Quantum numbers for hadronic states

Wilson-type fermions preserve flavour symmetries + charge conjugation

$\rightarrow$ here focus on space-time symmetries

In Minkowski space: boost to rest frame $\rightarrow$ get $p^\mu = (p^0, \vec{0})$

<u>Little group</u> with respect to p: Lorentz transformations satisfying $\Lambda p = p$

in rest frame: spatial rotations $O(3) = SO(3) \times \mathbb{Z}_2$ $\quad x^\mu = (x^0, \vec{x}) \rightarrow (x^0, R\vec{x})$

angular momentum $\hat{J}$ $\quad$ parity

fermions: use double cover $SU(2)$ of $SO(3)$

irreducible representations J^P of dimension $2J+1$

Both finite volume + nonzero lattice spacing break Lorentz or $O(4)$ symmetry

* Can't boost to rest frame
* Little group depends on $\vec{p}$. For $\vec{p} \neq 0$ it can be very small.

<u>Rest frame</u> ($\vec{p} = 0$). Little group is double cover O_h^D of octahedral group O_h

$$O_h = O \times \mathbb{Z}_2$$

rotational symmetries of cube or octahedron $\quad$ parity

$$(x_1, x_2, x_3) \rightarrow (s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, s_3 x_{\sigma(3)})$$

$\sigma \in S_3$ permutation

$s_i \in \{+1, -1\}$ $\qquad |O_h| = 48$

Each J^P subduces to one or more irreducible representations of O_h^D

infinitely many $\qquad$ finite

$\Rightarrow$ Each irrep of O_h^D contains infinite tower of J^P

1d:
- $A_1^\pm$: $0^\pm, 4^\pm, \ldots$
- $A_2^\pm$: $3^\pm, \ldots$

2d $\rightarrow$ $E^\pm$: $2^\pm, 4^\pm, \ldots$

3d:
- $T_1^\pm$: $1^\pm, 3^\pm, 4^\pm, \ldots$
- $T_2^\pm$: $2^\pm, 3^\pm, 4^\pm, \ldots$

+ half-integer-spin irreps

Moving frames: as first step, allow Lorentz transformations without boosts

we can take $\vec{p} = (0, 0, p_z)$

little group: double cover of $C_{\infty v} = O(2)$

symmetry of CO molecule or an egg/cone

→ rotation by any angle θ about $\hat{z}$ axis $R(\theta)$

→ reflection through any plane containing $\hat{z}$ axis σ_v

$$R(\theta) |J^P, J_z\rangle = e^{iJ_z\theta} |J^P, J_z\rangle$$

$$\sigma_v |J^P, J_z\rangle \propto |J^P, -J_z\rangle \quad \text{and} \quad \sigma_v |J^P, 0\rangle = (-1)^J P |J^P, 0\rangle$$

Each $|J_z| \neq 0$ is a 2d irrep.

Two 1d irreps for $J_z = 0$.

e.g.
$0^+ : |0^+, 0\rangle, |1^-, 0\rangle, |2^+, 0\rangle, \ldots$

$0^- : |0^-, 0\rangle, |1^+, 0\rangle, |2^-, 0\rangle, \ldots$

$1 : |1^\pm, \pm 1\rangle, |2^\pm, \pm 1\rangle, |3^\pm, \pm 1\rangle, \ldots$

$2 : |2^\pm, \pm 2\rangle, |3^\pm, \pm 2\rangle, \ldots$

$\vec{p} \propto (0, 0, 1)$ in finite volume or nonzero lattice spacing

Little group is double cover Dic_4 of C_{4v} : $(x_1, x_2, x_3) \to (s_1 x_{\sigma(1)}, s_2 x_{\sigma(2)}, x_3)$

symmetry of square pyramid

$\sigma \in S_2$
$s_i \in \{+1, -1\}$

Each irrep of Dic_4 contains infinite tower of irreps of $C_{\infty v}$

1d
$A_1 : 0^+, 4, \ldots$
$A_2 : 0^-, 4, \ldots$ + half-integer-spin irreps
$B_1 : 2, \ldots$
$B_2 : 2, \ldots$

2d → $E : 1, 3, \ldots$

e.g.
$$\underset{O(3)}{J^P = 2^+} \longrightarrow \underset{C_{\infty v}}{0^+, 1, 2} \longrightarrow \underset{C_{4v}}{A_1, B_1, B_2, E}$$

$$3^+ \longrightarrow 0^-, 1, 2, 3 \longrightarrow A_2, B_1, B_2, E, E$$

Interpolating operators

Simplest operators for mesons: quark bilinears $\bar{\Psi}(x) \ldots \Psi(x)$

insert gamma matrices, covariant derivatives

These transform under tensor representations of $SO(3)$:

$$T_{i_1 i_2 \ldots}(x) \rightarrow R_{i_1 j_1} R_{i_2 j_2} \cdots T_{j_1 j_2 \ldots}(R^{-1}x) \qquad R \in SO(3)$$

all spatial indices from γ_i or D_i

note: irrep J = rank J symmetric traceless tensor

example: $I=1$, $I_3=0$, $J^{PC}=1^{--}$ quantum numbers of $\rho^0(770)$

Simplest operator: use vector current

$$O_i(\vec{p}) = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \bar{\Psi}(\vec{x})\gamma_i \tau_3 \Psi(\vec{x}) \quad \text{where } \Psi = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \tau_3 \Psi = \begin{pmatrix} u \\ -d \end{pmatrix}$$

$$O_i(\vec{p}) \rightarrow R_{ij} O_j(R^{-1}\vec{p})$$

same is true for

$$\bar{\Psi}\gamma_4\gamma_i\tau_3\Psi$$

$$\bar{\Psi}\tau_3 \overleftrightarrow{D}_i \Psi \qquad \text{where } \overleftrightarrow{D}_i = \tfrac{1}{2}(\overrightarrow{D}_i - \overleftarrow{D}_i)$$

ensures correct C-symmetry

$$\varepsilon_{ijk}\bar{\Psi}\gamma_5\gamma_j\tau_3\overleftrightarrow{D}_j\Psi$$

$$\ldots$$

rest frame: $O_i(\vec{0})$ $i=1,2,3$ belongs to 3d irrep T_1^-

trivial subduction $J^P = 1^- \rightarrow T_1^-$

moving frame $\vec{p} \propto (0,0,1)$: $O_3(\vec{p})$ is invariant under little group

$\rightarrow$ trivial irrep A_1

we know helicity ± 1 belongs to 2d irrep E

$\Rightarrow O_1(\vec{p}),\ O_2(\vec{p})$ belong to this irrep

Smearing

Typical hadronic size ~ 0.5 fm

Interpolating operators with $\psi, \bar\psi$ at same point couple poorly to this.

→ use smeared quark fields: average nearby points

Generically, if $\psi_{sm}(x) = \int d^4y\, K(x,y)\psi(y)$ for some smearing kernel K (may depend on gauge field)

then $\psi_{sm}(x)\bar\psi_{sm}(y) = (K D^{-1} K^{+})(x,y)$ "smeared-to-smeared propagator"

One option: use Coulomb-gauge fixing (independent for each x_4)

→ K can be arbitrary function of $\vec{x}, \vec{y}$

extreme case: Coulomb wall source $K(x,y) = \delta_{x_4, y_4}$ independent of $\vec{x}, \vec{y}$

Without gauge fixing, typically make use of 3d Laplacian Δ

e.g. on the lattice $\Delta = \sum_i \nabla_i^* \nabla_i$, where smeared links can be used in covariant derivatives

Key property: Δ commutes with lattice translations, rotations, reflections

⇒ interpolating operator built from $\psi_{sm} = f(\Delta)\psi$ has same symmetry properties as operator built from ψ.

e.g. Wuppertal / Gaussian smearing (1990)

Consider free fermion in continuum: $\psi_{sm}(\vec{x}) = \int d^3\vec{y}\, e^{-\frac{|\vec{x}-\vec{y}|^2}{\sigma^2}} \psi(\vec{y})$

$\Rightarrow \tilde\psi_{sm}(\vec{k}) \propto e^{-\frac{\sigma^2}{4}k^2}\tilde\psi(\vec{k}) = e^{\frac{\sigma^2}{4}\Delta}\tilde\psi(\vec{k})$ since $\Delta\tilde\psi(\vec{k}) = -k^2\tilde\psi(\vec{k})$

using $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ we define for interacting field on the lattice:

$$\psi_{sm} = \left(1 + \frac{\sigma^2}{4n}\Delta\right)^n \psi \quad \text{"gauge-covariant approximate Gaussian"}$$

σ^2 controls smearing width

n controls smoothness