Mainz Institute for Theoretical Physics

Spectroscopy and Scattering: Formalism (Lecture 3/3)

Maxwell T. Hansen

July 21-22, 2025

THE UNIVERSITY of EDINBURGH

- [x] Warm-up and definitions
  - [x] Meaning of Euclidean
  - [x] Finite-volume set-up
- [x] e^{-mL} round one
  - [x] Mass in $\lambda\phi^4$
  - [x] Mass/matrix element in $g\phi^3$
- [x] $2 \to 2$ formalism
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- [ ] 3 → 3 formalism
  - [ ] New complications
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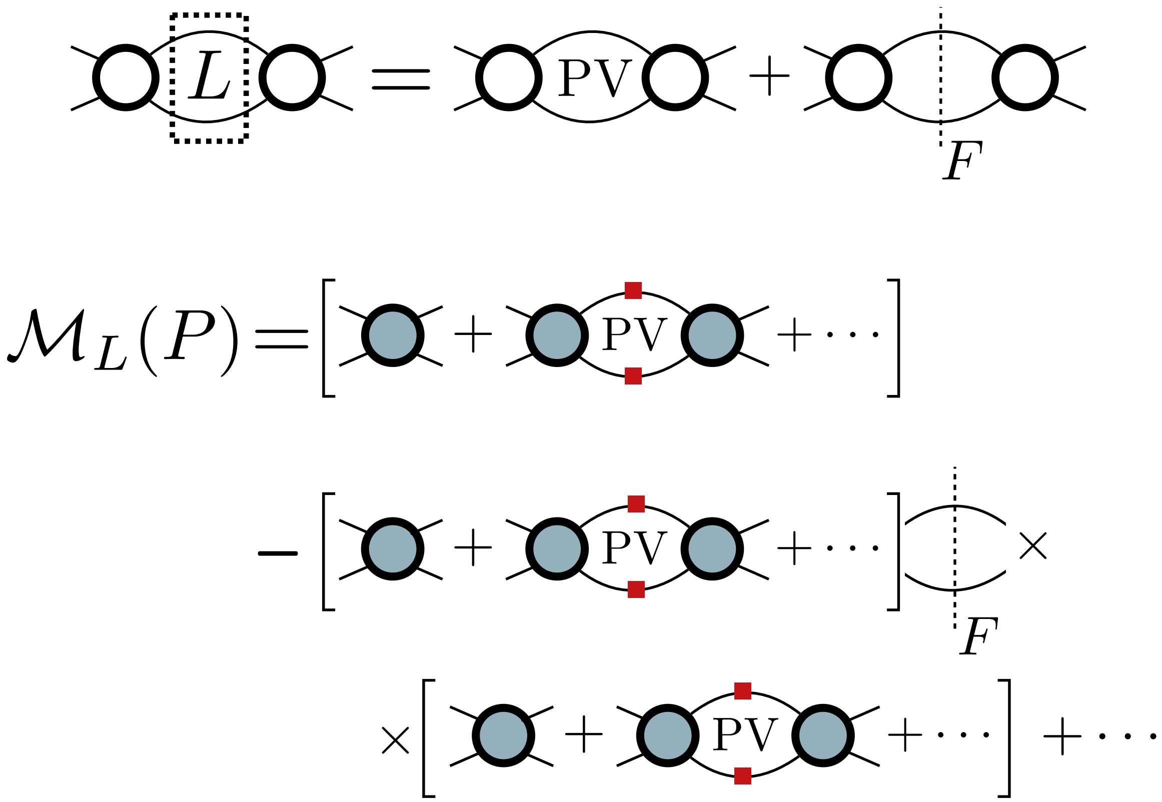
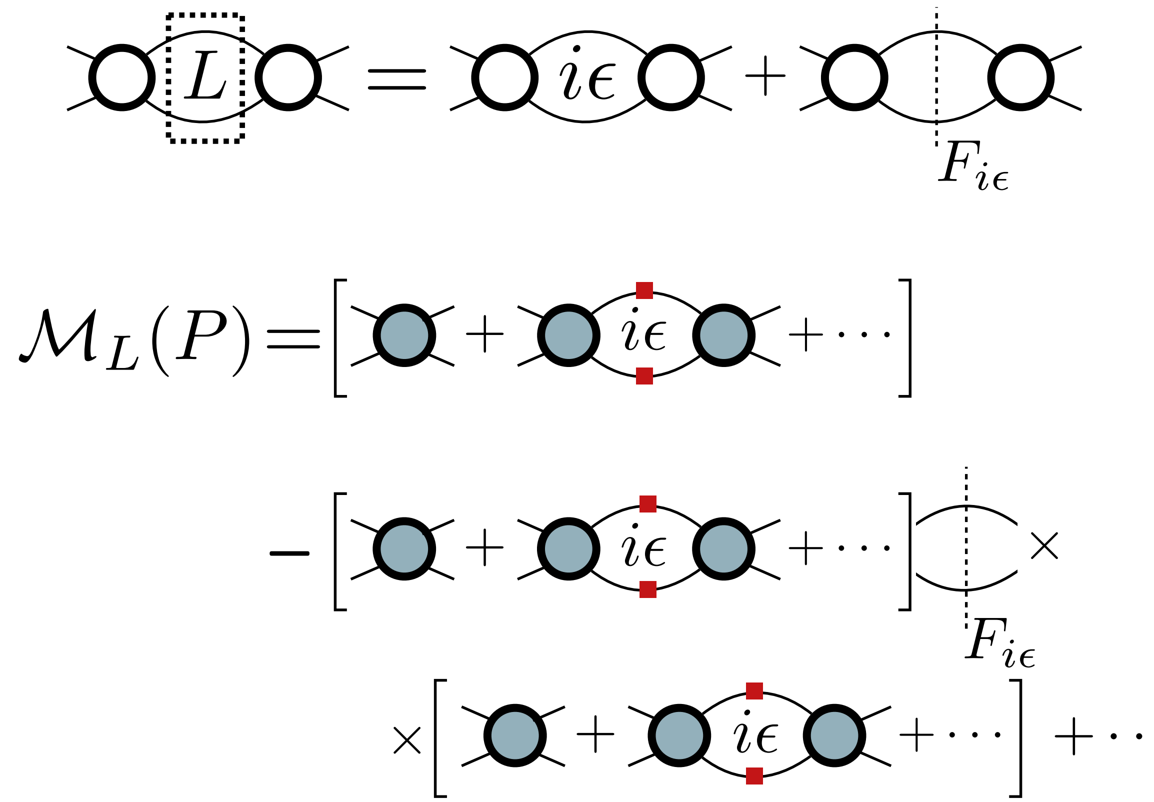
Pole prescription freedom



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

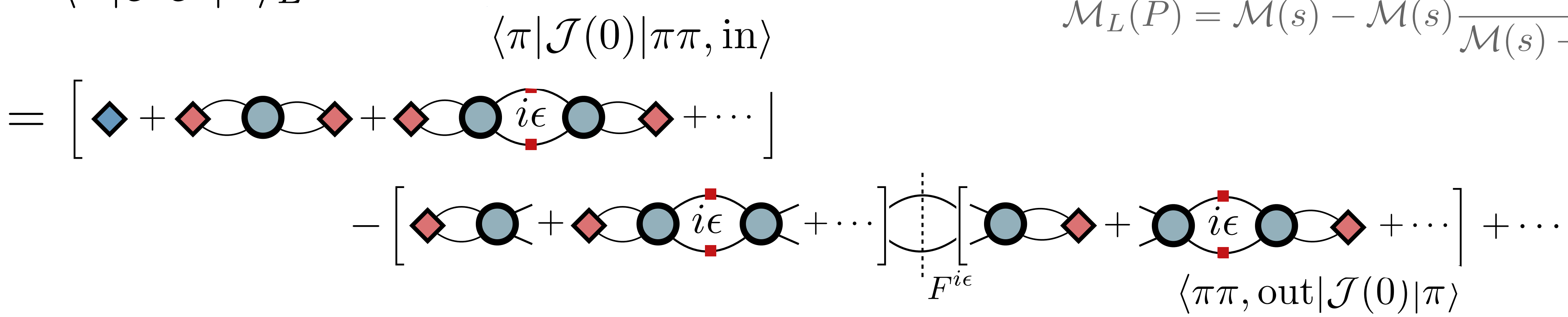


$$= \frac{1}{\mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)}$$

$$\mathcal{K}(s)^{-1} + F(P,L) = [\mathcal{K}(s)^{-1} - i\rho(s)] + [F(P,L) + i\rho(s)] = \mathcal{M}(s)^{-1} + F_{i\epsilon}(P,L)$$

$1 + \mathcal{J} \to 2$

$$C_L(P) = \langle \pi | \mathcal{J}\mathcal{J} | \pi \rangle_L$$



Instead of this…

$$\mathcal{M}_L(P) = \mathcal{M}(s) - \mathcal{M}(s) \frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} \mathcal{M}(s)$$

Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

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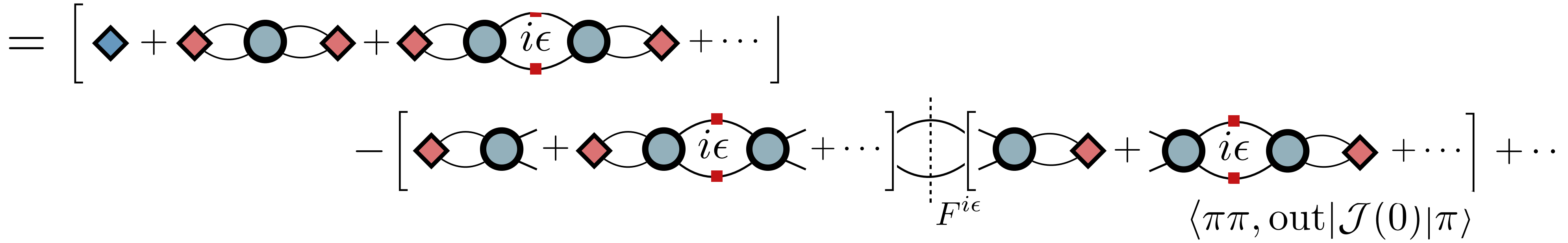
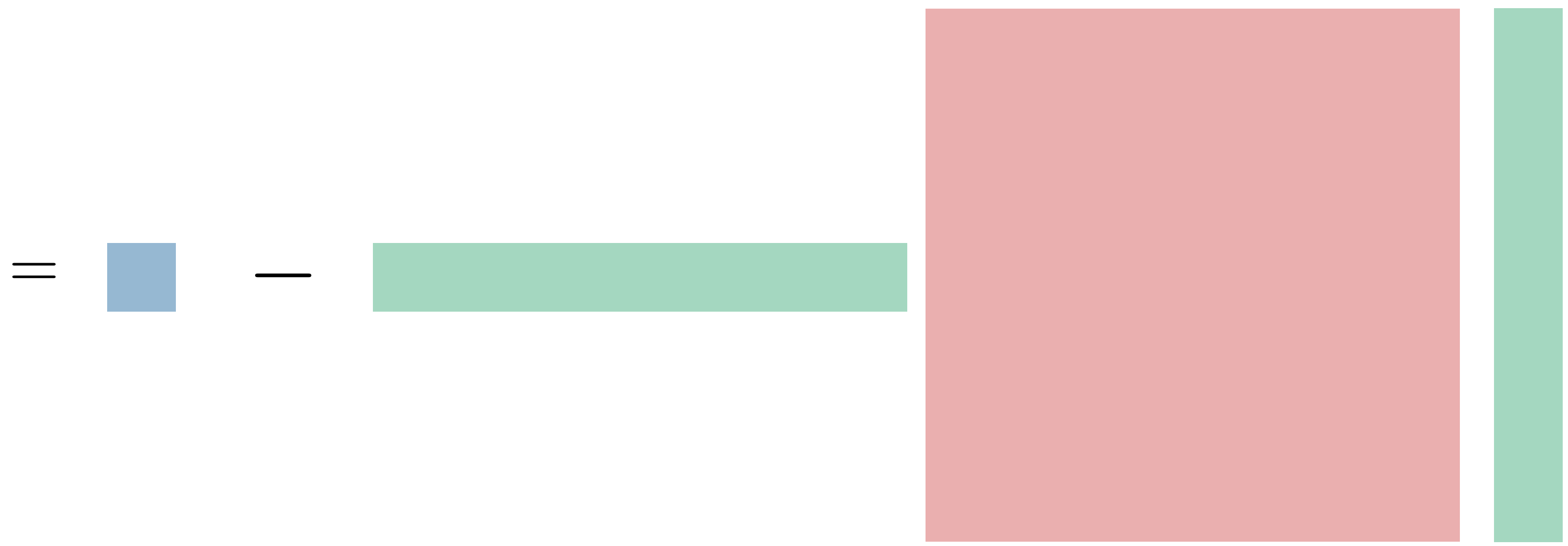
$$C_L(P) = \langle \pi | \mathcal{J}\mathcal{J} | \pi \rangle_L$$

$\langle \pi | \mathcal{J}(0) | \pi\pi, \text{in} \rangle$



$$= C_\infty^{i\epsilon}(s) - A_{\text{in}}^{i\epsilon}(s)\frac{1}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}}A_{\text{out}}^{i\epsilon}(s)$$

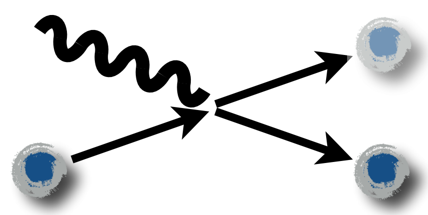


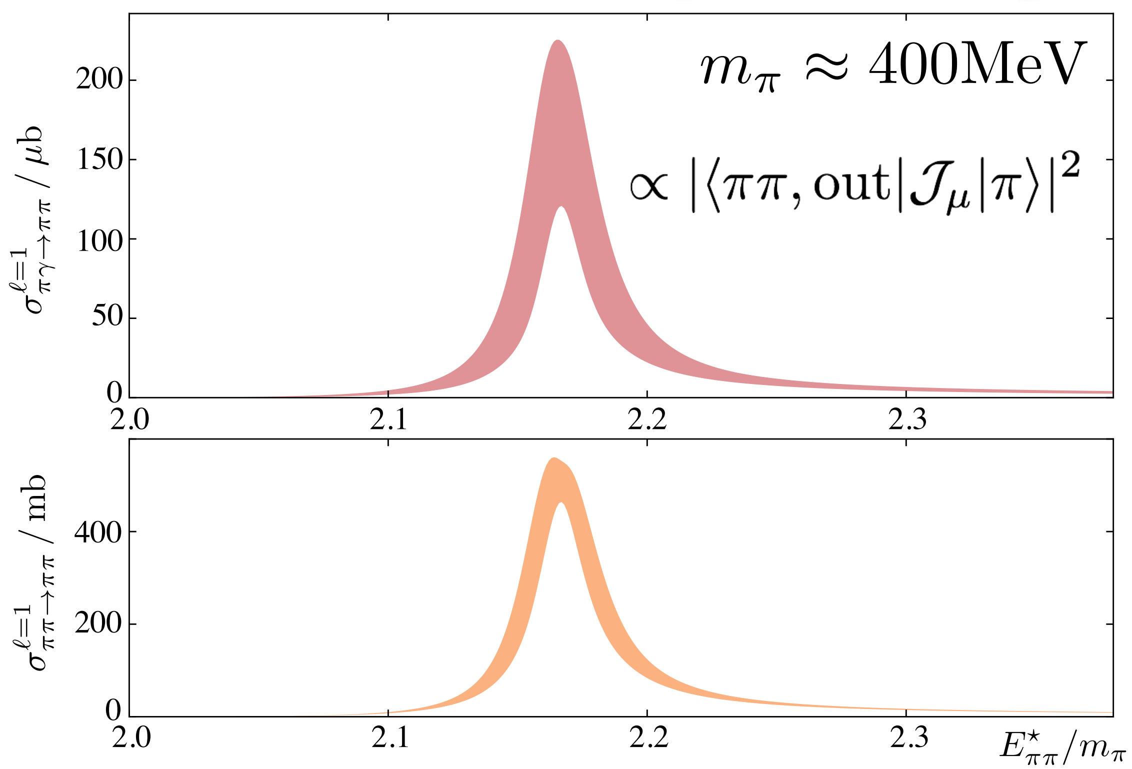
Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (2012) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

Transition amplitudes

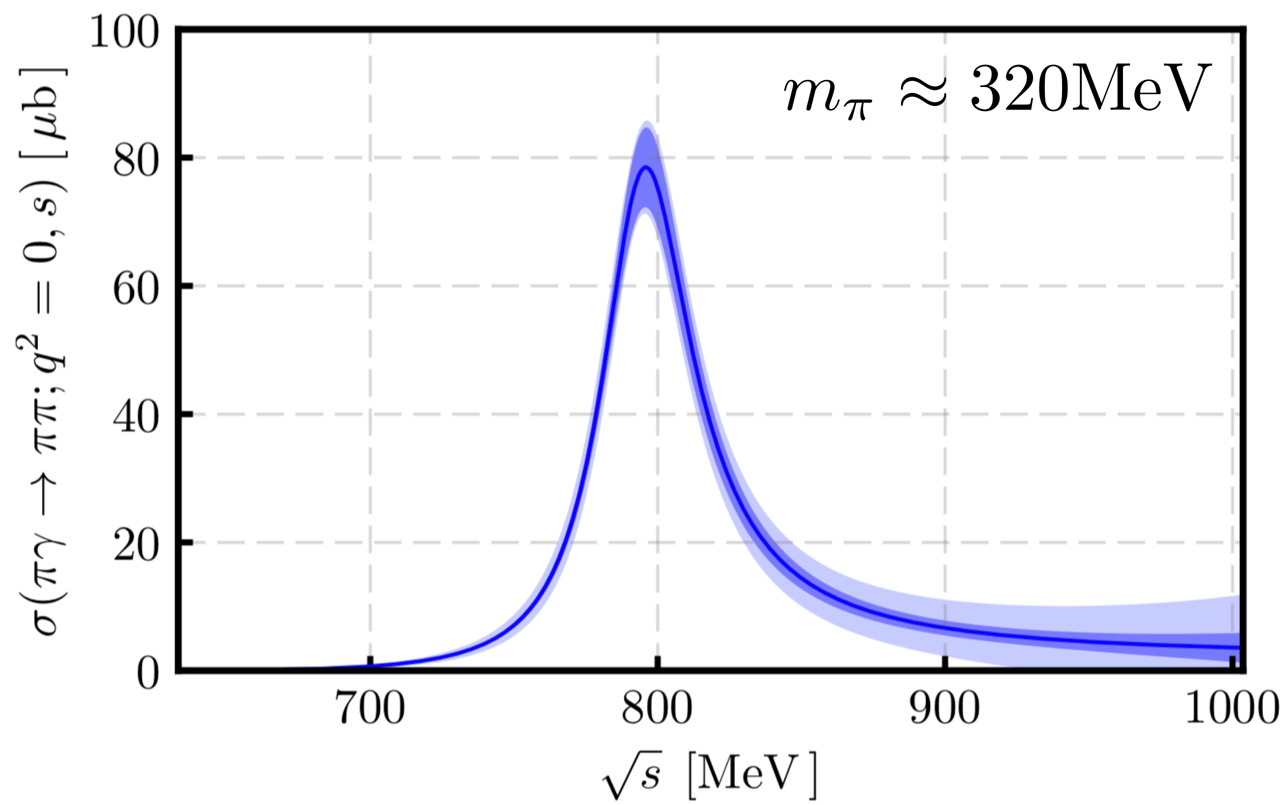
$$L^{3/2}|\langle n, L|\mathcal{J}(0)|\pi\rangle| = \frac{1}{\sqrt{\mu'(E)}}\left|\mathbf{v}_\alpha\langle\alpha, \text{out}|\mathcal{J}(0)|\pi\rangle\right|$$

$$\mathcal{R}(P,L) = \lim_{E\to E_n(L)} \frac{E - E_n(L)}{\mathcal{M}(s) + F^{i\epsilon}(P,L)^{-1}} = \frac{1}{\mu'(E)}\mathbf{v}^T\mathbf{v}$$

$\langle\pi\pi, \text{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 



Briceño et. al., Phys. Rev. D93, 114508 (2016)

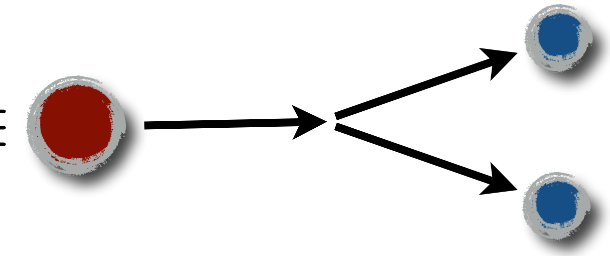


Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Same idea, many contexts

Kaon decay

$$\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$$

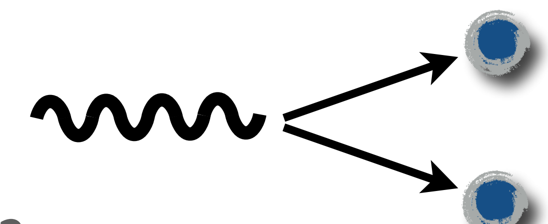


Implementation by RBC/UKQCD collaboration

Lellouch, Lüscher (2001) • Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005)

Time-like form factors

$$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$$

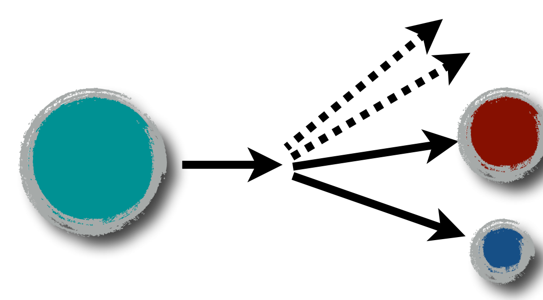


Relevant for muon HVP contribution to muon g-2

Meyer (2011)

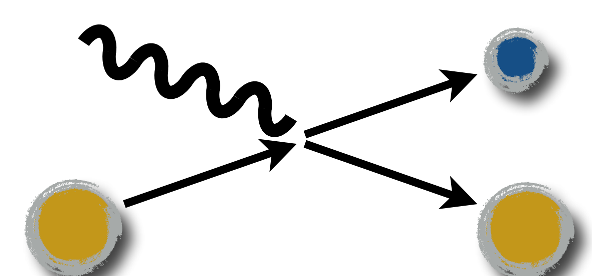
Resonance transition amplitudes

$$\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$$



Particles with spin

$$\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$$



Agadjanov *et al*. (2014) • Briceño, MTH, Walker-Loud (2015) • Briceño, MTH (2016)

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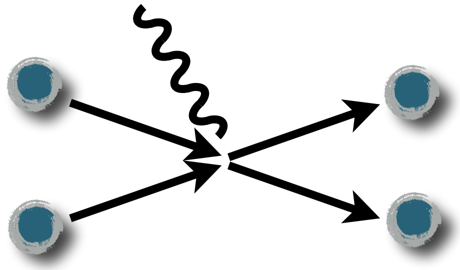
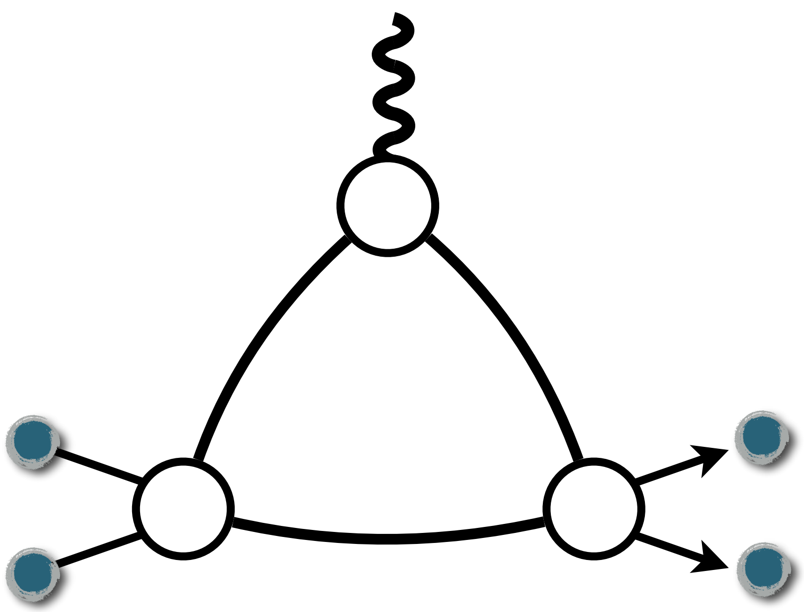
$2 + \mathscr{J} \to 2$

- Fully developed formalism for multi-hadron form factors

$$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \equiv$$



- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities



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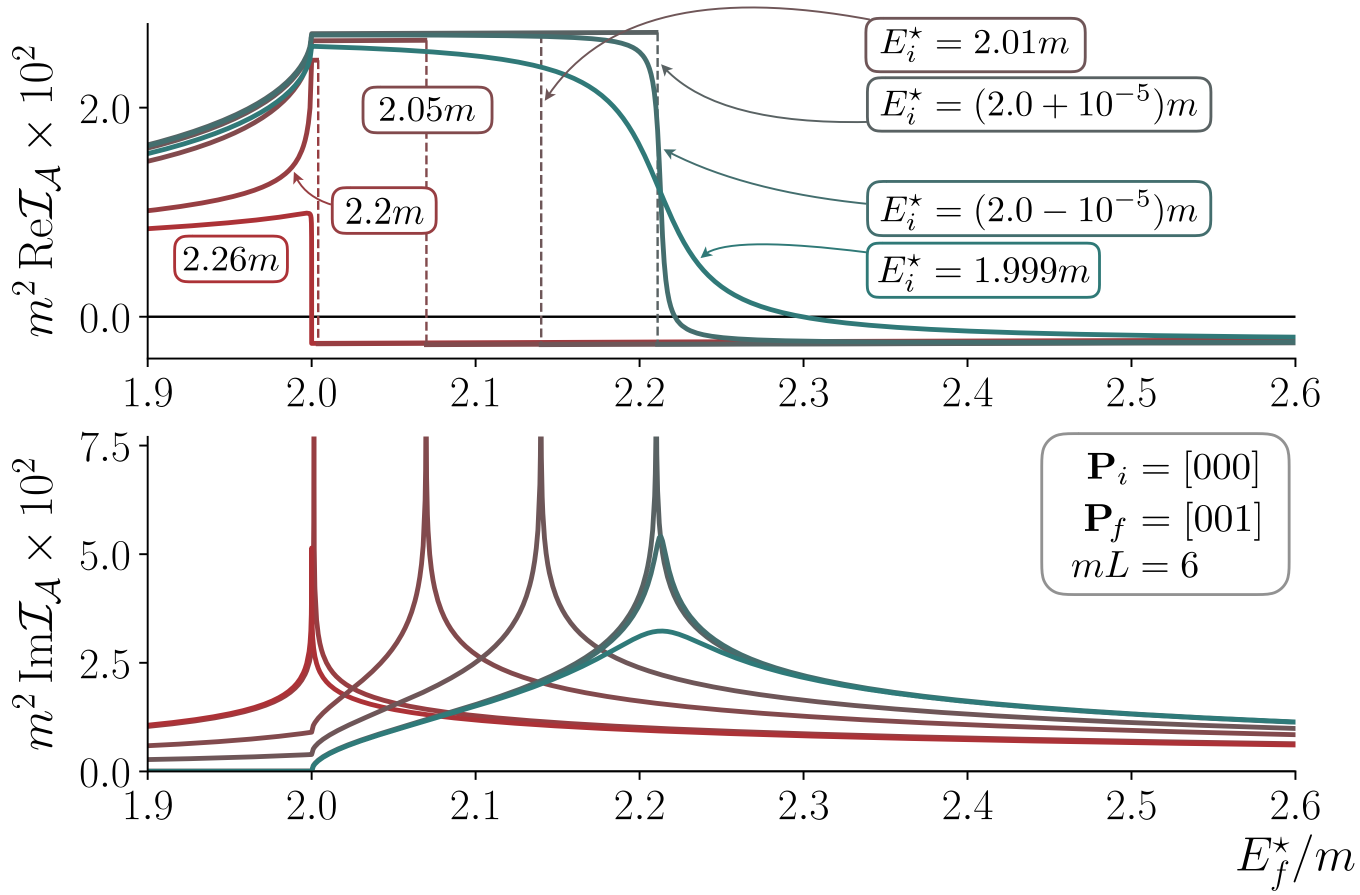
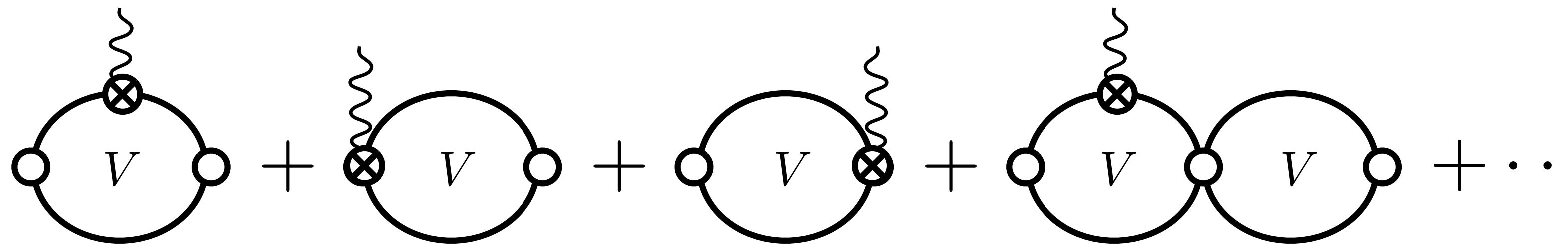


Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

In a nutshell

- By analysing an all orders skeleton expansion…

$$C_L^{2\to 2}(P_f, P_i) =$$



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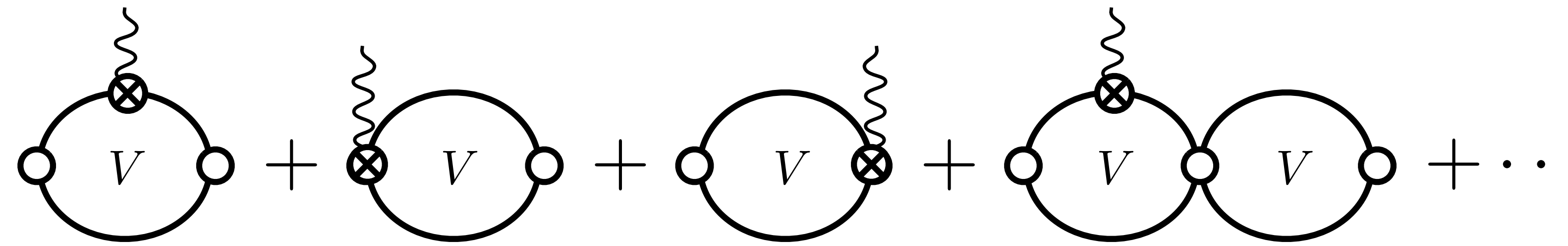
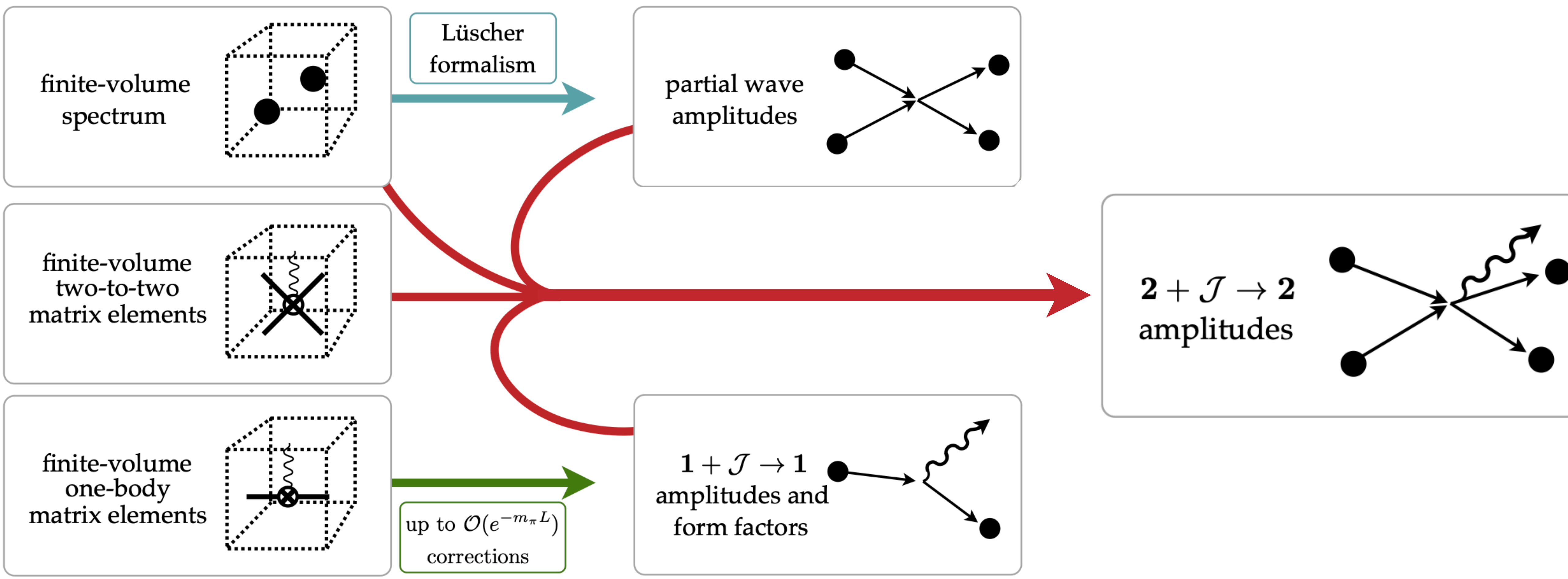


- … we derived a framework to calculate the $\mathbf{2} + \mathcal{J} \to \mathbf{2}$ amplitude



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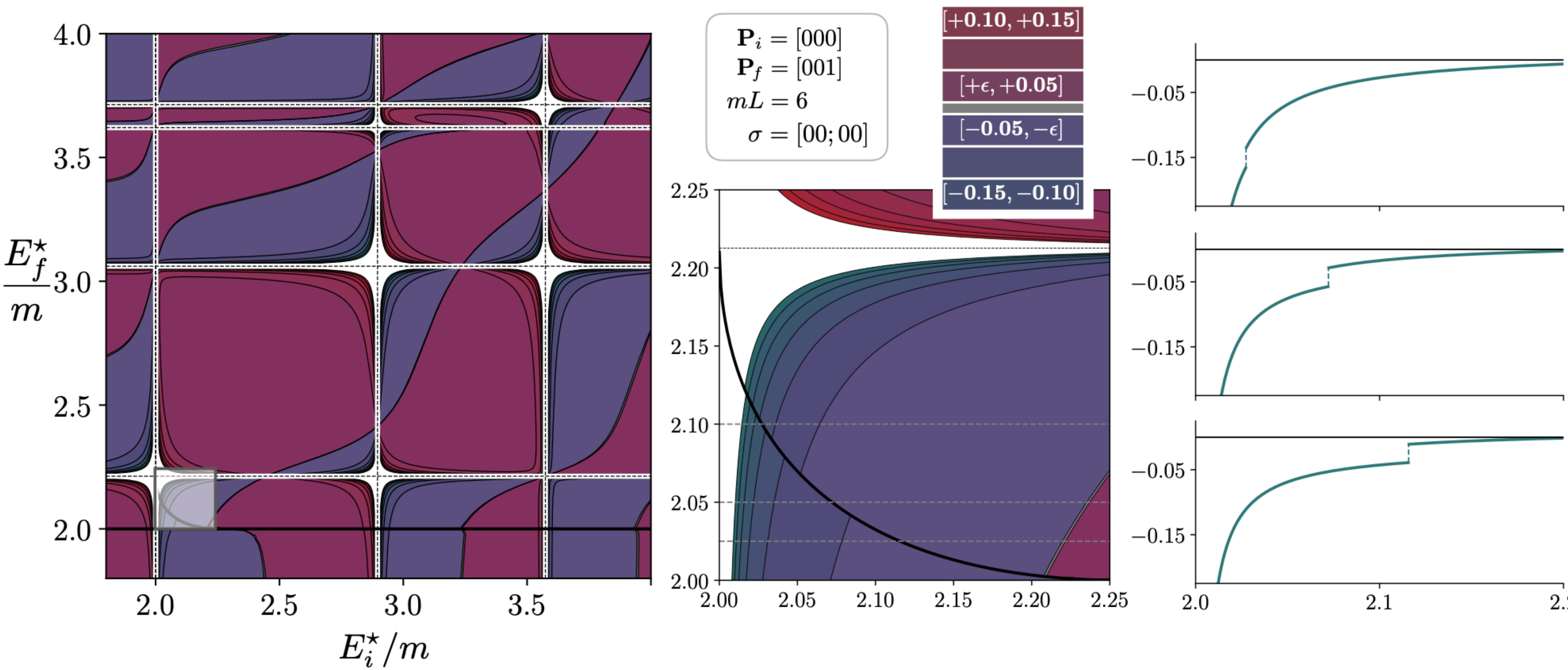
Visualizing G



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Formal & numerical progress: Long-distance matrix elements

- Formal method understood… *assuming only two-hadron intermediate states*

$$\Sigma^+ \underset{H_W}{\longrightarrow} N\pi \to p\gamma^* \qquad\qquad K^0 \underset{H_W}{\longrightarrow} \pi\pi \underset{H_W}{\longrightarrow} \overline{K}^0$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
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  - Issue of growing exponentials *(Christ et al.)*

$$\langle \overline{K} | \mathcal{H}_W(0) \mathcal{H}_W(-|\tau|) | K \rangle_L = \sum_n c_n(L) e^{-(E_n(L)-M_K)|\tau|} \quad \underset{\int_{-T}^{0} d\tau}{\Longrightarrow} \quad \sum_n c_n \frac{1-e^{-(E_n-M_K)T}}{M_K - E_n}$$

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- Issue of power-like finite-volume effects (after discarding exponential)

$$F_L = \sum_n \frac{c_n}{M_K - E_n}$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
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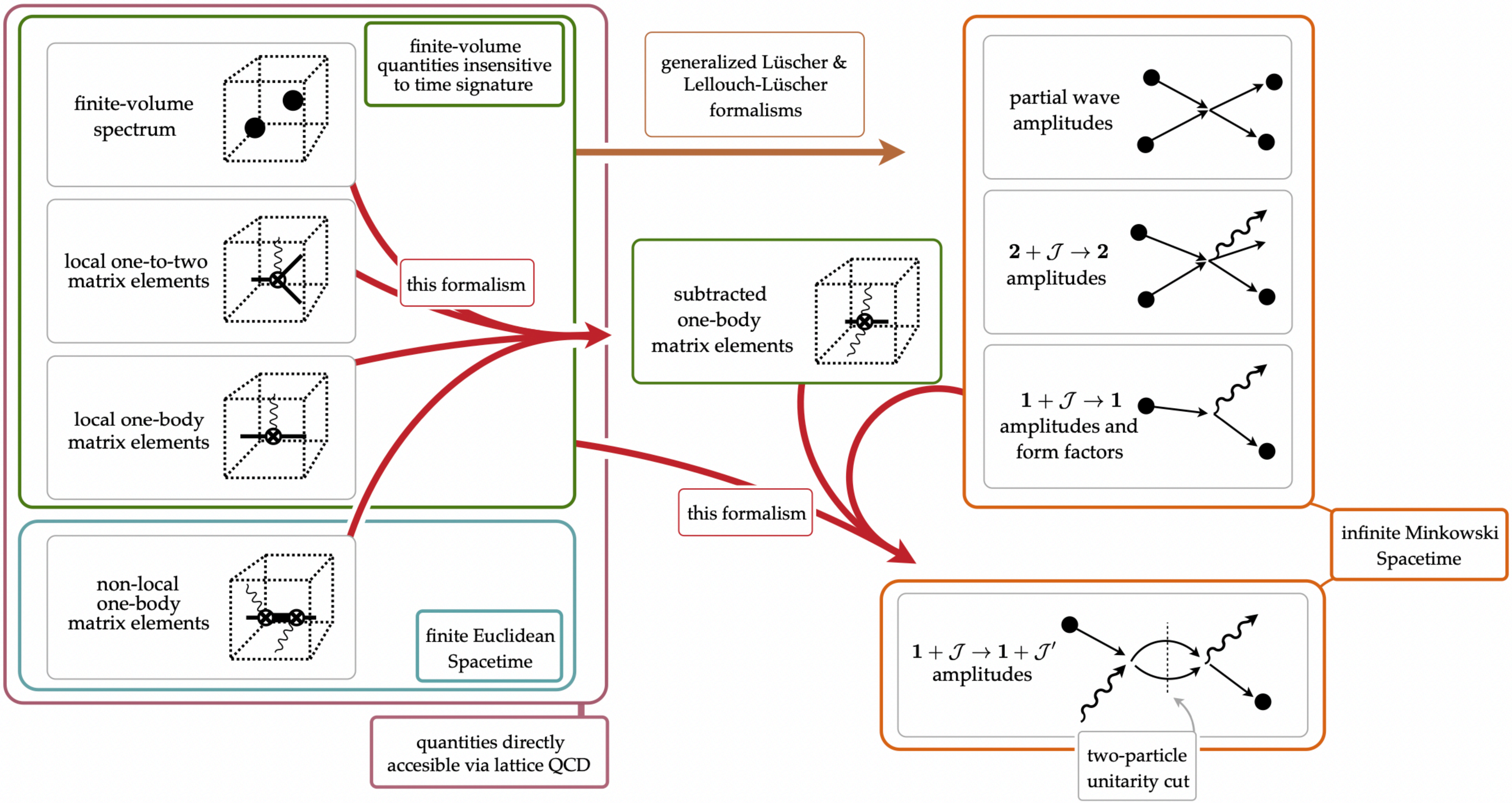

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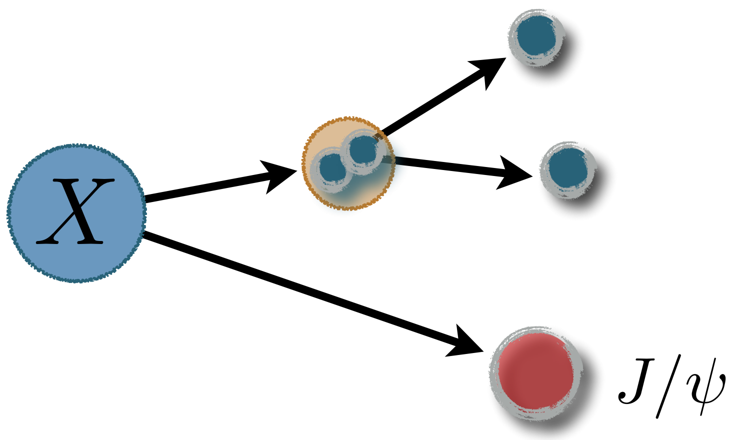
3-particle amplitudes

2-to-2 only samples $J^P \quad 0^+ \quad 1^- \quad 2^+ \cdots$

many interesting resonances have significant 3-body decays



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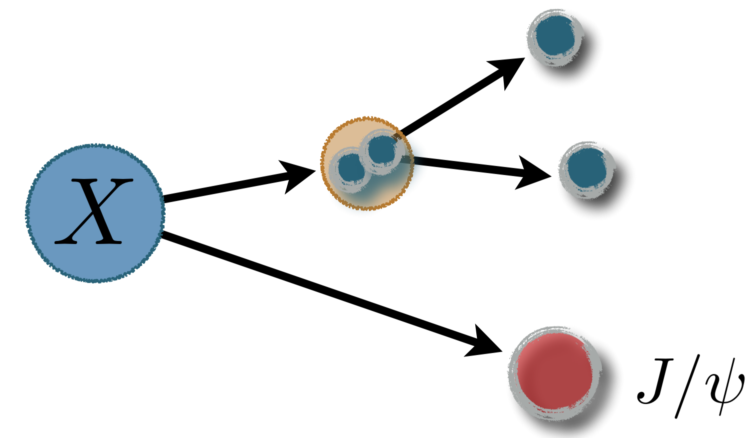
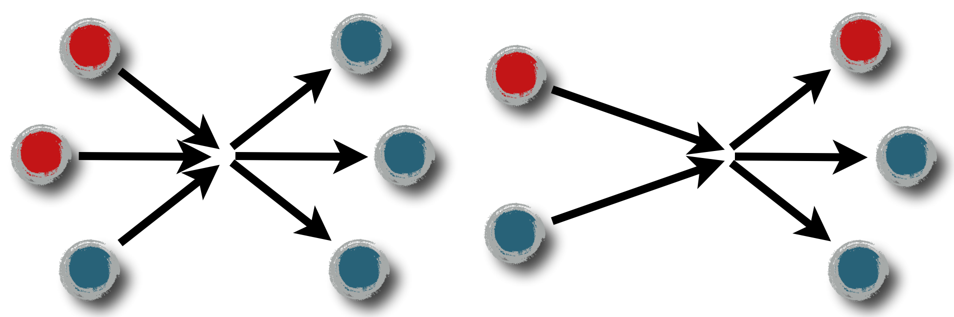


Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



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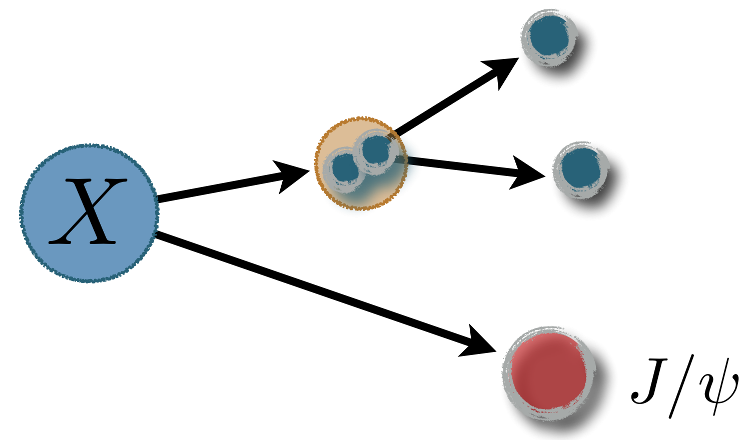

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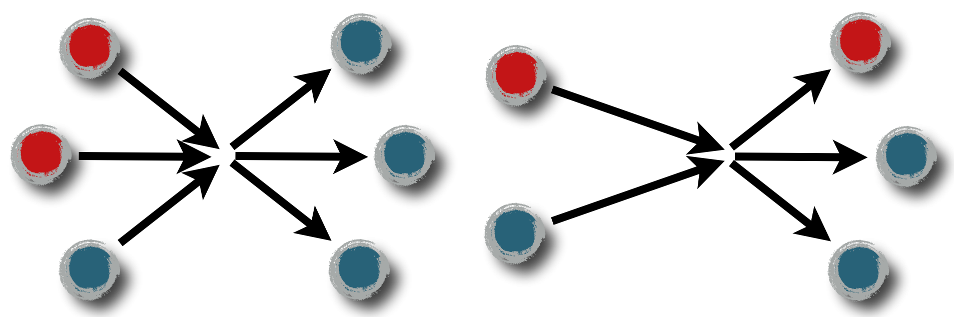


Applications…

exotic resonance pole positions, couplings, quantum numbers

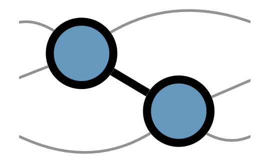
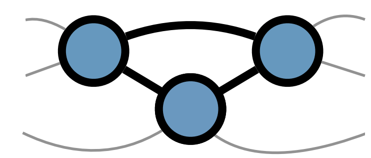
$\omega(782),\ a_1(1420) \to \pi\pi\pi \qquad X(3872) \to J/\psi\pi\pi \qquad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

The big idea

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as M_3

smooth real function

relation to M_3 = known

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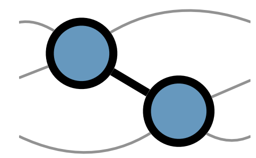
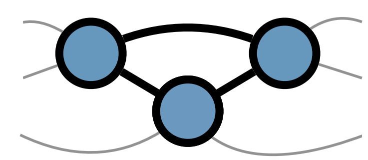
$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ *PV pole prescription* $-$  $+$  $+\cdots$

same degrees of freedom as *M*3 | smooth real function | relation to *M*3 = known

- $\mathcal{K}_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

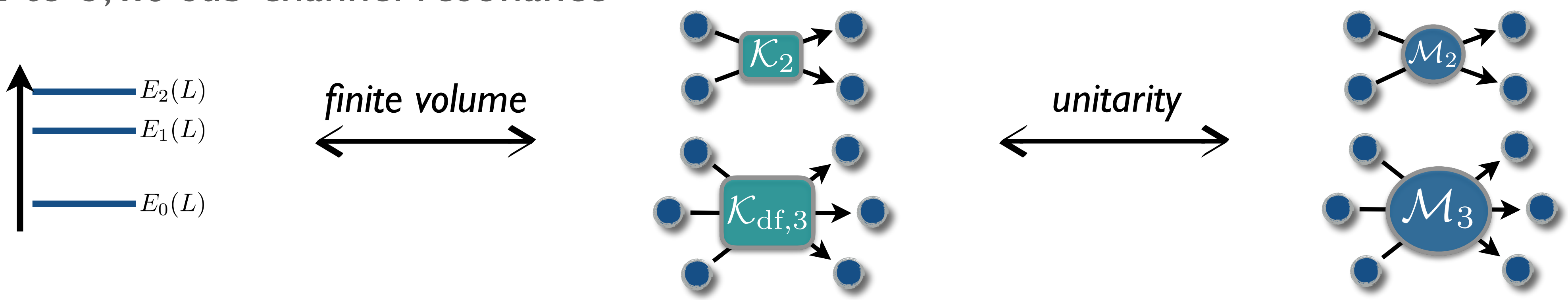
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

- General relation between *energies* and *two-and-three scalar scattering*

  No 2-to-3, *no* sub-channel resonance



• MTH, Sharpe (2014, 2015) •

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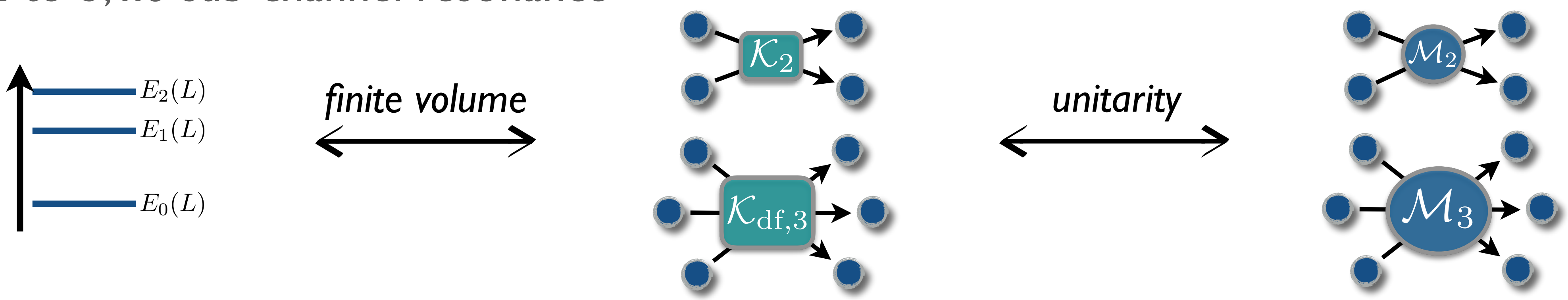

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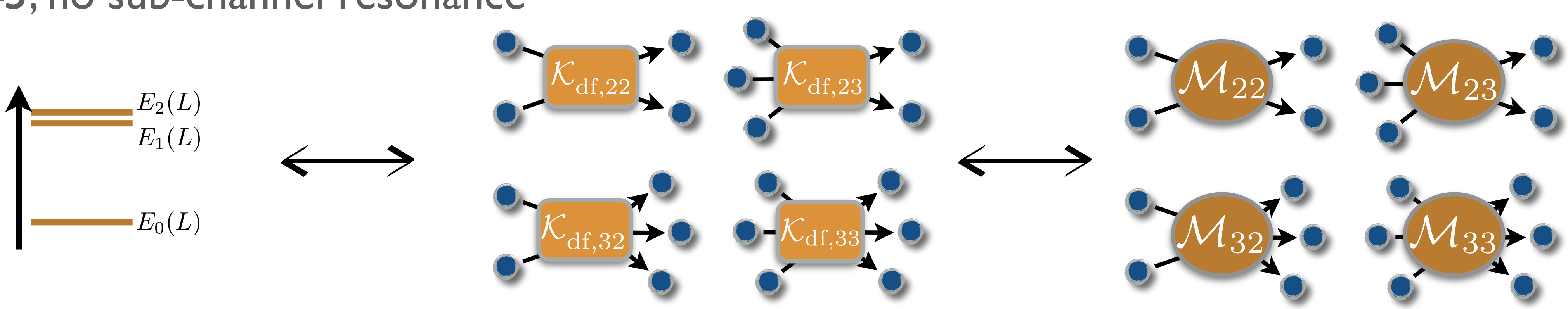
2-to-3, no sub-channel resonance



• Briceño, MTH, Sharpe (2017) •

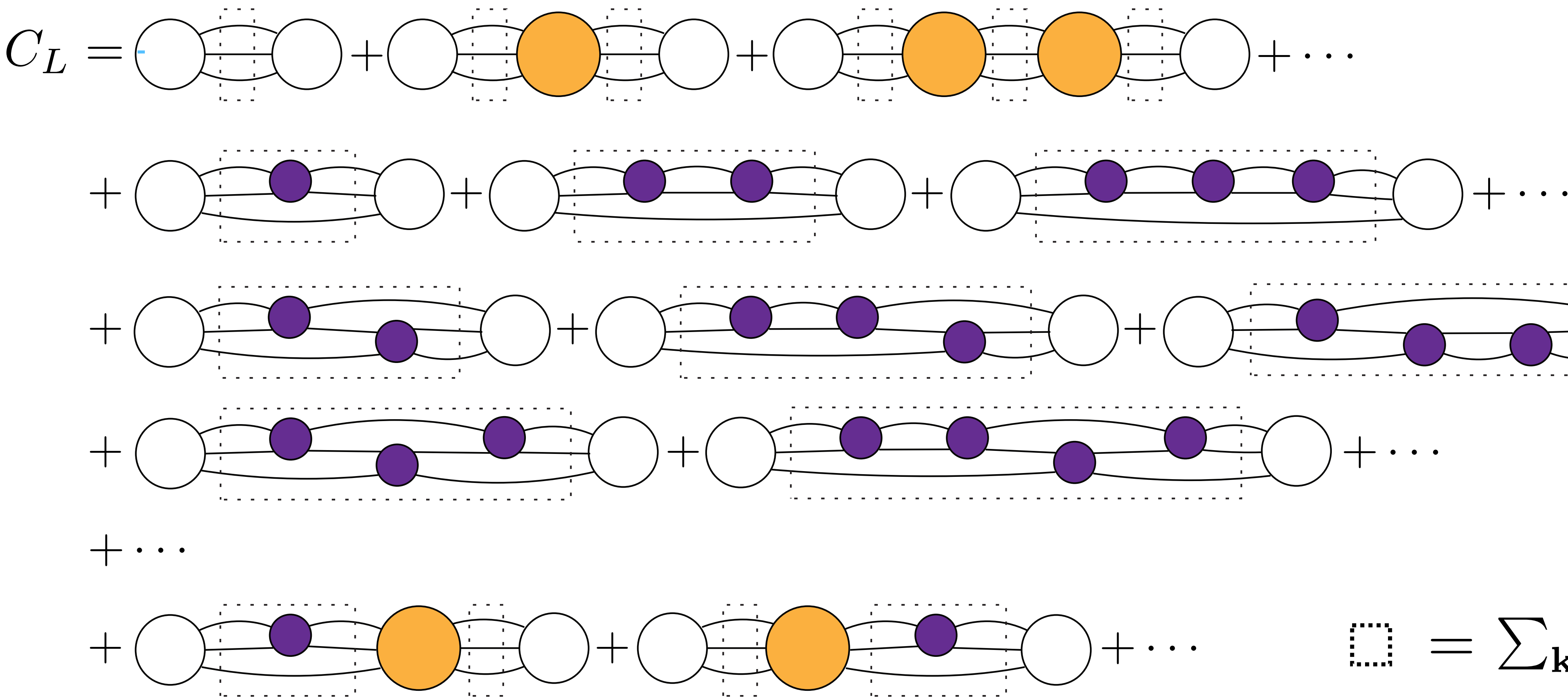
Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

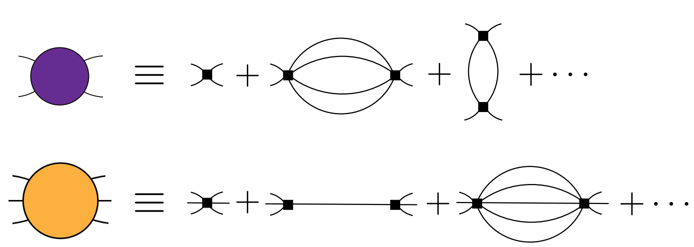
$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***



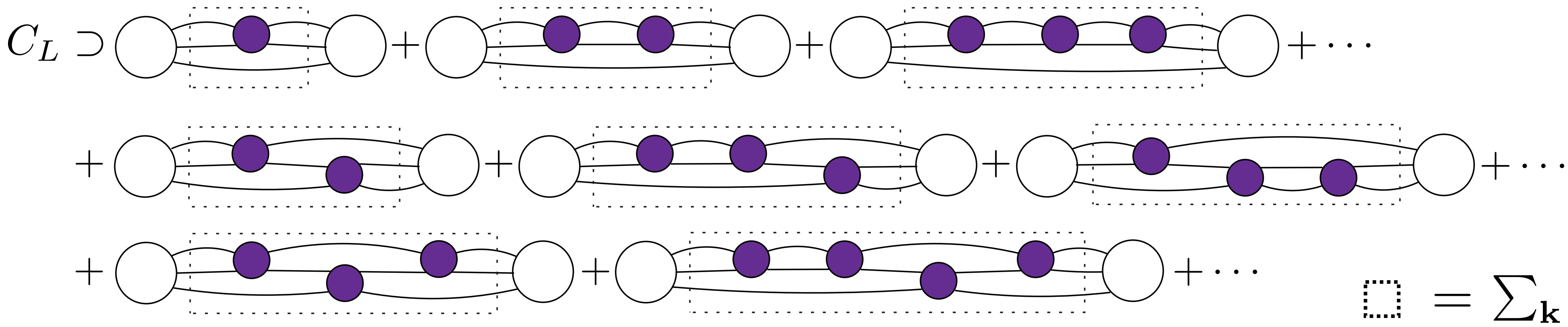


kernels have suppressed L dependence

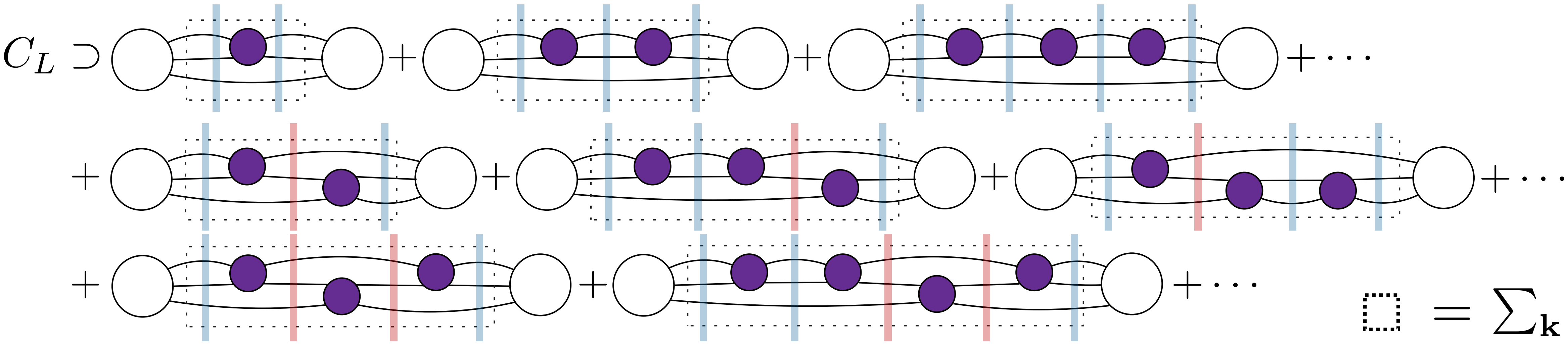
lines = fully dressed hadrons

• MTH, Sharpe (2014) •

Two types of cuts



Two types of cuts



$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^2 \mathbf{A}_3 + \mathbf{A}_3' \mathbf{F} [\mathbf{K}_2 \mathbf{F}]^3 \mathbf{A}_3 + \cdots = \mathbf{A}_3' \mathbf{F} \frac{1}{1 - \mathbf{K}_2 \mathbf{F}} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3$$

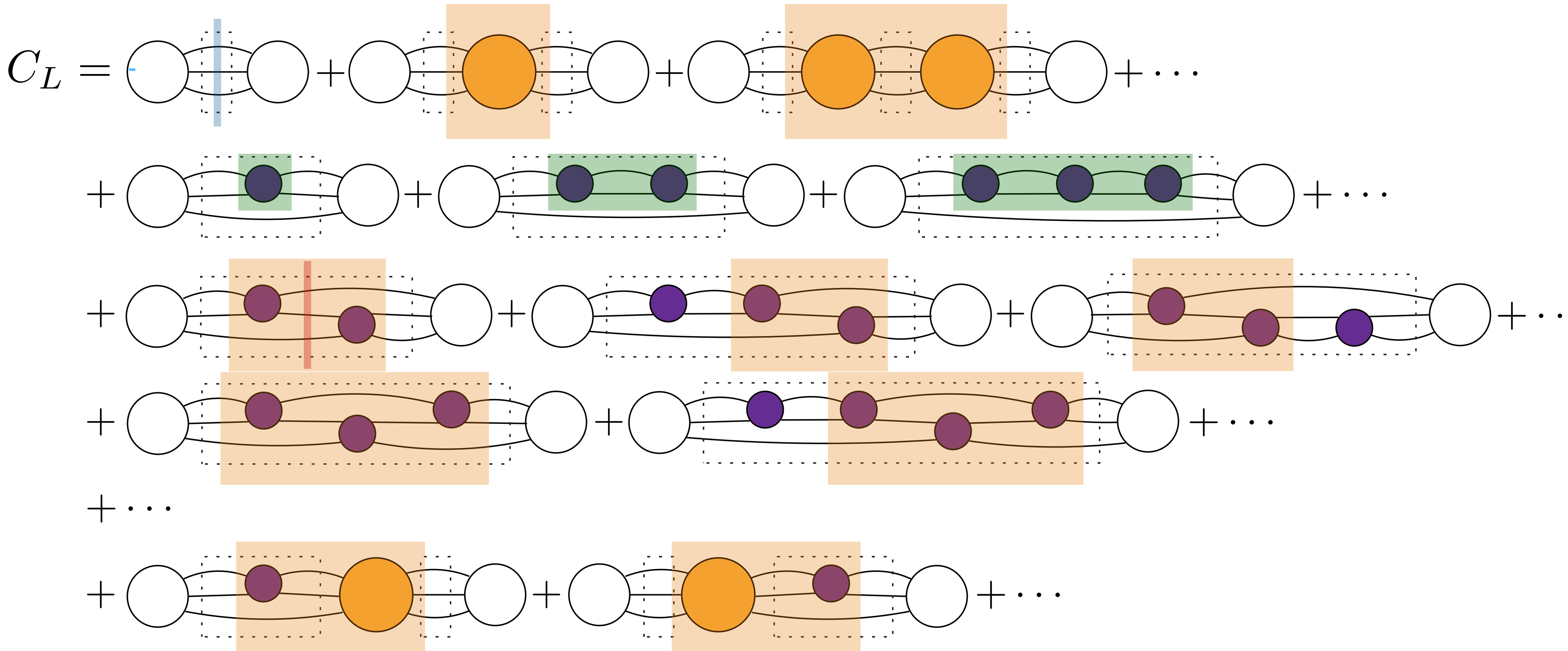
$$\mathbf{A}_3' \mathbf{F} \mathbf{K}_2 \mathbf{G} \mathbf{K}_2 \mathbf{F} \mathbf{A}_3 + \cdots$$

have not yet considered entire diagram contributions

missing contributions from ***off-shellness***

missing smooth terms (short-distance parts)

Short-distance parts & summation



$$C_L - C_\infty = \mathbf{A}_3' \, \mathbf{F}_{33} \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F}_{33} \, \mathbf{K}_{\mathrm{df},3} \, \mathbf{F}_{33} \, \mathbf{A}_3 + \cdots$$

$$= \mathbf{A}_3' \, \frac{1}{\mathbf{F}_{33}^{-1} + \mathbf{K}_{\mathrm{df},3}} \, \mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3}\mathbf{F} + \mathbf{F}\,\mathbf{K}_2 \, \frac{1}{1 - (\mathbf{F} + \mathbf{G})\,\mathbf{K}_2} \, \mathbf{F}$$

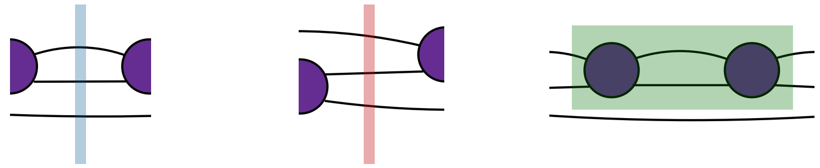
no term left behind

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P, L|\mathcal{K}_2)\right] = 0$$

$F_3(P, L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\mathcal{K}_2\frac{1}{1-(F+G)\mathcal{K}_2}F$$



Holds only for three-particle energies

Neglects e^{-mL}

General relation

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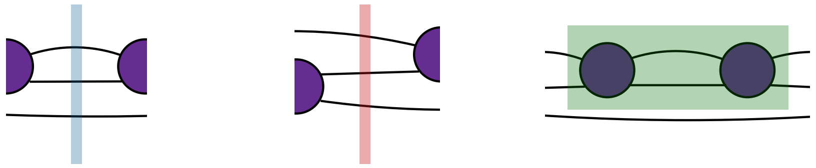


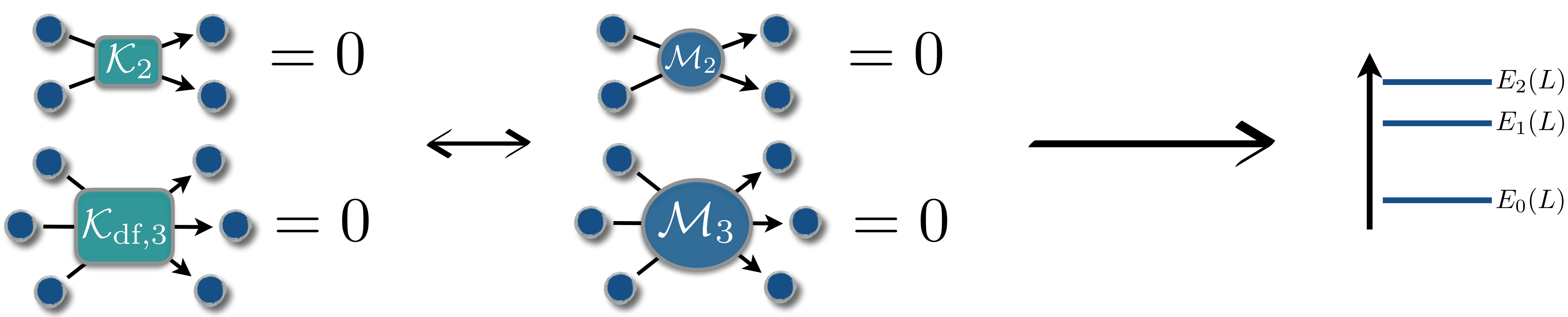
Holds only for three-particle energies

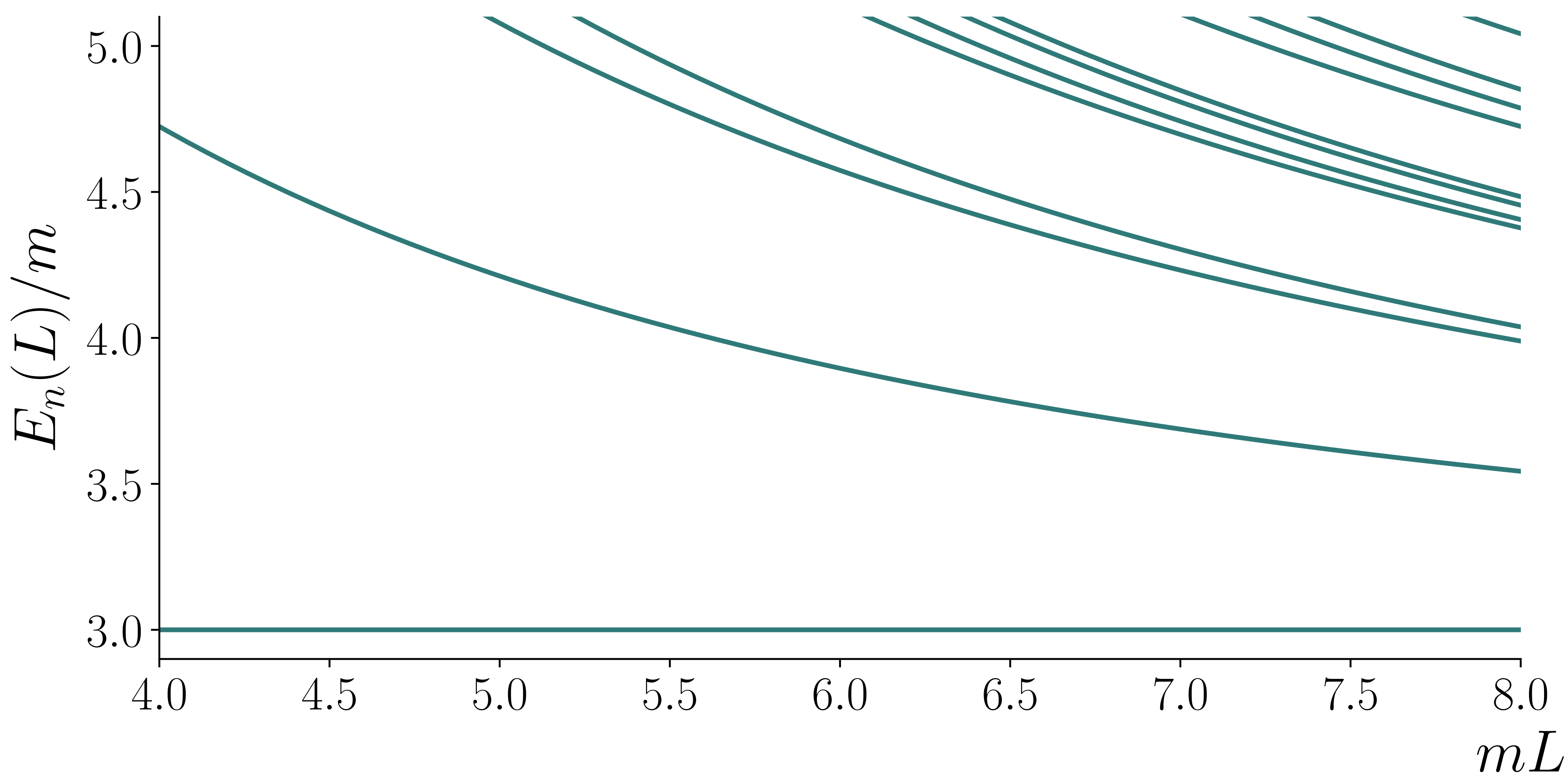
Neglects e^{-mL}

- MTH, Sharpe (2014-2016)
- *See also Döring, Mai, Hammer, Pang, Rusetsky*

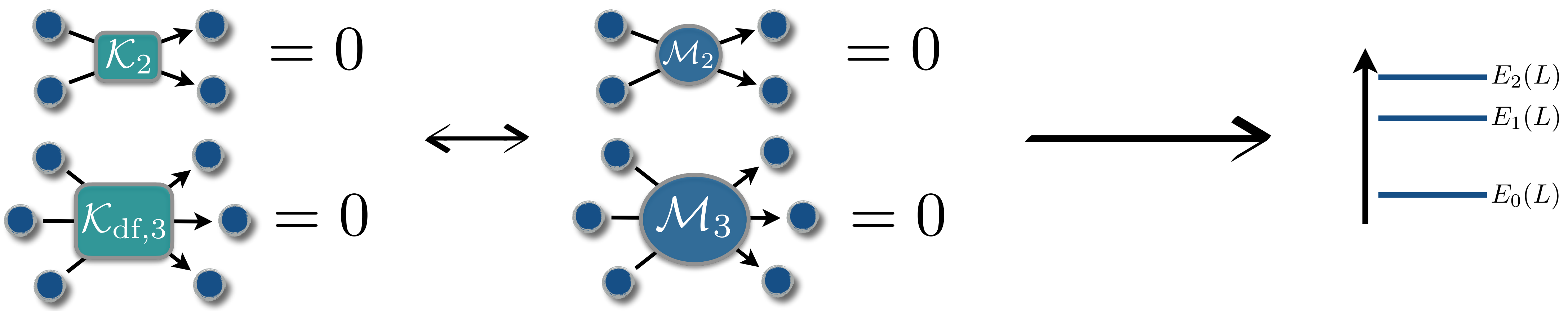
Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

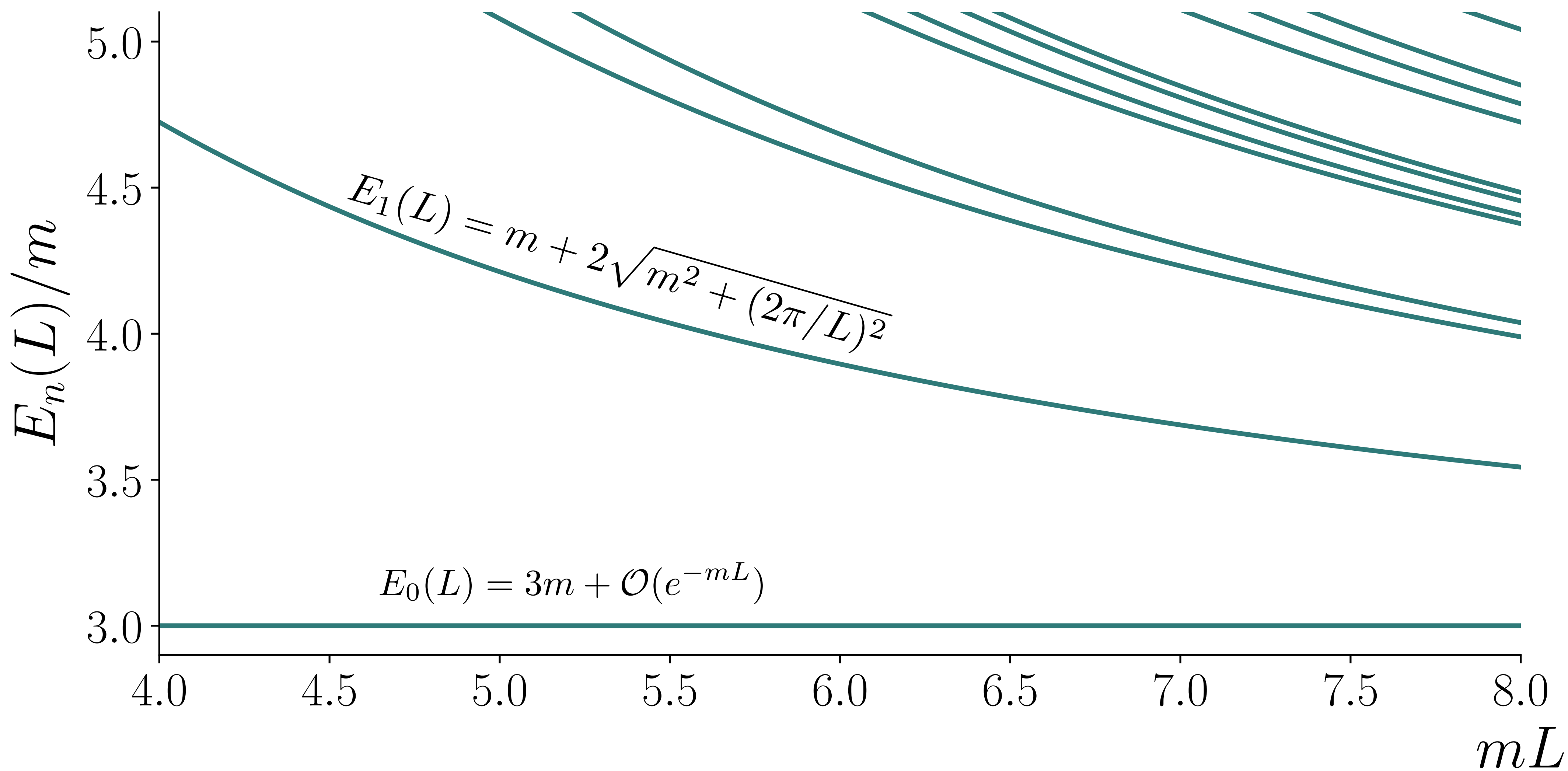
Non-interacting energies



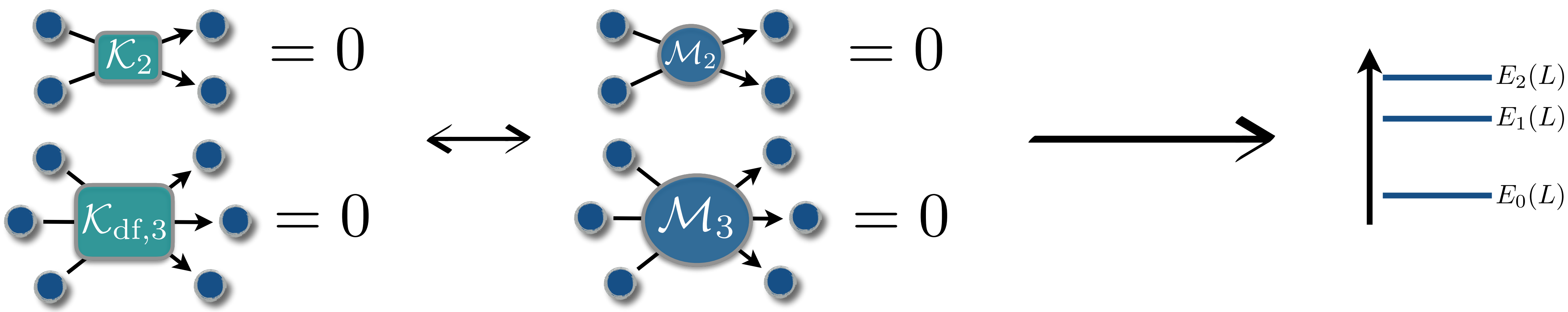


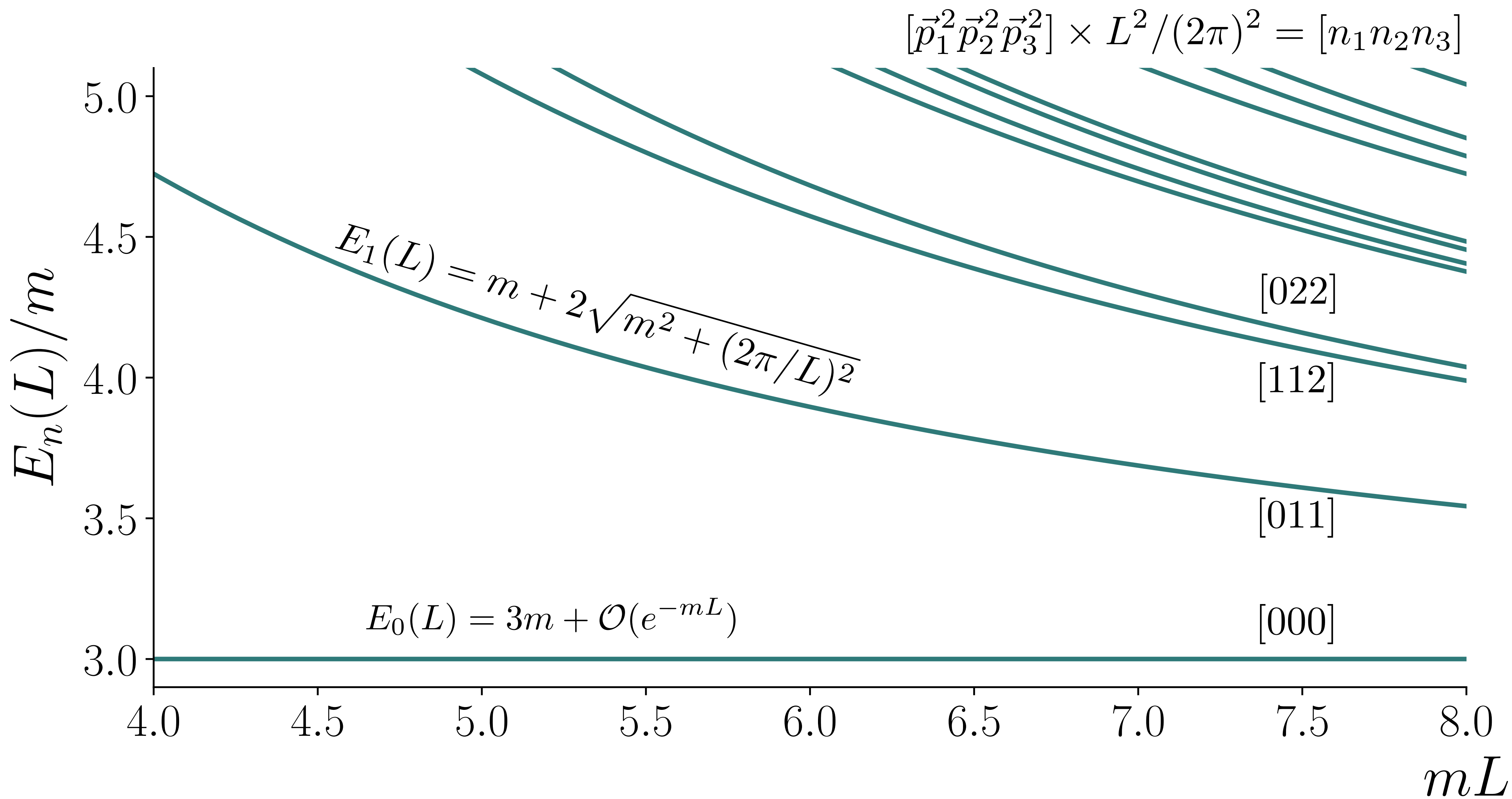
Non-interacting energies



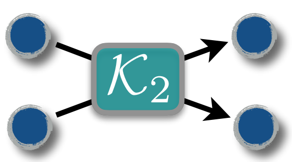


Non-interacting energies

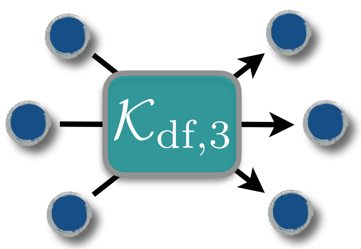




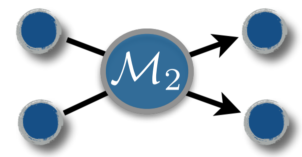
Two-particle interactions



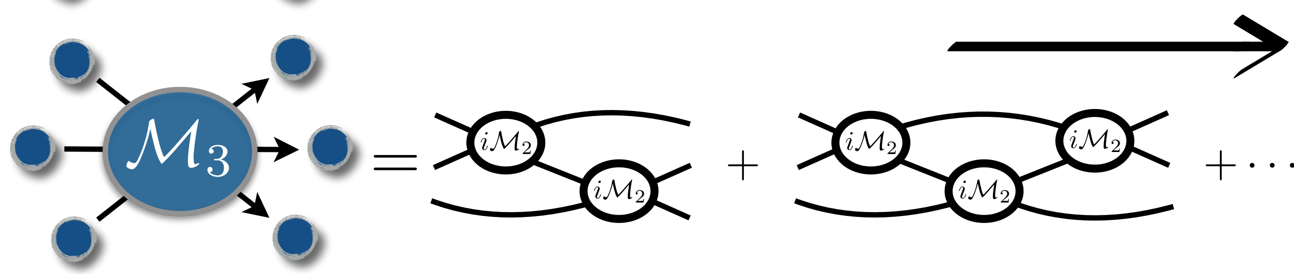
$= -16\pi\sqrt{s}\,a$

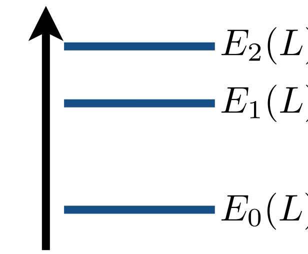


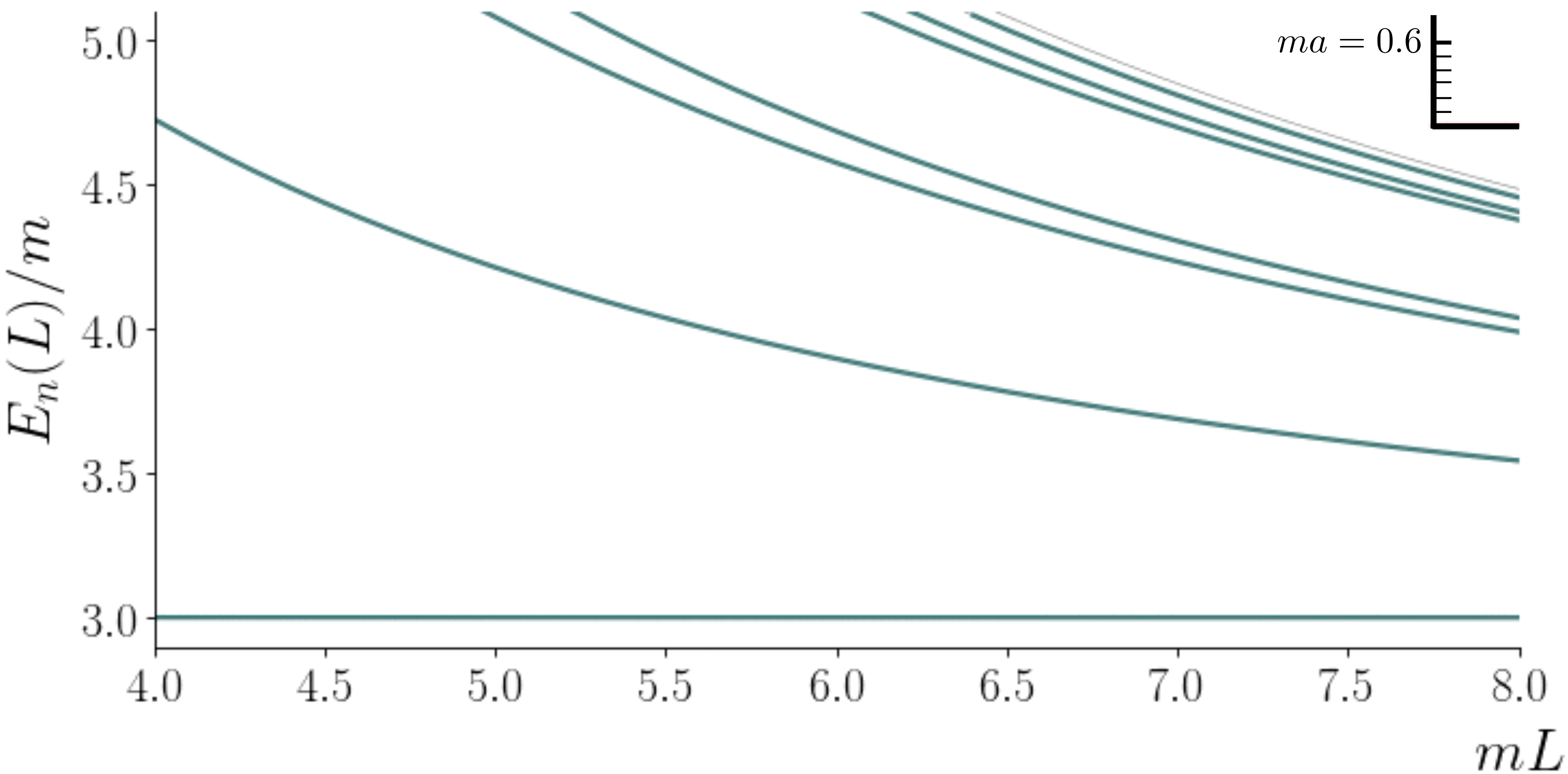
$= 0$



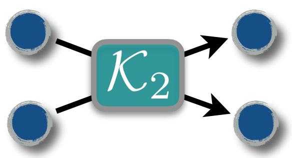
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



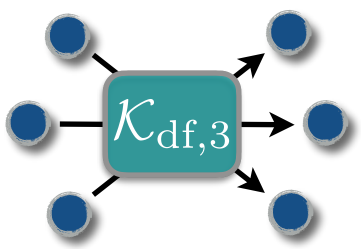




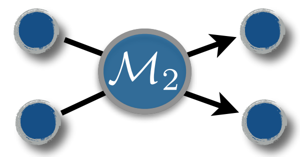
Two-particle interactions



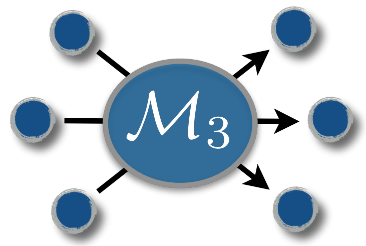
$= -16\pi\sqrt{s}\,a$



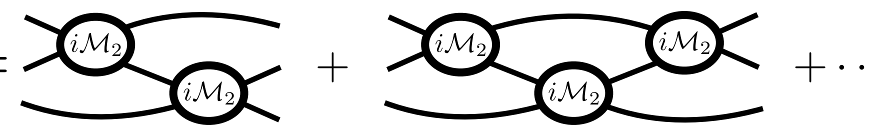
$= 0$

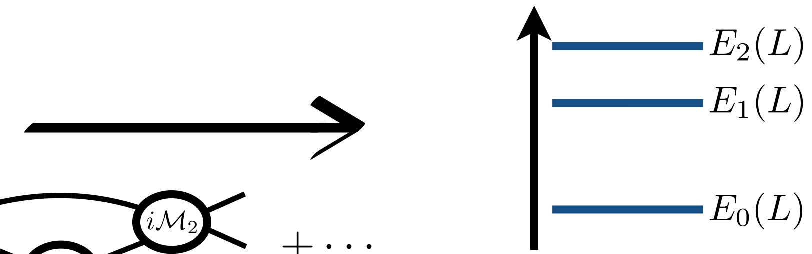


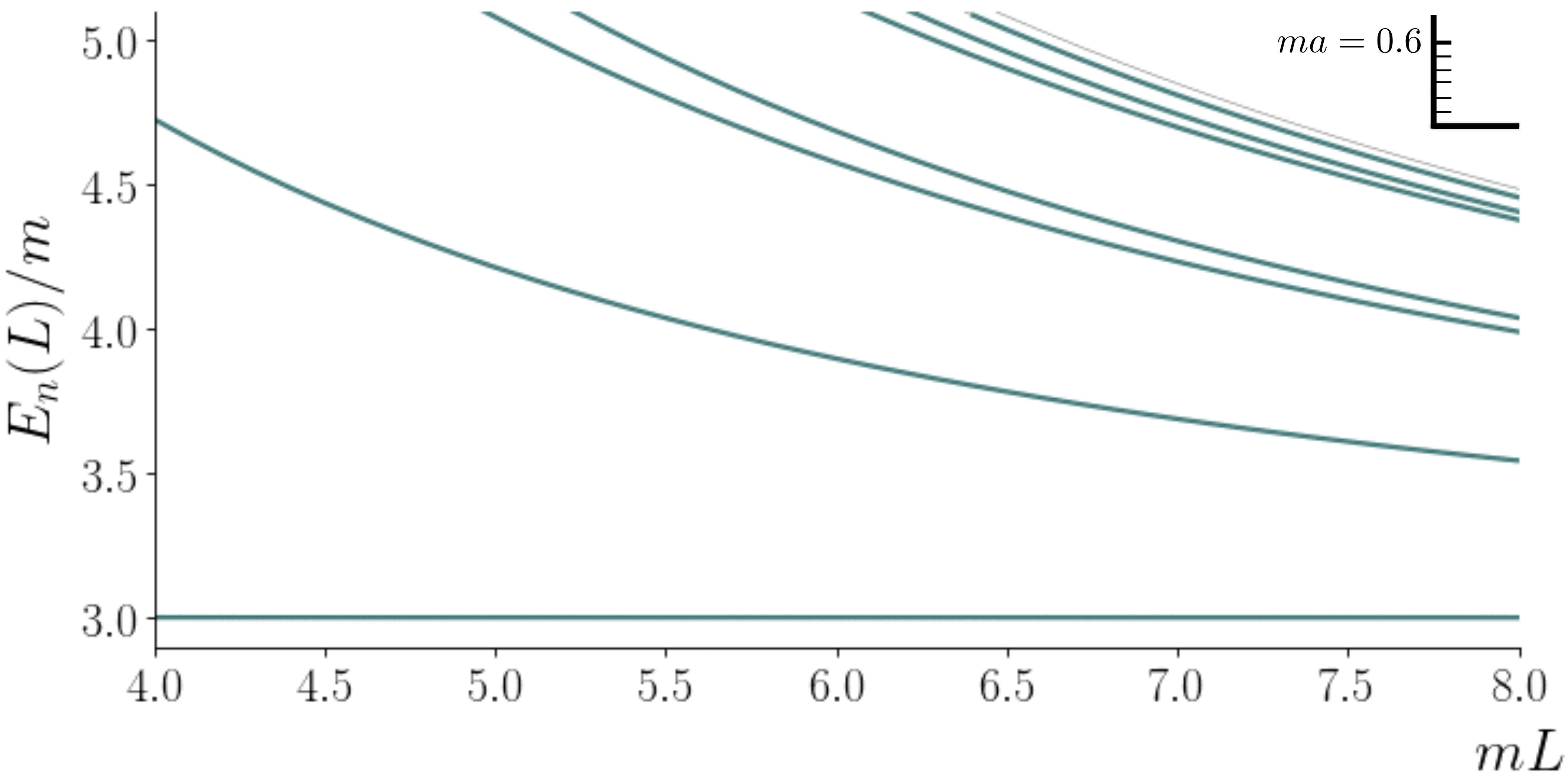
$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$



$=$

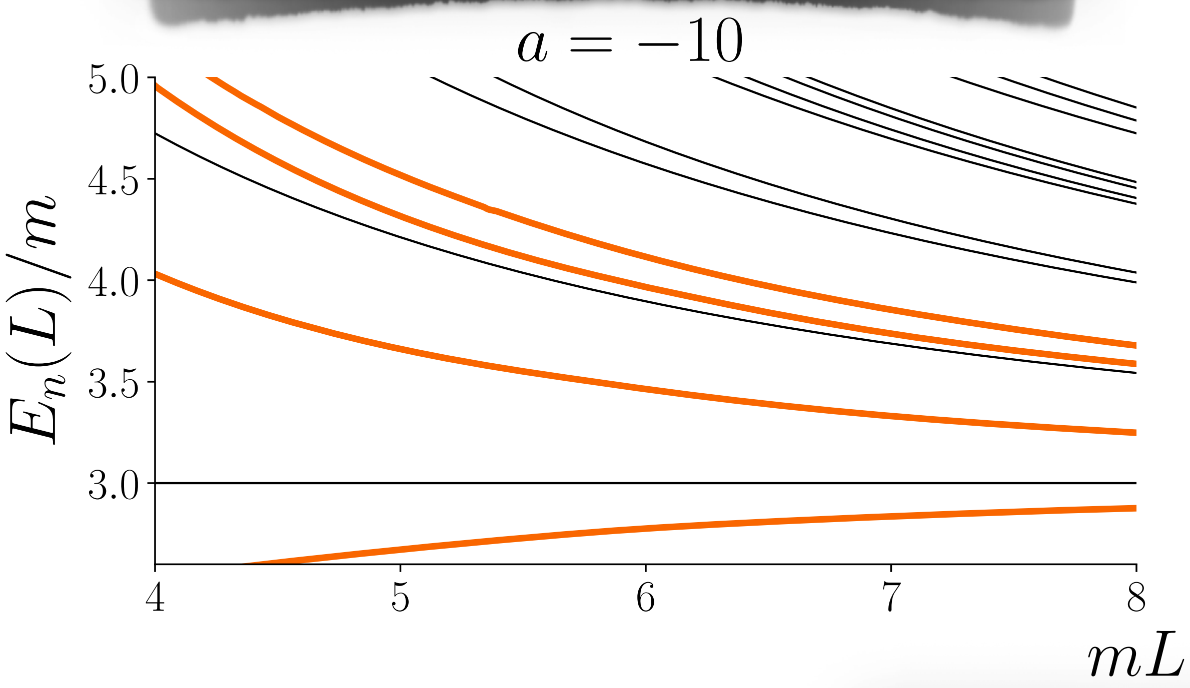




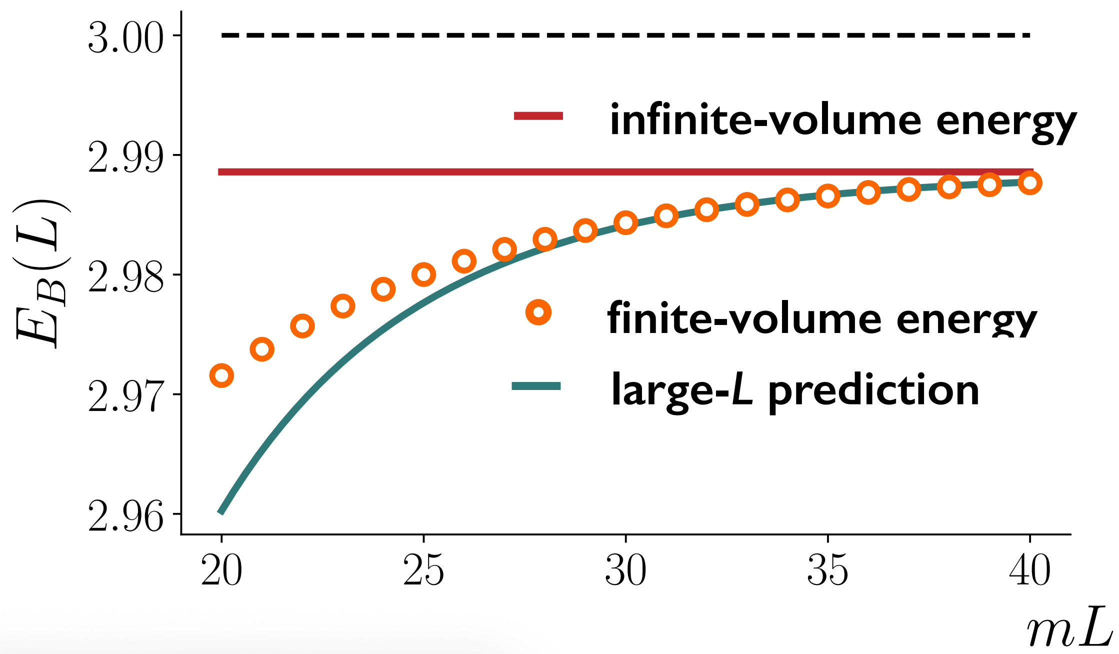


Many toy results

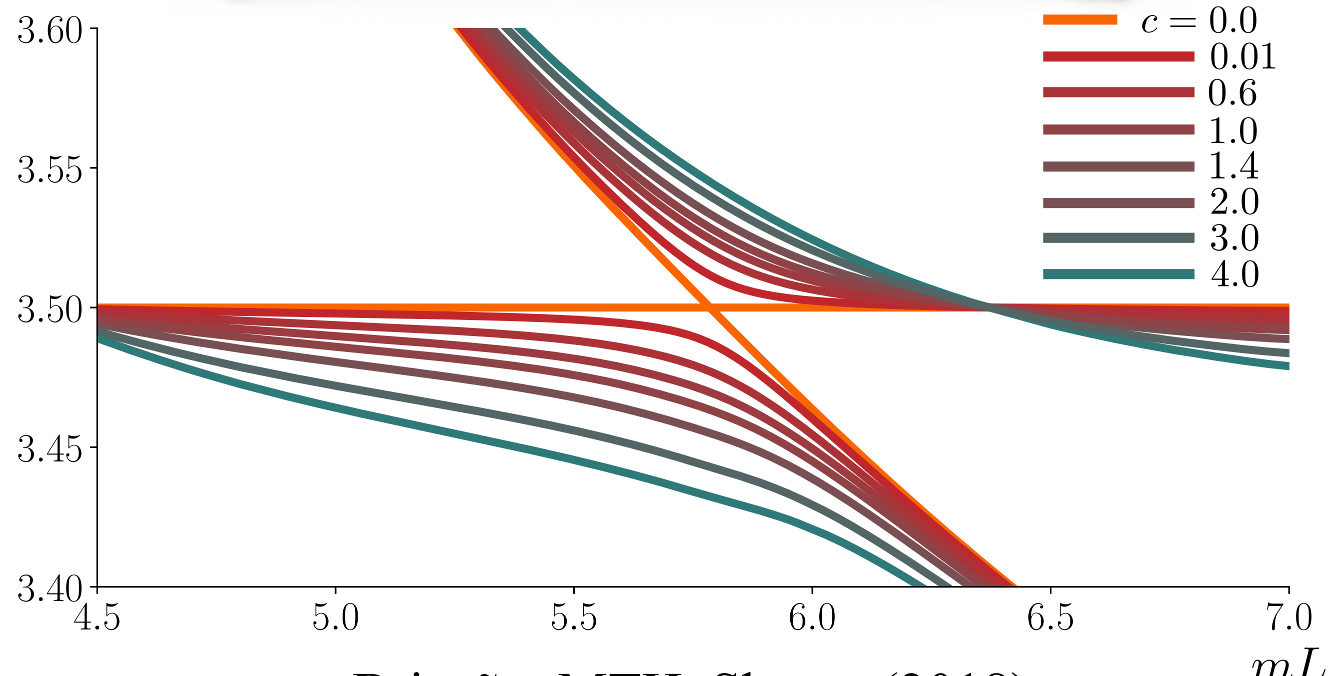
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



• Briceño, MTH, Sharpe (2018) •

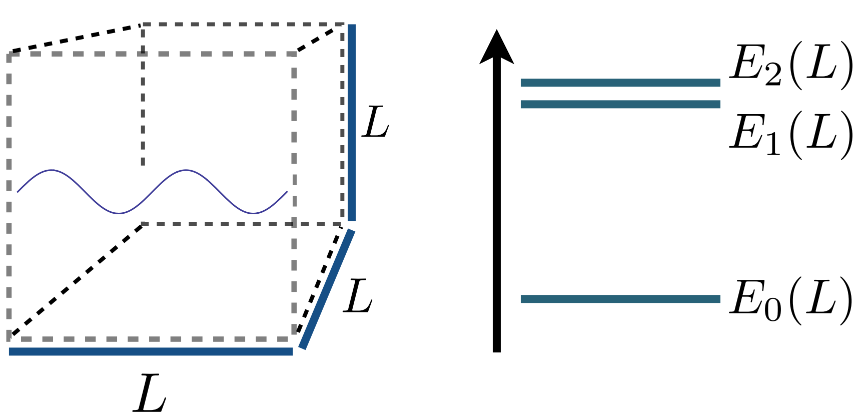
- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} round one
  - ☑ Mass in $\lambda\phi^4$
  - ☑ Mass/matrix element in $g\phi^3$
- ☑ $2 \to 2$ formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Example application
  - ☑ Generalizations
- ☑ e^{-mL} round two
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☑ $(1+)\mathscr{J} \to 2$ formalism
  - ☑ Derivation
  - ☑ Example application
- ☑ $2 + \mathscr{J} \to 2$ formalism
  - ☑ Derivation
  - ☑ Testing the result
  - ☑ Numerical explorations
- ☑ Non-local matrix elements
  - ☑ Derivation
  - ☑ Applications
- ☑ 3 → 3 formalism
  - ☑ New complications
  - ☑ Derivation ($E_n(L)$ to $\mathscr{K}_{\mathrm{df},3}$)
  - ☑ Integral equations ($\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$)
  - ☑ Testing the result
  - ☑ Numerical explorations/calculations
- ☐ Conclusion and outlook

Importance of the finite volume

$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock**

$|\pi\pi, \mathrm{out}\rangle, \quad |K\pi, \mathrm{out}\rangle, \quad \cdots \in$ **QCD Fock space (continuum of states)**

Relation is (highly) non-trivial



$\in$ **Discrete set of finite-volume states**

Not discussed in these lectures

- Three-hadron transitions ($K \to \pi\pi\pi, \;\; \gamma^\star \to \pi\pi\pi$)

  finite-volume methods exist

- Left-hand branch cuts

  finite-volume methods break on left-hand cuts (e.g. T_{cc}^+)

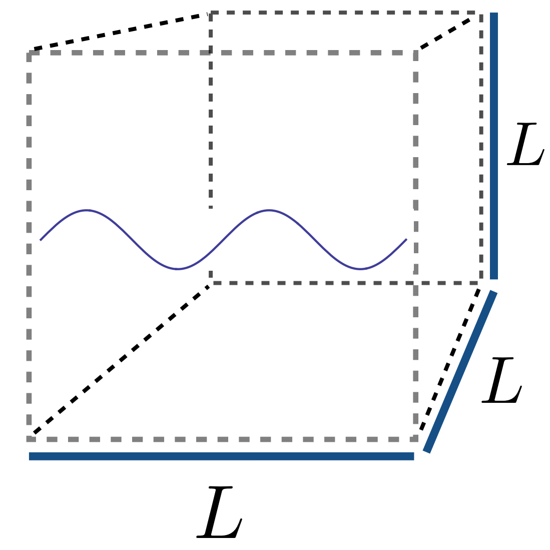
- Spectral densities from regulated inverse Laplace transform

Finite-volume setup

cubic, spatial volume (extent L)

periodic boundary conditions*

$$q(\tau, \mathbf{x}) = q(\tau, \mathbf{x} + L\mathbf{e}_i) \quad \Big| \quad \pi(\tau, \mathbf{x}) = \pi(\tau, \mathbf{x} + L\mathbf{e}_i)$$



time direction infinite*

continuum theory

*will also briefly consider finite T effects, alternative boundary conditions

$$\tilde{\pi}(\tau, \mathbf{p}) = \int_L d^3\mathbf{x}\; e^{-i\mathbf{p}\cdot\mathbf{x}}\, \pi(\tau, \mathbf{x})$$

$$= \int_L d^3\mathbf{x}\; e^{-i\mathbf{p}\cdot\mathbf{x}}\, \pi(\tau, \mathbf{x} + L\mathbf{e}_i)$$

$$= \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot(\mathbf{x} - L\mathbf{e}_i)}\, \pi(\tau, \mathbf{x})$$

$$e^{-i\mathbf{p}\cdot L\mathbf{e}_i} = 1 \quad \Longrightarrow \quad p_i L = 2\pi n_i$$

Quantization of momentum

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$$

□ Exponentially suppressed volume effects

$M(L)^2 = M^2 +$  $+ \mathcal{O}(\lambda^2)$

( = propogator

$\times$ = vertex)

 $= \frac{(-i\lambda)}{2}\frac{1}{i}\int\frac{dk^0}{2\pi}\frac{1}{L^3}\sum_{\underline{k}}\frac{1}{-(k^0)^2+\underline{k}^2+M^2-i\epsilon}$

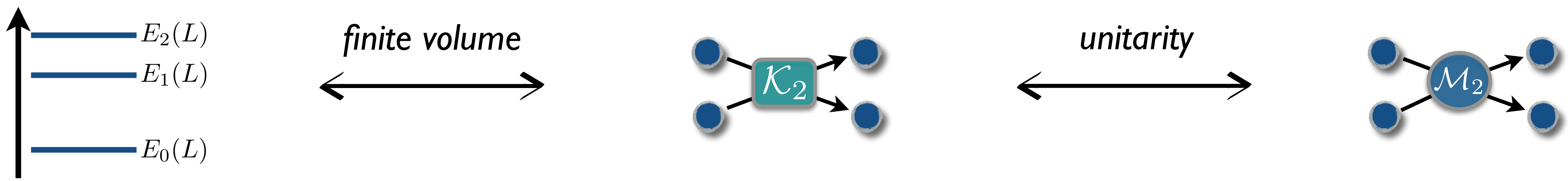
$$= \frac{\lambda}{2}\int\frac{dk_4}{2\pi}\frac{1}{L^3}\sum_{\underline{k}}\frac{1}{k^2+M^2} = \frac{\lambda}{2}\sum_{n}\int\frac{d^4k}{(2\pi)^4}e^{iLn\cdot k}\int_0^\infty d\alpha\, e^{-\alpha(k^2+M^2)}$$

$=$  $+ M^2\frac{\lambda}{32\pi^2}\sum_{\underline{n}\neq\underline{0}}\int_0^\infty d\alpha\,\frac{1}{\alpha^2}e^{-\alpha-\frac{1}{\alpha}\left[\frac{M^2L^2n^2}{4}\right]}$

$$\frac{M(L)^2-M_{phys}^2}{M_{phys}^2} = \frac{\lambda}{32\pi^2}\sum_{\underline{n}\neq\underline{0}}\frac{K_1(ML|\underline{n}|)}{ML|\underline{n}|} = \frac{6\lambda}{32\pi^2}\frac{1}{ML}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{ML}}e^{-ML}$$

Result

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

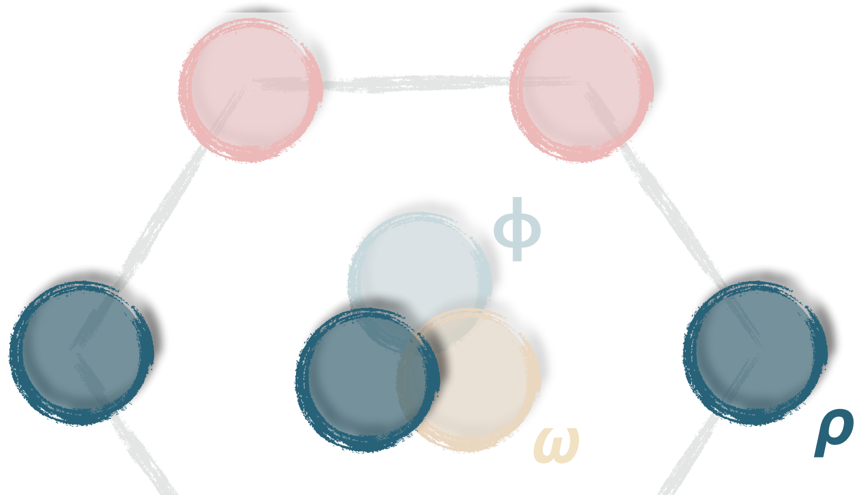
Generalized to *non-degenerate masses, multiple channels, spinning particles*

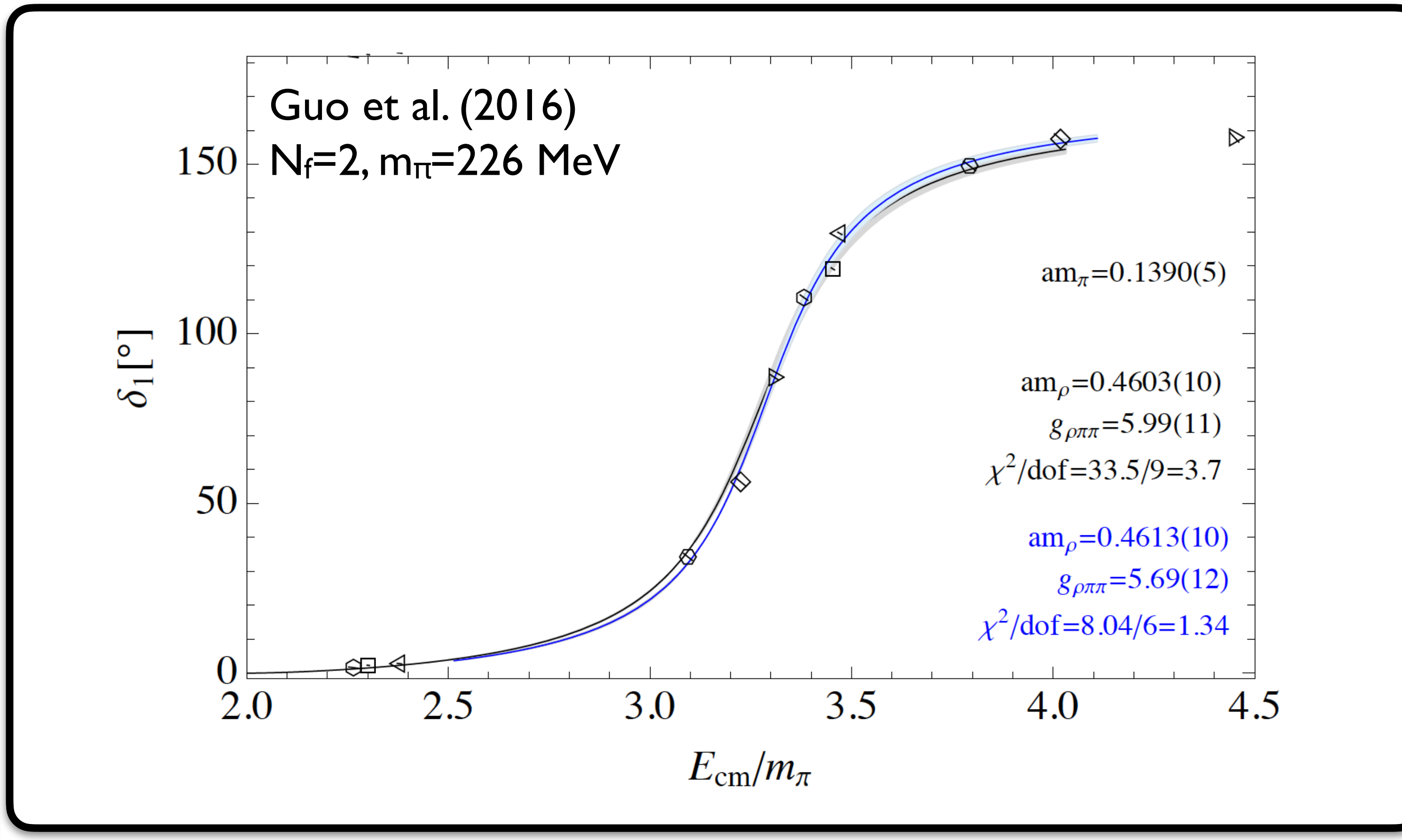
Encodes angular momentum mixing

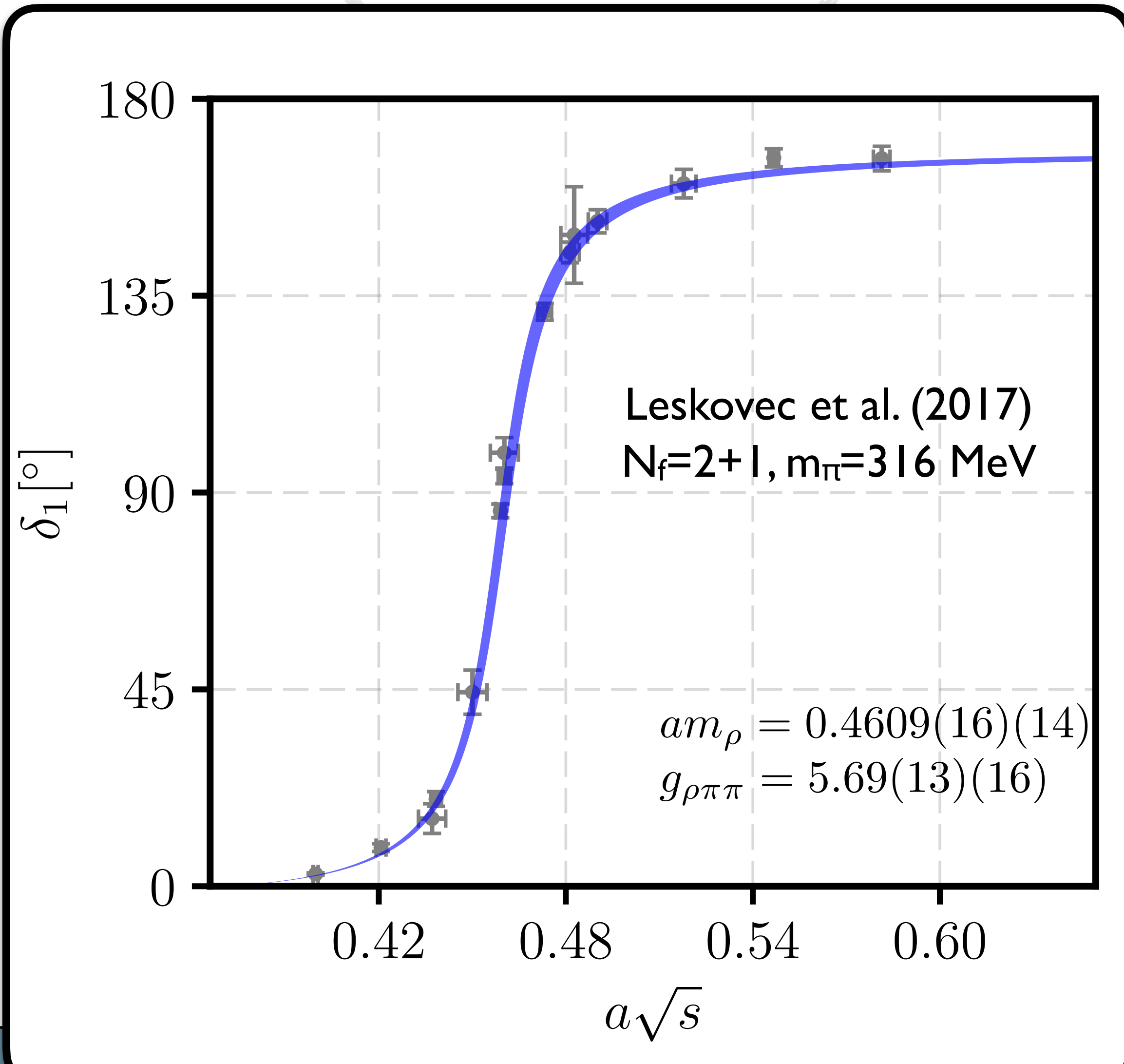
Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

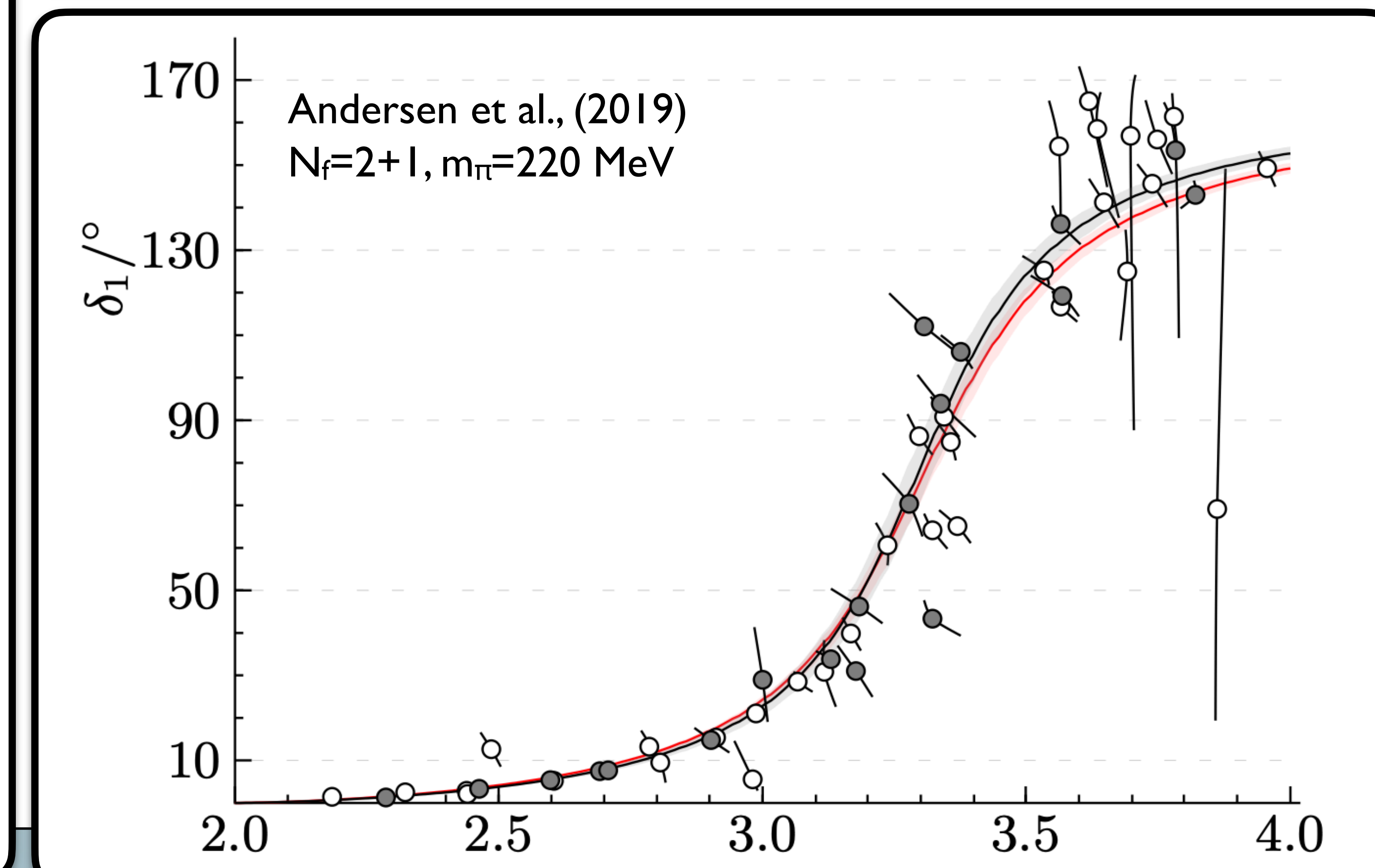
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$

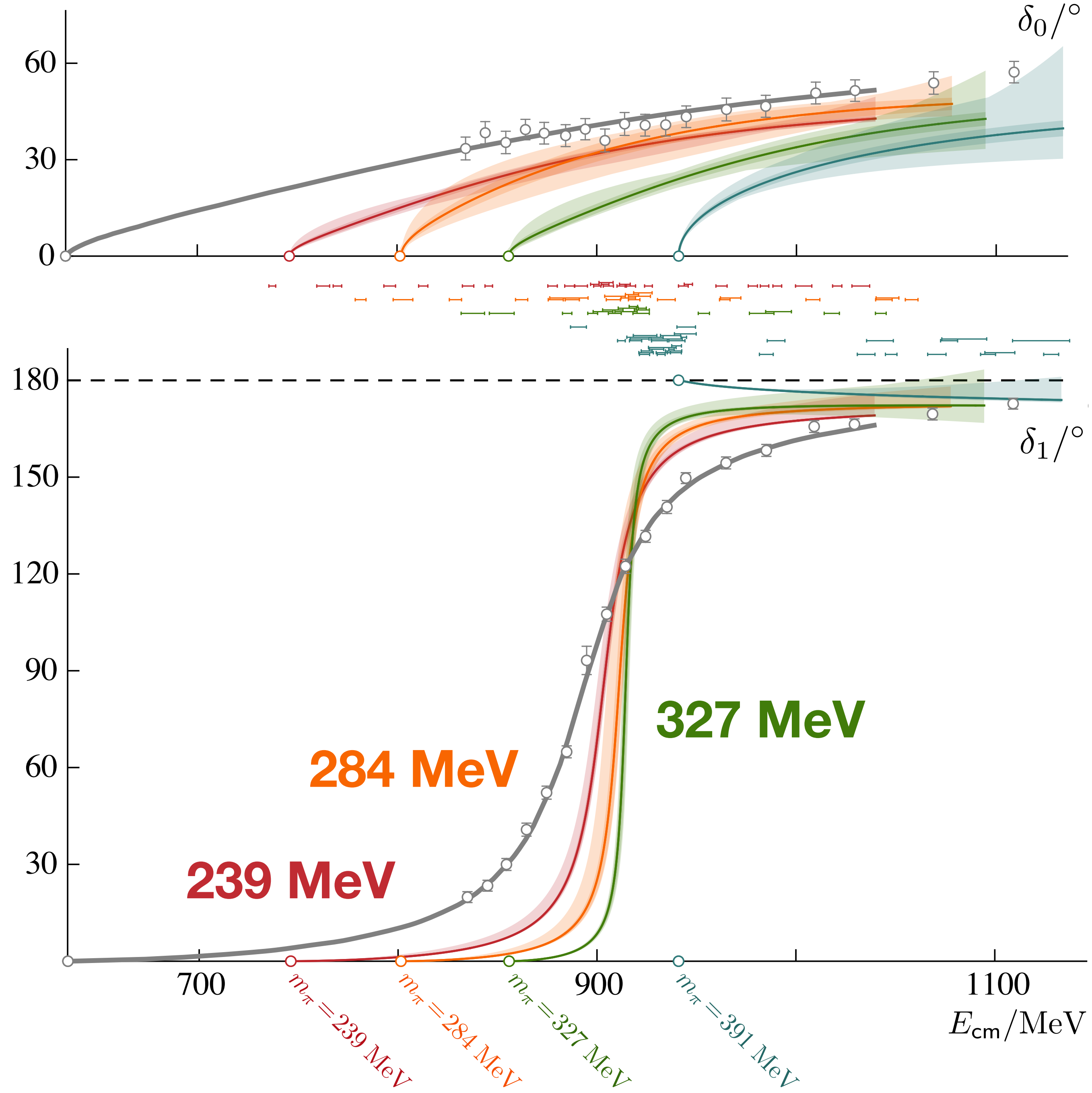


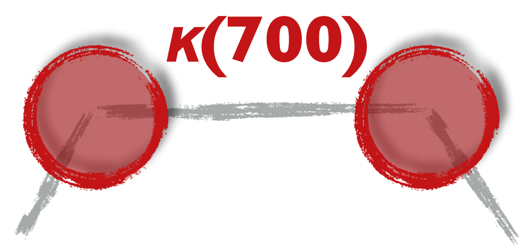




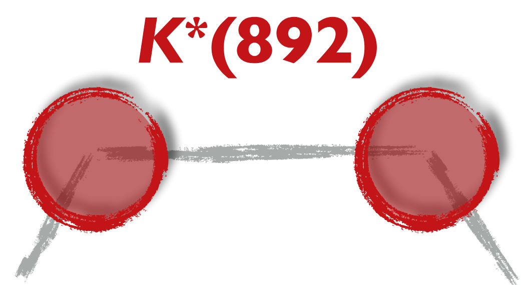


$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

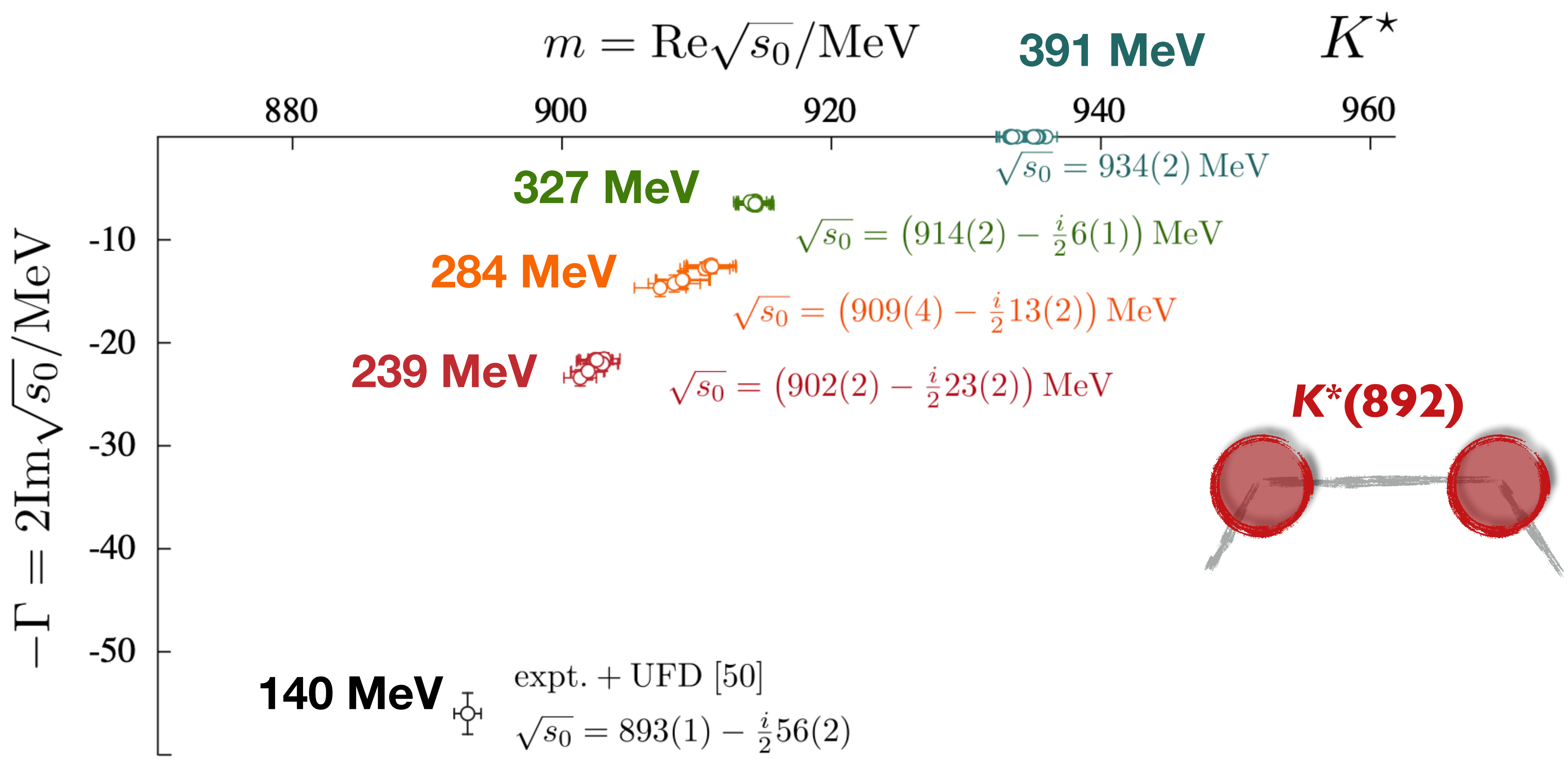


$$I(J^P) = 1/2(1^-)$$

- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

$\kappa, K^* \to K\pi$

$$I(J^P) = 1/2(1^-)$$



- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

Conclusions

- LQCD is in the era of 'rigorous resonance spectroscopy'
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → *ground work is now set*

  many calculations at unphysical quark masses → *physical-mass scattering now appearing*

  → *varying masses probes resonance structure*

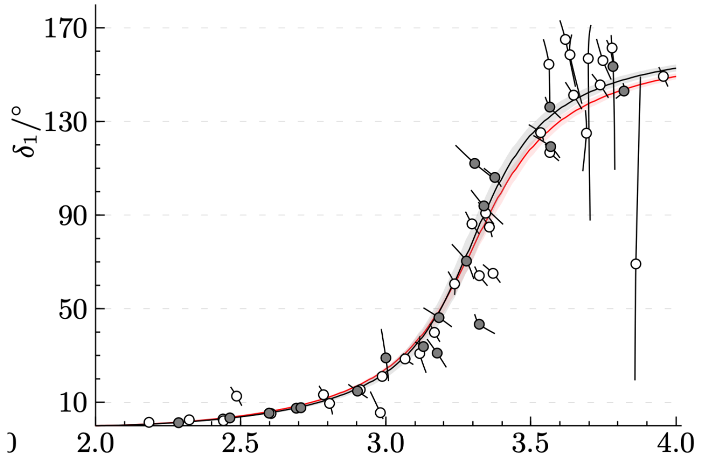
- Next steps…

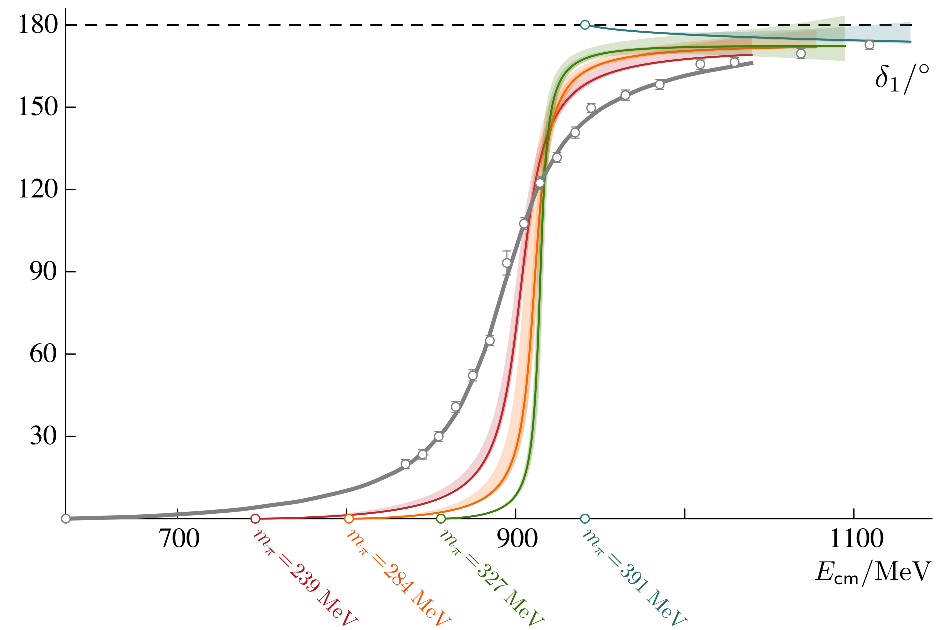
  complete 3-particle formalism → *extend to N-particle formalism*

  extend studies involving an external current

  push more channels into the precision regime

Big Picture





A thriving field, with much more to come…

Thanks for listening!