Bilocal / meson-meson interpolators

Consider a state with two hadrons in the noninteracting limit

$$E = E_1 + E_2$$

where $E_i = \sqrt{m_i^2 + |\vec{p}_i|^2}$, neglecting $O(a^2)$ discretization effects

and $O(e^{-m_\pi L})$ single-hadron finite-volume effects

in periodic box, $\vec{p}_i = \frac{2\pi}{L}\vec{n}_i$, $\vec{n}_i \in \mathbb{Z}^3$

* For $E_{cm} > m_1 + m_2$ there is an infinite tower of noninteracting states.
  two-particle threshold

* For any interval $[E_1, E_2]$ above threshold, as $L \to \infty$ the number of noninteracting states also grows infinitely large.

The number of interacting states (i.e. in QCD) will behave similarly.

→ we need a similar tower of interpolating operators

Simple strategy: use products of single-hadron interpolators $O_1(\vec{p}_1)\,O_2(\vec{p}_2)$

→ take suitable linear combinations to obtain desired quantum numbers

Each such operator corresponds to a noninteracting level.

example: $I=1$, $I_3=0$, $J^{PC} = 1^{--}$

lowest 2-hadron threshold: $\pi\pi$ (PDG: $\rho \to \pi\pi$ ~ 100% of decays)

start with isospin triplet of pion interpolators $(O_{\pi^+}(\vec{p}),\ O_{\pi^0}(\vec{p}),\ O_{\pi^-}(\vec{p}))$ $J^P = 0^-$

s.t. $O_\pi(\vec{p}) \to \det(R)\, O_\pi(R^{-1}\vec{p})$

e.g. $O_{\pi^-}(\vec{p}) = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \bar{u}(\vec{x})\gamma_5 d(\vec{x})$

flavour: apply Clebsch-Gordan coefficients for $1 \otimes 1 \to 1$

$$\Rightarrow \quad O^{I=1}_{\pi\pi}(\vec{p}_1,\vec{p}_2) \equiv \frac{1}{\sqrt{2}}\Big(O_{\pi^+}(\vec{p}_1)\, O_{\pi^-}(\vec{p}_2) - O_{\pi^-}(\vec{p}_1)\, O_{\pi^+}(\vec{p}_2) \Big)$$

Since pions are bosons and $I=1$ is antisymmetric in flavour $\Rightarrow O^{I=1}_{\pi\pi}(\vec{p}_1,\vec{p}_2) = -O^{I=1}_{\pi\pi}(\vec{p}_2,\vec{p}_1)$

for same reason, $\pi\pi$ $I=1$ only occurs in odd partial waves $\ell = 1, 3, 5, \ldots$

$$\Rightarrow J^P = 1^-, 3^-, 5^-, \ldots$$

$O_{\pi\pi}(\vec{p}_1,\vec{p}_2) \longrightarrow O_{\pi\pi}(R^{-1}\vec{p}_1, R^{-1}\vec{p}_2)$

Need to use finite-volume momenta $\vec{p}_i = \frac{2\pi}{L}\vec{n}_i$

+ find linear combinations belonging to specific irreps

note: (n_1^2, n_2^2) is invariant under rotations $\Rightarrow$ consider each (n_1^2, n_2^2) separately

rest frame: $\vec{P} \equiv \vec{p}_1 + \vec{p}_2 = (0,0,0)$

simple construction of some operators in T_1^-:

$$O_{\pi\pi,i}^{T_1^-(n,n)} \propto \sum_{\substack{\vec{p} \\ p^2 = n\left(\frac{2\pi}{L}\right)^2}} p_i \, O_{\pi\pi}(\vec{p}, -\vec{p})$$

can verify $O_{\pi\pi,i} \longrightarrow R_{ij}\, O_{\pi\pi,j}$ same transformation as vector current

$(n_1^2, n_2^2) = (0,0)$: no antisymmetric $(\vec{p}, -\vec{p})$

$(1,1)$: $\vec{p} = \pm\frac{2\pi}{L}\vec{e}_i$ 6 momenta $\longrightarrow$ 3 antisymmetric combinations

$\longrightarrow$ 3 elements of T_1^- $\quad O_{\pi\pi}\left(\frac{2\pi}{L}\vec{e}_i, -\frac{2\pi}{L}\vec{e}_i\right)$

$(2,2)$: $\vec{p} = \frac{2\pi}{L}(s_1\vec{e}_i + s_2\vec{e}_j)$, $s_i \in \{\pm 1\}$, $i<j$

12 momenta $\longrightarrow$ 6 antisymmetric combinations

$\longrightarrow$ 3 elements of T_1^- $\quad \sum_{j\neq i}\sum_{s=\pm1} O_{\pi\pi}\left(\frac{2\pi}{L}(\vec{e}_i + s\vec{e}_j), -\frac{2\pi}{L}(\vec{e}_i + s\vec{e}_j)\right)$

$\longrightarrow$ 3 elements of T_2^- $\quad \sum_{j=1}^{2}\sum_{s=\pm1} (-1)^j O_{\pi\pi}\left(\frac{2\pi}{L}(\vec{e}_i + s\vec{e}_{i+j}), -\frac{2\pi}{L}(\vec{e}_i + s\vec{e}_{i+j})\right)$

$\rightarrow$ sensitive to $\ell = 3$

$(3,3)$: $\vec{p} = \frac{2\pi}{L}(s_1, s_2, s_3)$ $\quad s_i \in \{\pm 1\}$

8 momenta $\longrightarrow$ 4 antisymmetric combinations

$\longrightarrow$ 3 elements of T_1^-

$\longrightarrow$ 1 element of A_2^-

moving frame $\vec{P} = \frac{2\pi}{L}(0,0,1)$

$(n_1^2, n_2^2) = (0,1)$ antisymmetric with $(1,0)$:

just one operator $O_{\pi\pi}(\vec{0}, \vec{P}) \rightarrow A_1$

$(n_1^2, n_2^2) = (1,2)$ antisymmetric with $(2,1)$: $\vec{p} = \pm \frac{2\pi}{L} \vec{e}_j$ $\quad j = 1,2 \quad$ 4 momenta

$\rightarrow A_1$: sum $\quad \sum_{j=1}^{2} \sum_{s=\pm 1} O_{\pi\pi}\left(\frac{2\pi}{L} s \vec{e}_j, \vec{P} - \frac{2\pi}{L} s \vec{e}_j\right)$

$\rightarrow E$: $i=1,2 \quad O_{\pi\pi}\left(\frac{2\pi}{L}\vec{e}_i, \vec{P} - \frac{2\pi}{L}\vec{e}_i\right) - O_{\pi\pi}\left(-\frac{2\pi}{L}\vec{e}_i, \vec{P} + \frac{2\pi}{L}\vec{e}_i\right)$

simple to verify transforms like vector current $\quad i = 1,2$

$\rightarrow B_1$: remaining combination $\quad \sum_{j=1}^{2} (-1)^j \sum_{s=\pm 1} O_{\pi\pi}\left(\frac{2\pi}{L} s \vec{e}_j, \vec{P} - \frac{2\pi}{L} s \vec{e}_j\right)$

Computing multi-hadron correlation functions

for $C_{ij}(t) = \langle O_i(t)\, O_j^\dagger(0)\rangle$ need all combinations of source+sink interpolator

consider simpler case $O_{\pi\pi}^{I=2}(\vec{p}_1,\vec{p}_2) = O_{\pi^+}(\vec{p}_1)\, O_{\pi^+}(\vec{p}_2)$

$$\langle O_{\pi\pi}^{I=2}(\vec{p}_1{}',\vec{p}_2{}',t)\, O_{\pi\pi}^{I=2}(\vec{p}_1,\vec{p}_2,0)^\dagger\rangle$$

$$= \int d^3\vec{x}_1\, d^3\vec{x}_2\, d^3\vec{x}_1{}'\, d^3\vec{x}_2{}'\, e^{-i\vec{p}_1{}'\cdot\vec{x}_1{}'}\, e^{-i\vec{p}_2{}'\cdot\vec{x}_2{}'}\, e^{i\vec{p}_1\cdot\vec{x}_1}\, e^{i\vec{p}_2\cdot\vec{x}_2} \langle \bar{d}\gamma_5 u(\vec{x}_1{}',t)\, \bar{d}\gamma_5 u(\vec{x}_2{}',t)\, \bar{u}\gamma_5 d(\vec{x}_1,0)\, \bar{u}\gamma_5 d(\vec{x}_2,0)\rangle$$

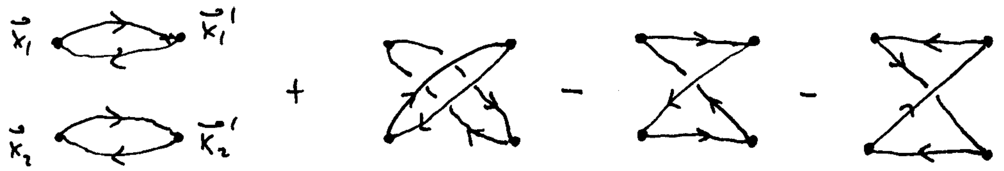
4 Wick contractions:

$$\langle\cdots\rangle = |D^{-1}(\vec{x}_1{}',t;\vec{x}_1,0)|^2\, |D^{-1}(\vec{x}_2{}',t;\vec{x}_2,0)|^2$$

$$+ |D^{-1}(\vec{x}_2{}',t;\vec{x}_1,0)|^2\, |D^{-1}(\vec{x}_1{}',t;\vec{x}_2,0)|^2$$

$$- \mathrm{tr}\left[D^{-1}(\vec{x}_1{}',t;\vec{x}_1,0)\, D^{-1}(\vec{x}_2{}',t;\vec{x}_1,0)^\dagger D^{-1}(\vec{x}_2{}',t;\vec{x}_2,0)\, D^{-1}(\vec{x}_1{}',t;\vec{x}_2,0)^\dagger\right]$$

$$- \mathrm{tr}\left[D^{-1}(\vec{x}_2{}',t;\vec{x}_1,0)\, D^{-1}(\vec{x}_1{}',t;\vec{x}_1,0)^\dagger\, D^{-1}(\vec{x}_1{}',t;\vec{x}_2,0)\, D^{-1}(\vec{x}_2{}',t;\vec{x}_2,0)^\dagger\right]$$



How to evaluate?

Can use translation symmetry once: e.g. set $\vec{x}_1 = 0$

still need more than point sources

Option 1: point, sequential, and stochastic sources

a) point sources for quark lines ending at $(\vec{x}_1, 0)$

b) sequential sources for quark lines connected indirectly to $(\vec{x}_1,0)$

c) stochastic sources for quark lines disconnected from $(\vec{x}_1,0)$

Disadvantages: * difficult to reuse propagators

* more interpolators requires more propagator solves

Option 2: distillation (2009)

method for timeslice-to-all or all-to-all smeared quark propagators key ingredient in distillation:

<u>Laplacian-Heaviside smearing</u> (LapH)

for each t, compute lowest eigenmodes of negative 3d Laplacian $-\Delta(t)$

$$-\Delta(t)\, v_i(t) = \lambda_i(t)\, v_i(t) \qquad \lambda_1(t) < \lambda_2(t) < \lambda_3(t) < \cdots$$

$$\psi_{sm}(\vec{x},t) = \sum_{i=1}^{N_{LapH}} v_i(\vec{x},t) \int d^3\vec{y}\; v_i^\dagger(\vec{y},t)\, \psi(\vec{y},t) \qquad \text{smearing width} \sim \frac{L}{N_{LapH}^{1/3}}$$

typically $N_{LapH} \ll N_c \left(\frac{L}{a}\right)^3$ i.e. smearing projects onto much lower-dimensional subspace

<u>Distillation</u>: use timeslice-to-all or all-to-all propagator in the LapH subspace

perambulator $\tau_{ij}(t',t) \equiv \int d^3\vec{x}'\, d^3\vec{x}\; v_i^\dagger(\vec{x}',t')\, D^{-1}(\vec{x}',t';\vec{x},t)\, v_j(\vec{x},t)$

$\rightarrow$ needs $4N_{LapH}$ solves for each source time t

for mesons, also need $\phi_{ij}(\vec{p},t) \equiv \int d^3\vec{x}\; e^{-i\vec{p}\cdot\vec{x}}\, v_i^\dagger(\vec{x},t)\, v_j(\vec{x},t)$

e.g. $\mathcal{O}(t) = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \bar{u}_{sm}(\vec{x},t)\, \Gamma\, d_{sm}(\vec{x},t)$

$$= \int d^3\vec{y}_1\, d^3\vec{y}_2 \sum_{i_1,i_2} \bar{u}(\vec{y}_1,t)\, v_i(\vec{y}_1,t)\, \phi_{ij}(\vec{p},t)\, v_j^\dagger(\vec{y}_2,t)\, \Gamma\, d(\vec{y}_2,t)$$

$$\Rightarrow \langle \mathcal{O}(t)\mathcal{O}^\dagger(0)\rangle = \langle \mathrm{tr}\left[\phi(\vec{p},t)\,\Gamma\,\tau(t,0)\,\phi^\dagger(\vec{p},0)\,\bar{\Gamma}\,\tau(0,t)\right]\rangle$$

trace over Dirac + Laplace-mode indices

After computing τ, ϕ (large upfront cost) can evaluate any multi-meson correlation function at cost $\propto N_{LapH}^3$.

Challenge: cost scaling with volume (at fixed smearing width).

Baryons are harder: N_{LapH}^4 for one or two baryons, worse for three baryons.