Spectroscopy and scattering: applications
Exercises

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Frontiers and Challenges in Lattice Gauge Theory
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1. **Stochastic wall source for two-point correlation function**
   Here we consider $C(t) = \langle \mathcal{O}(t)\mathcal{O}^\dagger(0)\rangle$ for the simple zero-momentum pion interpolator

   $$\mathcal{O}(t) = \int d^3\vec{x}\, \bar{u}(\vec{x},t)\gamma_5 d(\vec{x},t)$$

   with degenerate u and d quarks. Previously we looked at the point-source estimator; here we consider an alternative.

   (a) We introduce N_{src} independent auxiliary colour-spinor noise fields $\eta_i(x)$, $1 \le i \le N_{\text{src}}$ with support on $x_4 = 0$ and the key property

   $$\langle \eta_i(\vec{x},0)\eta_j^\dagger(\vec{y},0)\rangle = \delta_{ij}\delta_{\vec{x},\vec{y}}\mathbf{1}_{\text{colour,spin}}.$$

   The simplest way to achieve this is to choose every component $\eta_{ia\alpha}(\vec{x},0)$ randomly from $\{+1,-1\}$, i.e. "$\mathbb{Z}_2$ noise". Show how the solutions $\phi_i \equiv D^{-1}\eta_i$ can be used to define N_{src} estimators for $C(t)$.

   (b) Work out an expression for the variance of the average of these estimators. This is simplest for Gaussian noise, when the noise fields have action $S_{\text{noise}}[\eta] = \sum_{i=1}^{N_{\text{src}}} \eta_i^\dagger \eta_i$, in which case

   $$\begin{aligned}&\langle \eta_{ia\alpha}(\vec{x}_1,0)\eta^\dagger_{jb\beta}(\vec{x}_2,0)\eta_{kc\gamma}(\vec{x}_3,0)\eta^\dagger_{ld\delta}(\vec{x}_4,0)\rangle \\ &\qquad = \delta_{ij}\delta_{\vec{x}_1,\vec{x}_2}\delta_{ab}\delta_{\alpha\beta} \times \delta_{kl}\delta_{\vec{x}_3,\vec{x}_4}\delta_{cd}\delta_{\gamma\delta} + \delta_{il}\delta_{\vec{x}_1,\vec{x}_4}\delta_{ad}\delta_{\alpha\delta} \times \delta_{kj}\delta_{\vec{x}_3,\vec{x}_2}\delta_{cb}\delta_{\gamma\beta}.\end{aligned}$$

   Confirm that for $N_{\text{src}} \to \infty$, the result is the same as "brute force" solving on point sources for every $\vec{x}$. How does the signal-to-noise ratio scale for large L, compared with the single-point-source estimator?

2. **Neglecting the imaginary part**
   Let

   $$\mathcal{O}_\Gamma(t,\vec{p}) \equiv \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\bar{u}(\vec{x},t)\Gamma d(\vec{x},t)$$

   and consider the two-point correlation function

   $$C_{\Gamma'\Gamma}(t,\vec{p}) \equiv \langle \mathcal{O}_{\Gamma'}(t,\vec{p})\mathcal{O}_\Gamma^\dagger(0,\vec{p}\rangle.$$

   Use discrete lattice symmetries to relate its complex conjugate to itself. How can we ensure the correlation function is real, so that its imaginary part (a stochastic zero) can be neglected?

3. **GEVP as variational method**
Consider the two-point correlation function $C(t) \equiv \langle \bar{\mathcal{O}}(t)\bar{\mathcal{O}}^\dagger(0)\rangle$, where $\bar{\mathcal{O}}$ is a linear combination of N operators: $\bar{\mathcal{O}} = \sum_{i=1}^N v_i^\dagger \mathcal{O}_i$. Defining the effective mass

$$m_{\text{eff}}(t) \equiv \frac{1}{\delta t} \log \frac{C(t)}{C(t+\delta t)},$$

show that finding the minimum $m_{\text{eff}}(t)$ with respect to the coefficients v_i is equivalent to a generalized eigenvalue problem.

4. **Higher moving frames**
   (a) How many elements does the little group for $\vec{P} \propto (0,1,1)$ have? Find a set of transformations that generate this group.
   (b) Do the same for $(1,1,1)$.

5. **Rest-frame irreps**
The octahedral group O_h can be generated by the following three transformations:
   - $(x,y,z) \to (y,z,x)$,
   - $(x,y,z) \to (y,x,z)$,
   - $(x,y,z) \to (-x,y,z)$.

   The three components of the zero-momentum vector current, belonging to the T_1^- irrep, transform in this way.
   (a) Verify that the set of $I=1$ operators

$$\mathcal{O}_{\pi\pi,i}^{T_2^-(2,2)} = \sum_{j=1}^{2} \sum_{s=\pm 1} (-1)^j \mathcal{O}_{\pi\pi}^{I=1}\left(\tfrac{2\pi}{L}(\vec{e}_i + s\vec{e}_{i+j}), -\tfrac{2\pi}{L}(\vec{e}_i + s\vec{e}_{i+j})\right), \quad i = 1,2,3,$$

   where $i+j$ is taken modulo 3, is closed under these transformations and that these operators transform differently than the T_1^- irrep.
   (b) Find the $I=1$ $\pi\pi$ operator with $(n_1^2, n_2^2) = (3,3)$ that belongs to the A_2^- irrep: it should transform into itself, up to a phase. Show that this phase is not simply $\det R$, which would correspond to the A_1^- irrep.

6. **Moving frame (0,0,1)**
The little group C_{4v} of $\vec{P} = \frac{2\pi}{L}(0,0,1)$ can be generated by the two transformations
   - $(x,y,z) \to (y,x,z)$,
   - $(x,y,z) \to (-x,y,z)$.

   (a) Verify that the $I=1$ $\pi\pi$ operators

$$\mathcal{O}_{\pi\pi,i}^{E(1,2)} = \mathcal{O}_{\pi\pi}^{I=1}\left(\tfrac{2\pi}{L}\vec{e}_i, \vec{P} - \tfrac{2\pi}{L}\vec{e}_i\right) - \mathcal{O}_{\pi\pi}^{I=1}\left(-\tfrac{2\pi}{L}\vec{e}_i, \vec{P} + \tfrac{2\pi}{L}\vec{e}_i\right), \quad i = 1,2,$$

   transform the same as the corresponding components of the vector current with momentum $\vec{P}$.
   (b) Confirm that the operator

$$\mathcal{O}_{\pi\pi}^{B_1(1,2)} = \sum_{j=1}^{2} (-1)^j \sum_{s=\pm 1} \mathcal{O}_{\pi\pi}^{I=1}\left(\tfrac{2\pi}{L}s\vec{e}_j, \vec{P} - \tfrac{2\pi}{L}s\vec{e}_j\right)$$

   belongs to a one-dimensional irrep different from the trivial irrep A_1.