{M,E\}} e^{-iD_{x}^{(k)}t/2} \prod_{j=1}^{4} e^{-iT_{x}^{(j)}t/2} \right) e^{-iD_{N}^{(M)}t} \prod_{\substack{x \in \{E,M\} \\ \text{In reverse order}}} \left(\prod_{j=4}^{1} e^{-iT_{x}^{(j)}t/2} \prod_{k \in \{E,M\}} e^{-iD_{x}^{(k)}t/2} \right)$$

Example

This example concerns finding a quantum circuit for implementing

$$U^{(E)} = \prod_{i=1}^{N} U_i^{(E)} = e^{-it\sum_{i=1}^{N} E_i^2}$$

in the time-evolution of lattice Schwinger model in a near-term scenario that avoids introducing any ancilla qubits. Consider $E_i \in [-\Lambda, \Lambda]$ and encode the electric-field Hilbert space on each link *i* into $\eta \equiv \lceil \log_2(2\Lambda) + 1 \rceil$ qubits. Given this, find a circuit representation for $U_i^{(E)}$ in terms of only single-qubit rotations around the *z* axis of Bloch sphere as well as two-qubit CNOT gates. Verify your answer by explicitly working out a small example.

It is easy to show that the electric-field operator at each link acting on the computational (binary) basis is:

$$E = -\Lambda \mathbb{I} + \frac{1}{2} \left[(2^{\eta} - 1)\mathbb{I} - \sum_{j=0}^{\eta-1} 2^{j} \sigma_{j}^{z} \right]$$

Therefore,

$$E^{2} = \Lambda^{2} \mathbb{I} - \Lambda \left[(2^{\eta} - 1)\mathbb{I} - \sum_{j=0}^{\eta-1} 2^{j} \sigma_{j}^{z} \right] + \frac{1}{4} \left[(2^{\eta} - 1)^{2}\mathbb{I} - 2(2^{\eta} - 1)\sum_{j=0}^{\eta-1} 2^{j} \sigma_{j}^{z} + \sum_{j,j'=0}^{\eta-1} 2^{j+j'} \sigma_{j}^{z} \sigma_{j'}^{z} \right]$$

Consequently, the operator $U^{(E)}$ can be written as a product of $N\eta R^z$ rotations and $N\eta(\eta - 1)/2 R^{zz}$ rotations with rotation angles that can be read off from the expression above. Note that each R^{zz} gate amounts to two CNOT gates and one R^z gate.

Example

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The previous example requires $O(N\eta^2)$ number of R^z gates, which are costly operations in the fault-tolerant regime as they need to be synthesized up to accuracy ϵ using roughly $\log(1/\epsilon)$ T gates. Can one reduce the R^z cost of electric-field evolution to $O(N\eta)$? The answer is yes, but at the cost of extra $O(\eta)$ ancillas that are, nonetheless, available in the fault-tolerant era. One such circuit can be constructed using the so-called phase-kickback routine. For each $U_i^{(E)}$:



The logical copy and multiplication routines are known circuits and overall cost $O(\eta^2)$ T gates. The ancilla qubits are reset in the end and can be used in the remainder of the circuit.



How do you implement arbitrary diagonal operator $e^{-it\mathscr{D}}$ in the computational basis? [Think about two examples: i) $\mathscr{D} |n\rangle = n |n\rangle$ and ii) $\mathscr{D} |n\rangle = \sqrt{\frac{n+1}{n-1}} |n\rangle$!]



COMPARISON BETWEEN THE TWO FORMULATIONS FOR THE SECOND-ORDER FORMULA



*Defined at the required number of Trotter steps for simulation time t, system size $N \sim \Lambda$, and at fixed x and μ , given a fixed error tolerance.

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Explain the qubit and gate scalings of the second-order Trotter simulation of the lattice Schwinger model in both formulations, as given in the previous slide.

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Abelian vs. non-Abelian

Since we do not have the option of removing the gauge links generally, let us focus on the fermionic-bosonic formulations in the electric-field basis. So what are the major differences between simulating digitally Abelian and non-Abelian LGTs? Let is compare U(1) and SU(N) LGTs.

i) There are more degrees of freedom involved for SU(N) LGTs. For example, at each site, there are *N*-component fermions, and at each link there are multiple bosonic variables.

ii) As a result, there are more terms that need to be simulated, hence more complexity and generally more Trotter error.

iii) The diagnozalization procedure for hopping and magnetic terms generally follow the same rules but is more gate-intensive for SU(N).

iv) The diagonal operators in an Abelian theory like U(1) are trivial while for SU(N), they require evaluating phases that are non-trivial functions of bosonic occupation-number operators. These require expensive function-evaluation routines (in the E basis).

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Algorithms for simulating quantum field theories started from pioneering work of Jordan, Lee, Preskill.

Science 336, 6085 1130-1133 (2012).

Algorithmic progress for U(1), SU(2), and SU(3) quantum field theories include:

Byrnes and Yamamoto, Phys.Rev. A73 (2006) 022328.
Shaw, Lougovski, Stryker, Wiebe, Quantum 4, 306 (2020).
Ciavarella, Klco, and Savage, Phys. Rev. D 103, 094501 (2021).
Kan and Nam, arXiv:2107.12769 [quant-ph].
Lamm, Lawrence, and Yamauchi, Phys.Rev.D 100 (2019) 3, 034518.
Paulson et al, PRX Quantum 2 (2021) 030334.
Murairi, Cervia, Kumar, Bedaque, Alexandru, arXiv:2208.11789 [hep-lat].
ZD, Shaw, and Stryker, Quantum 7, 1213 (2023).
Sakamoto, Morisaki, Haruna, Itou, Fujii, Mitarai, Quantum 8, 1474 (2024).
M. Rhodes, M. Kreshchuk, S. Pathak, PRX Quantum 5:040347 (2024).
Lamm et al, arXiv:2405.12890 [hep-lat].

Second-order product formula Non-local fermion encoding

(2(10	53)	Γ	gates	5
	Kan,	Nam	(2021)	

Time evolving under QCD in a $10 \times 10 \times 10$ lattice in the irrep basis with $\Lambda = 10$ and $a = 10^{-2}$ at fixed parameters with error 10^{-3} (10^{10} qubits).



Starting from the Standard Model





Time evolving under QCD in a $10 \times 10 \times 10$ lattice in the irrep basis with $\Lambda = 10$ and $a = 10^{-2}$ at fixed parameters with error 10^{-3} (10^{10} qubits).

Second-order product formula (better term decomposition) Non-local fermion encoding

Starting from the Standard Model







