Spectroscopy

in Euclidean space, a) choose a time direction: e.g. $\hat{t} = (0,0,0,1)$

b) identify fermion fields $\psi^+ = \bar{\psi}\hat{\not{t}} = \bar{\psi}\gamma_4$

c) use a time extent L_t with (anti)periodic boundary conditions

if we let $\mathcal{O}(t)$ and $\mathcal{O}'(t)$ be functions of the fields at time t

then $$C(t) \equiv \langle \mathcal{O}'(t)\,\mathcal{O}^+(0)\rangle = \frac{\mathrm{tr}\left[e^{-(L_t-t)\hat{H}}\hat{\mathcal{O}}' e^{-t\hat{H}}\hat{\mathcal{O}}^+\right]}{\mathrm{tr}\left[e^{-L_t\hat{H}}\right]} \quad \text{for } t>0$$

$$\xrightarrow{L_t\to\infty} \sum_{n=0}^{\infty} \langle\Omega|\hat{\mathcal{O}}'|n\rangle\langle\Omega|\hat{\mathcal{O}}|n\rangle^* e^{-E_n t}$$

vacuum energy eigenstates (discrete in finite volume)

restrict sum to states with $\langle\Omega|\hat{\mathcal{O}}'|n\rangle \neq 0 \neq \langle\Omega|\hat{\mathcal{O}}|n\rangle$

and order energies $E_0 < E_1 < E_2 < \cdots$

$\Rightarrow \quad -\frac{d}{dt}\log C(t) = E_0 + \mathcal{O}\left(e^{-(E_1-E_0)t}\right)$ obtain E_0 at large t

"effective energy" $E_{\mathrm{eff}}(t)$

if $\mathcal{O} = \mathcal{O}'$ then $C(t)$ is positive and convex

$\Rightarrow E_{\mathrm{eff}}(t)$ approaches E_0 monotonically from above

Example: single isovector meson at zero momentum

let $\mathcal{O}(t) = \int d^3\vec{x}\; \bar{u}(\vec{x},t)\,\Gamma\, d(\vec{x},t)$ e.g. $\Gamma = \gamma_5 \longrightarrow \pi \qquad J^P = 0^-$

$\Gamma = \gamma_i \longrightarrow \rho \qquad 1^-$

$\Rightarrow \mathcal{O}^+(t) = \int d^3\vec{x}\; \bar{d}(\vec{x},t)\,\bar{\Gamma}\, u(\vec{x},t)\,, \quad \bar{\Gamma} \equiv \gamma_4\Gamma^+\gamma_4$

$$C(t) = \langle \mathcal{O}(t)\,\mathcal{O}^+(0)\rangle$$
$$= \int d^3\vec{x}\, d^3\vec{y}\; \langle \bar{u}(\vec{x},t)\,\Gamma\, d(\vec{x},t)\,\bar{d}(\vec{y},0)\,\bar{\Gamma}\, u(\vec{y},0)\rangle \qquad \text{one Wick contraction}$$

$$= \int d^3\vec{x}\, d^3\vec{y}\; \langle -\mathrm{tr}\left[\Gamma\, D_d^{-1}(\vec{x},t;\vec{y},0)\,\bar{\Gamma}\, D_u^{-1}(\vec{y},0;\vec{x},t)\right]\rangle$$

Abuse of notation! $\langle\mathcal{O}(t)\mathcal{O}^+(0)\rangle$ refers to QCD path integral (over gauge + fermion fields) but now $\langle\cdots\rangle$ refers to Monte Carlo integral over gauge fields how to compute? simplest method: use translation symmetry + γ_5-hermiticity with degenerate u and d:

$$C(t) = -L^3 \int d^3\vec{x} \, \langle \mathrm{tr}\left[\Gamma D^{-1}(\vec{x},t;\vec{0},0) \bar{\Gamma} \gamma_5 D^{-1}(\vec{x},t;\vec{0},0)^{\dagger} \gamma_5 \right] \rangle$$

point-source quark propagator

Variance of C(t)

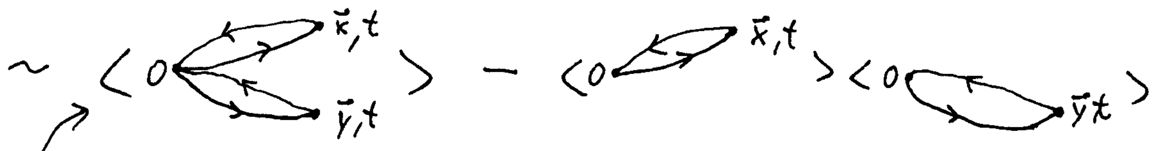
In general: for estimator X, $\mathrm{var}(X) = \langle X^2 \rangle - \langle X \rangle^2$

Monte Carlo expectation

Variance depends on how we compute $C(t)$.

point-source estimator yields

$$\text{variance} = L^6 \int d^3\vec{x}\, d^3\vec{y} \, \langle \mathrm{tr}\left[\Gamma D^{-1}(\vec{x},t;\vec{0},0) \bar{\Gamma} \gamma_5 D^{-1}(\vec{x},t;\vec{0},0) \gamma_5 \right] \times \mathrm{tr}\left[\Gamma D^{-1}(\vec{y},t;\vec{0},0) \bar{\Gamma} \gamma_5 D^{-1}(\vec{y},t;\vec{0},0) \gamma_5 \right] \rangle - C(t)^2$$



what is this expectation value?

$$\langle (\bar{u}\Gamma d)(\vec{x},t)\, (\bar{u}'\Gamma d')(\vec{y},t)\, (\bar{d}\bar{\Gamma} u\, \bar{d}'\bar{\Gamma} u')(\vec{0},0) \rangle$$

Need 4 different degenerate valence flavours u, d, u', d' to produce this diagram

→ expectation value in partially quenched theory

What states can propagate between 0 and t?

smallest energy: 2 pseudoscalar mesons ⇒ variance $\sim e^{-2m_\pi t}$

Signal-to-noise ratio $\dfrac{C(t)}{\sqrt{\text{variance}}} \sim e^{-(E_0 - m_\pi)t}$ good for $E_0 = m_\pi$

bad for heavier mesons

problem is severe for many quarks: A nucleons $\rightarrow \sim e^{-A(m_N - \frac{3}{2} m_\pi)t}$

Combining multiple operators

let $\mathcal{O}_i(t)$, $1 \le i \le N$, be linearly independent interpolating operators with chosen quantum numbers

define the correlator matrix $C_{ij}(t) \equiv \langle \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) \rangle$ $N \times N$

hermitian, positive definite

$$\xrightarrow{L_t \to \infty} \sum_{n=0}^{\infty} Z_{in} Z_{jn}^* e^{-E_n t}, \quad Z_{in} = \langle \Omega | \hat{\mathcal{O}}_i | n \rangle$$

what do we gain from this?

consider extreme scenario of ν degenerate states: $E_0 = E_1 = \cdots = E_{\nu-1}$

then correlator has $\left(\sum_{n=0}^{\nu-1} Z_{in} Z_{jn}^* \right) e^{-E_0 t}$ + excited states

matrix of rank $\min(N, \nu)$

for a single correlator $(N=1)$ it is impossible to identify degenerate states and very difficult to identify near-degenerate states

if $N \ge \nu$ then the states can be identified in principle and in practice

Variational method

want to find linear combination $\tilde{\mathcal{O}}_n = \sum_{i=1}^{N} (v_n^\dagger)_i\, \mathcal{O}_i$ s.t. $\tilde{\mathcal{O}}_n^\dagger |\Omega\rangle \approx |n\rangle$

if states $n \ge N$ can be neglected (only N states contribute)

then solution satisfies $\sum_{j=1}^{N} Z_{jn}^* (v_m)_j = \delta_{nm}$

v_n is a solution to generalized eigenvalue problem (GEVP) $C(t) v_n = \lambda_n C(t_0) v_n$

with eigenvalue $\lambda_n = e^{-E_n (t - t_0)}$

neglected part of $C_{ij}(t)$: $\sum_{n=N}^{\infty} Z_{in} Z_{jn}^* e^{-E_n t} = \mathcal{O}(e^{-E_N t})$

Corrections from neglected states (ALPHA collab. 2009):

let $E_n^{\text{eff}}(t, t_0) = -\frac{\partial}{\partial t} \log \lambda_n(t, t_0)$

if $t_0 \ge \frac{t}{2}$ then $E_n^{\text{eff}}(t, t_0) = E_n + \mathcal{O}(e^{-(E_N - E_n)t})$

more operators (increase N) $\rightarrow$ larger E_N $\longrightarrow$ faster convergence to E_n

Notes:

* Proven result is asymptotic behaviour for large t.

  Need to let t_0 depend on t in some way: e.g. $t_0 = \frac{t}{2}$ or $t_0 = t - \delta t$.

* Solving GEVP yields set $\{\lambda_n\}_{n=0}^{N-1}$ but ordering is arbitrary.

  For each n, need to pick λ_n out of this set for each t, each resample.

  If energies well separated $\longrightarrow$ sort by value.

  Otherwise consider using eigenvectors to match different t, different resamples.

* Alternative approach: for fixed (t_0, t_D) solve $C(t_D) v_n = \lambda_n C(t_0) v_n$,

  then let $\bar{C}_{nm}(t) = v_n^\dagger C(t) v_m$.

  $\bar{C}(t)$ is approximately diagonal; let $E_n^{eff}(t) = -\frac{d}{dt} \log \bar{C}_{nn}(t)$.

  Simpler but need to verify stability as (t_0, t_D) varied.

  Downside: lacks proven good asymptotic behaviour.