Quantum Simulation of the Lattice Schwinger Model

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Part I: Exact numerics for spectrum and dynamics

The goal of this part of the project is to obtain, numerically, the energy spectrum of the Schwinger model with open boundary conditions (OBC) for a given set of parameters, and to evaluate both the continuous and the Trotterized evolution of the Schwinger model. Consider the Schwinger Hamiltonian in the purely fermionic formulation:

$$H = \mathrm{x} \sum_{n=0}^{N-2} \left[\psi^\dagger(n)\psi(n+1) + \text{h.c.} \right] + \mu \sum_{n=0}^{N-1} (-1)^n \psi^\dagger(n)\psi(n) +$$
$$\sum_{n=0}^{N-2} \left\{ \epsilon_0 + \sum_{m=0}^{n} \left[-\psi^\dagger(m)\psi(m) + \frac{1-(-1)^m}{2} \right] \right\}^2 . \tag{1}$$

Part (a) Show that after a Jordan-Wigner transformation, one can map the fermionic Hamiltonian to a qubit Hamiltonian of the form:

$$H = \mathrm{x} \sum_{n=0}^{N-2} \left(\sigma_n^+ \sigma_{n+1}^- + \text{h.c.} \right) + \frac{\mu}{2} \sum_{n=0}^{N-1} (-1)^{n+1} \sigma_n^{\mathrm{z}} +$$
$$\sum_{n=0}^{N-2} \left\{ \epsilon_0 + \frac{1}{2} \sum_{m=0}^{n} [\sigma_m^{\mathrm{z}} - (-1)^m] \right\}^2 \tag{2}$$
$$:= \sum_{n=0}^{N-2} H_{n,n+1}^{(\mathrm{XX})} + \sum_{n=0}^{N-2} H_{n,n+1}^{(\mathrm{YY})} + H^{(\mathrm{ZZ})} + H^{(\mathrm{Z})}. \tag{3}$$

Make sure to identify $H_{n,n+1}^{(\mathrm{XX})}$, $H_{n,n+1}^{(\mathrm{YY})}$, $H^{(\mathrm{ZZ})}$, and $H^{(\mathrm{Z})}$, where $H_{n,n+1}^{(\mathrm{XX/YY})}$ are terms proportional to the product of two Pauli-X/Y matrices on qubits n and $n+1$, $H^{(\mathrm{ZZ})}$ are all terms proportional to the product of two Pauli-Z matrices on two distinct qubits, while $H^{(\mathrm{Z})}$ are all terms proportional to a Pauli-Z matrix on a single qubit.

Part (b) Consider a system of $N = 4$ staggered sites. Construct the Hamiltonian matrix associated with the Hamiltonian in Part (a). This means that you have to find all the matrix elements in terms of x, μ, and ϵ_0. Now try to diagonalize the Hamiltonian to find the energy eigenvalues for the following model parameters: $\epsilon_0 = 0$, $\mathrm{x} = 0.6$, and $\mu = 0.1$.

Part (c) Consider the strong-coupling vacuum (i.e., the eigenstate of the Hamiltonian in the limit $\mathrm{x} = 0$) and call it $|\psi(0)\rangle$. Apply the time-evolution operator e^{-itH} onto this

state for the total evolution time $t = 5$. For all other parameters, use the values given in Part (b). This procedure gives you $|\psi(t)\rangle = e^{-itH} |\psi(0)\rangle$. Evaluate and plot, as a function of time, the Loschmidt echo, i.e., the survival probability the initial state, defined as

$$\mathcal{P}(t) := |\langle\psi(0)|\psi(t)\rangle|^2, \tag{4}$$

What do you learn from this quantity?

Part (c) For the same procedure and parameters as in the previous part, evaluate and plot, as a function of time, the staggered fermion density defined as

$$\nu(t) := \frac{1}{N} \sum_{n=0}^{N-1} \nu_n(t), \tag{5}$$

where

$$\nu_n(t) := \langle\psi(t)| \frac{(-1)^{n+1}\sigma_n^z + 1}{2} |\psi(t)\rangle. \tag{6}$$

How many electron-positron pairs do you have in the initial state? (Recall the mapping we talked about in the lecture.) Does the dynamics generate electron-positron pairs? The phenomenon of pair production out of "vacuum" is one of the hallmarks of a relativistic quantum field theory, which can be seen in nonequilibrium dynamics as simple as the one you just studied! Compare your results to the lower panel in fig.6 of 2112.14262.

Part (d) Now divide the time evolution for duration $t = 5$ into $N_T = 10$ Trotter steps. Hence, the evolution time for each step is $\delta t = 0.5$. Apply the first-order Trotter-Suzuki approximation

$$V(t) = \prod_{i=1}^{N_T} \left(e^{-i\delta t H^{(Z)}} e^{-i\delta t H^{(ZZ)}} \prod_{n=0}^{N-2} e^{-i\delta t H_{n,n+1}^{(XX)}} \prod_{n=0}^{N-2} e^{-i\delta t H_{n,n+1}^{(YY)}} \right) \tag{7}$$

to evolve the strong-coupling vacuum state $|\psi(0)\rangle$ as in Part (c) for time t. Notice that there are more than one way to do the term decomposition, but we are going to adopt the form above. Plot both the Loschmidt echo and the staggered fermion density defined in the previous parts as a function of Trotterized time. Overlay your plots with the continuous time evolution you saw previously to clearly see any deviation from exact result. If you are up for it, change the Trotter step size to explore how Trotter error responds to this change.

Part (e) [**Bonus**] First show that the total charge operator

$$Q := \sum_{n=0}^{N-1} \frac{\sigma_n^z + (-1)^{n+1}}{2} \tag{8}$$

commutes with the Hamiltonian. What are the eigenvalues of this operator? Since $[H, Q] = 0$, the eigenvalues of Q are conserved quantities: if you start the evolution in a given Q-eigenvalue sector, Hamiltonian time dynamics should not take you out of this sector. Does the Trotter-Suzuki expansion we picked in the previous part conserve the total charge Q? If yes, argue why. If no, can you come up with a decomposition that conserves the total charge? This example shows that approximate algorithms can break the symmetries of the model!

Part II: Circuit design and implementation of time dynamics

For this part of the project, we continue to work with the qubit Hamiltonian:

$$H = x \sum_{n=0}^{N-2} \left(\sigma_n^+ \sigma_{n+1}^- + \text{h.c.} \right) + \frac{\mu}{2} \sum_{n=0}^{N-1} (-1)^{n+1} \sigma_n^z +$$

$$\sum_{n=0}^{N-2} \left\{ \epsilon_0 + \frac{1}{2} \sum_{m=0}^{n} [\sigma_m^z - (-1)^m] \right\}^2 \tag{9}$$

$$:= \sum_{n=0}^{N-2} H_{n,n+1}^{(XX)} + \sum_{n=0}^{N-2} H_{n,n+1}^{(YY)} + H^{(ZZ)} + H^{(Z)}, \tag{10}$$

associated with the lattice Schwinger model with open boundary conditions (OBC). We further consider Trotterized time evolution using the first-order Trotter-Suzuki formula using the decomposition:

$$V(t) = \prod_{i=1}^{N_T = t/\delta t} \left(e^{-i\delta t H^{(Z)}} e^{-i\delta t H^{(ZZ)}} \prod_{n=0}^{N-2} e^{-i\delta t H_{n,n+1}^{(XX)}} \prod_{n=0}^{N-2} e^{-i\delta t H_{n,n+1}^{(YY)}} \right). \tag{11}$$

The initial state, $|\psi(0)\rangle$, of this evolution is the strong-coupling vacuum, and the time-evolved state is $|\psi(t)\rangle := V(t)|\psi(0)\rangle$. The system size and parameters are as in the previous exercise: $N = 4$, $\epsilon_0 = 0$, $x = 0.6$, $\mu = 0.1$, $t = 5$, $\delta t = 0.5$ (or $N_T = 10$).

Part (a) Pictorially draw the quantum-circuit elements that evaluate each Trotter step of the evolution. Make sure to identify the gates to the level they can be implemented on a quantum hardware. Identify all the gate angles.

Part (b) Write a `Qiskit` code that implements your circuit in part (a). If you are not yet comfortable with writing the code from scratch, use the IBMQ automated tool.

Part (c) Evaluate the Loschmidt echo

$$\mathcal{P}(t) := |\langle \psi(0)|\psi(t)\rangle|^2, \tag{12}$$

and the staggered fermion density

$$\nu(t) := \frac{1}{N} \sum_{n=0}^{N-1} \langle \psi(t)| \frac{(-1)^{n+1} \sigma_n^z + 1}{2} |\psi(t)\rangle, \tag{13}$$

as defined in the previous exercise. To do this, run the circuit for 1,000 times and measure the outcome in the computational basis (the Pauli-Z basis). Then reconstruct the observables using these measurements. Note that the labeling in `Qiskit` is such that the first (0th) qubit represents the least significant digit. Try to explore the dependence of observables on the number of shots by changing the shot count. In particular, you should be able to see the convergence to the numerically evaluated Trotterized time evolution in the previous exercise by increasing the shot count. Think about how you would you assign statistical uncertainty to the observables.

Part (d) [Bonus] Make appropriate measurements to obtain the energy of the time-evolved state:

$$E(t) := \langle \psi(t)|H|\psi(t)\rangle, \tag{14}$$

at $t = 5$. To do this, you need to measure all the Hamiltonian operators and sum them up. For non-diagonal operators in the computational basis, i.e., $H_{n,n+1}^{(\mathrm{XX/YY})}$, you need to first perform a change of basis from Z to X/Y, then measure the associated qubits. Compare your results to the numerically evaluated result. Importantly note that energy is a conserved quantity, so any deviation from the initial-state's energy should be attributed to the errors in the implementation. Explore the dependence of the error on the shot count and Trotter error.