QUANTUM COMPUTATION OF LATTICE GAUGE THEORIES

Based on https://arxiv.org/abs/2507.15840

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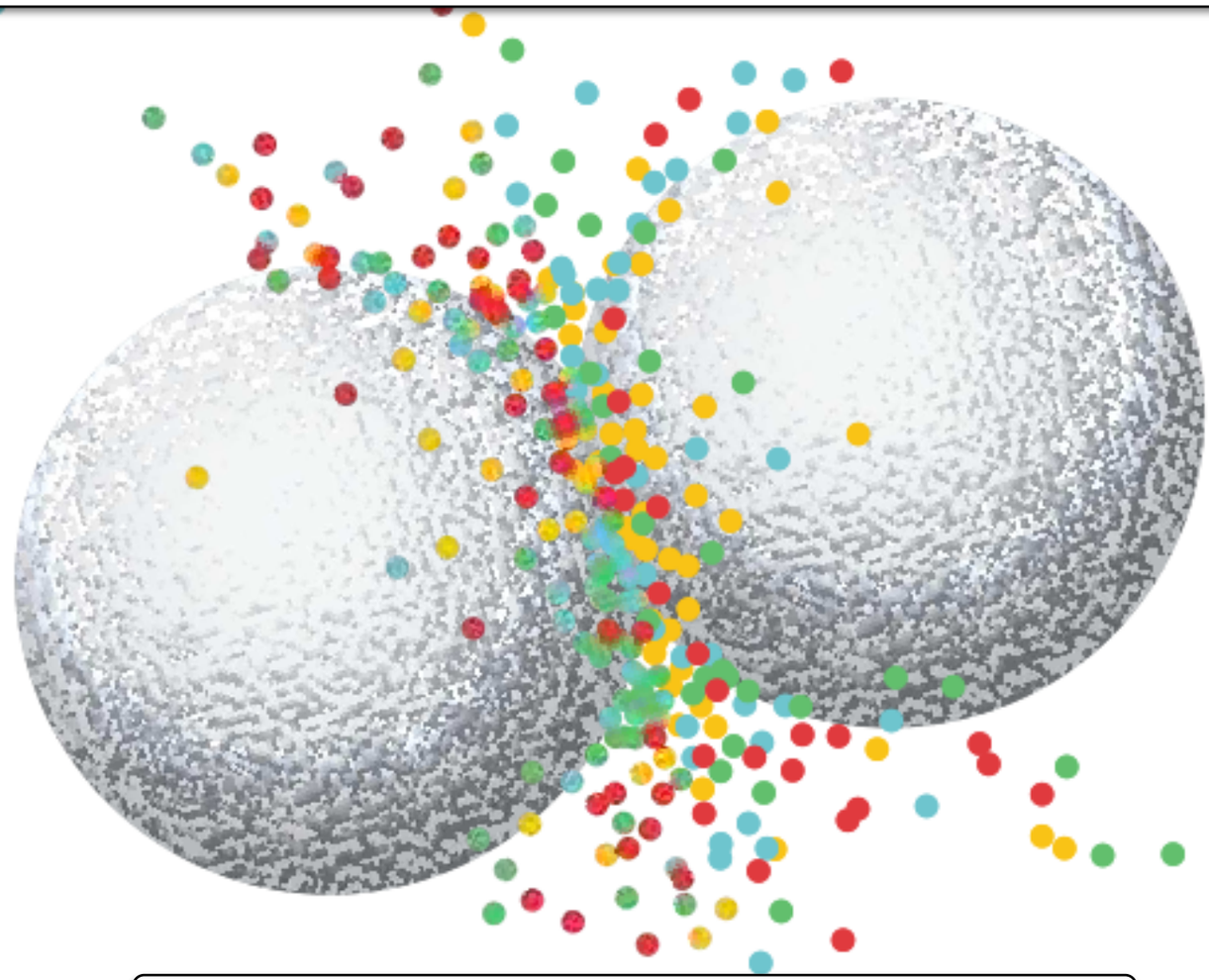

The MITP Summer School 2025 on "Frontiers and Challenges in Lattice Gauge Theory"
MITP, Mainz, Germany
July 2025

OUTLINE OF PART II:
DIGITAL QUANTUM COMPUTING TIME EVOLUTION IN LGTs

i) A general algorithmic strategy

ii) Time evolution in the Schwinger model

- In purely fermionic formulation
- In fermion-boson formulation

iii) Outlining the differences between Abelian and non-Abelian algorithms

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SIMULATION STRATEGIES: FEW QUESTIONS

Q1: What is the best breakdown of Hamiltonian term to H_i terms such that:

i) each term can be simulated with the least resources,
ii) the number of terms to be simulated is minimized,
iii) the Trotter error is minimized,
iv) as many symmetries as possible are retained?

We may not be able to simultaneously satisfy all these conditions so we need to seek a balance.

Q2: How to simulate each e^{-itH_i} ? This amounts to:

i) finding the unitary transformation that diagonalizes e^{-itH_i} in the computational basis, i.e., $e^{-itH_i} = \mathcal{U}_i e^{-it\mathcal{D}_i}\mathcal{U}_i^\dagger$.
ii) circuitizing the unitary transformation $\mathcal{U}_i$,
iii) circuitizing the diagonal form $e^{-it\mathcal{D}_i}$.

If e^{-itH_i} is already diagonal, steps i) and ii) are not needed.

Q3: What quantum resources should we minimize given those choices in the previous Qs?

i) In the near-term scenario,
- the hardware systems are small so the **less ancillary qubits** the better,
- single-qubit gates are almost free but **two-qubit gates (CNOT)** are of low fidelity.

ii) In the far-term scenario,
- we likely do not have qubit-resource constraints,
- compilation of all Clifford gates (including CNOT) is less costly but non-Clifford (**T gates**) have high fault-tolerant implementation cost.

Q4: Given all these consideration, which Hamiltonian formulation and basis states of the theory are most suitable? We may need to consider formulations that:

i) give the desired continuum physics faster with the least resources,
ii) have the least encoding overhead,
ii) have less complex terms,
iii) respect more symmetries by construction.

We are not considering state preparation and measurements here, but those often enter our considerations of what the most suitable formulation is given the observable of interest.

RESOURCE ANALYSIS

$$\left\| V_p(T) - e^{-iTH} \right\| \leq \varepsilon$$

Given the accuracy ε on the time evolution operator, how many ancilla qubits and costly gates are needed for simulating a Hamiltonian with given parameters for time T using the p^{th}-order product formula?

For a LGT Hamiltonian, these are volume, lattice spacing, couplings, masses, and truncation scale of the bosonic fields.

The errors that accumulate to add up to the total error ϵ include:

i) Trotter error,
ii) function-evaluation approximation error,
iii) gate-synthesis error,
iv) measurement error, and
v) theoretical errors (finite-volume, discretization, truncation, etc.).

The p^{th}-order product formula requires $O\left(\frac{T^{(p+1)/p}}{\epsilon^{1/p}}\right)$ Trotter steps. Near-optimal algorithms based on completely different digitization strategies achieve $O\left(T, \log\left(\frac{1}{\varepsilon}\right)\right)$.

(IMPROVED) THEORY OF PRODUCT FORMULAS

Consider the Hamiltonian $$H = \sum_{i=1}^{\Gamma} H_i$$

First-order product formula

$$V_1(t) = e^{-itH_1} e^{-itH_2} \cdots e^{-itH_\Gamma}$$

is bounded by:

$$\| V_1(t) - e^{-itH} \| \leq \frac{t^2}{2} \sum_{i=1}^{\Gamma} \left\| \left[\sum_{j=i+1}^{\Gamma} H_j, H_i \right] \right\|$$

Second-order formula

$$V_2(t) = (e^{-itH_\Gamma/2} \cdots e^{-itH_2/2} e^{-itH_1/2})(e^{-itH_1/2} e^{-itH_2/2} \cdots e^{-itH_\Gamma/2})$$

is bounded by:

$$\| V_2(t) - e^{-itH} \| \leq \frac{t^3}{12} \sum_{i=1}^{\Gamma} \left\| \left[\sum_{k=i+1}^{\Gamma} H_k, \left[\sum_{j=i+1}^{\Gamma} H_j, H_i \right] \right] \right\| + \frac{t^3}{24} \sum_{i=1}^{\Gamma} \left\| \left[H_i, \left[H_i, \sum_{j=i+1}^{\Gamma} H_j \right] \right] \right\|$$

A general bound also exist, see: **Childs, Su, Tran, Wiebe, Zhu, Phys. Rev. X 11, 011020 (2021).**

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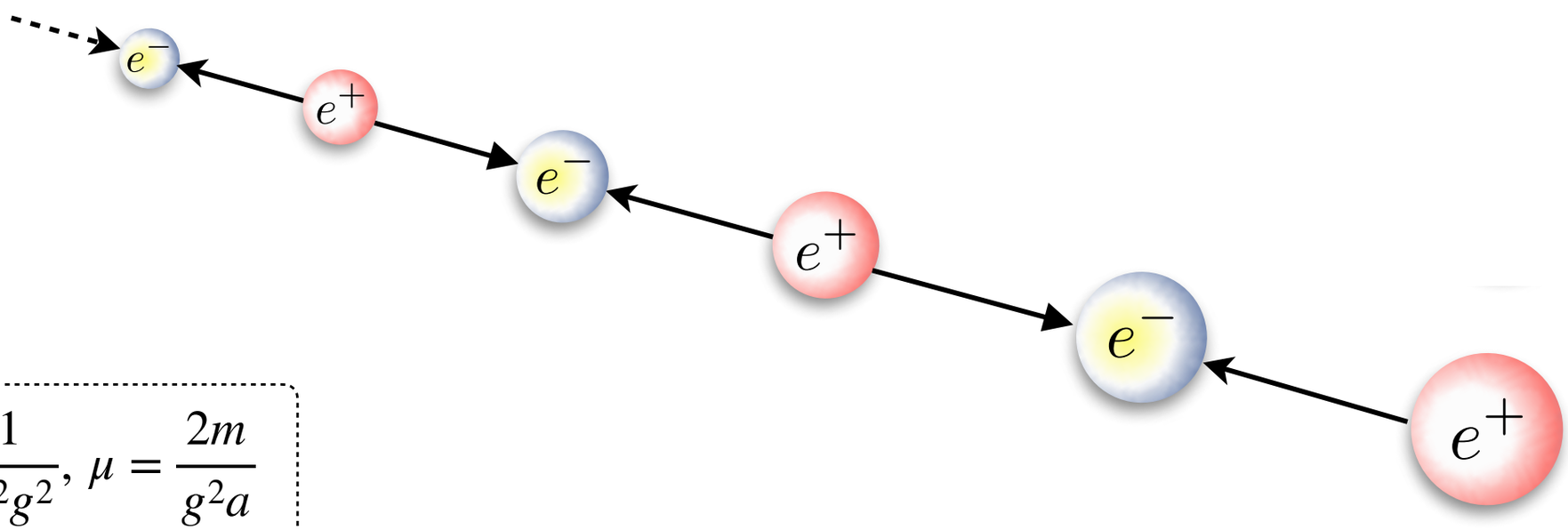
(RESCALED HENCE DIMENSIONLESS)
LATTICE SCHWINGER MODEL HAMILTONIAN

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left[\psi^\dagger(n)\psi(n+1)+\mathrm{h.c.}\right] + \mu\sum_{n=0}^{N-1}(-1)^n\psi^\dagger(n)\psi(n) + \sum_{n=0}^{N-2}\left\{\epsilon_0+\sum_{m=0}^{n}\left[-\psi^\dagger(m)\psi(m)+\frac{1-(-1)^m}{2}\right]\right\}^2.$$

Fermion hopping via gauge links

Staggered mass term

Electric field energy



$$\mathrm{x} = \frac{1}{a^2g^2},\ \mu = \frac{2m}{g^2a}$$

(RESCALED HENCE DIMENSIONLESS)
LATTICE SCHWINGER MODEL HAMILTONIAN

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left[\psi^\dagger(n)\psi(n+1)+\mathrm{h.c.}\right] + \mu\sum_{n=0}^{N-1}(-1)^n\psi^\dagger(n)\psi(n) + \sum_{n=0}^{N-2}\left\{\epsilon_0+\sum_{m=0}^{n}\left[-\psi^\dagger(m)\psi(m)+\frac{1-(-1)^m}{2}\right]\right\}^2.$$

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Electric field energy

Jordan-Wigner transformation

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left(\sigma_n^+\sigma_{n+1}^- + \mathrm{h.c.}\right) + \frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n+1}\sigma_n^{\mathrm{z}} + \sum_{n=0}^{N-2}\left\{\epsilon_0+\frac{1}{2}\sum_{m=0}^{n}\left[\sigma_m^{\mathrm{z}}-(-1)^m\right]\right\}^2$$

Nearest neighbor spin-spin interactions

An effective magnetic field

Long range spin-spin interactions plus an effective magnetic field

LET'S START SIMPLE: THE **FULLY FERMIONIC** REPRESENTATION WITH **FIRTS-ORDER** PRODUCT FORMULA.

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left(\sigma_n^+\sigma_{n+1}^- + \mathrm{h.c.}\right) + \frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n+1}\sigma_n^{\mathrm{z}} + \sum_{n=0}^{N-2}\left\{\epsilon_0 + \frac{1}{2}\sum_{m=0}^{n}\left[\sigma_m^{\mathrm{z}} - (-1)^m\right]\right\}^2$$

$$= H^{\mathrm{x}} + H^{ZZ} + H^{Z} \quad \text{or} \quad H^{(XX)} + H^{(YY)} + H^{ZZ} + H^{Z}$$

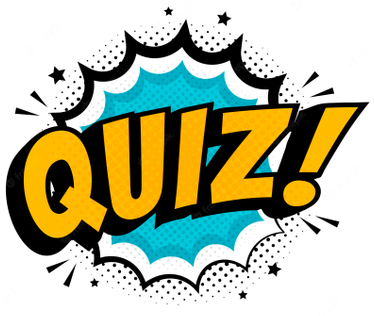
Two time orderings, one that respects the global charge conservation:

$$V_1(\delta t) = e^{-i\delta t\,\hat{H}^Z}\, e^{-i\delta t\,\hat{H}^{ZZ}} \prod_{k=1}^{(N/2)-1} e^{-i\delta t\,\hat{H}^x_{2k,2k+1}} \prod_{k=1}^{N/2} e^{-i\delta t\,\hat{H}^x_{2k-1,2k}}$$

and one that breaks it!

$$V_1'(\delta t) = e^{-i\delta t\,\hat{H}^Z}\, e^{-i\delta t\,\hat{H}^{ZZ}} \prod_{k=1}^{N-1} e^{-i\delta t\,\hat{H}^{(YY)}_{k,k+1}} \prod_{k=1}^{N-1} e^{-i\delta t\,\hat{H}^{(XX)}_{k,k+1}}$$

and many more!

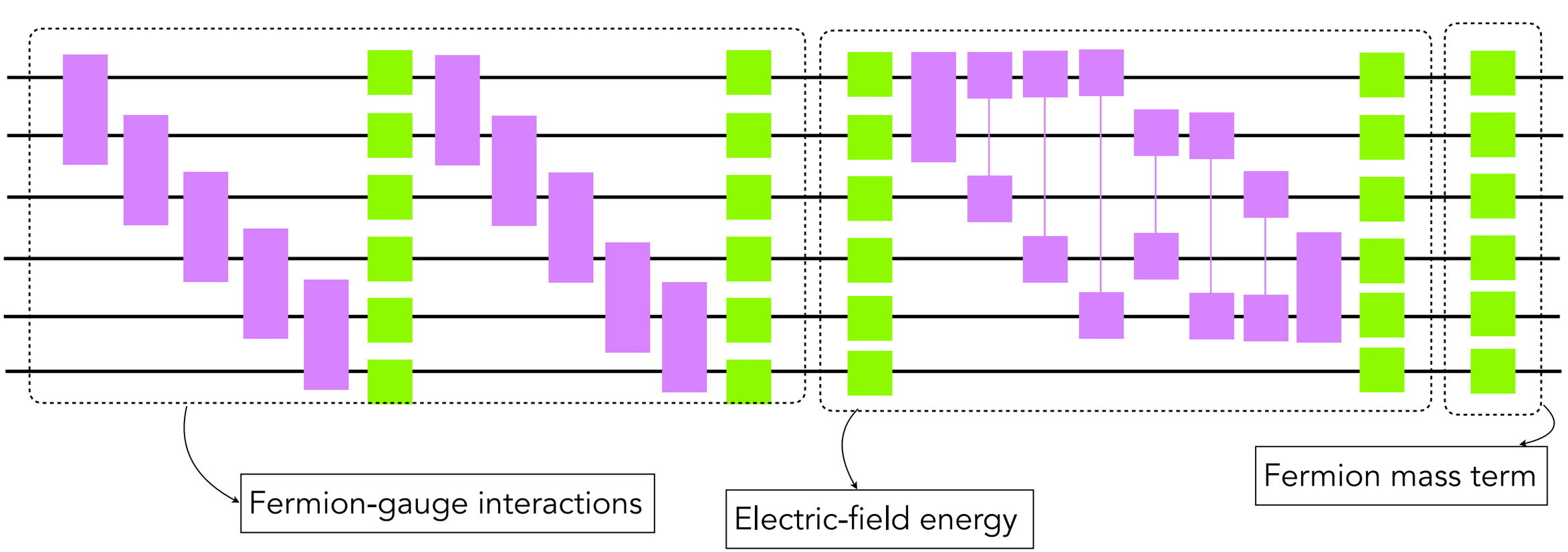


What is the global conserved charge in the Schwinger-model Hamiltonian?
Why is one of the schemes in the previous slide conserves the global charge and the other does not?

- $R_{ij}^{zz} \equiv e^{-i\theta\sigma_i^z\sigma_j^z}$ can be implemented either directly (like in trapped ions) or by two CNOTs and one single-qubit rotation since $e^{-i\theta\sigma_i^z\sigma_j^z} = \mathrm{CNOT}_{ij}\, R_i^z(\theta)\, \mathrm{CNOT}_{ij}$.
- $e^{-i\theta\sigma_i^x\sigma_j^x}$ and $e^{-i\theta\sigma_i^y\sigma_j^y}$ can be implemented similarly by rotating to the eigenstates of σ^z.
- $e^{-i\theta\sigma_i^z}$ is already an elementary gate and can be applied directly.

$V_1'(\delta t)$

Example of circuit structure for a six-site theory in each Trotter step:

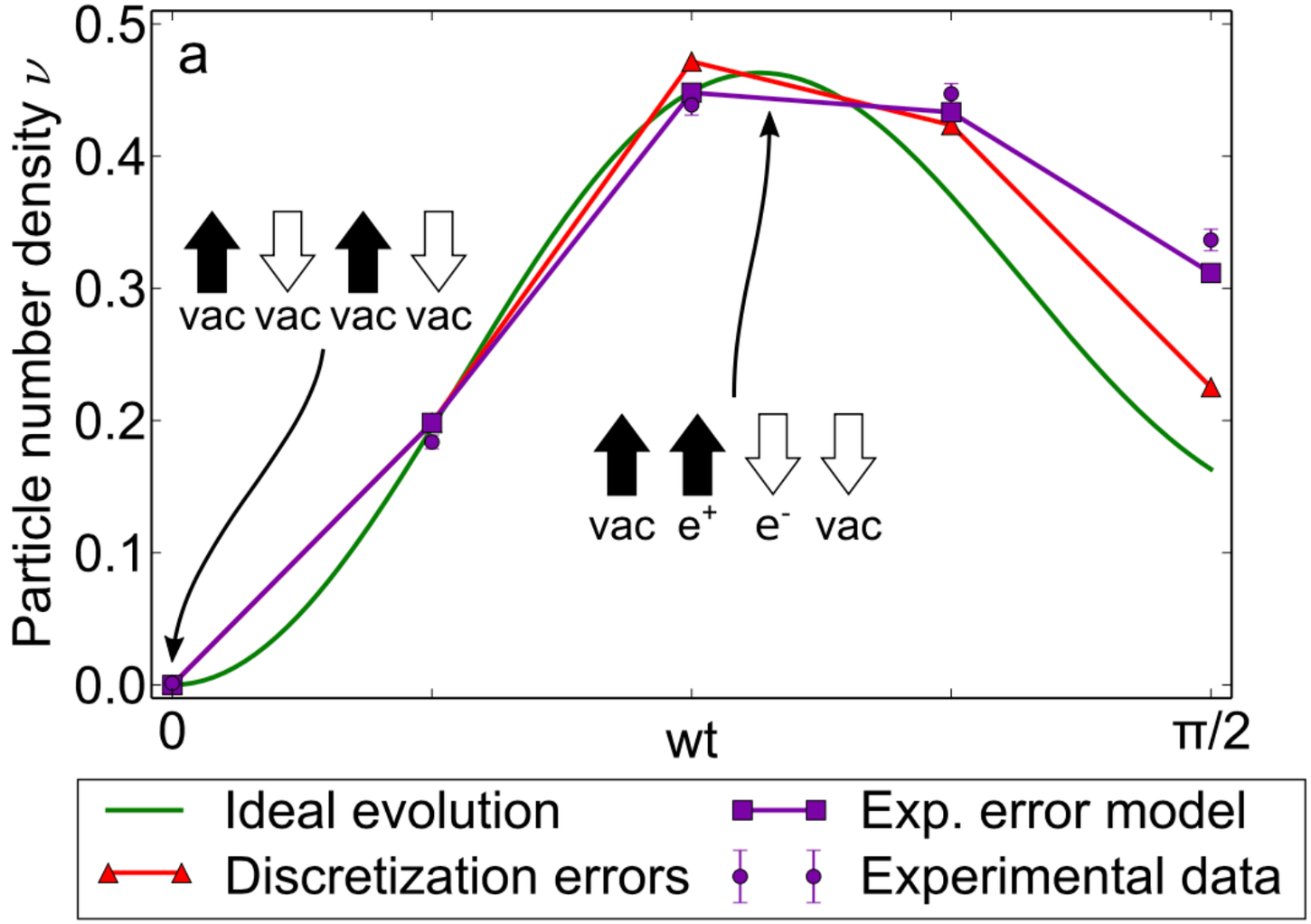


This Trotter block will be repeated $N_T = t/\delta t$ times.

RELA-DEVICE IMPLEMENTATIONS

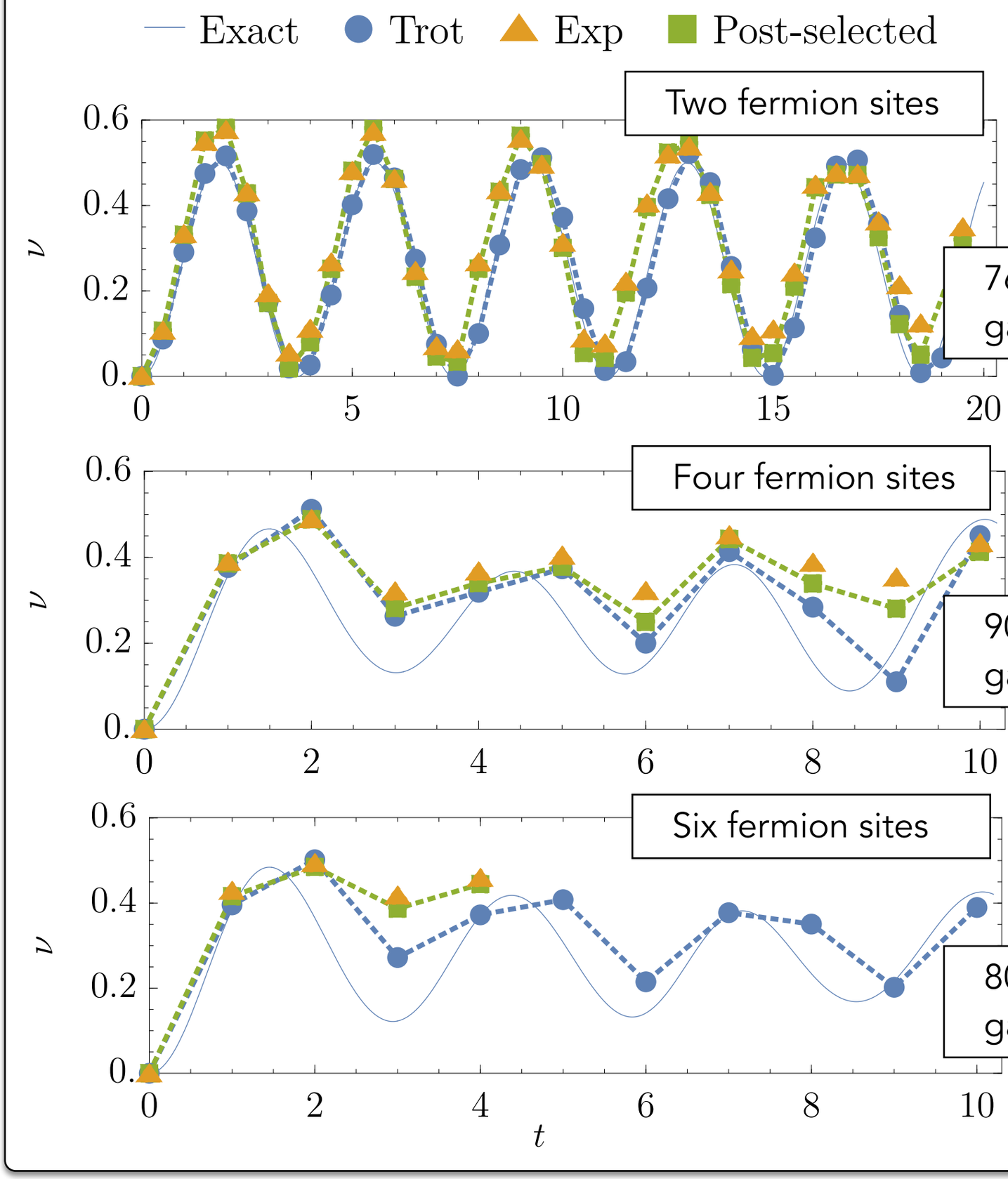
Martinez et al, Nature 534, 516 EP (2016).

universität innsbruck



RELA-DEVICE IMPLEMENTATIONS

Nguyen, Tran, Zhu, Green, Huerta Alderete, ZD, Linke, *PRX Quantum* 3 (2022) 2, 020324.

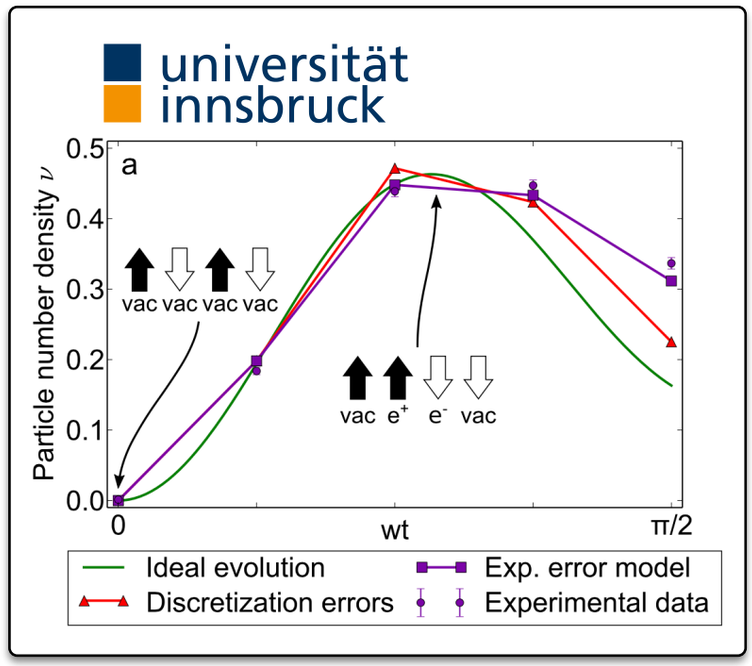


76 entangling gates!

90 entangling gates!

80 entangling gates!

Martinez et al, Nature 534, 516 EP (2016).



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NOW WHAT ABOUT **FERMIONIC-BOSONIC** REPRESENTATION WITH THE **SECOND-ORDER** PRODUCT FORMULA?

$$H = \mathrm{x}\sum_n \left[\sigma_n^+ U_n \sigma_{n-1}^- + \mathrm{h.c.}\right] + \frac{\mu}{2}\sum_n (-1)^{n+1}\sigma_n^z + \sum_n E_n^2$$

One can split the terms in the Hamiltonian as:

Shaw, Lougovski, Stryker, Wiebe, Quantum 4, 306 (2020).

$$H = \sum_x \left(T_x + D_x\right), \quad \text{with} \quad D_x := D_x^{(M)} + D_x^{(E)}.$$

$$T_x := \mathrm{x}\left(\frac{1}{4}(U_x + U_x^\dagger)(X_x X_{x+1} + Y_x Y_{x+1}) + \frac{i}{4}(U_x - U_x^\dagger)(X_x Y_{x+1} - Y_x X_{x+1})\right)$$

$$D_x^{(M)} := \frac{\mu}{2}(-1)^x Z_x \quad \text{and} \quad D_x^{(E)} := E_x^2$$

and do the following ordering of the terms

$$V_2(t) = \prod_x \left(\prod_{k\in\{M,E\}} e^{-iD_x^{(k)}t/2} \prod_{j=1}^{4} e^{-iT_x^{(j)}t/2}\right) e^{-iD_N^{(M)}t} \prod_x \left(\prod_{j=4}^{1} e^{-iT_x^{(j)}t/2} \prod_{k\in\{E,M\}} e^{-iD_x^{(k)}t/2}\right)$$

In reverse order

Example

This example concerns finding a quantum circuit for implementing

$$U^{(E)} = \prod_{i-1}^{N} U_i^{(E)} = e^{-it\sum_{i=1}^{N} E_i^2}$$

in the time-evolution of lattice Schwinger model in a near-term scenario that avoids introducing any ancilla qubits. Consider $E_i \in [-\Lambda, \Lambda]$ and encode the electric-field Hilbert space on each link i into $\eta \equiv \lceil \log_2(2\Lambda) + 1 \rceil$ qubits. Given this, find a circuit representation for $U_i^{(E)}$ in terms of only single-qubit rotations around the z axis of Bloch sphere as well as two-qubit CNOT gates. Verify your answer by explicitly working out a small example.

It is easy to show that the electric-field operator at each link acting on the computational (binary) basis is:

$$E = -\Lambda\, \mathbb{I} + \frac{1}{2}\left[(2^\eta - 1)\mathbb{I} - \sum_{j=0}^{\eta-1} 2^j\, \sigma_j^z\right]$$

Therefore,

$$E^2 = \Lambda^2\, \mathbb{I} - \Lambda\left[(2^\eta - 1)\mathbb{I} - \sum_{j=0}^{\eta-1} 2^j\, \sigma_j^z\right] + \frac{1}{4}\left[(2^\eta - 1)^2\mathbb{I} - 2(2^\eta - 1)\sum_{j=0}^{\eta-1} 2^j\, \sigma_j^z + \sum_{j,j'=0}^{\eta-1} 2^{j+j'}\, \sigma_j^z \sigma_{j'}^z\right]$$

Consequently, the operator $U^{(E)}$ can be written as a product of $N\eta$ R^z rotations and $N\eta(\eta-1)/2$ R^{zz} rotations with rotation angles that can be read off from the expression above. Note that each R^{zz} gate amounts to two CNOT gates and one R^z gate.

Example

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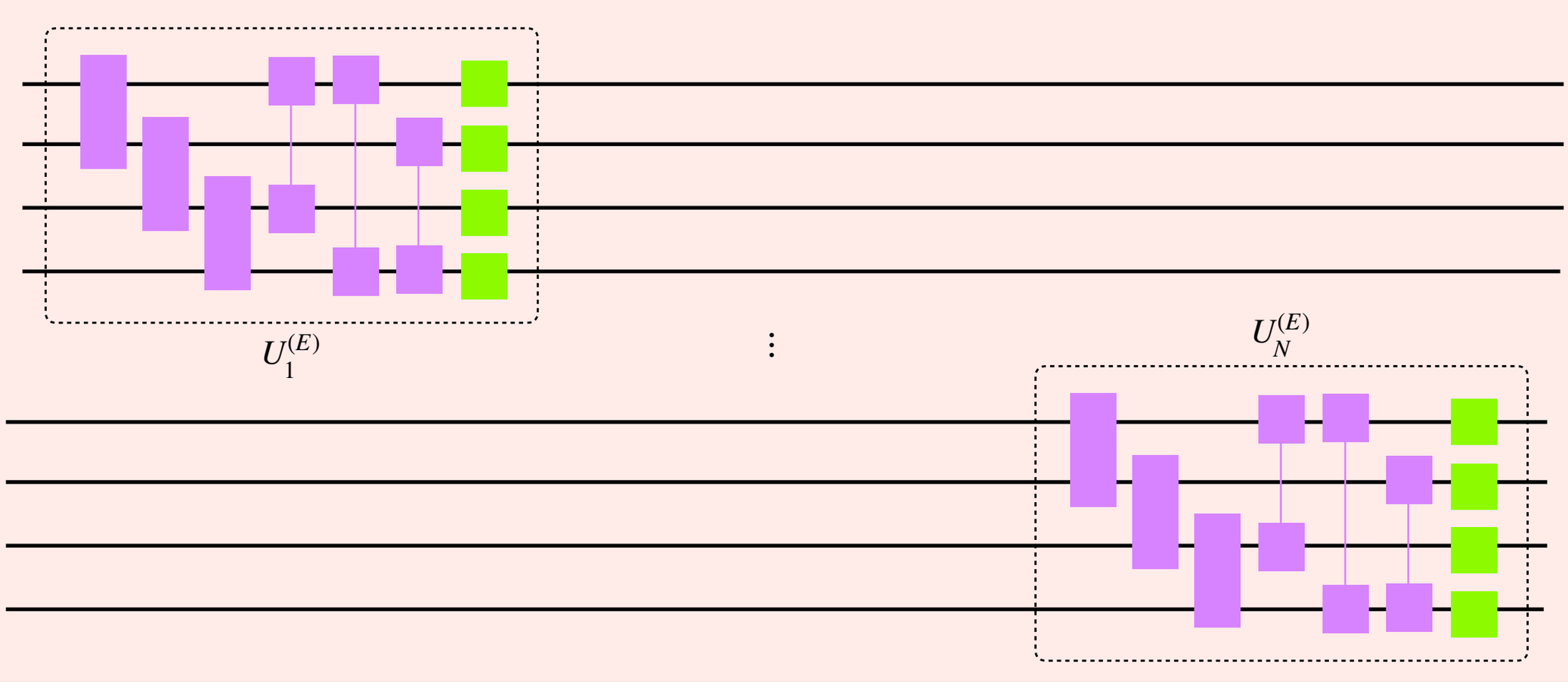
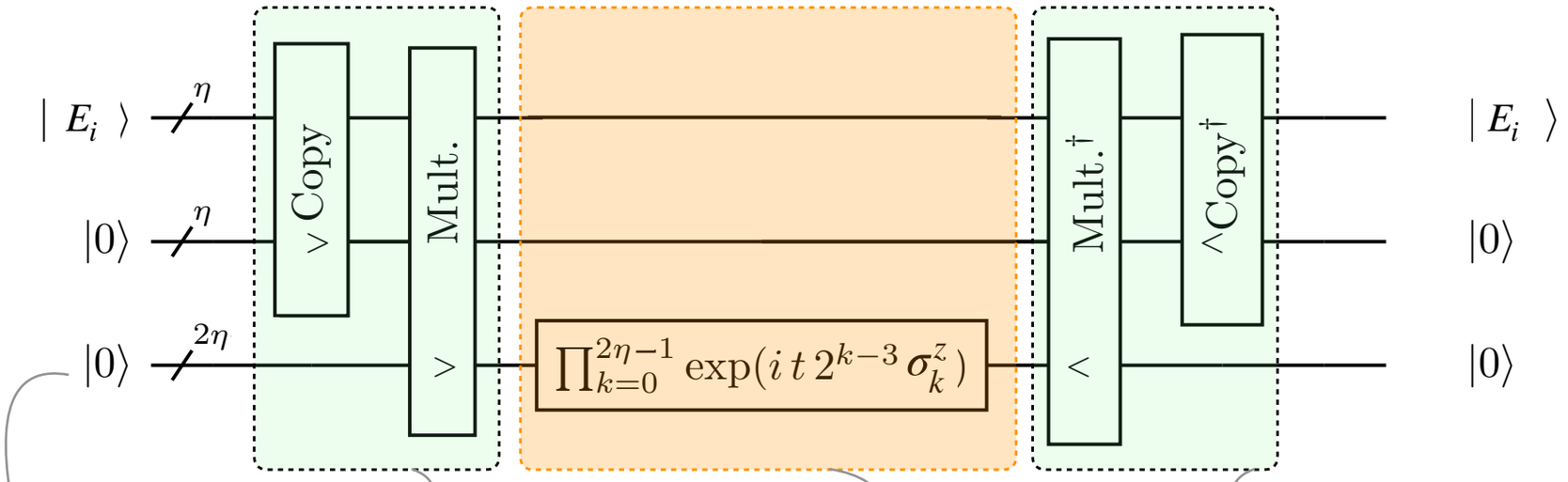
An $\eta = 4$ example:



The previous example requires $O(N\eta^2)$ number of R^z gates, which are costly operations in the fault-tolerant regime as they need to be synthesized up to accuracy ϵ using roughly $\log(1/\epsilon)$ T gates. Can one reduce the R^z cost of electric-field evolution to $O(N\eta)$? The answer is yes, but at the cost of extra $O(\eta)$ ancillas that are, nonetheless, available in the fault-tolerant era. One such circuit can be constructed using the so-called phase-kickback routine. For each $U_i^{(E)}$:



Register that temporarily holds the E_i^2 value at each link

Logic gates computing E_i^2.

Phase gets implemented here based on the E_i^2 value

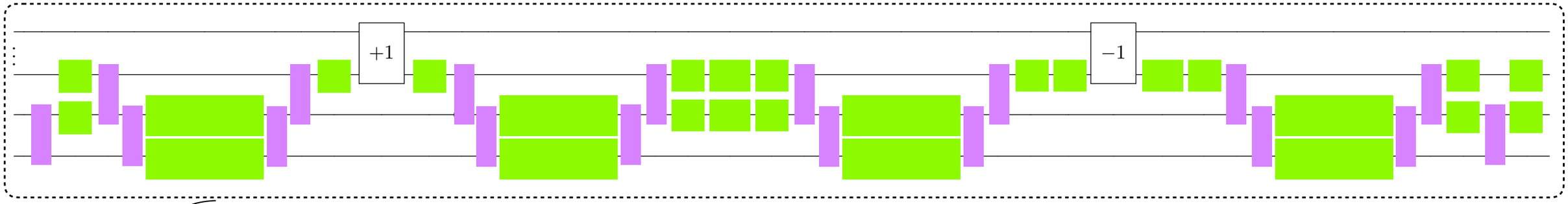
Logic gates uncomputing E_i^2.

The logical copy and multiplication routines are known circuits and overall cost $O(\eta^2)$ T gates. The ancilla qubits are reset in the end and can be used in the remainder of the circuit.

How do you implement arbitrary diagonal operator $e^{-it\mathscr{D}}$ in the computational basis? [Think about two examples: i) $\mathscr{D}\,|n\rangle = n\,|n\rangle$ and ii) $\mathscr{D}\,|n\rangle = \sqrt{\frac{n+1}{n-1}}\,|n\rangle$!]

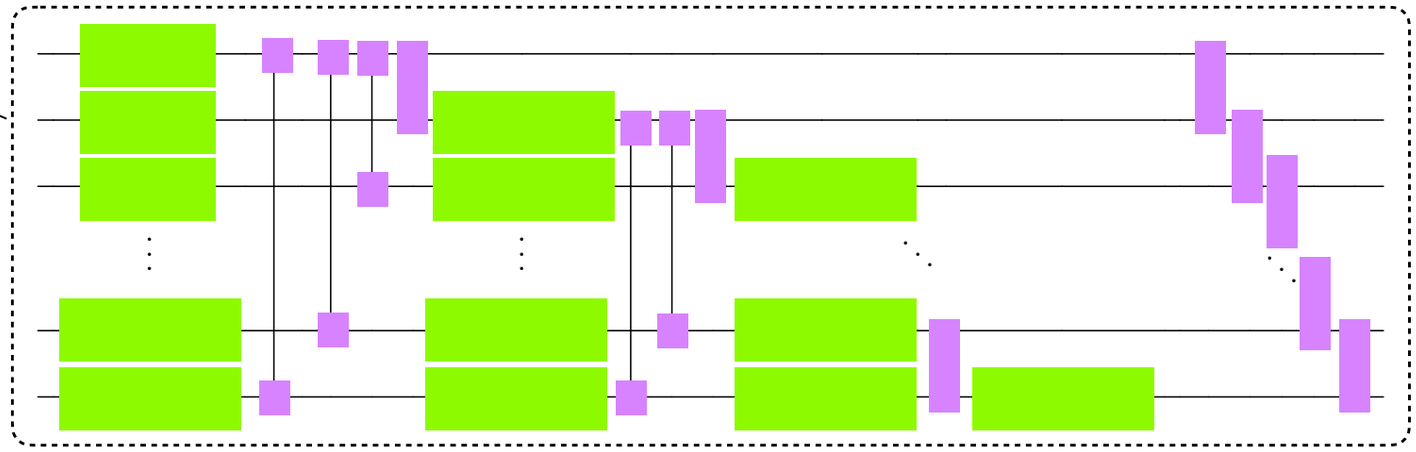
Circuit and recourse analysis

Shaw, Lougovski, Stryker, Wiebe, Quantum 4, 306 (2020).



Sample gauge-fermion interaction block

Part of electric field interactions acting on gauge DOF registers



Near term cost

	$\delta_g = 10^{-3}$		$\delta_g = 10^{-4}$		$\delta_g = 10^{-5}$		$\delta_g = 10^{-6}$		$\delta_g = 10^{-7}$	
	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT
$x = 10^{-2}$	—	7.3e4	—	1.6e5	—	3.4e5	—	7.3e5	5.6e-2	1.6e6
$x = 10^{-1}$	—	1.6e4	—	3.5e4	—	7.5e4	5.9e-2	1.6e5	2.7e-3	3.5e5
$x = 1$	—	4.6e3	—	9.9e3	1.0e-1	2.1e4	4.7e-3	4.6e4	2.2e-4	9.9e4
$x = 10^2$	—	2.8e3	8.3e-1	6.1e3	3.8e-2	1.3e4	1.8e-3	2.8e4	8.2e-5	6.0e4

COMPARISON BETWEEN THE TWO FORMULATIONS FOR THE SECOND-ORDER FORMULA

	Purely fermionic (non-local)	Fermionic-bosonic (local)
Qubit cost	N	$N + N\log_2(N)$
Gate complexity*	$\mathcal{O}(N^{9/2}t^{3/2})$	$\mathcal{O}(N^{5/2}t^{3/2})$

*Defined at the required number of Trotter steps for simulation time t, system size $N \sim \Lambda$, and at fixed x and μ, given a fixed error tolerance.

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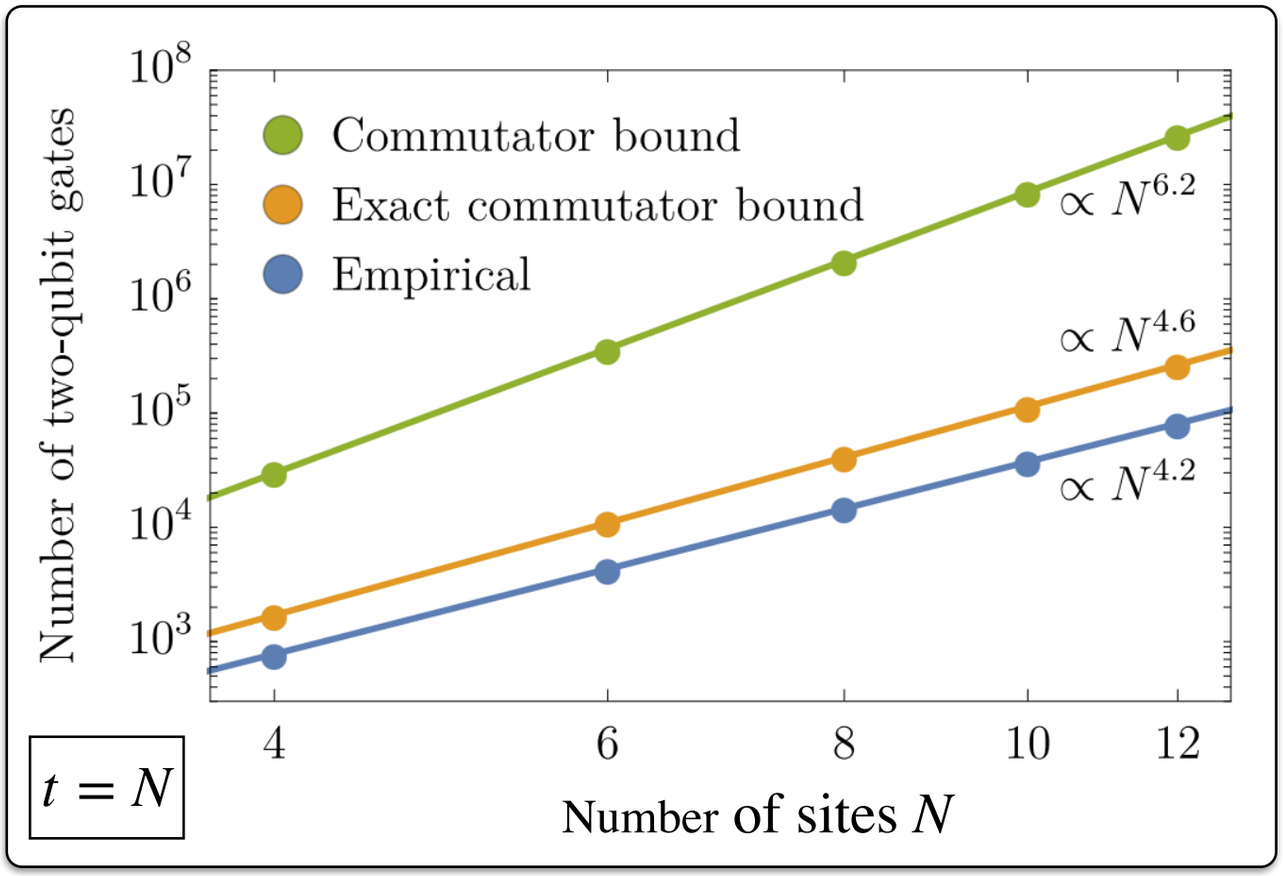
Nonetheless, empirically it seems like the non-local formulation performs as well as the bound on the local formulation!



Nguyen, Tran, Zhu, Green, Huerta Alderete, ZD, Linke, *PRX Quantum* 3 (2022) 2, 020324.

Explain the qubit and gate scalings of the second-order Trotter simulation of the lattice Schwinger model in both formulations, as given in the previous slide.

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Abelian vs. non-Abelian

Since we do not have the option of removing the gauge links generally, let us focus on the fermionic-bosonic formulations in the electric-field basis. So what are the major differences between simulating digitally Abelian and non-Abelian LGTs? Let is compare U(1) and SU(N) LGTs.

i) There are more degrees of freedom involved for SU(N) LGTs. For example, at each site, there are *N*-component fermions, and at each link there are multiple bosonic variables.

ii) As a result, there are more terms that need to be simulated, hence more complexity and generally more Trotter error.

iii) The diagnozalization procedure for hopping and magnetic terms generally follow the same rules but is more gate-intensive for SU(N).

iv) The diagonal operators in an Abelian theory like U(1) are trivial while for SU(N), they require evaluating phases that are non-trivial functions of bosonic occupation-number operators. These require expensive function-evaluation routines (in the E basis).

Algorithmic progress for U(1), SU(2), and SU(3) theories can be found in, e.g.:

Shaw, Lougovski, Stryker, Wiebe, Quantum 4, 306 (2020).
Ciavarella, Klco, and Savage, Phys. Rev. D 103, 094501 (2021).
Kan and Nam, arXiv:2107.12769 [quant-ph].
ZD, Shaw, and Stryker, Quantum 7, 1213 (2023),
Rhodes, Kreshchuk, Pathak, arXiv:2405.10416 [quant-ph]

QUESTIONS?