

Spectroscopy and Scattering: Formalism ***(Lecture 2/3)***

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July 21-22, 2025

☒ Warm-up and definitions

- ☒ Meaning of Euclidean
- ☒ Finite-volume set-up

☒ e^{-mL} round one

- ☒ Mass in $\lambda\phi^4$
- ☒ Mass/matrix element in $g\phi^3$

☒ $2 \rightarrow 2$ formalism

- ☒ Scattering basics
- ☐ Derivation
- ☐ Example application
- ☐ Generalizations

☐ e^{-mL} round two

- ☐ LO-HVP for $(g - 2)_\mu$
- ☐ Bethe-Salpeter kernel

☐ $(1+)\mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Example application

☐ $2 + \mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Testing the result
- ☐ Numerical explorations

☐ Non-local matrix elements

- ☐ Derivation
- ☐ Applications

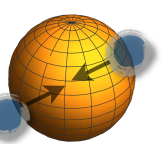
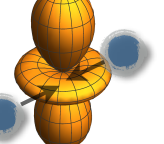
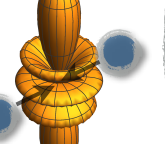
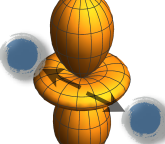
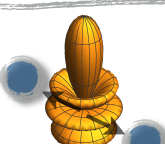
☐ $3 \rightarrow 3$ formalism

- ☐ New complications
- ☐ Derivation ($E_n(L)$ to $\mathcal{K}_{\text{df},3}$)
- ☐ Integral equations ($\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$)
- ☐ Testing the result
- ☐ Numerical explorations/calculations

☐ Conclusion and outlook

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d$, $K \sim \bar{s}u$, $p \sim uud$
- Overlaps of multi-hadron *asymptotic states* $\rightarrow$ S matrix

		$ \pi\pi, \text{in}\rangle$			
					
		$e^{2i\delta_0(s)}$	0	0	depends on $s = E_{\text{cm}}^2$ and angular variables
$S(s) \equiv \langle \pi\pi, \text{out} $		0	$e^{2i\delta_1(s)}$	0	diagonal in angular momentum
		0	0	$e^{2i\delta_2(s)}$	
		0	0	$e^{2i\delta_2(s)}$	

$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

- An enormous space of information

$$|\pi\pi\pi\pi, \text{in}\rangle \quad |K\bar{K}, \text{in}\rangle \quad \dots$$

Bethe Salpeter equation

□ All orders diagrammatic expansion

$$\begin{aligned} \mathcal{M} = & \text{[diagrammatic expansion of } \mathcal{M} \text{ in terms of loops and vertices]} \\ = & \text{[series expansion in terms of } \textcircled{\hspace{0.5cm}} \text{ and } \text{---}\bullet\text{---}] + \dots \end{aligned}$$

□ Construct $\textcircled{\hspace{0.5cm}}$ and $\text{---}\bullet\text{---}$ such that this is true

$$\textcircled{\hspace{0.5cm}} =$$

$$\text{---}\bullet\text{---} =$$

Bethe Salpeter equation

□ All orders diagrammatic expansion

$$\begin{aligned} \mathcal{M} = & \text{[diagrammatic expansion of } \mathcal{M} \text{ in terms of various loop and tree diagrams]} \\ = & \text{[diagrammatic expansion of } \mathcal{M} \text{ in terms of } \textcircled{\bullet} \text{ and } \text{---}\bullet\text{---}] \end{aligned}$$

□ Construct $\textcircled{\bullet}$ and $\text{---}\bullet\text{---}$ such that this is true

$$\textcircled{\bullet} = \text{[diagrammatic representation]}$$

$$\text{---}\bullet\text{---} =$$

Bethe Salpeter equation

□ All orders diagrammatic expansion

$$\mathcal{M} = \begin{array}{c} \text{[Diagrammatic expansion of } \mathcal{M} \text{ in terms of } \text{---}\blacksquare\text{---} \text{ and } \text{---}\bullet\text{---}] \\ \text{---}\blacksquare\text{---} + \boxed{\text{---}\blacksquare\text{---}\bullet\text{---}\blacksquare\text{---}} + \text{---}\blacksquare\text{---}\bullet\text{---}\bullet\text{---}\blacksquare\text{---} + \dots \end{array}$$

□ Construct  and  such that this is true

$$\text{blue circle} = \text{---}\blacksquare\text{---}$$

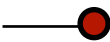
$$\text{---}\bullet\text{---} = \text{---}\text{---}$$

Bethe Salpeter equation

□ All orders diagrammatic expansion

$$\mathcal{M} = \begin{array}{ccccccccc} \text{[diagram 1]} & \text{[diagram 2]} & \boxed{\text{[diagram 3]} \quad \text{[diagram 4]}} & \text{[diagram 5]} & \text{[diagram 6]} & \text{[diagram 7]} & \text{[diagram 8]} & \text{[diagram 9]} & \text{[diagram 10]} \\ \text{[diagram 11]} & \text{[diagram 12]} & \text{[diagram 13]} & \text{[diagram 14]} & \text{[diagram 15]} & \text{[diagram 16]} & \text{[diagram 17]} & \text{[diagram 18]} & \text{[diagram 19]} \\ \text{[diagram 20]} & \text{[diagram 21]} & \text{[diagram 22]} & \text{[diagram 23]} & \text{[diagram 24]} & \text{[diagram 25]} & \text{[diagram 26]} & \text{[diagram 27]} & \text{[diagram 28]} \end{array}$$

$$= \boxed{\text{[diagram 3]}} + \text{[diagram 29]} + \text{[diagram 30]} + \text{[diagram 31]} + \dots$$

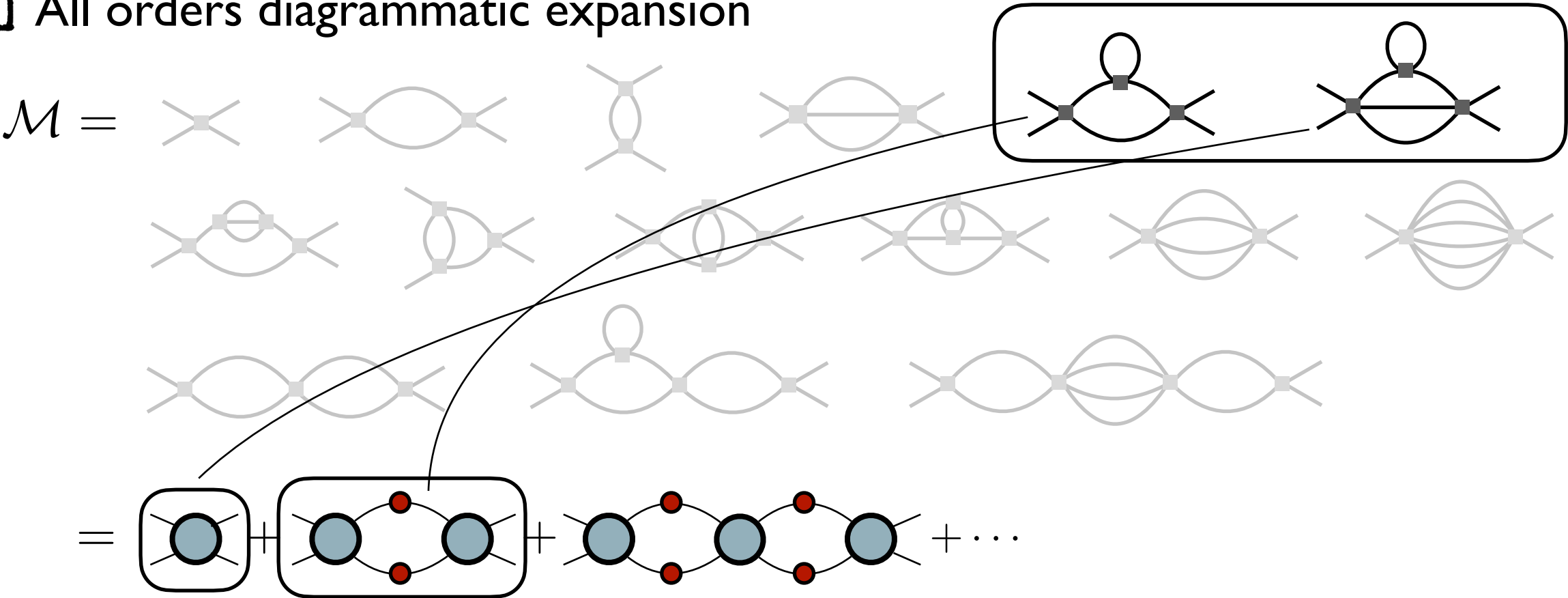
□ Construct  and  such that this is true

$$\text{[blue circle]} = \text{[diagram 3]} + \text{[diagram 32]} + \text{[diagram 33]}$$

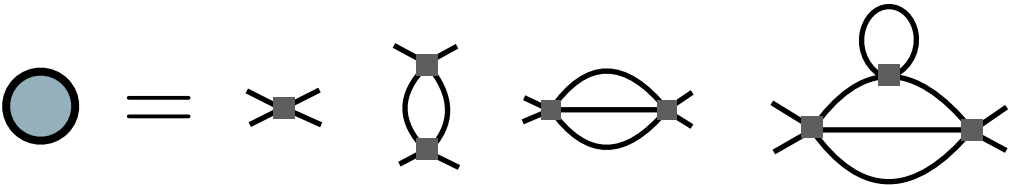
$$\text{[line with red dot]} = \text{[line]}$$

Bethe Salpeter equation

□ All orders diagrammatic expansion



□ Construct and such that this is true



Bethe Salpeter equation

□ All orders diagrammatic expansion

$$\mathcal{M} = \begin{array}{cccccc} \text{[diagram 1]} & \text{[diagram 2]} & \text{[diagram 3]} & \text{[diagram 4]} & \text{[diagram 5]} & \text{[diagram 6]} \\ \text{[diagram 7]} & \text{[diagram 8]} & \text{[diagram 9]} & \text{[diagram 10]} & \text{[diagram 11]} & \text{[diagram 12]} \\ \text{[diagram 13]} & \text{[diagram 14]} & \text{[diagram 15]} & & & \end{array}$$

$$= \text{[blue circle]} + \text{[blue circle]} \text{---} \text{[red dot]} \text{---} \text{[blue circle]} + \text{[blue circle]} \text{---} \text{[red dot]} \text{---} \text{[blue circle]} \text{---} \text{[red dot]} \text{---} \text{[blue circle]} + \dots$$

□ Construct  and  such that this is true

$$\text{[blue circle]} = \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} + \text{[diagram 4]}$$

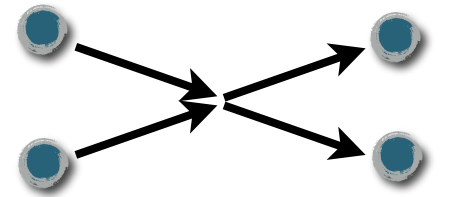
Bethe Salpeter kernel
(2PI in the s-channel)

$$\text{[line with red dot]} = \text{[line]} + \text{[line with loop]} + \text{[line with two loops]} + \text{[line with bubble]}$$

Fully dressed propagator

Analyticity of $\mathcal{M}$ (from B.S. kernel)

For two-particle energies $(2m)^2 < s < (3m)^2$, what is the analytic structure?



$$\mathcal{M}(s) \equiv \text{[diagram of a single vertex]} + \text{[diagram of two vertices connected by a loop with } i\epsilon \text{]} + \text{[diagram of three vertices connected by two loops with } i\epsilon \text{]} + \dots$$

non-analytic:
on-shell particles = singularities

— propagating pion

$$\text{[diagram of two vertices connected by a loop with } i\epsilon \text{]} = \text{[diagram of two vertices connected by a loop with PV]} + \text{[diagram of two vertices connected by a loop with a dashed line and } \rho(s) \text{]}$$

$\rho(s) \propto \sqrt{s - (2m)^2}$

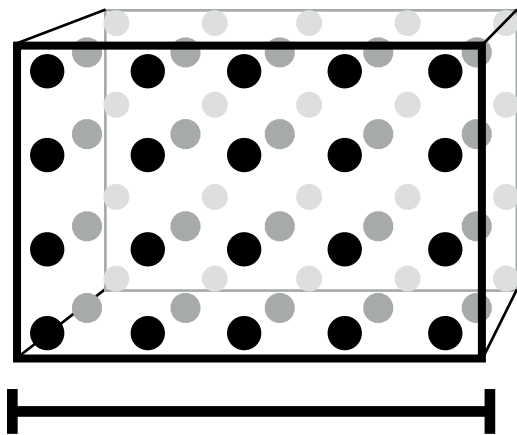
cutting rule

defines the K matrix

$$= \left[\text{[diagram of a single vertex]} + \text{[diagram of two vertices connected by a loop with PV]} + \dots \right] + \left[\text{[diagram of a single vertex]} + \text{[diagram of two vertices connected by a loop with PV]} + \dots \right] \text{[diagram of a loop with a dashed line and } \rho(s) \text{]} \left[\text{[diagram of a single vertex]} + \text{[diagram of two vertices connected by a loop with PV]} + \dots \right] + \dots$$

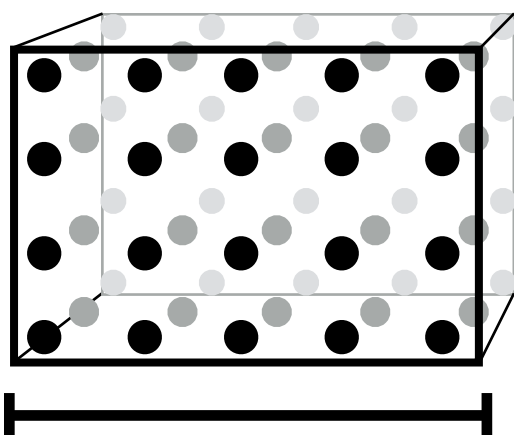
$$= \mathcal{K}(s) + \mathcal{K}(s)i\rho(s)\mathcal{K}(s) + \dots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

branch-cut singularity
 $\sqrt{s - (2m)^2}$

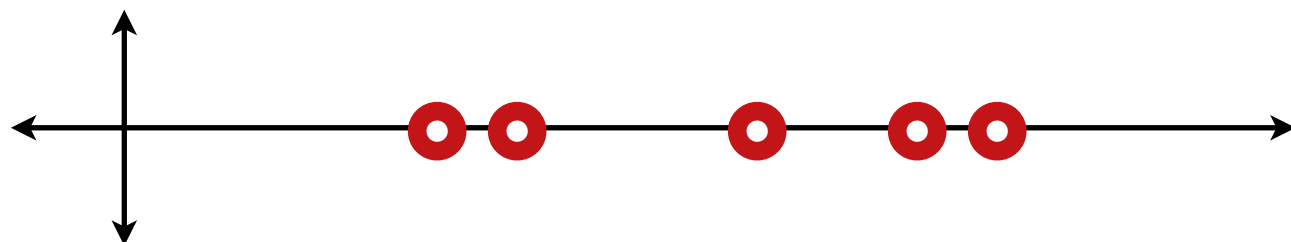


□ The *finite volume*...

- *Discretizes* the spectrum
- *Eliminates* the branch cuts and extra sheets
- *Hides* the resonance poles



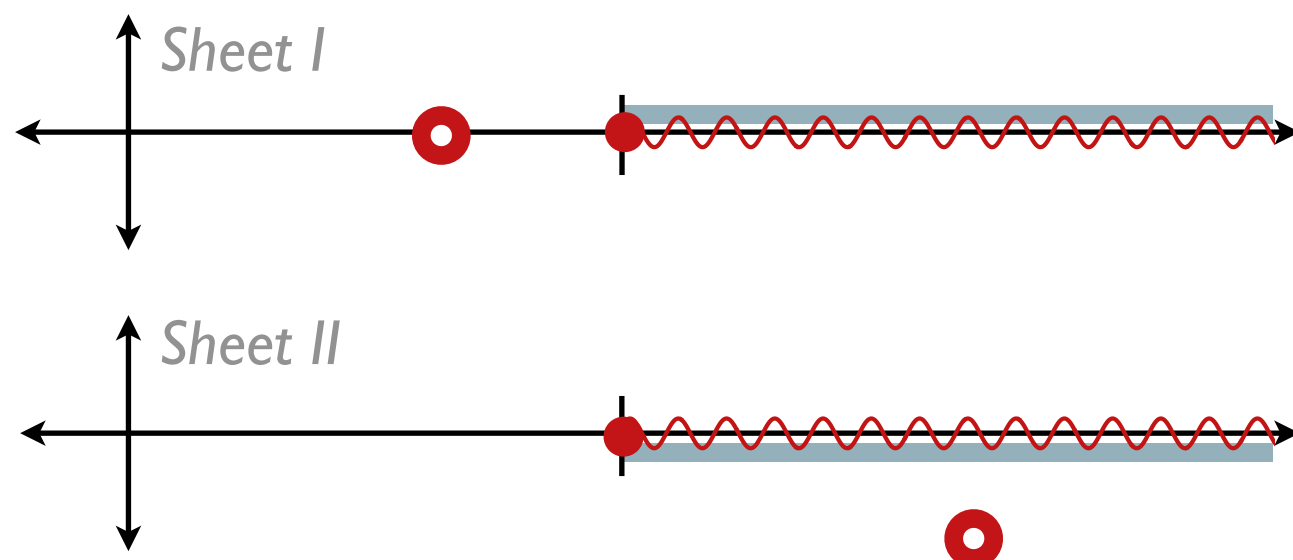
Finite-volume analytic structure

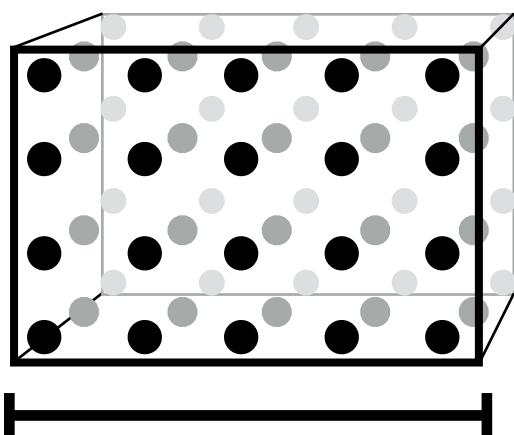


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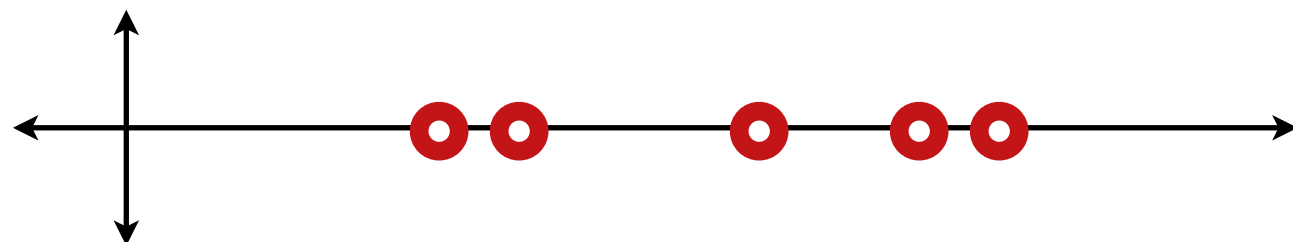
- *Discretizes* the spectrum
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Infinite-volume analytic structure





Finite-volume analytic structure



□ LQCD → Energies and matrix elements

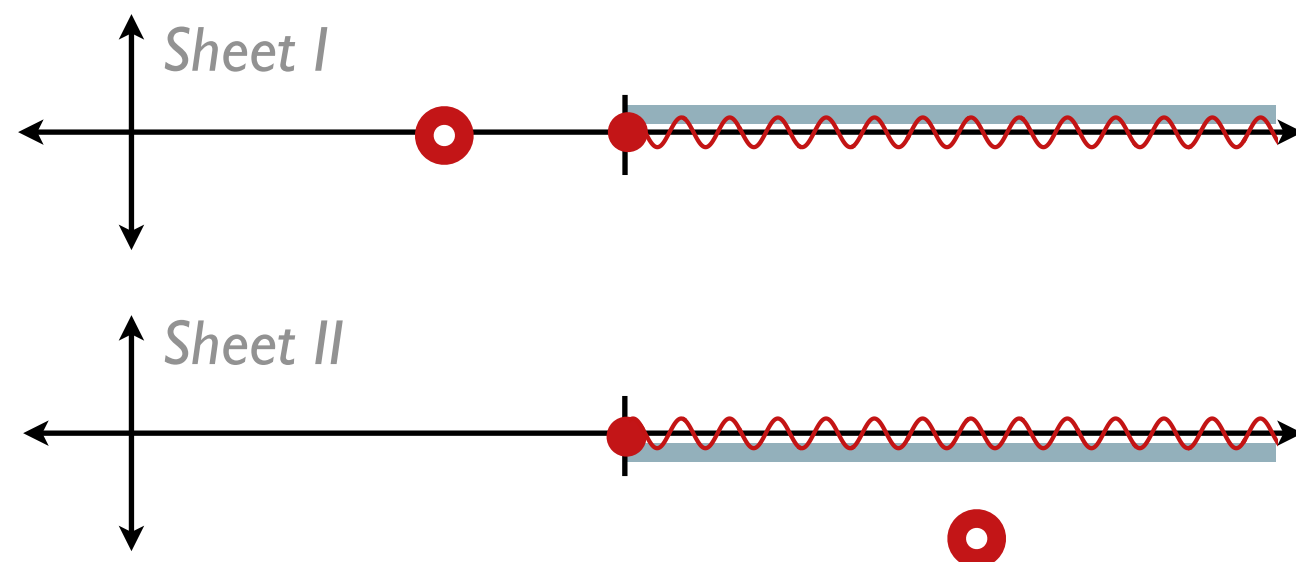
$$\langle \mathcal{O}_j(\tau) \mathcal{O}_i^\dagger(0) \rangle = \sum_n \langle 0 | \mathcal{O}_j(\tau) | E_n \rangle \langle E_n | \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

□ Our task is relate $E_n(L)$ and $\langle E_{m'} | \mathcal{J}(0) | E_m \rangle$ to **experimental observables**

□ The *finite volume*...

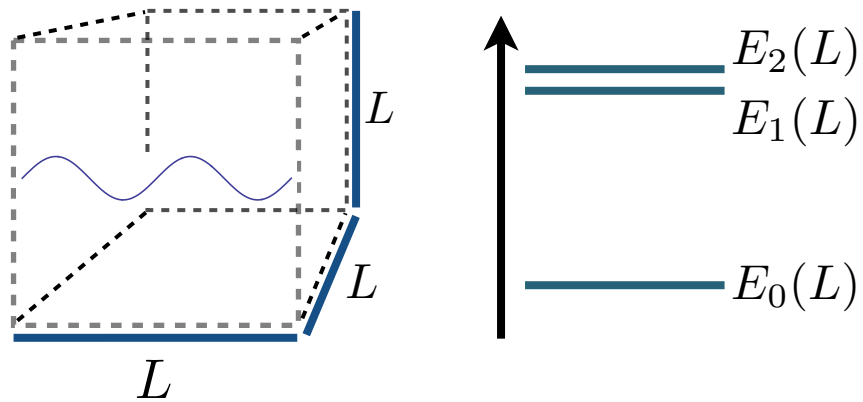
- *Discretizes* the spectrum
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Infinite-volume analytic structure



The finite-volume as a tool

□ Finite-volume set-up



□ **cubic**, spatial volume (extent L)

□ **periodic**

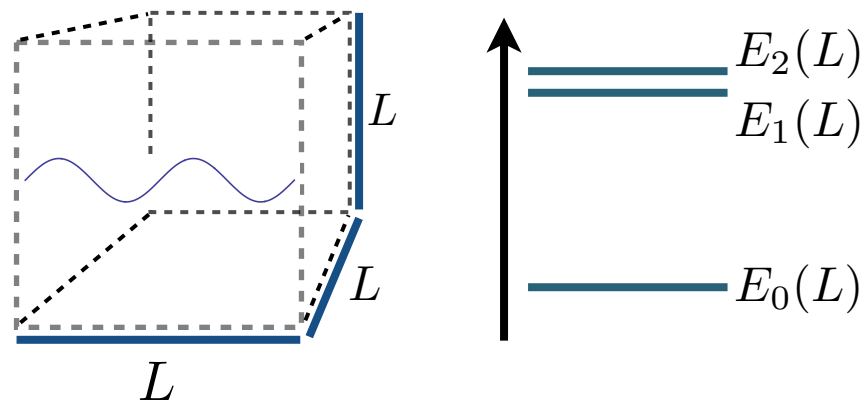
$$\vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3$$

□ L is large enough to neglect $e^{-M_\pi L}$

□ T and lattice also negligible

The finite-volume as a tool

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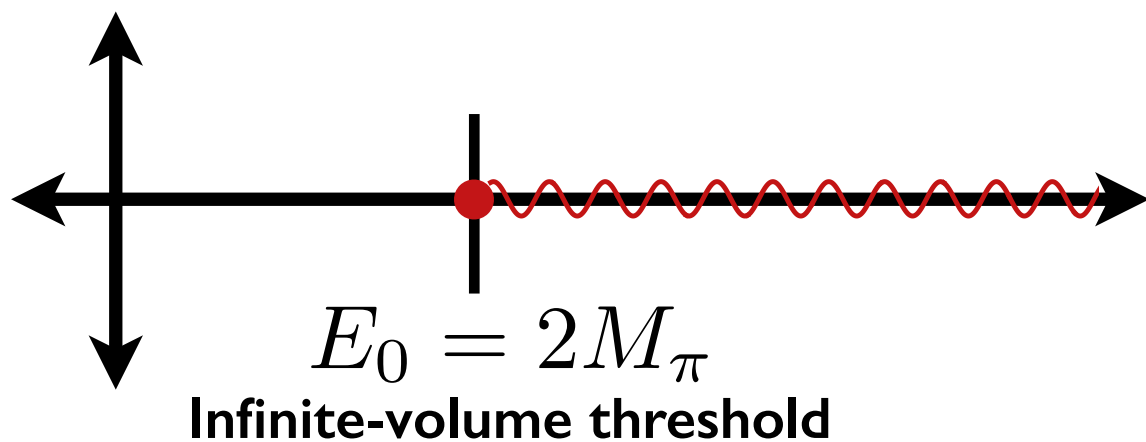
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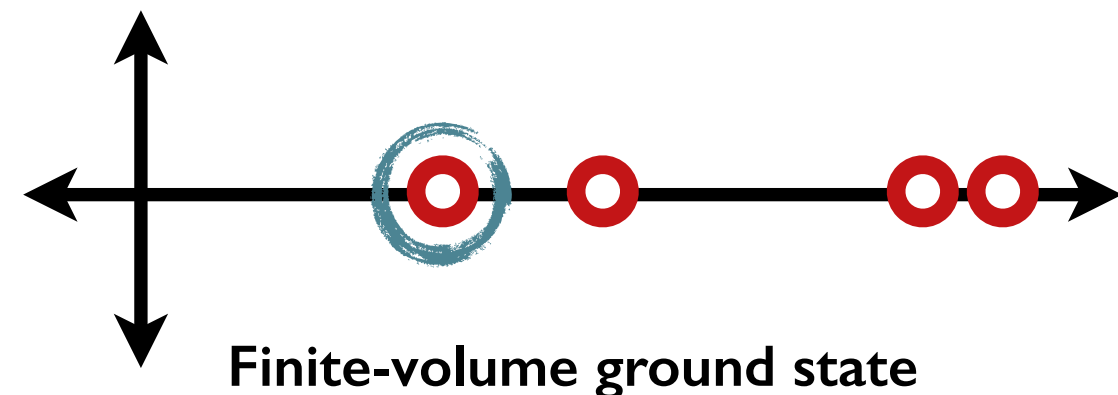
□ L is large enough to neglect $e^{-M_\pi L}$

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□ Scattering leaves an *imprint* on finite-volume quantities



$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

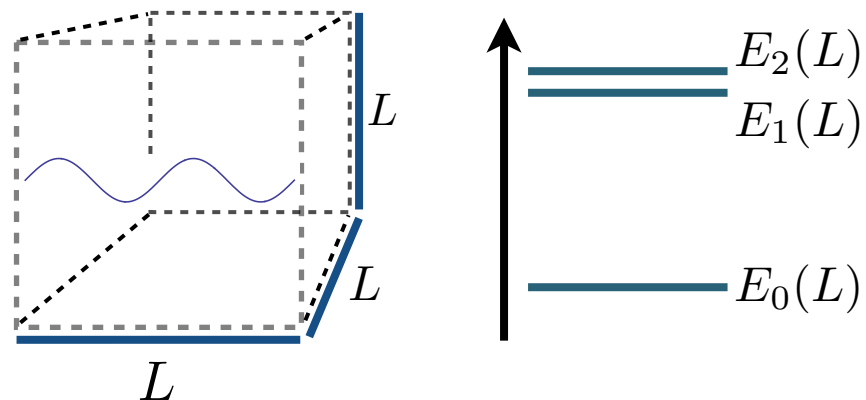


$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

The finite-volume as a tool

Finite-volume set-up



□ **cubic**, spatial volume (extent L)

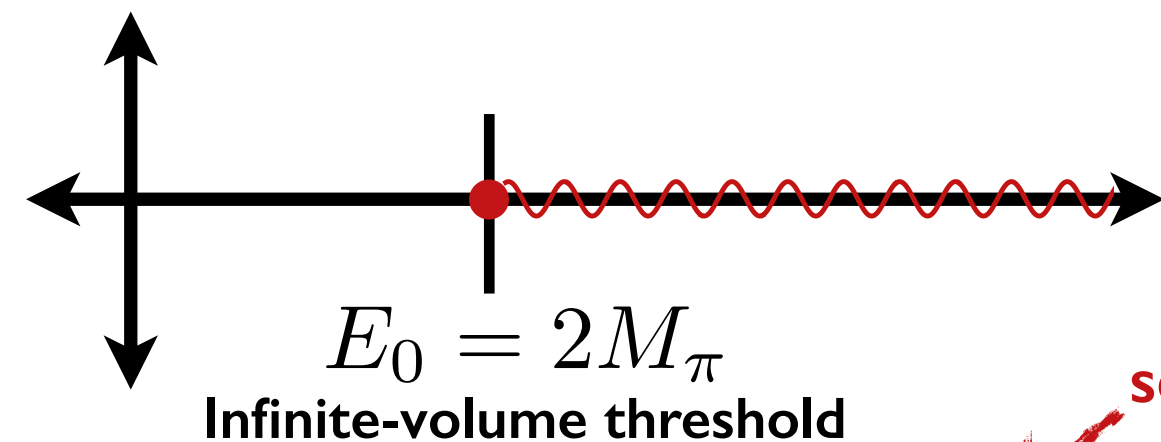
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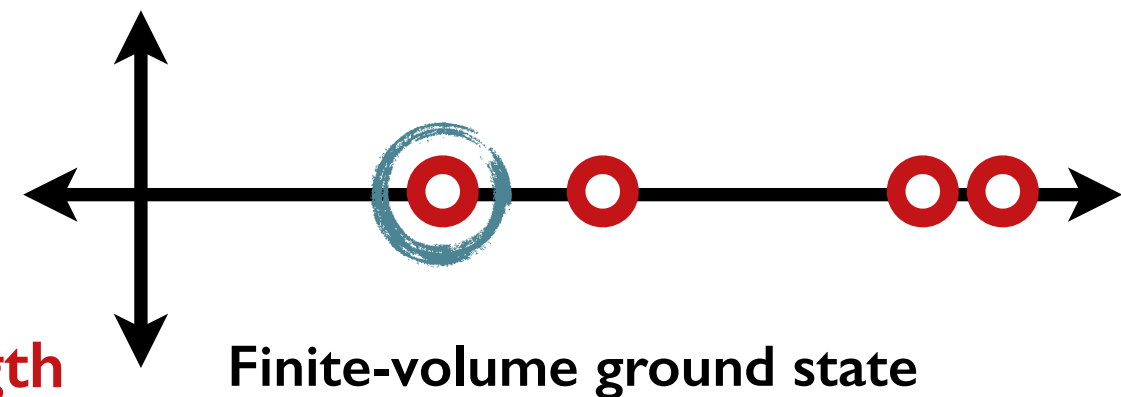
□ T and lattice also negligible

□ Scattering leaves an *imprint* on finite-volume quantities



$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

scattering length



$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

Derivation

□ Consider the finite-volume correlator:

$$\mathcal{M}_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots \quad \square = \sum_{\mathbf{k}}$$

e^{-mL} L $1/L^n$

probability amplitude

$$\mathcal{M}_L(P)$$

poles give f.v.
spectrum

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

Derivation

□ Consider the finite-volume correlator:

$$\mathcal{M}(s)$$

probability amplitude

$$\mathcal{M}_L(P)$$

poles give f.v.
spectrum

$$\mathcal{M}_L(P) = \text{diagram with one blob} + \text{diagram with two blobs and a loop} + \text{diagram with three blobs and two loops} + \dots$$

e^{-mL} $1/L^n$ $\square = \sum_{\mathbf{k}}$

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

$$\text{diagram with two blobs and a loop} = \text{diagram with two blobs and a loop labeled PV} + \text{diagram with two blobs and a loop labeled F}$$

Cut projects loop to **on-shell energies**
 F = matrix of known geometric functions

Derivation

□ Consider the finite-volume correlator:

$\mathcal{M}(s)$

probability amplitude

$\mathcal{M}_L(P)$

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$$\mathcal{M}_L(P) = \text{diagram with one blob} + \text{diagram with two blobs and a loop} + \text{diagram with three blobs and two loops} + \dots \quad \square = \sum_{\mathbf{k}}$$

e^{-mL} $1/L^n$

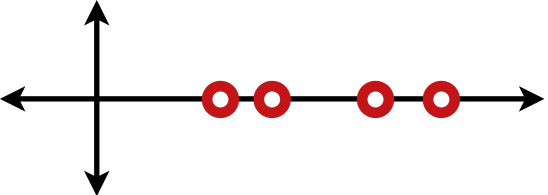
For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

$$\text{diagram with two blobs and a loop} = \text{diagram with two blobs and a PV loop} + \text{diagram with two blobs and a loop cut by } F$$

Cut projects loop to **on-shell energies**
 F = matrix of known geometric functions

Defines the K matrix

$$= \left[\text{diagram with one blob} + \text{diagram with two blobs and a PV loop} + \dots \right] - \left[\text{diagram with one blob} + \text{diagram with two blobs and a PV loop} + \dots \right] \text{ loop cut by } F \left[\text{diagram with one blob} + \text{diagram with two blobs and a PV loop} + \dots \right] + \dots$$

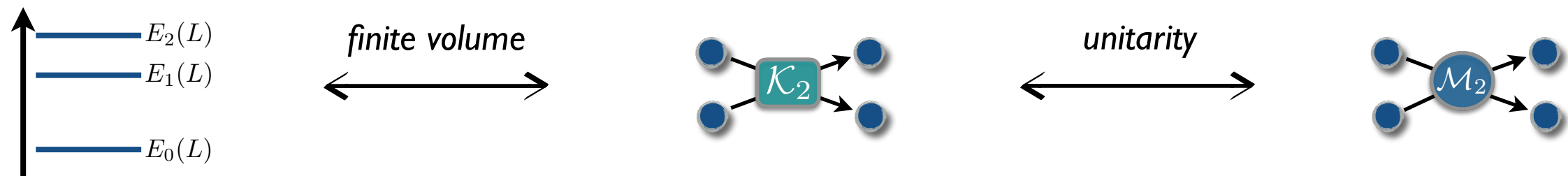
$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P, L)}$$


- Lüscher (1986)
- Kim, Sachrajda, Sharpe (2005)
- MTH, Sharpe (*coupled channels*, 2012)
-

Result

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0$$

$F(P, L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

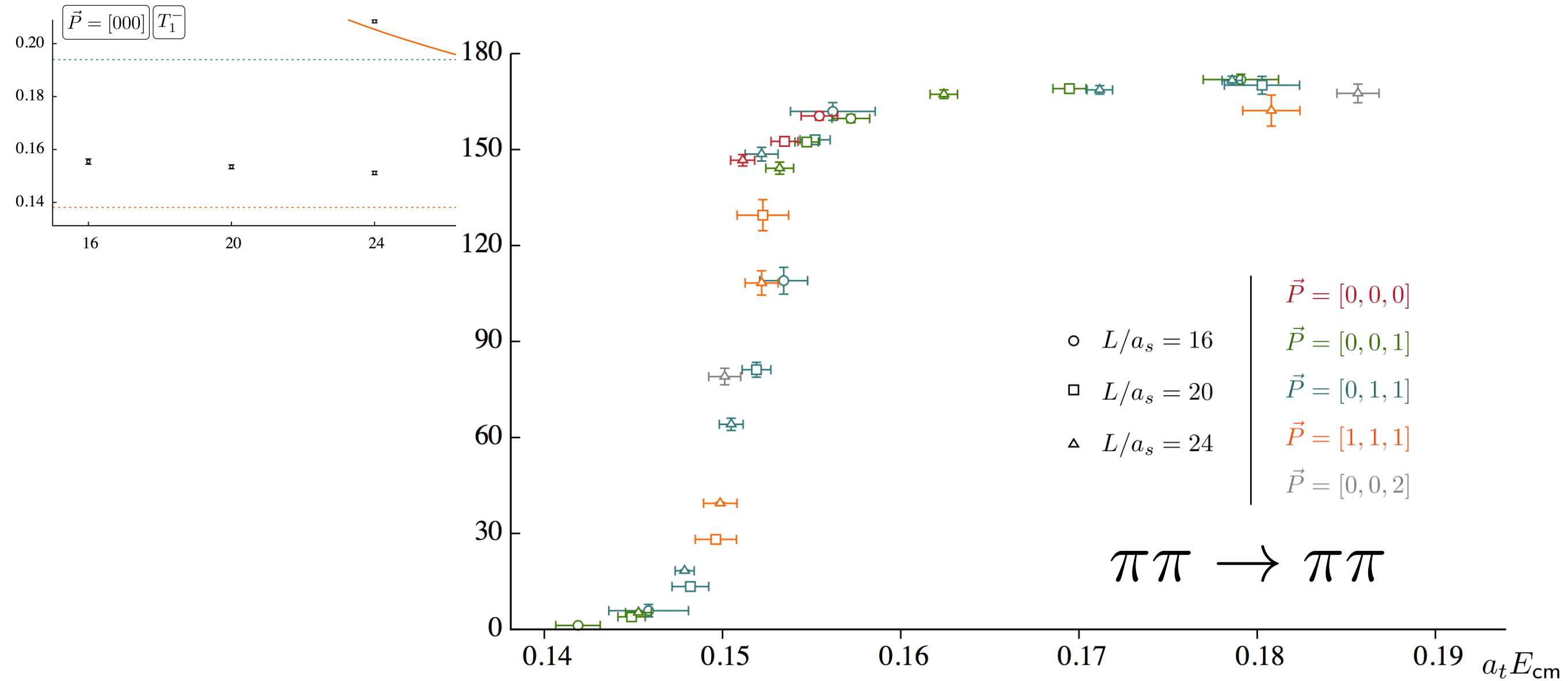
Encodes angular momentum mixing

- Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
 Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
 Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
 Li, Liu (2013) • Briceño (2014)

Using the result

□ Single-channel case (*pions in a p-wave*)

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$

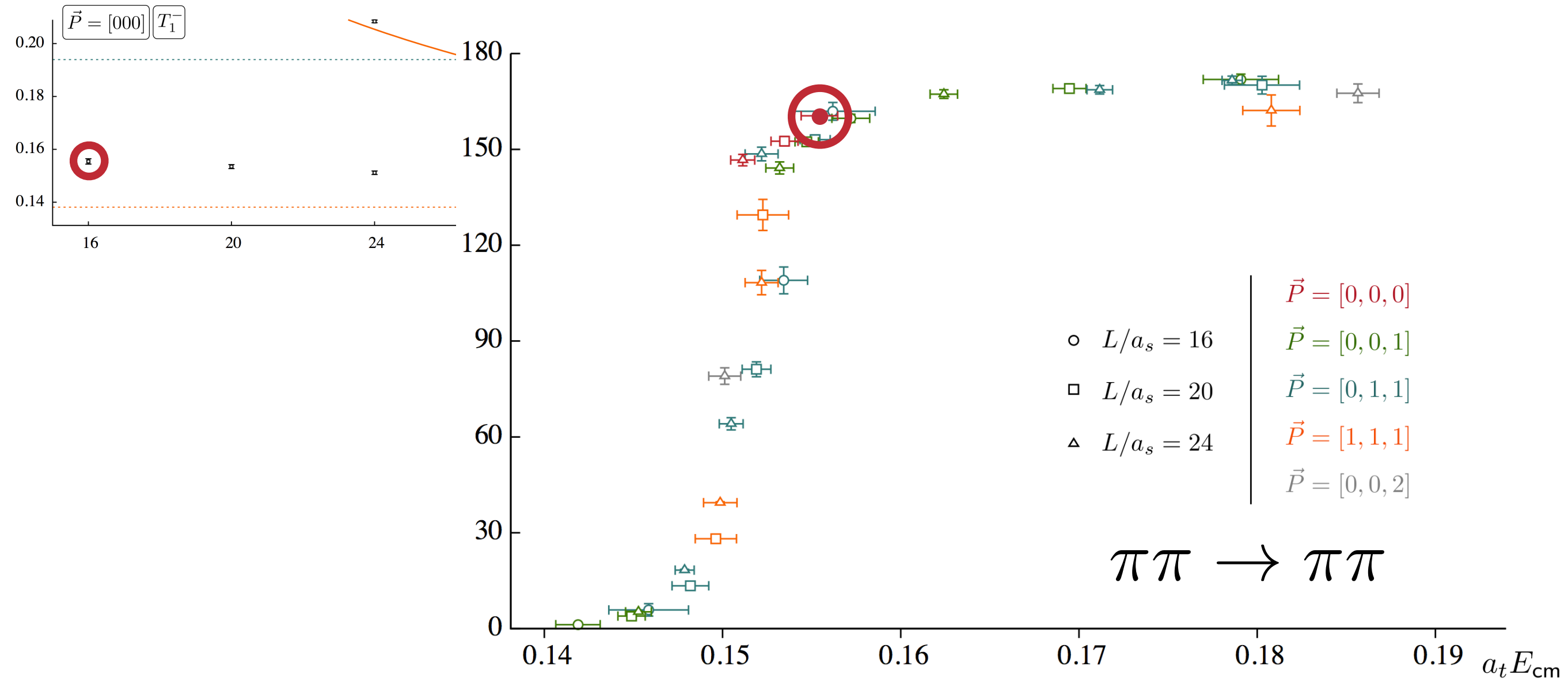


- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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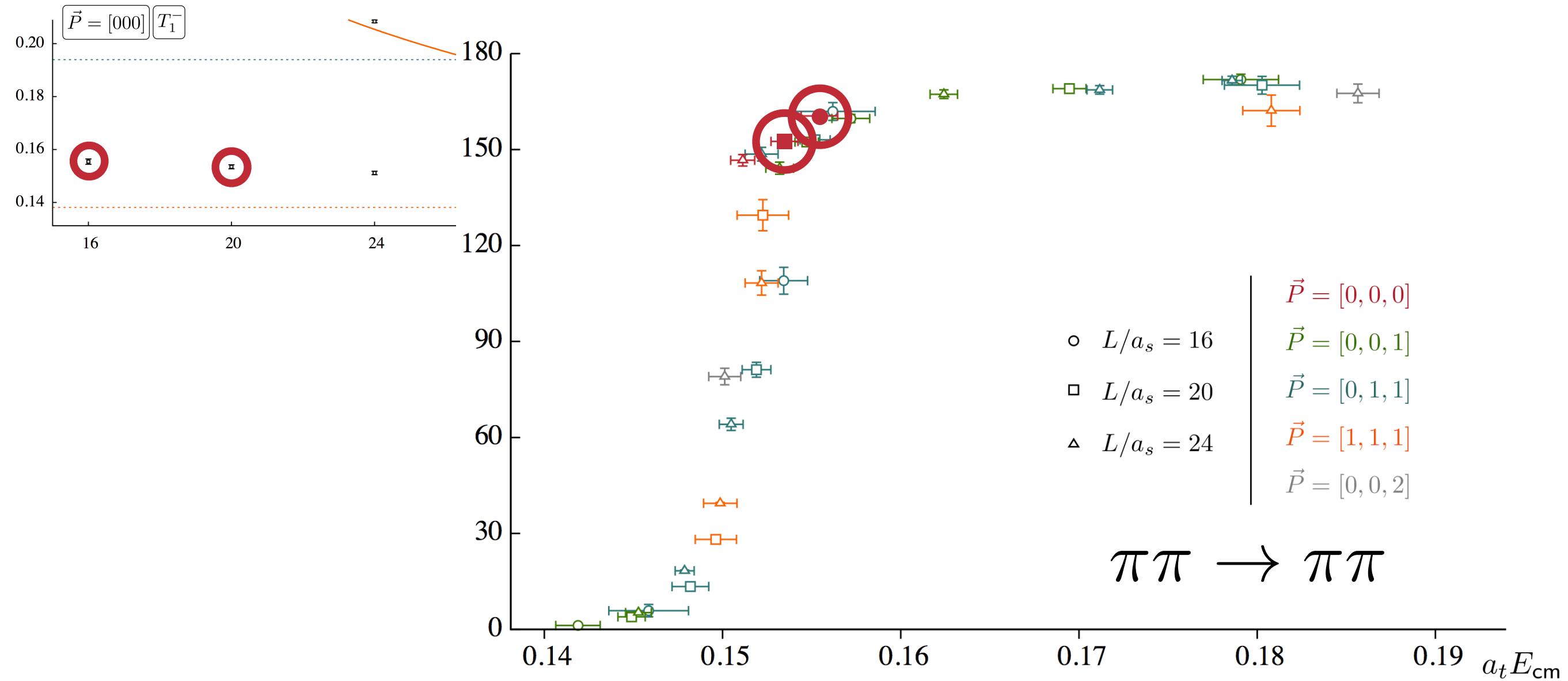


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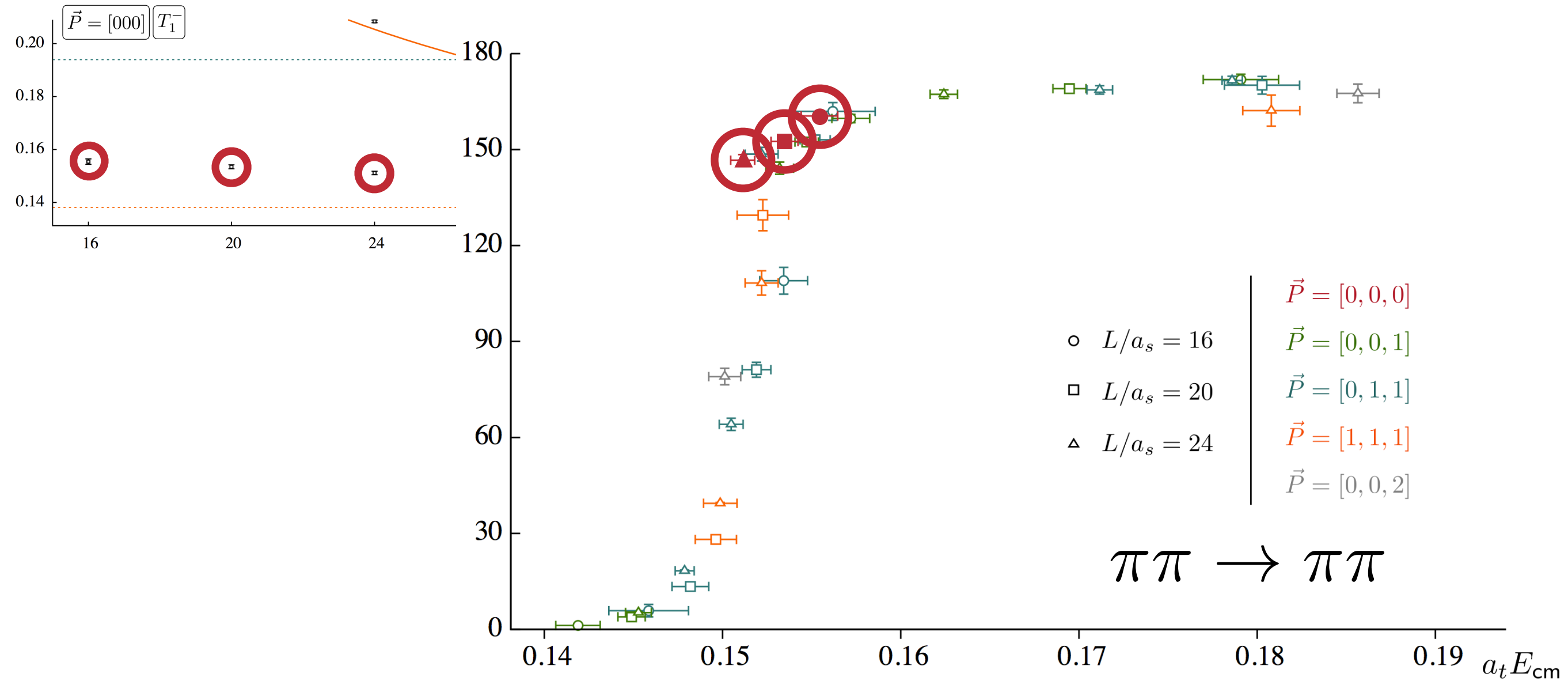


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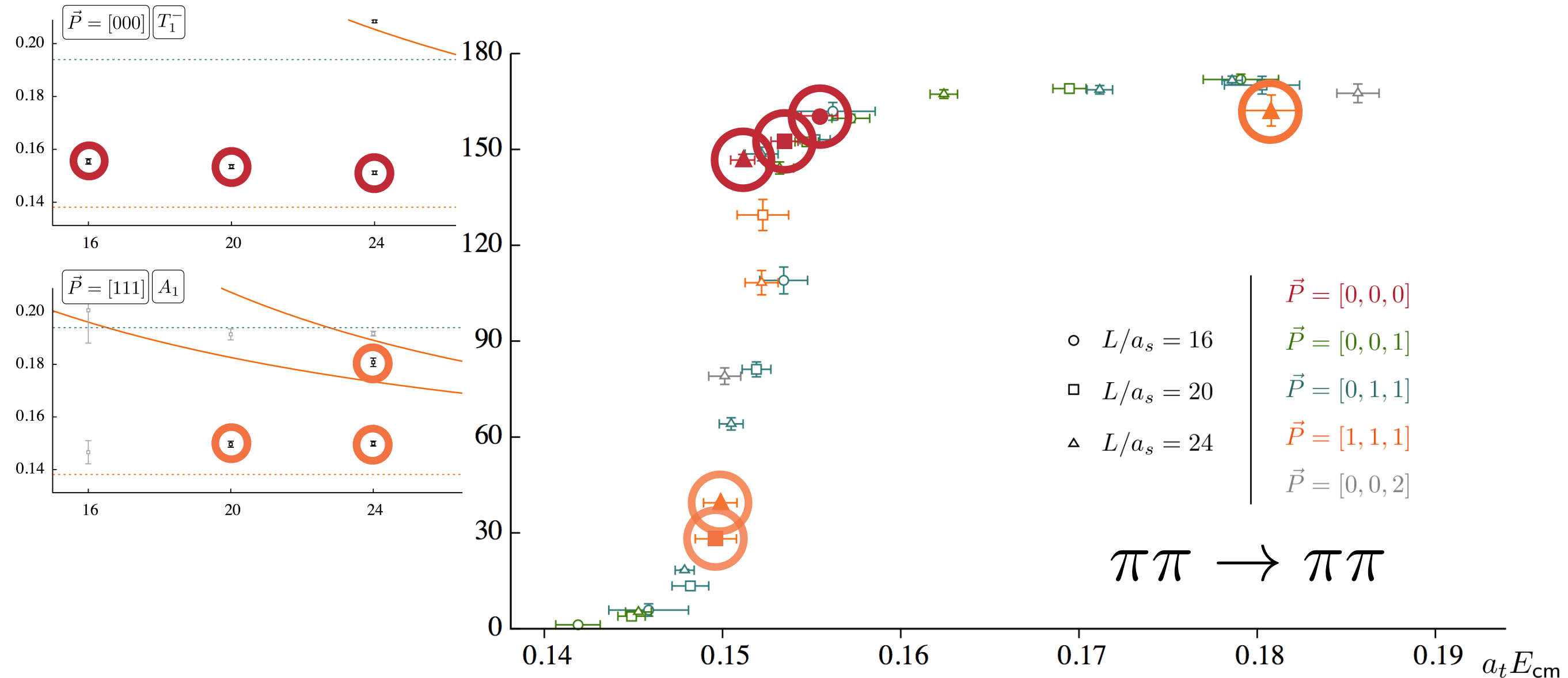


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Bethe-Salpeter kernel

$$\text{blue circle} = \text{cross} + \text{vertical oval} + \text{horizontal oval} + \text{loop oval} = B(s)$$

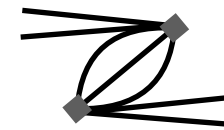
Bethe-Salpeter kernel

$$\text{blue circle} = \text{cross} \quad \text{vertical oval} \quad \text{horizontal oval} \quad \text{loop oval} = B(s)$$

- Exponentially suppressed volume effects e^{-mL} for $(2m)^2 < s < (3m)^2$
time-ordered (old fashioned) perturbation theory



$$\frac{1}{L^6} \sum_{\mathbf{p}_1, \mathbf{p}_2} \frac{1}{E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2}}$$



$$\frac{1}{L^6} \sum_{\mathbf{p}_1, \mathbf{p}_2} \frac{1}{-E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2} - \omega_{\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2}}$$

- Recall...

$$\int_k [\text{smooth}] e^{iLk \cdot n} = \mathcal{O}(e^{-mL})$$

Finite-volume cutting rule

$$\text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}$$

The diagrammatic equation represents the finite-volume cutting rule. The first term on the left is a two-point function consisting of two vertices (circles) connected by two internal lines (an upper and a lower arc). A vertical dashed line, labeled F below it, passes through the middle of the diagram, representing a cut. The second term is a two-point function where the two vertices are connected by two internal lines, and the central region is enclosed in a dotted rectangular box labeled L . The third term is a two-point function where the two vertices are connected by two internal lines, and the central region is labeled PV, representing a principal value contribution.

Finite-volume cutting rule

$$\text{Diagram with two circles and a dashed line } F = \text{Diagram with two circles and a dashed box } L - \text{Diagram with two circles and a solid oval } PV$$

$$\begin{aligned}
 B \otimes F \otimes B &= -i \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \int \frac{dk^0}{2\pi} B(s, \mathbf{p}, \mathbf{k}) D(k) D(P - k) B(s, \mathbf{k}, \mathbf{p}') \\
 &= -\frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{2\omega_{\mathbf{k}}} \frac{B(s, \mathbf{p}, \mathbf{k}) B(s, \mathbf{k}, \mathbf{p}')}{(P - k)^2 + m^2 - \Sigma(P - k)} \Big|_{k^0 = \omega_{\mathbf{k}}} + \mathcal{O}(e^{-mL})
 \end{aligned}$$

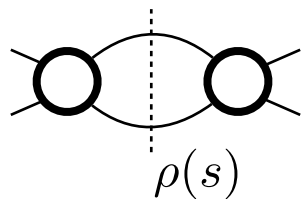
Finite-volume cutting rule

$$\text{Bubble with } F = \text{Bubble with } L - \text{Bubble with PV}$$

$$\begin{aligned}
 B \otimes F \otimes B &= -i \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \int \frac{dk^0}{2\pi} B(s, \mathbf{p}, \mathbf{k}) D(k) D(P - k) B(s, \mathbf{k}, \mathbf{p}') \\
 &= -\frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{2\omega_{\mathbf{k}}} \frac{B(s, \mathbf{p}, \mathbf{k}) B(s, \mathbf{k}, \mathbf{p}')}{(P - k)^2 + m^2 - \Sigma(P - k)} \Big|_{k^0 = \omega_{\mathbf{k}}} + \mathcal{O}(e^{-mL}) \\
 &= \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{(2\omega_{\mathbf{k}})^2} \frac{B_{\text{os}}(s, \hat{\mathbf{p}}, \hat{\mathbf{k}}) B_{\text{os}}(s, \hat{\mathbf{k}}, \hat{\mathbf{p}}')}{E - 2\omega_{\mathbf{k}}} + \mathcal{O}(e^{-mL}) \\
 B_{\text{os}}(s, \hat{\mathbf{k}}, \hat{\mathbf{p}}) &= B(s, \sqrt{s/4 - m^2} \hat{\mathbf{k}}, \sqrt{s/4 - m^2} \hat{\mathbf{p}})
 \end{aligned}$$

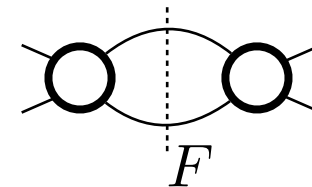
$$= \mathcal{Y}^*(\hat{\mathbf{p}}) \cdot B_{\text{os}}(s) \cdot \left[\frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} \text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{(2\omega_{\mathbf{k}})^2} \frac{\mathcal{Y}(\hat{\mathbf{k}}) \mathcal{Y}^*(\hat{\mathbf{k}})}{E - 2\omega_{\mathbf{k}}} \right] \cdot B_{\text{os}}(s) \cdot \mathcal{Y}^*(\hat{\mathbf{p}}') + \mathcal{O}(e^{-mL})$$

Comparison of cuts



$$\rho(s) = \mathbb{I} \frac{\sqrt{1 - 4m^2/s}}{32\pi}$$

No volume dependence



$$F(E, L) = \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} -\text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{(2\omega_{\mathbf{k}})^2} \frac{\mathcal{Y}(\hat{\mathbf{k}}) \mathcal{Y}^*(\hat{\mathbf{k}})}{E - 2\omega_{\mathbf{k}}}$$

Volume dependent

Both can be understood as matrices in $\ell m, \ell' m'$
 proportional to the identity on- and off-diagonal elements

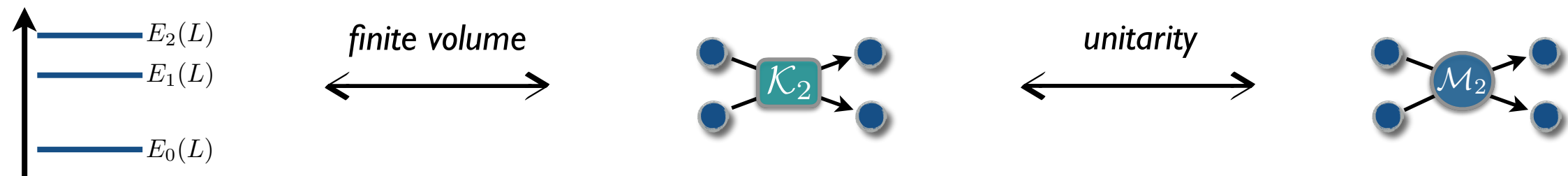
Exact difference between
 $i\epsilon$ and PV

Approximate difference
 between sum and PV

Result

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0$$

$F(P, L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
 Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
 Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
 Li, Liu (2013) • Briceño (2014)

Spin-half particles

- Essentially the same Bethe-Salpeter expansion

$$\mathcal{M}_L(P) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

The diagrams represent terms in the Bethe-Salpeter expansion. The first term is a single vertex (blue circle) with four external lines, labeled e^{-mL} below it. The second term is a vertex connected to a loop (two blue circles connected by two lines, with red squares at the vertices) and another vertex, labeled $1/L^n$ below the loop. The third term is a vertex connected to two such loops in series, each labeled L inside the loop.

- Now the two-particle loops have spin... and are still projected on-shell

$$\frac{1}{i} \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon} \implies \frac{1}{i} \frac{\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p})}{p^2 + m^2 - i\epsilon}$$

Spin-half particles

- Essentially the same Bethe-Salpeter expansion

$$\mathcal{M}_L(P) = \text{diagram with one blob} + \text{diagram with two blobs and a loop labeled } L \text{ with } 1/L^n \text{ below it} + \text{diagram with three blobs and two loops labeled } L \text{ with } 1/L^n \text{ below each} + \dots$$

The first diagram is a single blob with four external lines, labeled e^{-mL} below it. The second diagram consists of two blobs connected by a loop. The loop is labeled L and has four red squares at its vertices. Below the loop is the label $1/L^n$. The third diagram consists of three blobs connected by two loops, each labeled L with red squares at vertices and $1/L^n$ below it.

- Now the two-particle loops have spin... and are still projected on-shell

$$\frac{1}{i} \frac{-\not{p} + m}{p^2 + m^2 - i\epsilon} \implies \frac{1}{i} \frac{\sum_s u_s(\mathbf{p}) \bar{u}_s(\mathbf{p})}{p^2 + m^2 - i\epsilon}$$

- Spinors project B and thus $\mathcal{M}$ to definite spin components
- Final result is the same quantisation condition with a new F

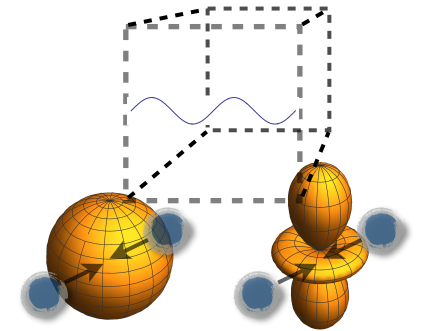
$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0$$

$$F_{\ell' m' s'_1 s'_2, \ell m s_1 s_2}(E, L) = \frac{1}{2} \left[\frac{1}{L^3} \sum_{\mathbf{k}} -\text{p.v.} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \frac{1}{(2\omega_{\mathbf{k}})^2} \frac{\mathcal{Y}_{\ell' m'}(\hat{\mathbf{k}}) \mathcal{Y}_{\ell m}^*(\hat{\mathbf{k}})}{E - 2\omega_{\mathbf{k}}} \delta_{s'_1 s_1} \delta_{s'_2 s_2}$$

Coupled channels

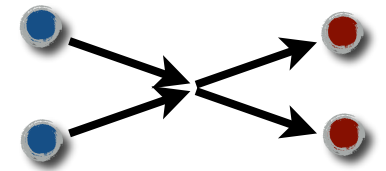
□ The cubic volume mixes different partial waves...

$$\text{e.g. } \begin{matrix} K\pi \rightarrow K\pi \\ \vec{P} \neq 0 \end{matrix} \longrightarrow \det \left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$$



...as well as different flavor channels...

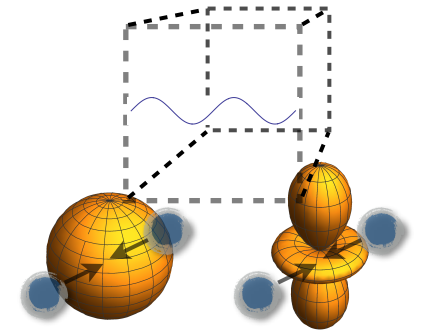
$$\text{e.g. } \begin{matrix} a = \pi\pi \\ b = K\bar{K} \end{matrix} \longrightarrow \det \left[\begin{pmatrix} \mathcal{K}_{a \rightarrow a} & \mathcal{K}_{a \rightarrow b} \\ \mathcal{K}_{b \rightarrow a} & \mathcal{K}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$



Coupled channels

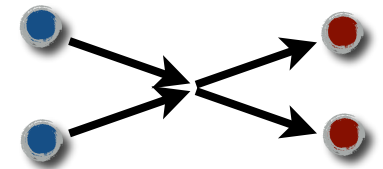
□ The cubic volume mixes different partial waves...

$$\text{e.g. } K\pi \rightarrow K\pi \xrightarrow{\vec{P} \neq 0} \det \left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$$



...as well as different flavor channels...

$$\text{e.g. } \begin{matrix} a = \pi\pi \\ b = K\bar{K} \end{matrix} \xrightarrow{} \det \left[\begin{pmatrix} \mathcal{K}_{a \rightarrow a} & \mathcal{K}_{a \rightarrow b} \\ \mathcal{K}_{b \rightarrow a} & \mathcal{K}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$$



□ Workflow...

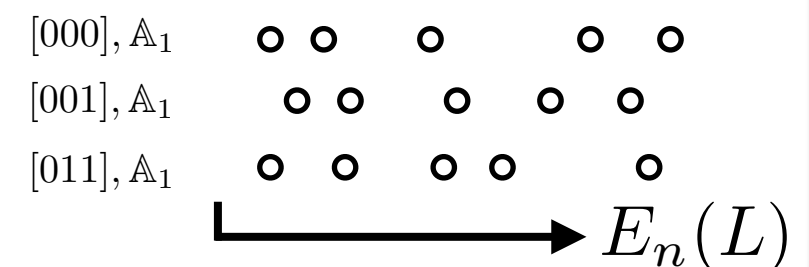
Correlators with a large operator basis

$$\langle \mathcal{O}_a(\tau) \mathcal{O}_b^\dagger(0) \rangle$$

Reliably extract finite-volume energies

$$\langle \Omega_m(\tau) \Omega_m^\dagger(0) \rangle \sim e^{-E_m(L)\tau}$$

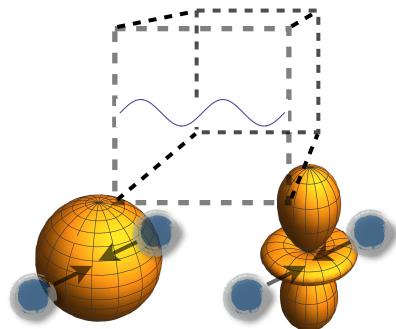
Vary L and P to recover a dense set of energies



Coupled channels

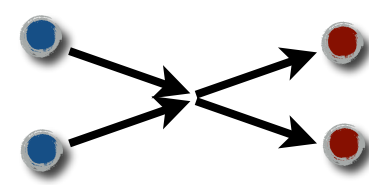
❑ The cubic volume mixes different partial waves...

e.g. $K\pi \rightarrow K\pi$
 $\vec{P} \neq 0 \longrightarrow \det \left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix} \right] = 0$

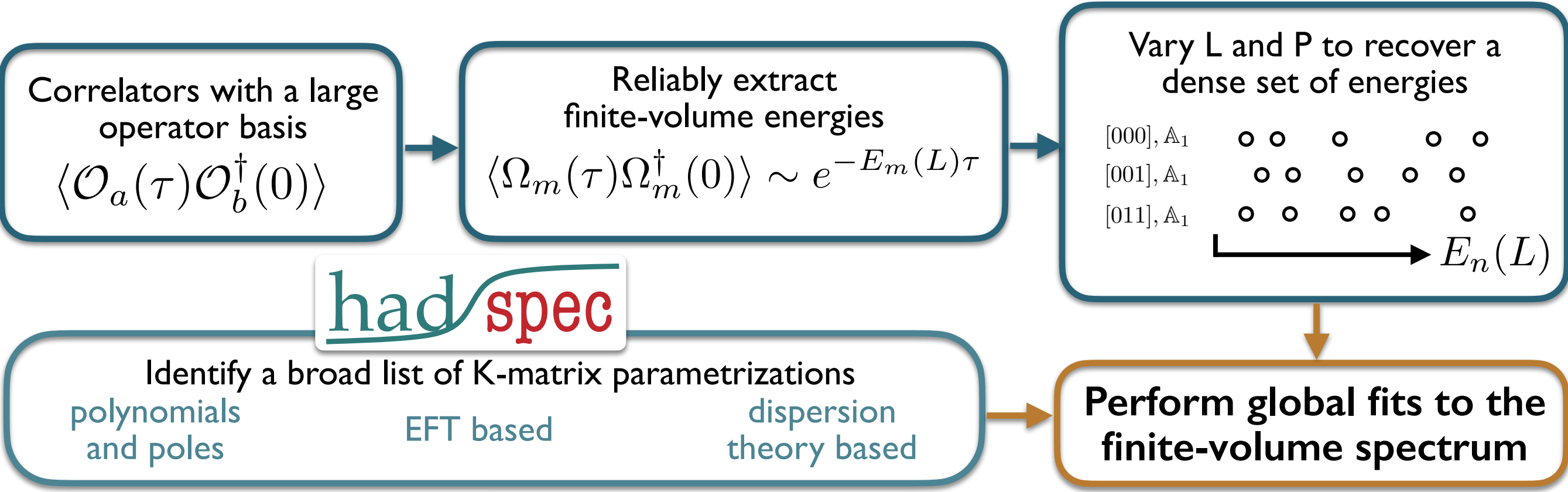


...as well as different flavor channels...

e.g. $a = \pi\pi$
 $b = K\bar{K} \longrightarrow \det \left[\begin{pmatrix} \mathcal{K}_{a \rightarrow a} & \mathcal{K}_{a \rightarrow b} \\ \mathcal{K}_{b \rightarrow a} & \mathcal{K}_{b \rightarrow b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix} \right] = 0$



❑ Workflow...



☒ Warm-up and definitions

- ☒ Meaning of Euclidean
- ☒ Finite-volume set-up

☒ e^{-mL} round one

- ☒ Mass in $\lambda\phi^4$
- ☒ Mass/matrix element in $g\phi^3$

☒ $2 \rightarrow 2$ formalism

- ☒ Scattering basics
- ☒ Derivation
- ☒ Example application
- ☒ Generalizations

☐ e^{-mL} round two

- ☐ LO-HVP for $(g - 2)_\mu$
- ☒ Bethe-Salpeter kernel

☐ $(1+)\mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Example application

☐ $2 + \mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Testing the result
- ☐ Numerical explorations

☐ Non-local matrix elements

- ☐ Derivation
- ☐ Applications

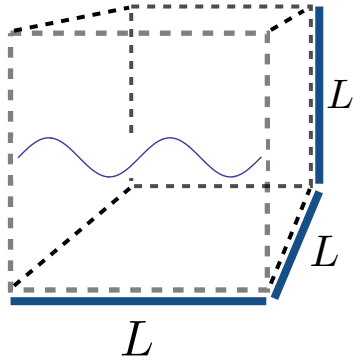
☐ $3 \rightarrow 3$ formalism

- ☐ New complications
- ☐ Derivation ($E_n(L)$ to $\mathcal{K}_{\text{df},3}$)
- ☐ Integral equations ($\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$)
- ☐ Testing the result
- ☐ Numerical explorations/calculations

☐ Conclusion and outlook

LO HVP (hadronic vacuum polarization)

- $T \times L^3$ periodic, Euclidean signature

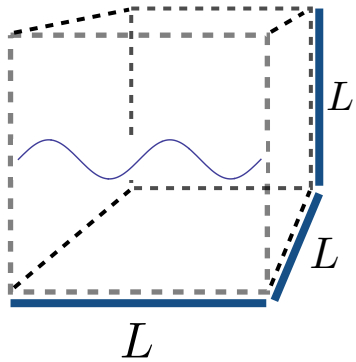


$$a_{\mu}^{\text{LO,HVP}}(L, T) \equiv \int_0^{T/2} dx_0 \mathcal{K}(x_0) \int_{L^3} d^3 \mathbf{x} \langle j_k^{\dagger}(x_0, \mathbf{x}) j_k(0) \rangle_{L, T}$$

- Continuous kernel function: $\mathcal{K}(x_0) \stackrel{x_0 \rightarrow \infty}{\propto} x_0^2$

LO HVP (hadronic vacuum polarization)

- $T \times L^3$ periodic, Euclidean signature



$$a_{\mu}^{\text{LO,HVP}}(L, T) \equiv \int_0^{T/2} dx_0 \mathcal{K}(x_0) \int_{L^3} d^3 \mathbf{x} \langle j_k^{\dagger}(x_0, \mathbf{x}) j_k(0) \rangle_{L, T}$$

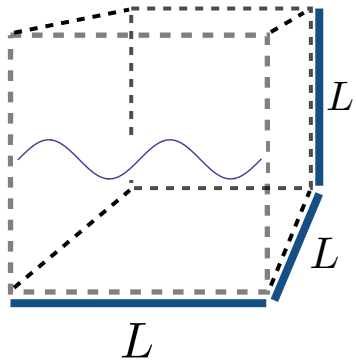
- Continuous kernel function: $\mathcal{K}(x_0) \stackrel{x_0 \rightarrow \infty}{\propto} x_0^2$
- For asymptotically large $T, L \dots$

$$\begin{aligned} a_{\mu}^{\text{LO,HVP}}(L, T) - a_{\mu}^{\text{LO,HVP}} &= \mathcal{O}(e^{-M_{\pi}L}) + \mathcal{O}(e^{-\sqrt{2}M_{\pi}L}) + \mathcal{O}(e^{-\sqrt{3}M_{\pi}L}) + \mathcal{O}(e^{-1.9M_{\pi}L}) + \dots \\ &+ \mathcal{O}(e^{-M_{\pi}T}) + \mathcal{O}(e^{-\frac{3}{2}M_{\pi}T}) + \dots \\ &+ \mathcal{O}(e^{-M_{\pi}\sqrt{T^2+L^2}}) + \dots \\ &+ \mathcal{O}(e^{-M_K L}) + \dots \end{aligned}$$

$\nearrow \sqrt{2 + \sqrt{3}} \approx 1.92$

LO HVP (hadronic vacuum polarization)

- $T \times L^3$ periodic, Euclidean signature



$$a_{\mu}^{\text{LO,HVP}}(L, T) \equiv \int_0^{T/2} dx_0 \mathcal{K}(x_0) \int_{L^3} d^3 \mathbf{x} \langle j_k^{\dagger}(x_0, \mathbf{x}) j_k(0) \rangle_{L, T}$$

- Continuous kernel function: $\mathcal{K}(x_0) \stackrel{x_0 \rightarrow \infty}{\propto} x_0^2$
- For asymptotically large $T, L \dots$

$$\begin{aligned} a_{\mu}^{\text{LO,HVP}}(L, T) - a_{\mu}^{\text{LO,HVP}} = & \underbrace{\mathcal{O}(e^{-M_{\pi} L})}_{\text{1904.10010}} + \underbrace{\mathcal{O}(e^{-\sqrt{2} M_{\pi} L}) + \mathcal{O}(e^{-\sqrt{3} M_{\pi} L})}_{\text{2004.03935}} + \underbrace{\mathcal{O}(e^{-1.9 M_{\pi} L})}_{\text{for now}} + \dots \\ & + \underbrace{\mathcal{O}(e^{-M_{\pi} T})}_{\text{2004.03935}} + \mathcal{O}(e^{-\frac{3}{2} M_{\pi} T}) + \dots \\ & + \mathcal{O}(e^{-M_{\pi} \sqrt{T^2 + L^2}}) + \dots \\ & + \mathcal{O}(e^{-M_K L}) + \dots \end{aligned}$$

$\sqrt{2 + \sqrt{3}} \approx 1.92$

1904.10010

2004.03935

for now

MTH and A. Patella • *strong inspiration from...* Lüscher 1986

Diagrammatic expansion

□ All orders diagrammatic expansion

$$a_\mu(L) =$$

The diagrammatic expansion shows two rows of Feynman diagrams. Each diagram consists of two wavy lines (external) connected by a loop structure (internal). The internal structures are as follows:

- Row 1, Diagram 1: A bubble with a horizontal line connecting the two vertices.
- Row 1, Diagram 2: A bubble with a vertical line connecting the two vertices.
- Row 1, Diagram 3: A bubble with a horizontal line and a small loop on top.
- Row 1, Diagram 4: A bubble with a horizontal line, a small loop on top, and a vertical line.
- Row 1, Diagram 5: A bubble with two horizontal lines.
- Row 2, Diagram 1: A bubble with a horizontal line and a small loop on top.
- Row 2, Diagram 2: A bubble with a horizontal line, a small loop on top, and a vertical line.
- Row 2, Diagram 3: A bubble with a horizontal line, a small loop on top, and a vertical line, and a horizontal line.
- Row 2, Diagram 4: A bubble with two horizontal lines and a vertical line.
- Row 2, Diagram 5: A bubble with three horizontal lines.

The expansion ends with a plus sign and three dots, indicating an infinite series.

Diagrammatic expansion

□ All orders diagrammatic expansion

$$a_\mu(L) = \begin{array}{ccccccccc} \text{diagram 1} & \text{diagram 2} & \text{diagram 3} & \text{diagram 4} & \text{diagram 5} & \text{diagram 6} & \text{diagram 7} & \text{diagram 8} & \text{diagram 9} & + \dots \end{array}$$

The diagrams represent various Feynman diagrams for a loop expansion. They include a single loop, a bubble diagram, a tadpole diagram, a sunset diagram, a sunrise diagram, a triangle diagram, a box diagram, and higher-order diagrams with multiple loops and internal lines. Each diagram is connected to external wavy lines.

□ Loop momenta are summed... rewrite using *Poisson summation*

$$\frac{1}{L^3} \sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{n}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{iL\mathbf{n} \cdot \mathbf{k}}$$

$\mathbf{n} = (n_x, n_y, n_z)$ can be interpreted as loop pion wrapping the torus

Diagrammatic expansion

- All orders diagrammatic expansion

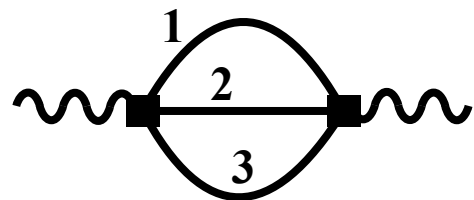
$$a_\mu(L) = \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} + \text{[diagram 4]} + \text{[diagram 5]} + \text{[diagram 6]} + \text{[diagram 7]} + \text{[diagram 8]} + \text{[diagram 9]} + \dots$$

- Loop momenta are summed... rewrite using *Poisson summation*

$$\frac{1}{L^3} \sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{n}} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{iL\mathbf{n} \cdot \mathbf{k}}$$

$\mathbf{n} = (n_x, n_y, n_z)$ can be interpreted as loop pion wrapping the torus

- Some subtleties in assigning *Poisson modes*



$$\begin{aligned} \mathbf{n}_1 &= (0, 0, 1) \\ \mathbf{n}_2 &= (0, 0, 1) \end{aligned} \equiv \begin{aligned} \mathbf{n}_2 &= (0, 0, 0) \\ \mathbf{n}_3 &= (0, 0, 1) \end{aligned}$$

understood as a gauge redundancy $\rightarrow$ gauge fixing to catalogue effects

Lüscher (1986)

Extension (all orders diagrammatic)

- Leading contributions: *only one pion wraps the torus*

$$\Delta a_\mu(L) = \int_0^\infty dx_0 \mathcal{K}(x_0) \left[\text{one-particle irreducible vertices} + \frac{1}{2} \text{dressed propagators} \right] + \mathcal{O}(e^{-1.9M_\pi L})$$

leading 2-wrap contribution

- All L dependence inside...

$$\text{---} \boxed{L} \text{---} \equiv \sum_n \frac{e^{iL\mathbf{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$$

Extension (all orders diagrammatic)

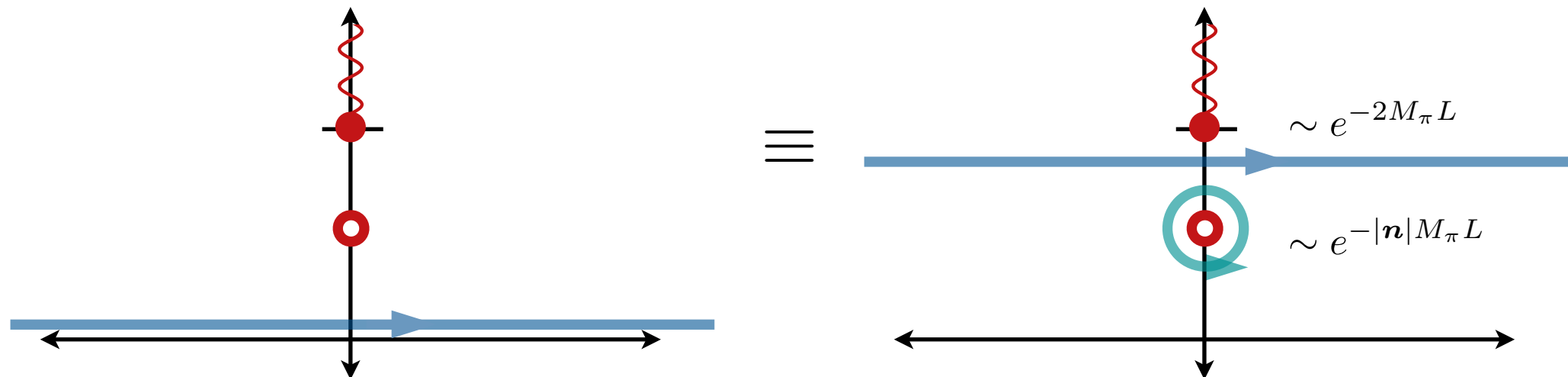
- Leading contributions: *only one pion wraps the torus*

$$\Delta a_\mu(L) = \int_0^\infty dx_0 \mathcal{K}(x_0) \left[\text{one-particle irreducible vertices} + \frac{1}{2} \text{dressed propagators} \right] + \mathcal{O}(e^{-1.9M_\pi L})$$

leading 2-wrap contribution

- All L dependence inside... $\text{---} \boxed{L} \text{---} \equiv \sum_n \frac{e^{iL\mathbf{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$

- Exponential decay?



pole contributions $\rightarrow$ on-shell pions $\rightarrow$ *physical Compton amplitude*

Result

□ Sum of *all single-winding terms* gives

$$\Delta a_\mu(L) = -\frac{\alpha^2}{m_\mu^2} \sum_{\mathbf{n} \neq 0} \int \frac{dp_3}{2\pi} \underbrace{\frac{e^{-|\mathbf{n}|L\sqrt{M_\pi^2+p_3^2}}}{2\pi L|\mathbf{n}|}}_{\text{volume dependence}} \int_0^\infty dx_0 \underbrace{\hat{\mathcal{K}}(m_\mu x_0)}_{\text{kernel}} \int \frac{dk_3}{2\pi} \cos(x_0 k_3) \underbrace{\text{Re } T(-k_3^2, -k_3 p_3)}_{\text{Compton amplitude}}$$

reference to EFT has disappeared

requires only spacelike photons

$$T(k^2, k \cdot p) \equiv i \lim_{\mathbf{p}' \rightarrow \mathbf{p}} \sum_{q=0, \pm 1} \int d^4x e^{ikx} \langle \mathbf{p}', q | T \mathcal{J}_\rho(x) \mathcal{J}^\rho(0) | \mathbf{p}, q \rangle$$

$$= \text{[Feynman diagrams]} = T^{\text{reg}}(k^2, k \cdot p)$$

The Feynman diagrams show a horizontal line with two vertices (green circles) connected by a curved line labeled $F_\pi(Q^2)$. Each vertex has a wavy line (photon) attached. The second diagram is similar but with a different internal structure. The third diagram shows a single vertex (blue circle) with two wavy lines attached, labeled $T^{\text{reg}}(k^2, k \cdot p)$.

Numerical estimates

$$M_\pi = 137 \text{ MeV} \quad m_\mu = 106 \text{ MeV} \quad F_\pi(Q^2) = [1 + Q^2/M^2]^{-1} \quad M = 727 \text{ MeV}$$

$$-100 \times \Delta a_\mu(L)/(700 \times 10^{-10})$$

$M_\pi L$	$ \mathbf{n} = 1$	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$2\sqrt{2}$	3	Σ_n
4	1.26	1.16	0.317	0.104	0.193	0.0944	0.0128	0.0174	3.17
5	0.852	0.428	0.0813	0.0199	0.0287	0.0112	0.00102	0.00117	1.42
6	0.461	0.141	0.0189	0.00349	0.00394	0.00124	0.0000764	0.0000735	0.630
7	0.226	0.0433	0.00417	0.000582	0.000515	0.000130	5.46×10^{-6}	4.41×10^{-6}	0.274
8	0.104	0.0128	0.000883	0.0000936	0.0000652	0.0000132	3.79×10^{-7}	2.57×10^{-7}	0.118

- ☐ Slow convergence of $e^{-\alpha_n M_\pi L}$ series
- ☐ Last column estimates %-finite volume effect
- ☐ 1% precision goal for theoretical relevance

☒ Warm-up and definitions

- ☒ Meaning of Euclidean
- ☒ Finite-volume set-up

☒ e^{-mL} round one

- ☒ Mass in $\lambda\phi^4$
- ☒ Mass/matrix element in $g\phi^3$

☒ $2 \rightarrow 2$ formalism

- ☒ Scattering basics
- ☒ Derivation
- ☒ Example application
- ☒ Generalizations

☒ e^{-mL} round two

- ☒ LO-HVP for $(g - 2)_\mu$
- ☒ Bethe-Salpeter kernel

☐ $(1+)\mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Example application

☐ $2 + \mathcal{J} \rightarrow 2$ formalism

- ☐ Derivation
- ☐ Testing the result
- ☐ Numerical explorations

☐ Non-local matrix elements

- ☐ Derivation
- ☐ Applications

☐ $3 \rightarrow 3$ formalism

- ☐ New complications
- ☐ Derivation ($E_n(L)$ to $\mathcal{K}_{\text{df},3}$)
- ☐ Integral equations ($\mathcal{K}_{\text{df},3}$ to $\mathcal{M}_3$)
- ☐ Testing the result
- ☐ Numerical explorations/calculations

☐ Conclusion and outlook