Spectroscopy and Scattering: Formalism (Lecture 2/3)

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Mainz Institute for Theoretical Physics

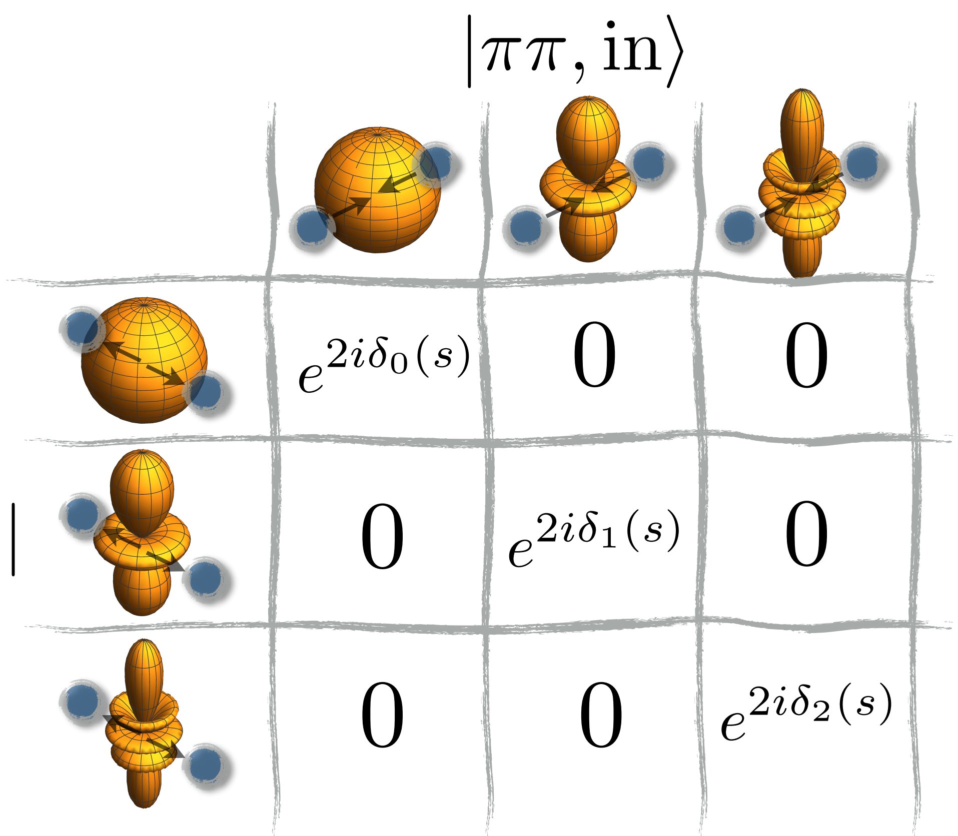
THE UNIVERSITY of EDINBURGH

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} round one
  - ☑ Mass in $\lambda\phi^4$
  - ☑ Mass/matrix element in $g\phi^3$
- ☑ $2 \to 2$ formalism
  - ☑ Scattering basics
  - ☐ Derivation
  - ☐ Example application
  - ☐ Generalizations
- ☐ e^{-mL} round two
  - ☐ LO-HVP for $(g-2)_\mu$
  - ☐ Bethe-Salpeter kernel
- ☐ $(1+)\mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Example application
- ☐ $2 + \mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\mathrm{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\mathrm{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Conclusion and outlook

QCD Fock space

- ❑ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- ❑ Overlaps of multi-hadron ***asymptotic states*** → S matrix

$S(s) \equiv \langle \pi\pi, \text{out}|$



depends on $s = E_{\text{cm}}^2$
and angular variables

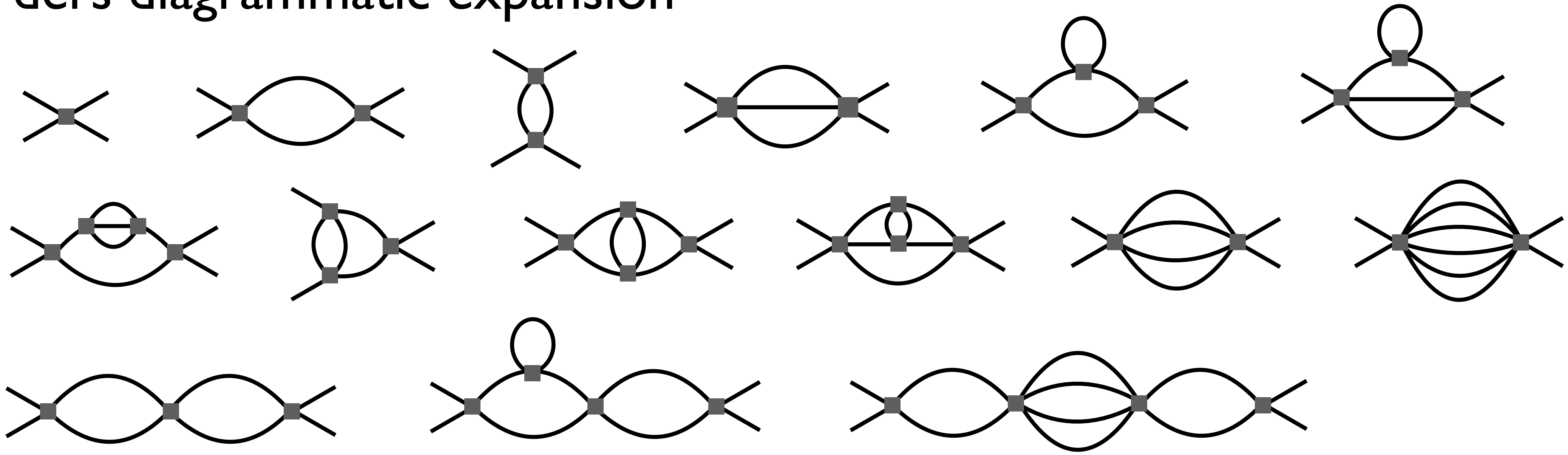
diagonal in angular momentum

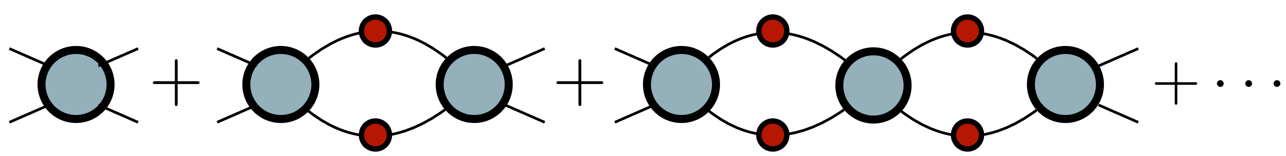
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

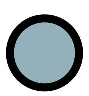
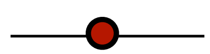
- ❑ ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

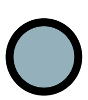
Bethe Salpeter equation

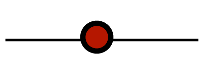
- All orders diagrammatic expansion

$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

 $=$

 $=$

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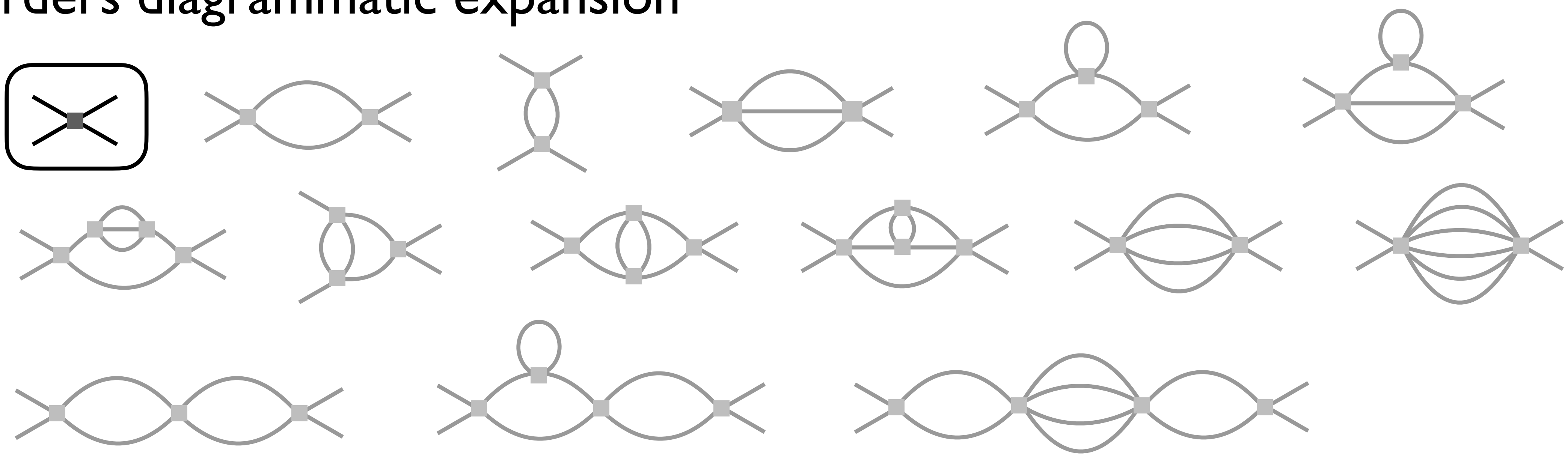
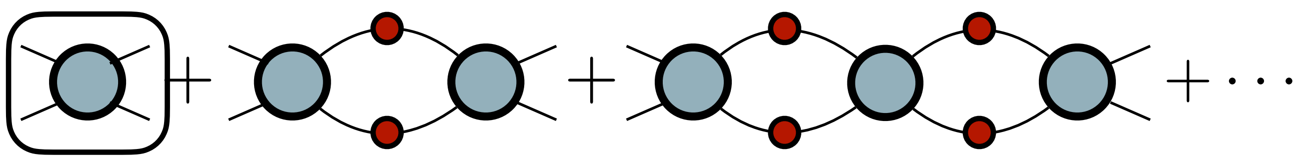
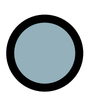
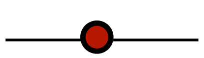
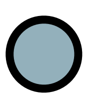
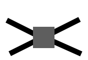
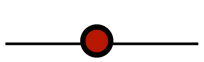
$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

 $=$ 

 $=$

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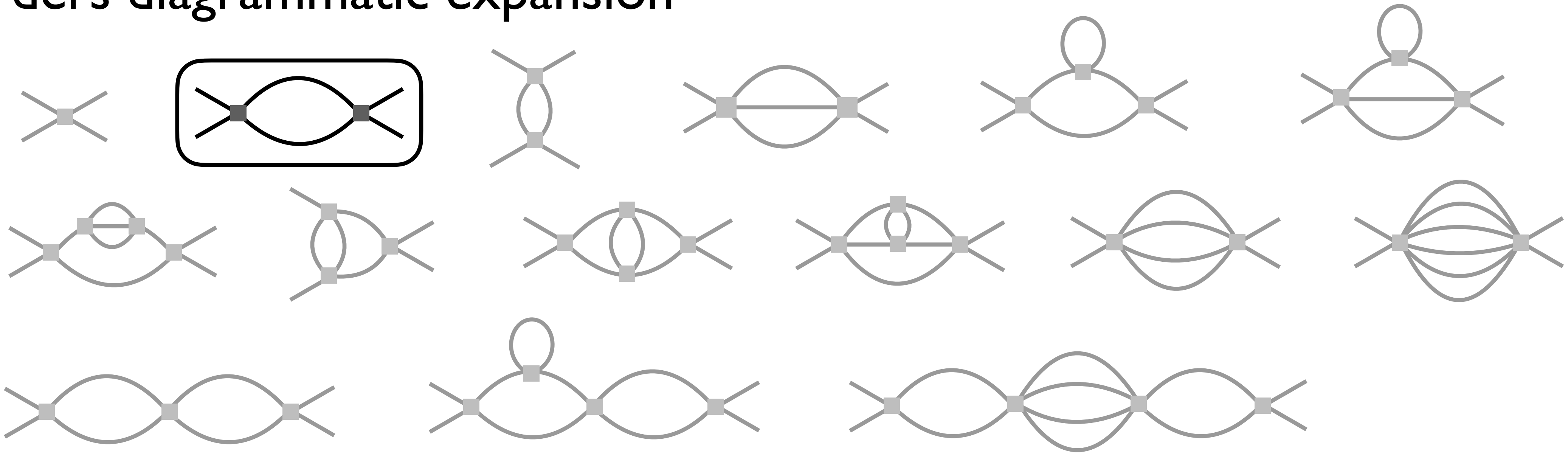
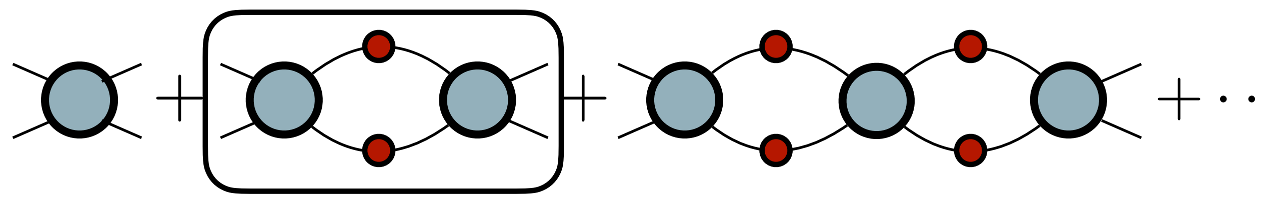
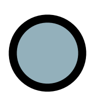
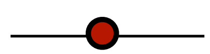
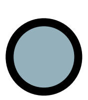
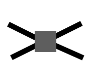
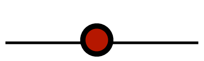
$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

 $=$ 

 $=$

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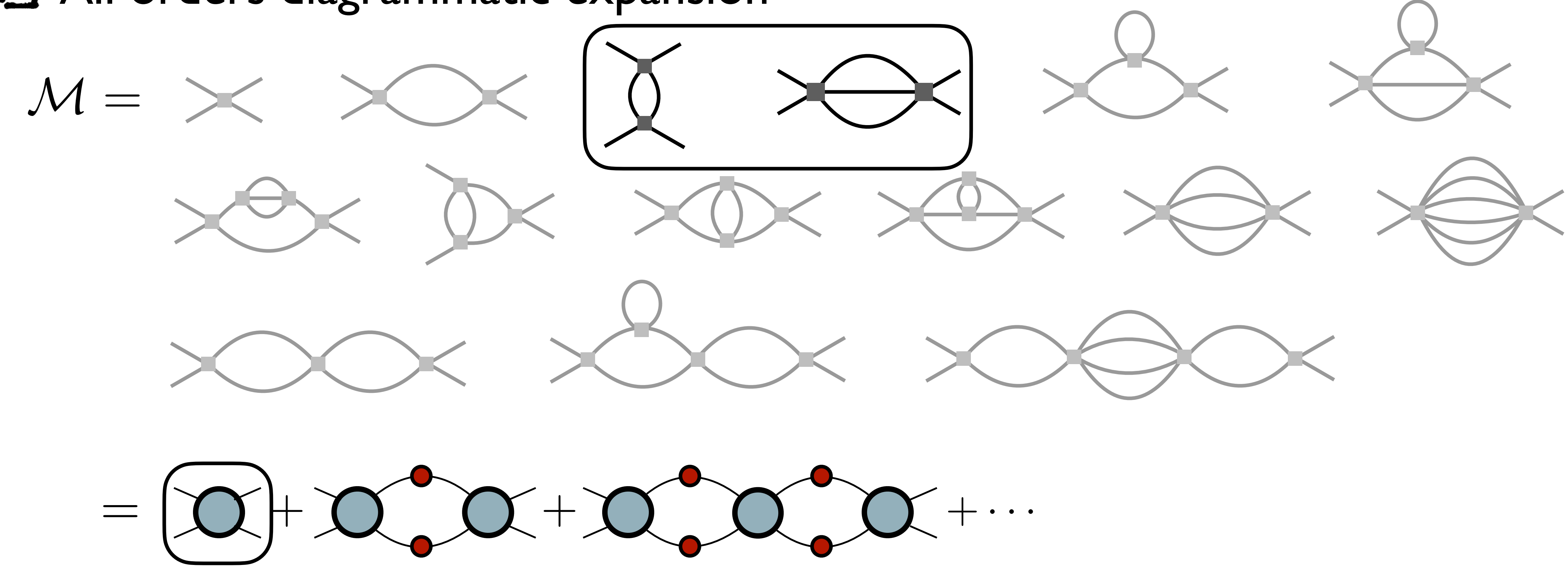
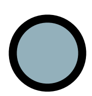
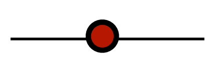
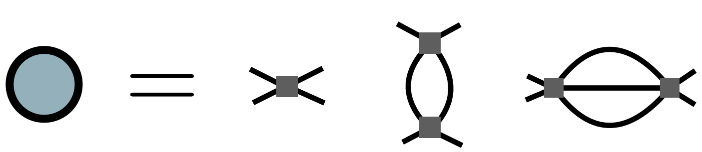


- Construct  and  such that this is true



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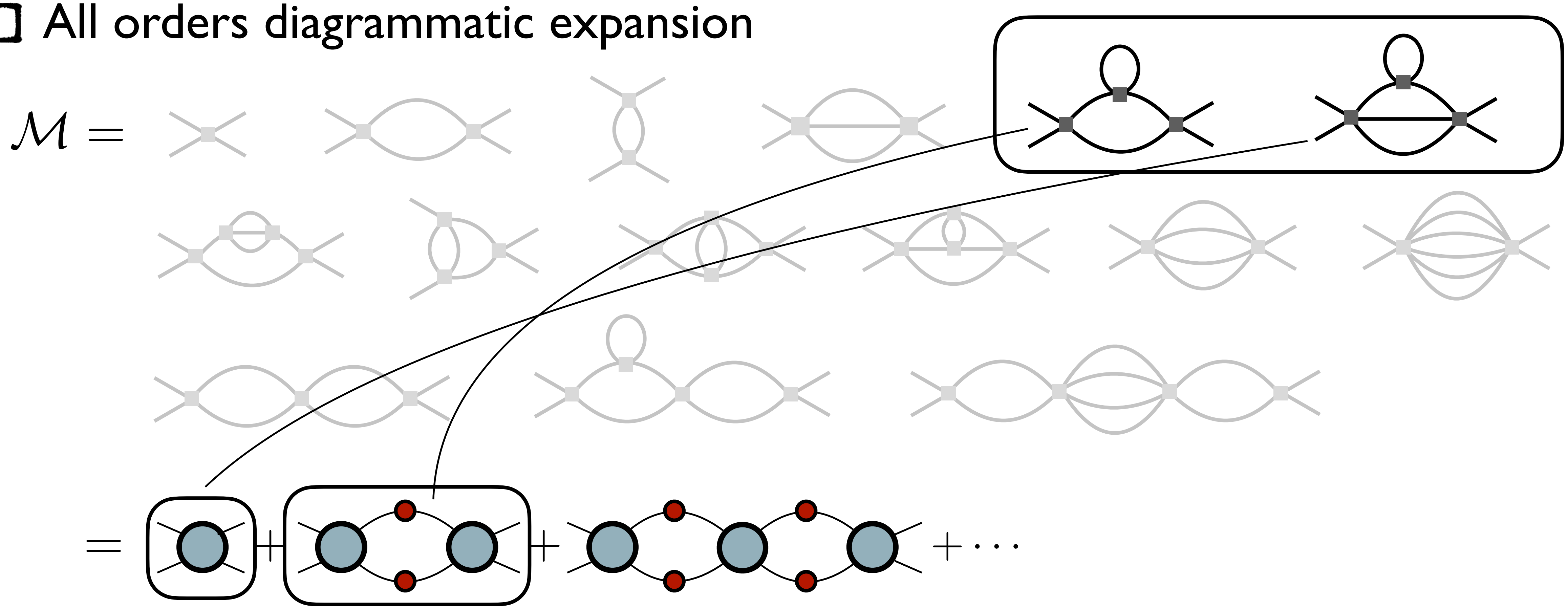

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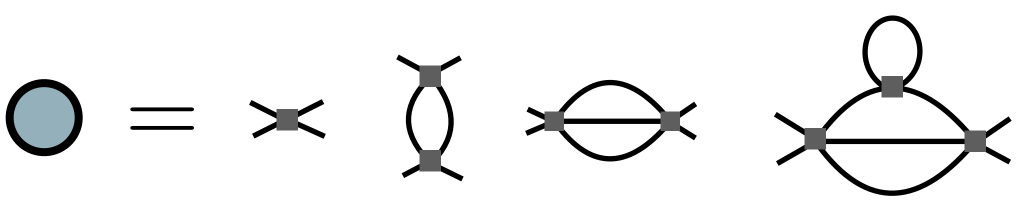

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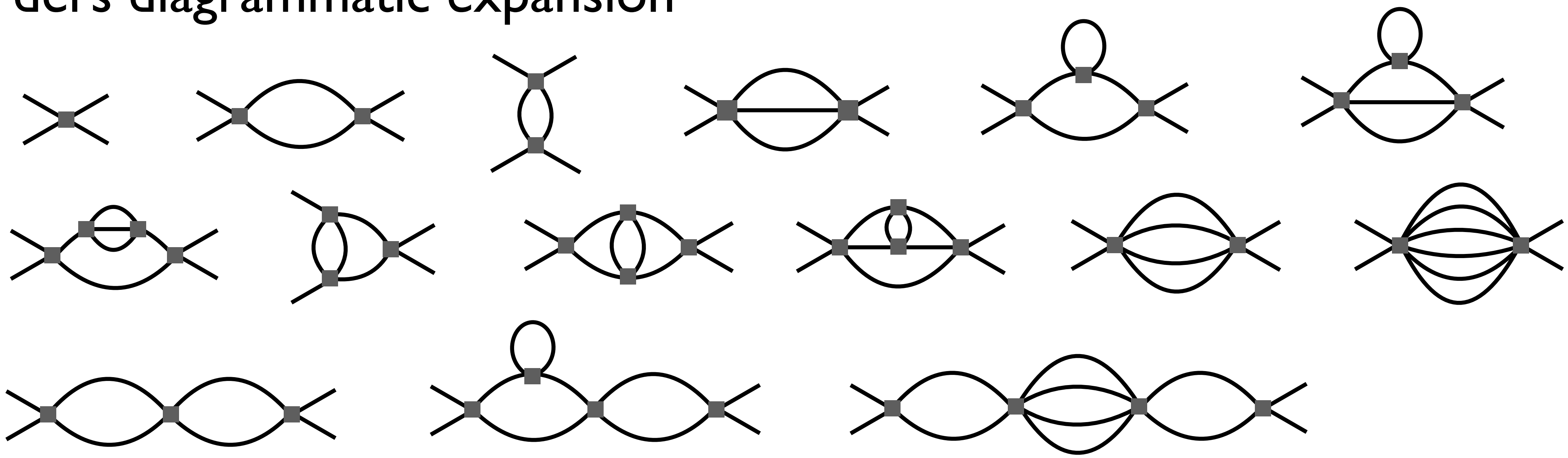
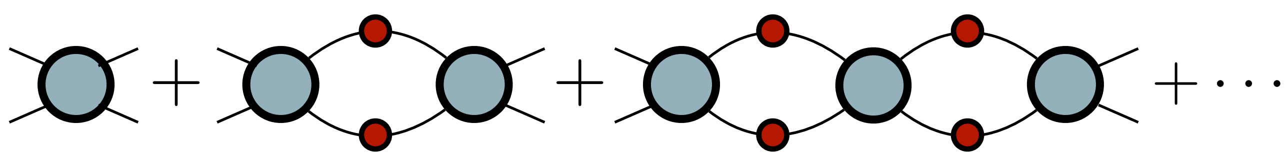
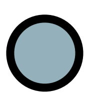
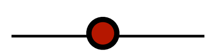
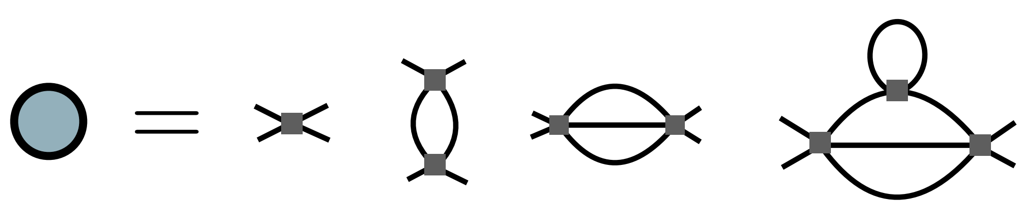
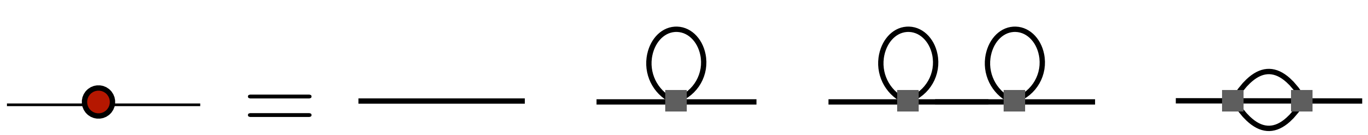
$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

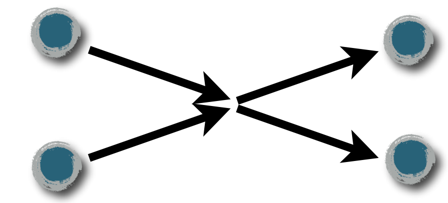


Bethe Salpeter kernel
(2PI in the s-channel)



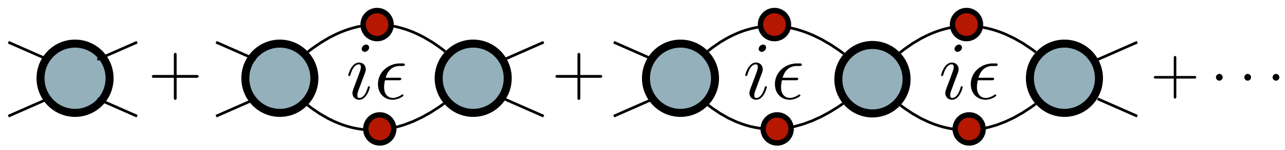
Fully dressed propagator

Analyticity of $\mathscr{M}$ (from B.S. kernel)



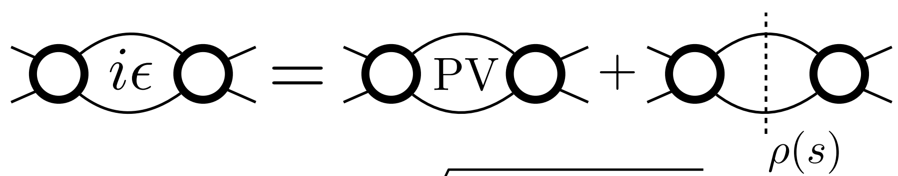
For two-particle energies $(2m)^2 < s < (3m)^2$, what is the analytic structure?

$\mathcal{M}(s) \equiv$



non-analytic:
on-shell particles = singularities

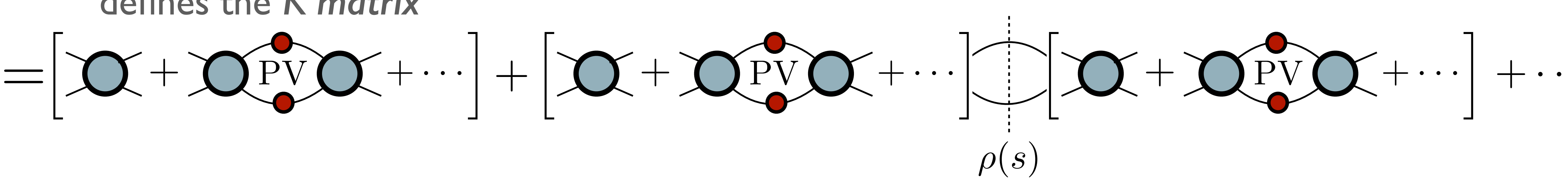
propagating pion



$$\rho(s) \propto \sqrt{s-(2m)^2}$$

cutting rule

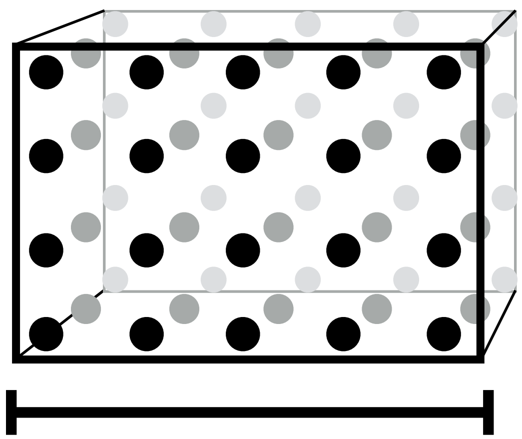
defines the *K* ***matrix***



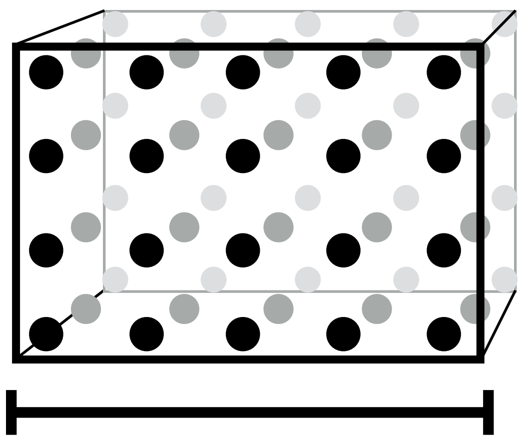
$$= \mathcal{K}(s) + \mathcal{K}(s) i\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

branch-cut singularity

$\sqrt{s-(2m)^2}$



- The *finite volume*…
  - *Discretizes* the spectrum
  - *Eliminates* the branch cuts and extra sheets
  - *Hides* the resonance poles



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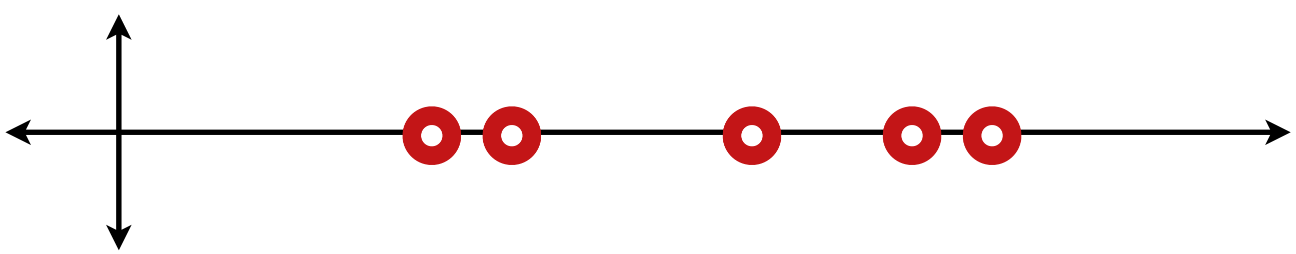
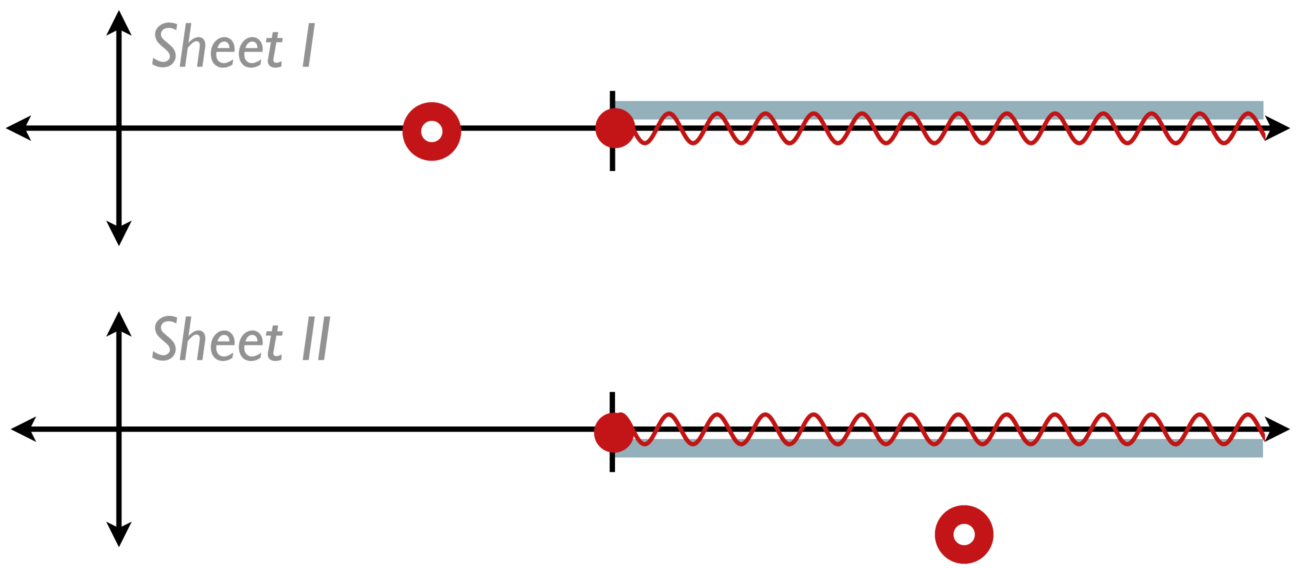
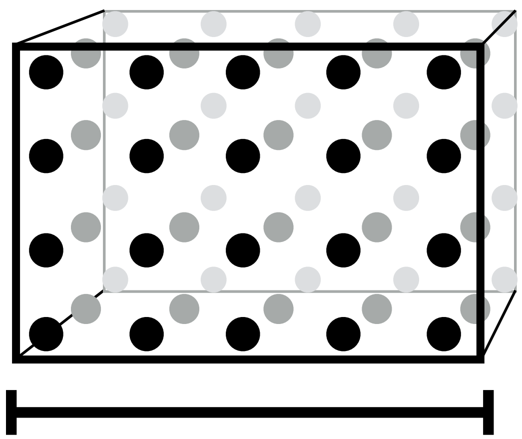
Finite-volume analytic structure



Infinite-volume analytic structure





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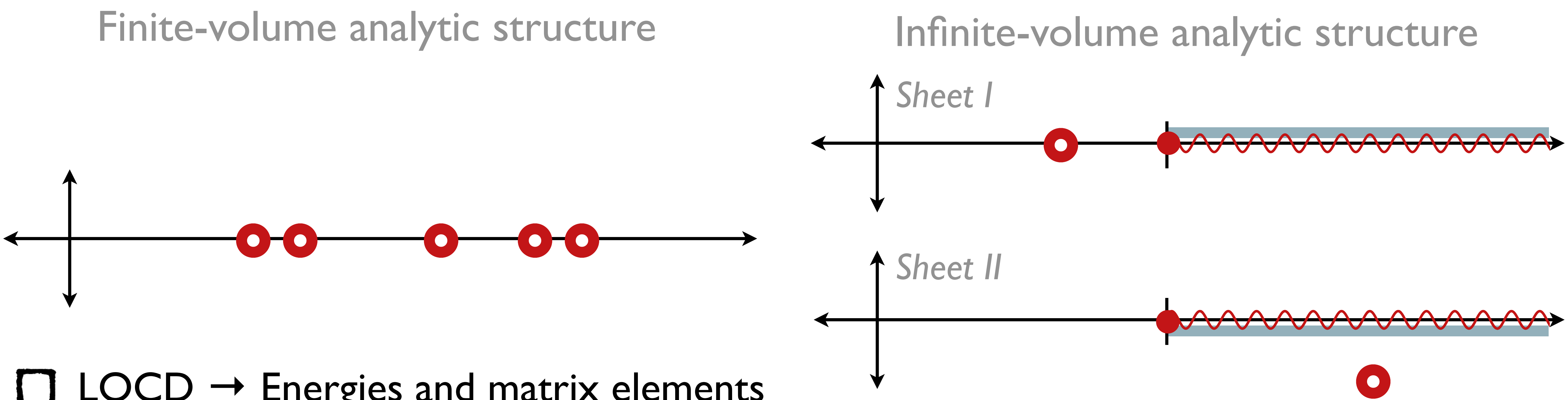


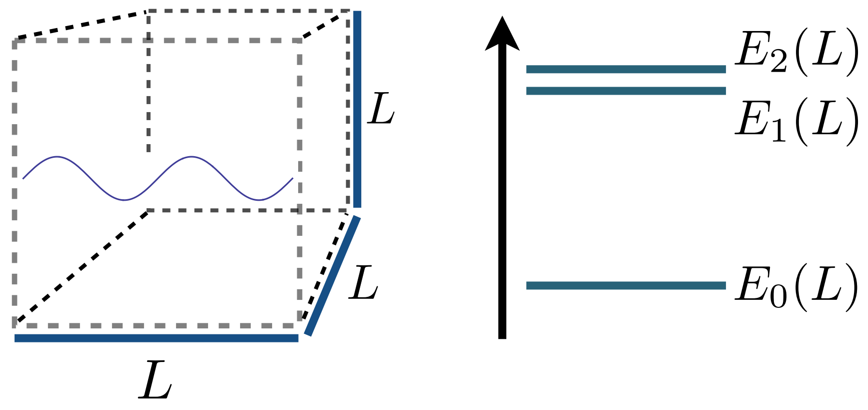
- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

The finite-volume as a tool

- Finite-volume set-up



- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$
- *T* and lattice also negligible

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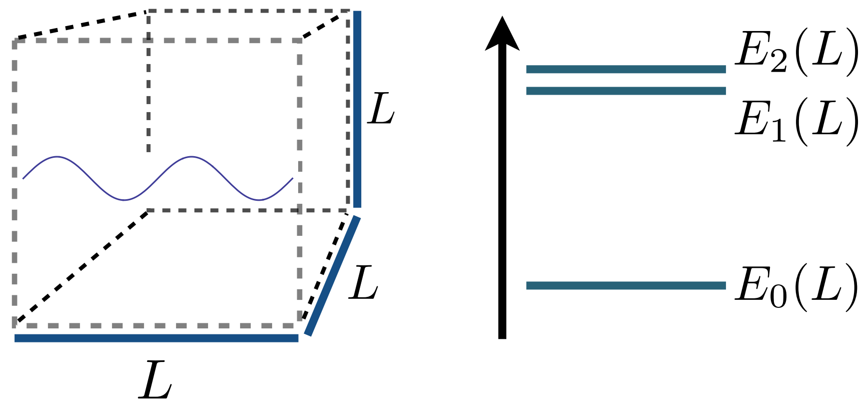

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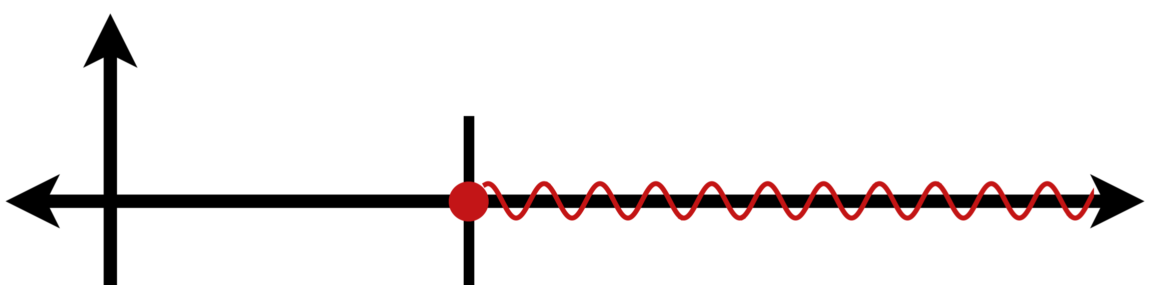
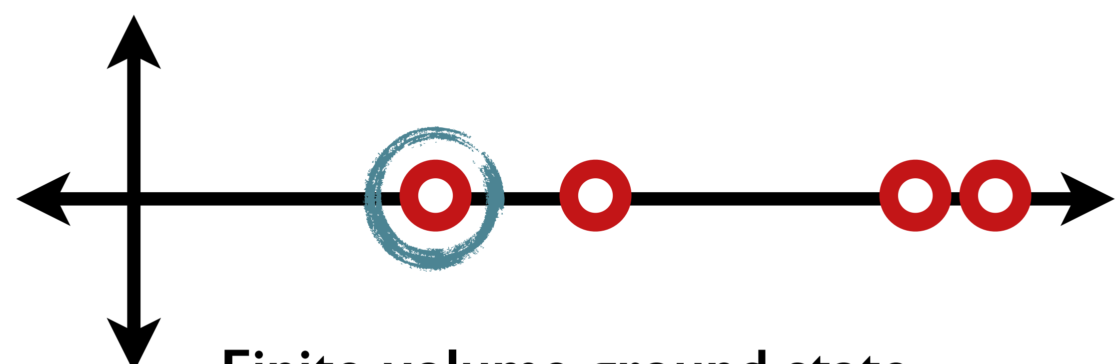
- Scattering leaves an *imprint* on finite-volume quantities



$E_0 = 2M_\pi$

Infinite-volume threshold



Finite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

The finite-volume as a tool

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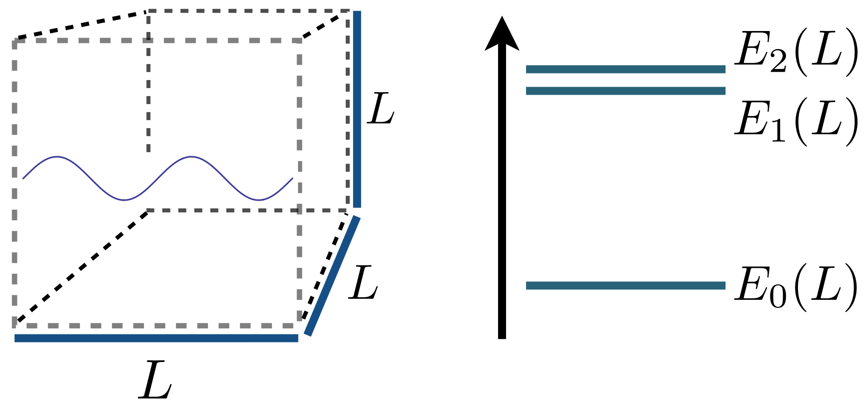

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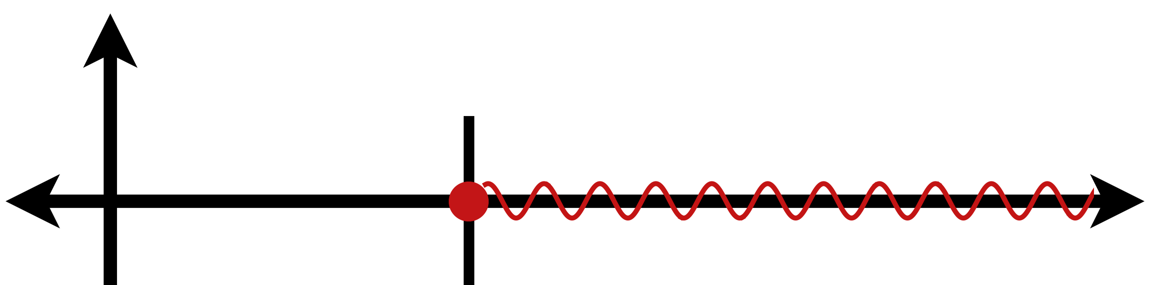

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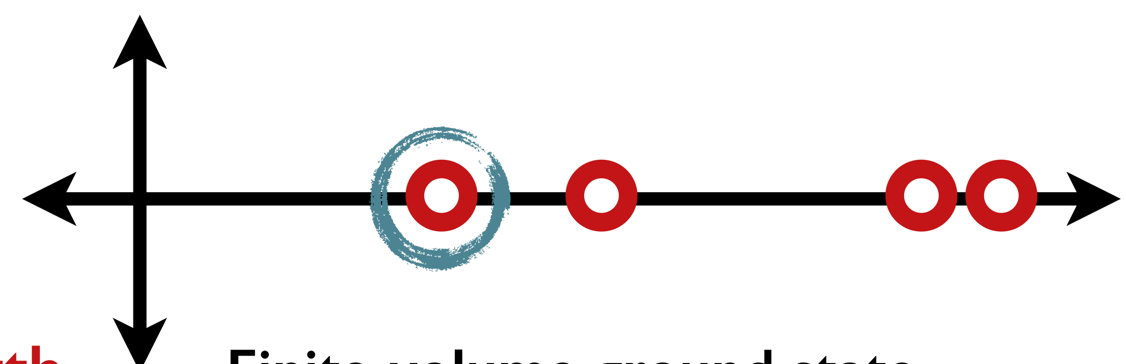

Finite-volume ground state

scattering length

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• Huang, Yang (1958) •

Derivation

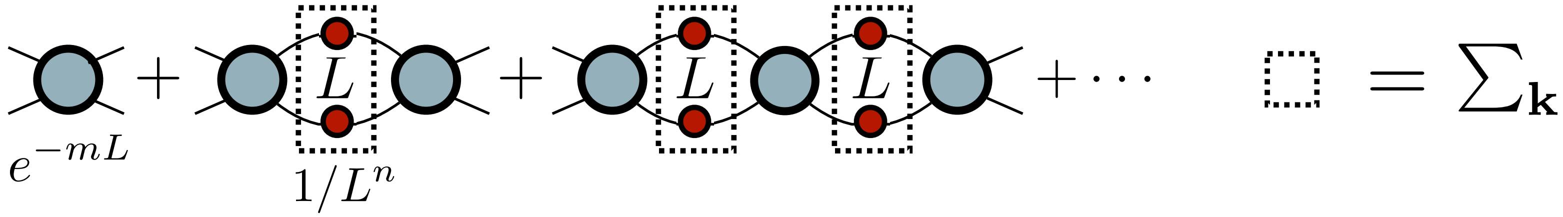
$\mathcal{M}(s)$

probability amplitude

$\mathcal{M}_L(P)$

poles give f.v. spectrum

- Consider the finite-volume correlator:

$\mathcal{M}_L(P) =$ 

For two-particle energies $(2m)^2 < s < (4m)^2$, what is the L dependence?

Derivation

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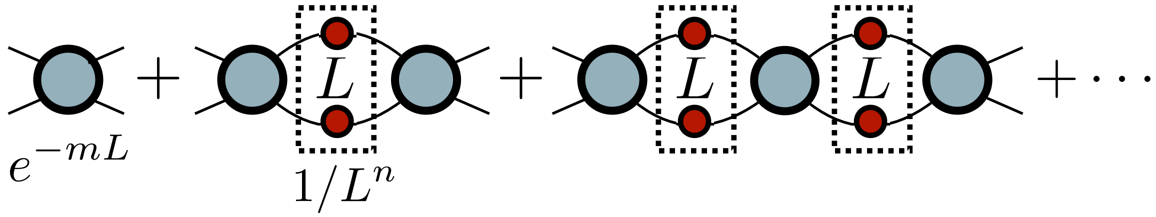
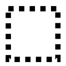
$\mathcal{M}_L(P) =$ 

 $= \sum_{\mathbf{k}}$

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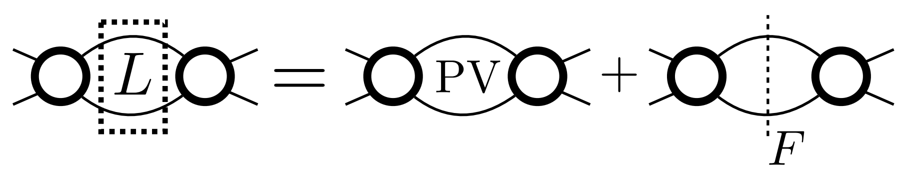

Cut projects loop to ***on-shell energies***

F = matrix of known geometric functions

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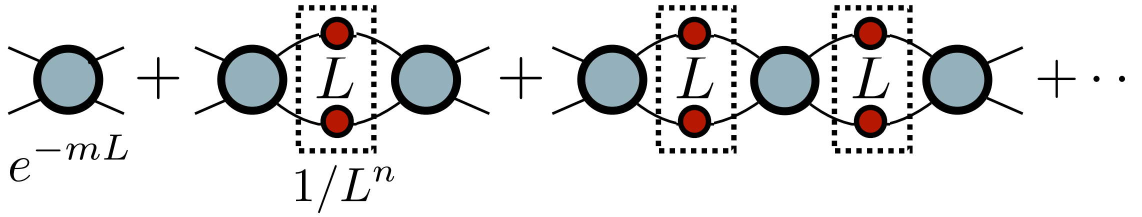
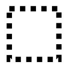
$\mathcal{M}_L(P) =$ 

 $= \sum_{\mathbf{k}}$

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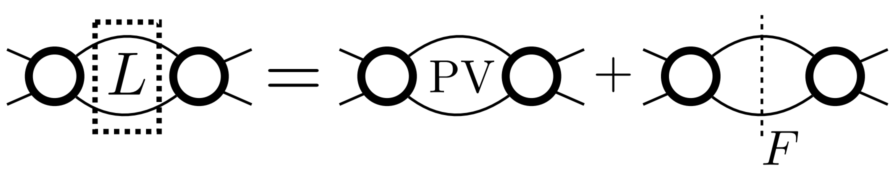
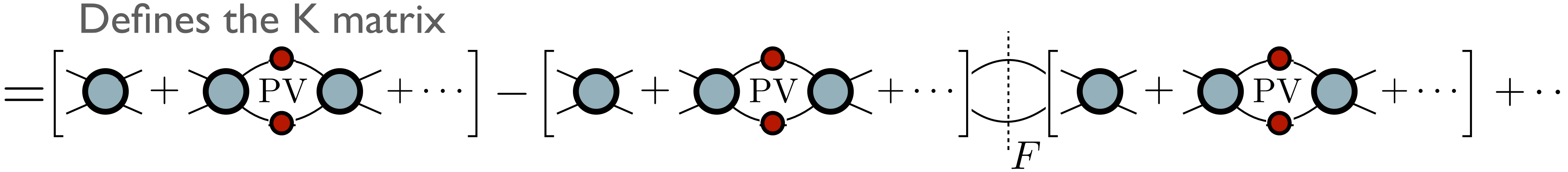


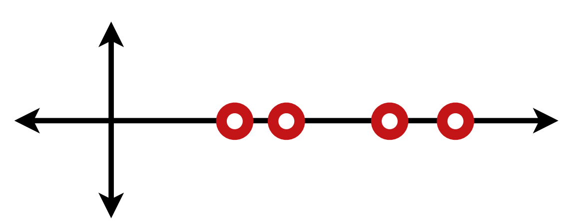
Cut projects loop to ***on-shell energies***

F = matrix of known geometric functions

Defines the K matrix



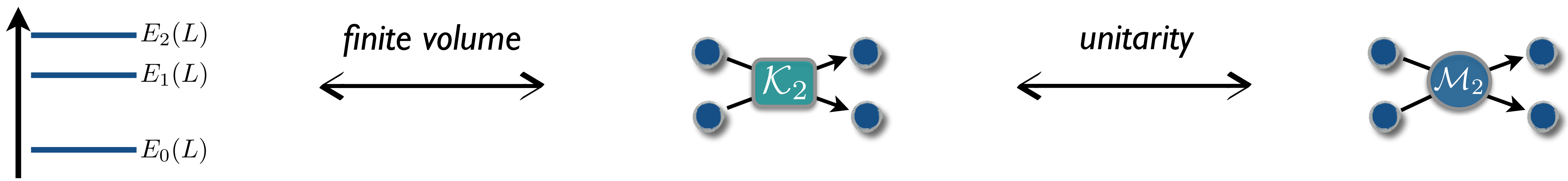
$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P, L)}$$



• Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

Result

$$\det[\mathcal{K}^{-1}(s) + F(P, L)] = 0 \qquad F(P, L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

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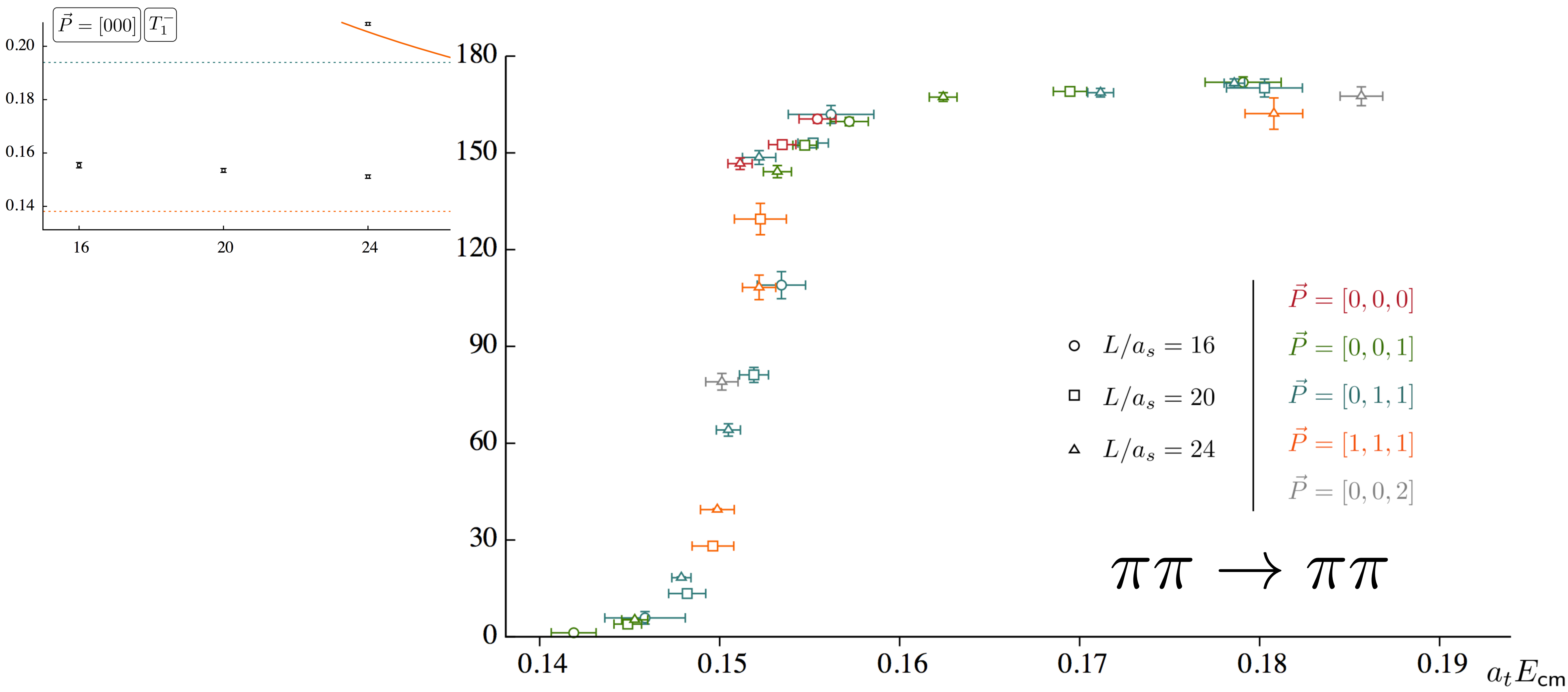
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505 •

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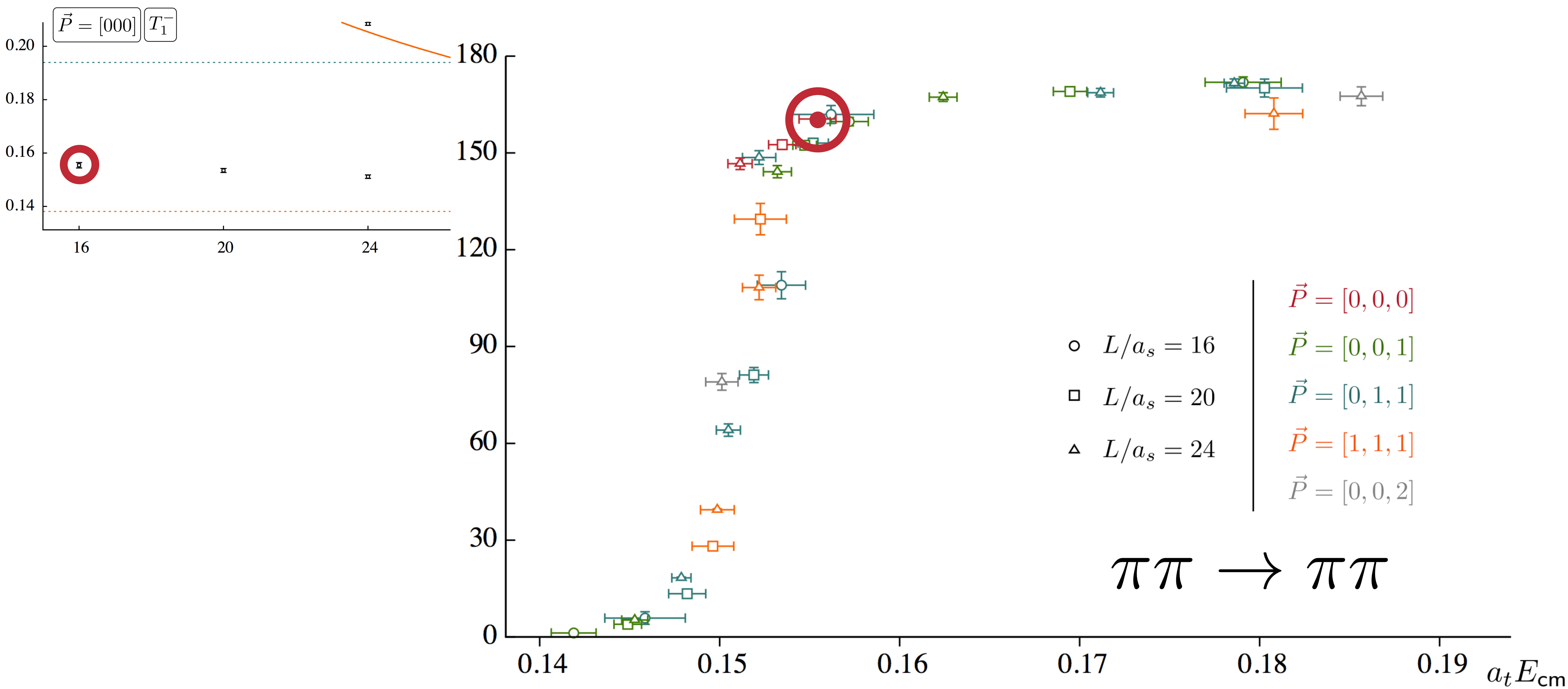

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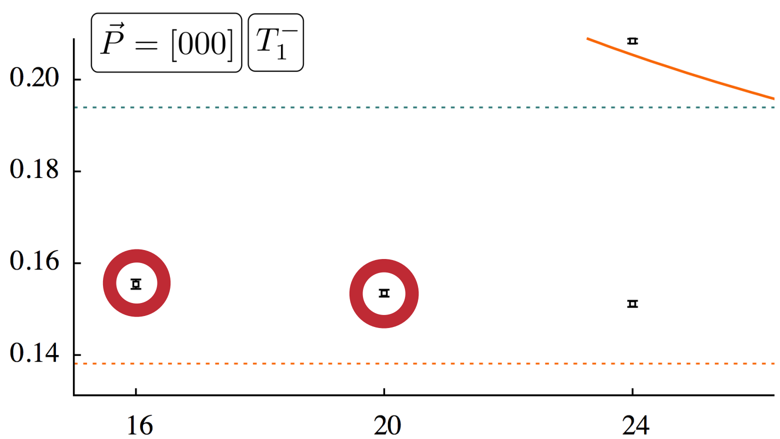

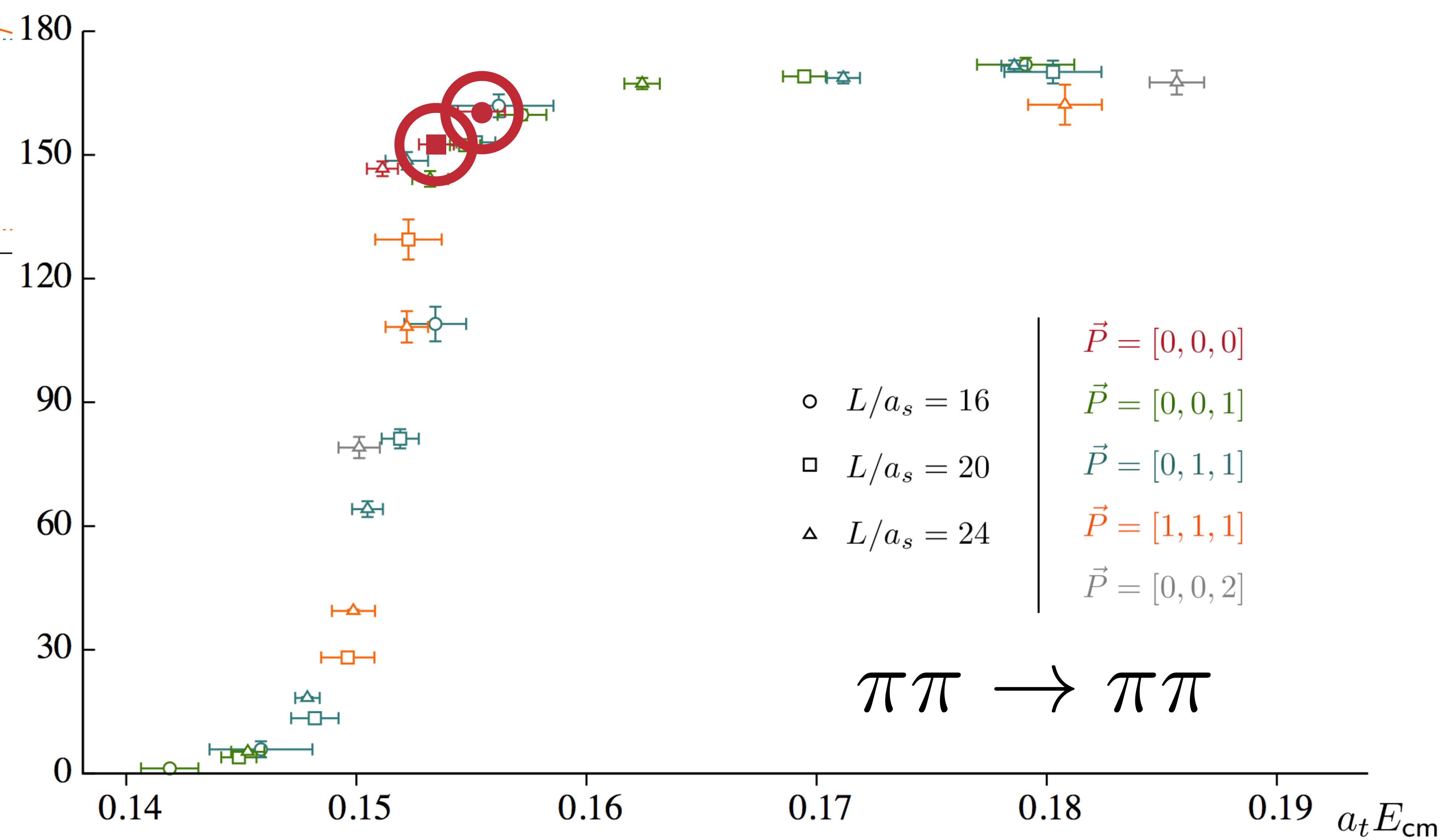

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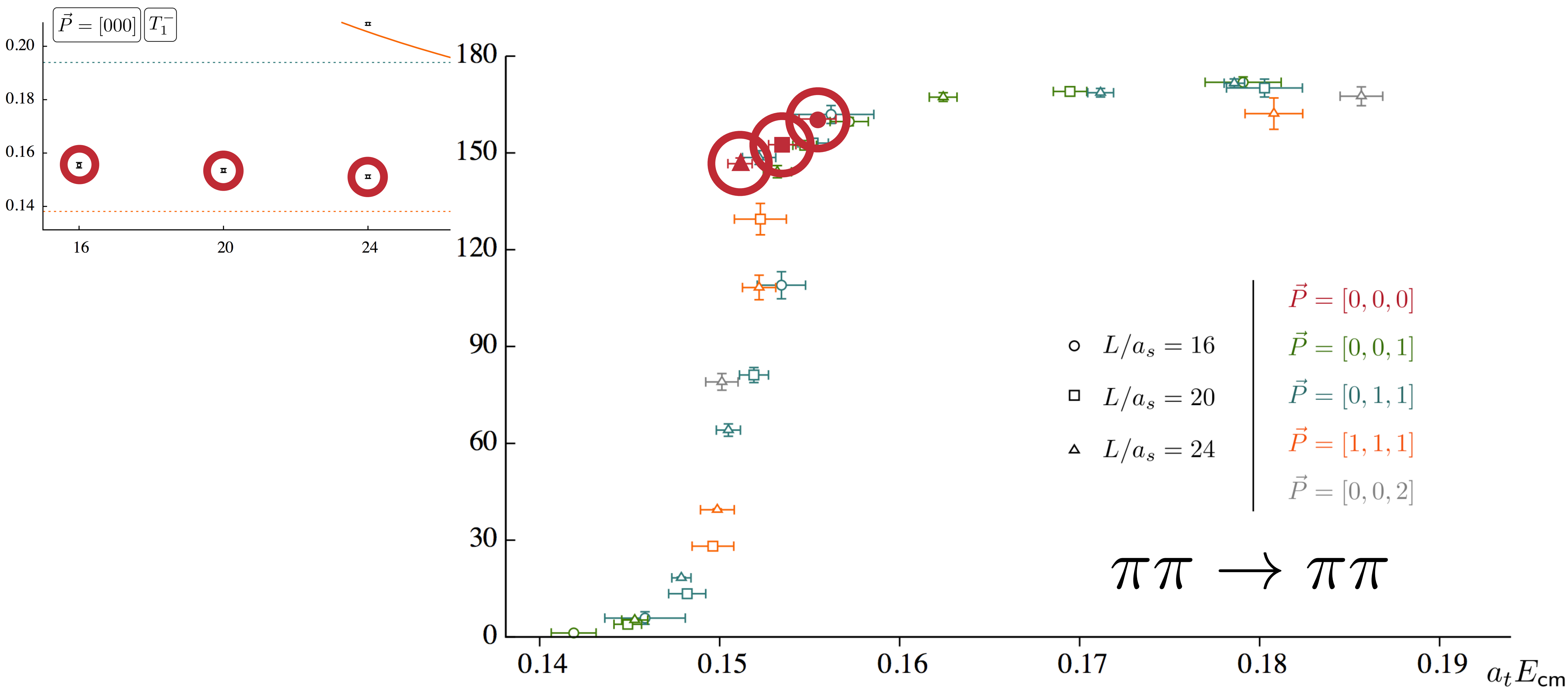

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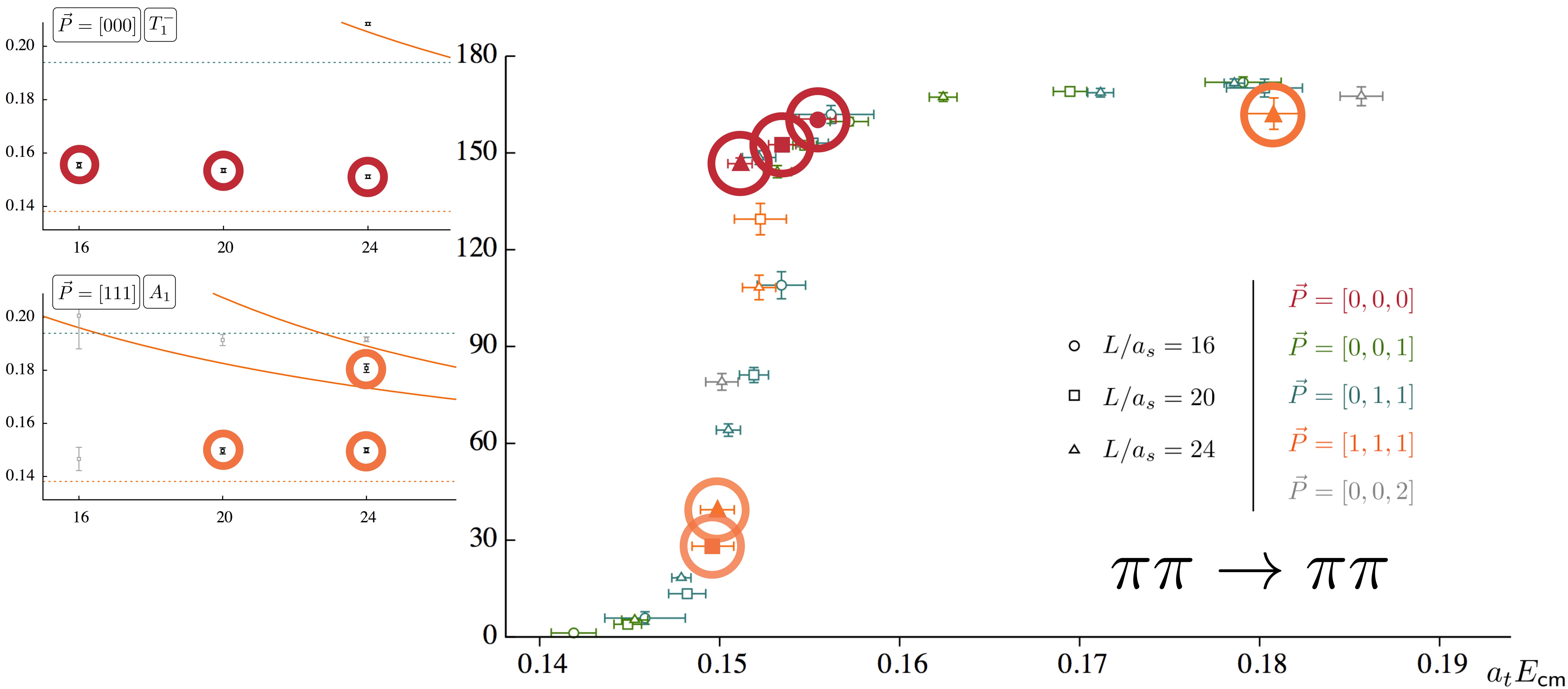

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- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
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  - ☑ Mass in $\lambda\phi^4$
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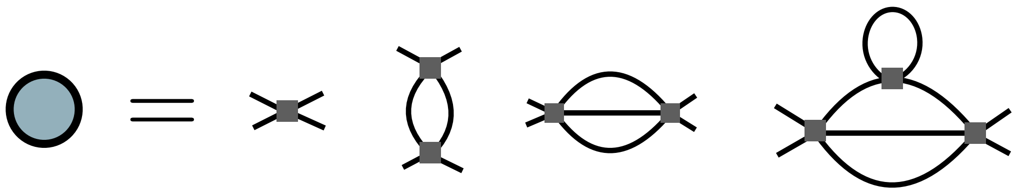
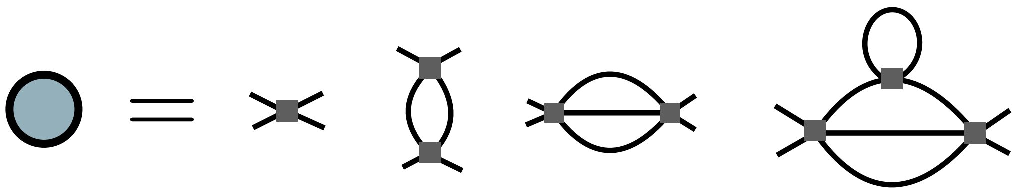
Bethe-Salpeter kernel



$$= B(s)$$

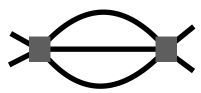
Bethe-Salpeter kernel



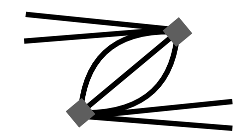
$= B(s)$

- Exponentially suppressed volume effects e^{-mL} for $(2m)^2 < s < (3m)^2$

  time-ordered (old fashioned) perturbation theory



$$\frac{1}{L^6}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\frac{1}{E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

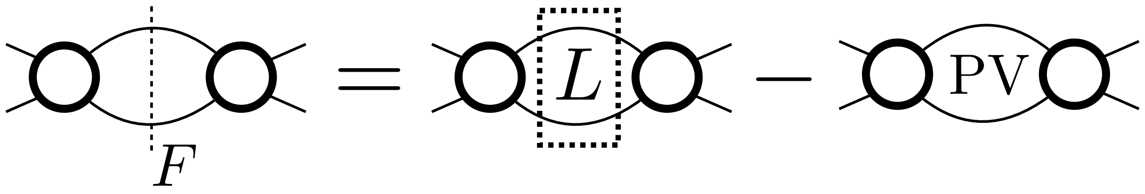


$$\frac{1}{L^6}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2}\frac{1}{-E-\omega_{\boldsymbol{p}_1}-\omega_{\boldsymbol{p}_2}-\omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

- Recall…

$$\int_k [\text{smooth}]\ e^{iLk\cdot n} = \mathcal{O}(e^{-mL})$$

Finite-volume cutting rule



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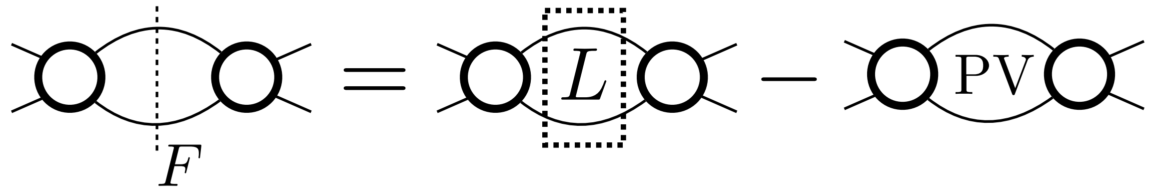
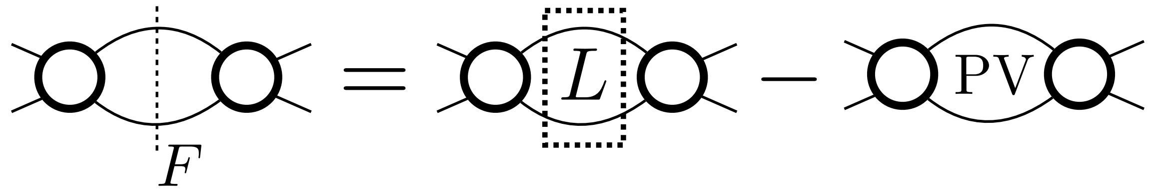


$$B \otimes F \otimes B = -i\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] \int \frac{dk^0}{2\pi} B(s,\boldsymbol{p},\boldsymbol{k}) D(k) D(P-k) B(s,\boldsymbol{k},\boldsymbol{p}')$$

$$= -\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{\boldsymbol{k}}} \frac{B(s,\boldsymbol{p},\boldsymbol{k})\, B(s,\boldsymbol{k},\boldsymbol{p}')}{(P-k)^2 + m^2 - \Sigma(P-k)}\bigg|_{k^0=\omega_{\boldsymbol{k}}} + \mathcal{O}(e^{-mL})$$

Finite-volume cutting rule



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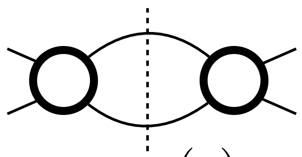
$$= \frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{B_{\mathsf{os}}(s,\hat{\boldsymbol{p}},\hat{\boldsymbol{k}})\,B_{\mathsf{os}}(s,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}')}{E-2\omega_{\boldsymbol{k}}} + \mathcal{O}(e^{-mL})$$

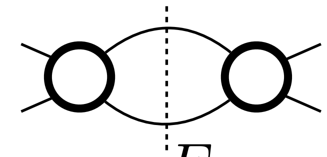
$$B_{\mathsf{os}}(s,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}) = B(s,\sqrt{s/4-m^2}\,\hat{\boldsymbol{k}},\sqrt{s/4-m^2}\,\hat{\boldsymbol{p}})$$

$$= \mathcal{Y}^*(\hat{\boldsymbol{p}})\cdot B_{\mathsf{os}}(s)\cdot\left[\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{E-2\omega_{\boldsymbol{k}}}\right]\cdot B_{\mathsf{os}}(s)\cdot\mathcal{Y}^*(\hat{\boldsymbol{p}}') + \mathcal{O}(e^{-mL})$$

Comparison of cuts



$\rho(s)$



F

$$\rho(s) = \mathbb{I}\frac{\sqrt{1-4m^2/s}}{32\pi}$$

$$F(E,L) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \text{p.v.}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\boldsymbol{k}})^2}\frac{\mathcal{Y}(\hat{\boldsymbol{k}})\mathcal{Y}^*(\hat{\boldsymbol{k}})}{E-2\omega_{\boldsymbol{k}}}$$

No volume dependence

Volume dependent

Both can be understood as matrices in $\ell m, \ell' m'$

proportional to the identity

on- and off-diagonal elements

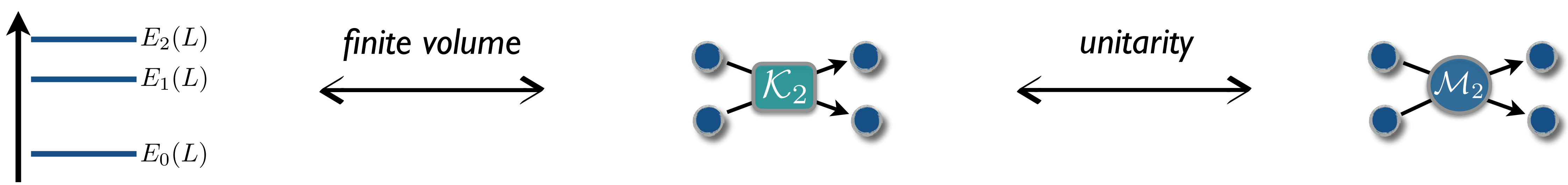
Exact difference between $i\epsilon$ and PV

Approximate difference between sum and PV

Result

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



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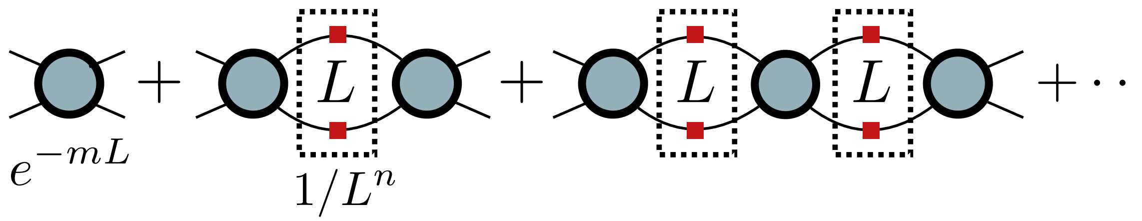
Spin-half particles

- Essentially the same Bethe-Salpeter expansion

$\mathcal{M}_L(P) =$



- Now the two-particle loops have spin… and are still projected on-shell

$$\frac{1}{i}\frac{-\not{p}+m}{p^2+m^2-i\epsilon} \quad \Longrightarrow \quad \frac{1}{i}\frac{\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p})}{p^2+m^2-i\epsilon}$$

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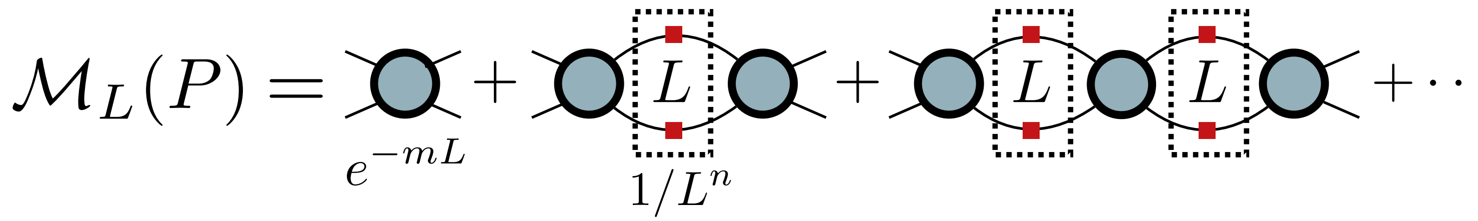

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- Spinors project B and thus $\mathscr{M}$ to definite spin components
- Final result is the same quantisation condition with a new F

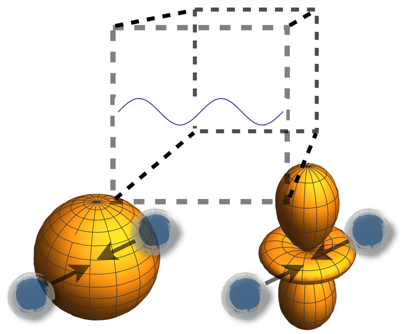
$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$$F_{\ell' m' s_1' s_2', \ell m s_1 s_2}(E,L) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \text{p.v.}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{1}{(2\omega_{\mathbf{k}})^2}\frac{\mathcal{Y}_{\ell' m'}(\hat{\mathbf{k}})\mathcal{Y}^*_{\ell m}(\hat{\mathbf{k}})}{E-2\omega_{\mathbf{k}}}\delta_{s_1' s_1}\delta_{s_2' s_2}$$

Coupled channels

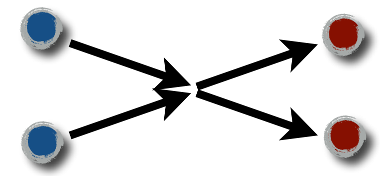
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $$\det\left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $$\det\left[\begin{pmatrix} \mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$$



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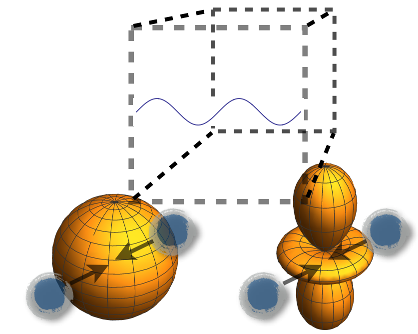

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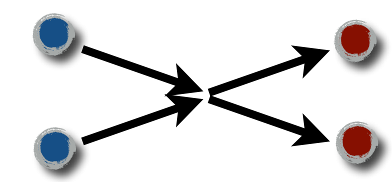


- Workflow…

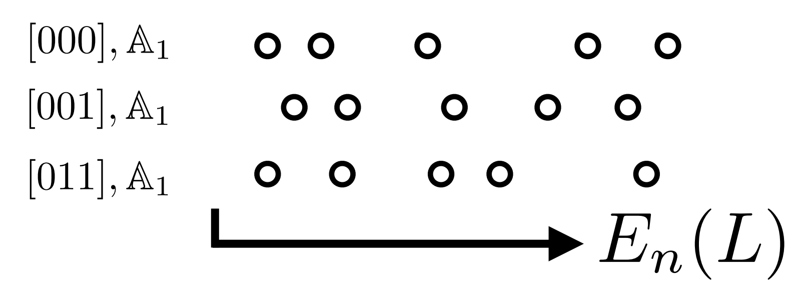
Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→

Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary L and P to recover a dense set of energies



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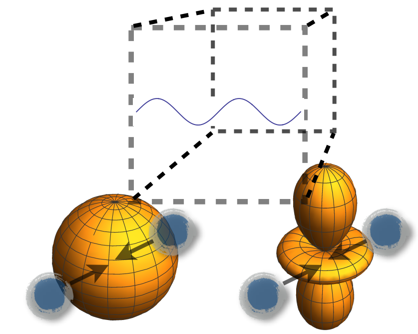

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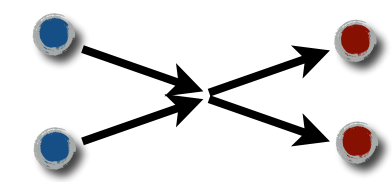
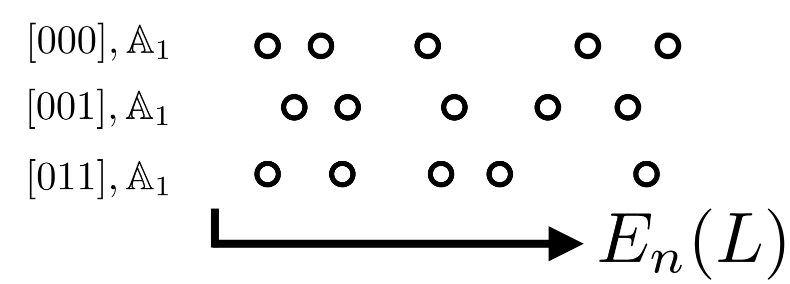


- Workflow…

Correlators with a large operator basis $\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→ Reliably extract finite-volume energies $\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→ Vary L and P to recover a dense set of energies



hadspec

Identify a broad list of K-matrix parametrizations

polynomials and poles | EFT based | dispersion theory based

→ **Perform global fits to the finite-volume spectrum**

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} round one
  - ☑ Mass in $\lambda\phi^4$
  - ☑ Mass/matrix element in $g\phi^3$
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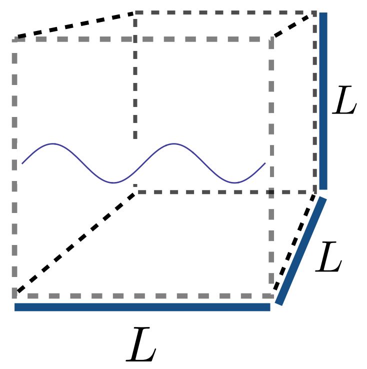
LO HVP (hadronic vacuum polarization)

- *T x L*³ periodic, Euclidean signature

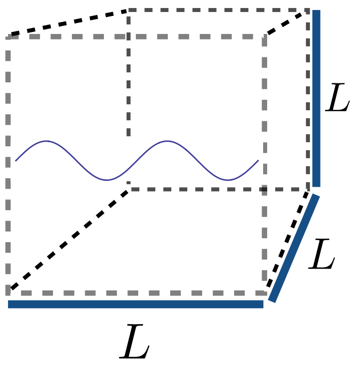


$$a_\mu^{\mathrm{LO,HVP}}(L,T) \equiv \int_0^{T/2} dx_0\, \mathcal{K}(x_0) \int_{L^3} d^3\boldsymbol{x}\, \langle j_k^\dagger(x_0,\boldsymbol{x}) j_k(0)\rangle_{L,T}$$

- Continuous kernel function: $\mathcal{K}(x_0) \overset{x_0\to\infty}{\propto} x_0^2$

LO HVP (hadronic vacuum polarization)

- *T x L*[3] periodic, Euclidean signature



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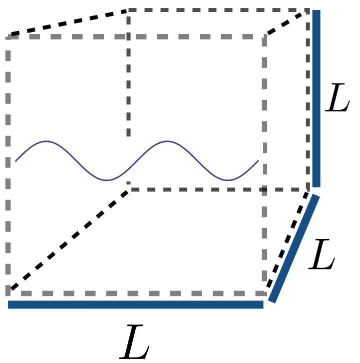
- Continuous kernel function: $\mathcal{K}(x_0) \overset{x_0 \to \infty}{\propto} x_0^2$
- For asymptotically large *T, L* …

$$\begin{aligned} a_\mu^{\mathrm{LO,HVP}}(L,T) - a_\mu^{\mathrm{LO,HVP}} &= \mathcal{O}(e^{-M_\pi L}) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) + \mathcal{O}(e^{-\sqrt{3}M_\pi L}) + \mathcal{O}(e^{-1.9 M_\pi L}) + \cdots \\ &+ \mathcal{O}(e^{-M_\pi T}) + \mathcal{O}(e^{-\frac{3}{2}M_\pi T}) + \cdots \\ &+ \mathcal{O}(e^{-M_\pi \sqrt{T^2+L^2}}) + \cdots \\ &+ \mathcal{O}(e^{-M_K L}) + \cdots \end{aligned}$$

$\sqrt{2+\sqrt{3}} \approx 1.92$

LO HVP (hadronic vacuum polarization)

- *T x L*³ periodic, Euclidean signature



$$a_\mu^{\mathrm{LO,HVP}}(L,T) \equiv \int_0^{T/2} dx_0\, \mathcal{K}(x_0) \int_{L^3} d^3\boldsymbol{x}\, \langle j_k^\dagger(x_0,\boldsymbol{x}) j_k(0) \rangle_{L,T}$$

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$$\begin{aligned} a_\mu^{\mathrm{LO,HVP}}(L,T) - a_\mu^{\mathrm{LO,HVP}} = {} & \mathcal{O}(e^{-M_\pi L}) + \mathcal{O}(e^{-\sqrt{2}M_\pi L}) + \mathcal{O}(e^{-\sqrt{3}M_\pi L}) + \mathcal{O}(e^{-1.9 M_\pi L}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi T}) + \mathcal{O}(e^{-\frac{3}{2}M_\pi T}) + \cdots \\ & + \mathcal{O}(e^{-M_\pi \sqrt{T^2+L^2}}) + \cdots \\ & + \mathcal{O}(e^{-M_K L}) + \cdots \end{aligned}$$

$\sqrt{2+\sqrt{3}} \approx 1.92$

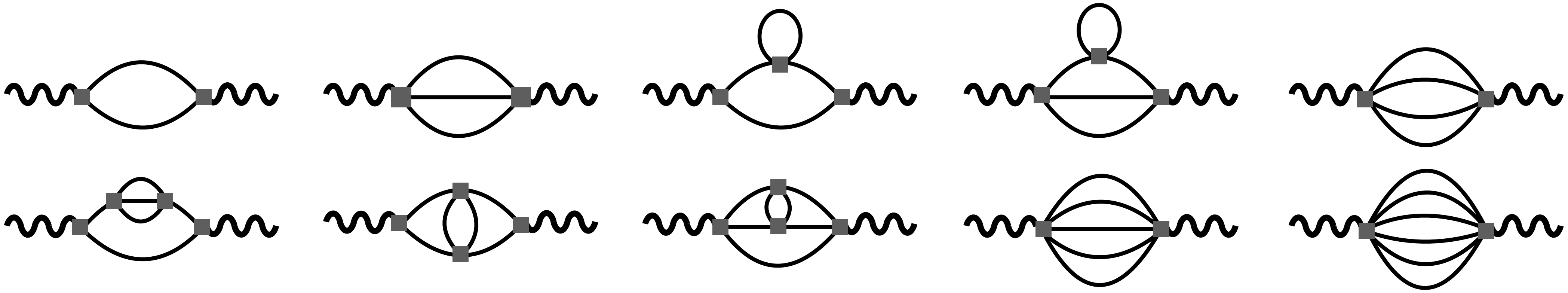
1904.10010 2004.03935 for now

MTH and A. Patella • *strong inspiration from…* Lüscher 1986

Diagrammatic expansion

- All orders diagrammatic expansion

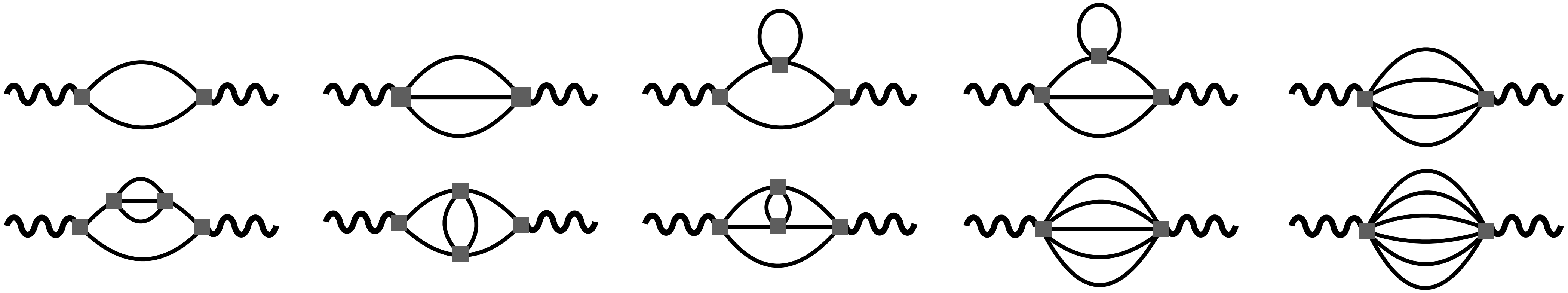
$$a_\mu(L) =$$



$$+ \cdots$$

Diagrammatic expansion

- All orders diagrammatic expansion

$a_\mu(L) =$  $+ \cdots$

- Loop momenta are summed… rewrite using ***Poisson summation***

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \longrightarrow \sum_{\boldsymbol{n}}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}e^{iL\boldsymbol{n}\cdot\boldsymbol{k}}$$

$\boldsymbol{n} = (n_x, n_y, n_z)$ can be interpreted as loop pion wrapping the torus

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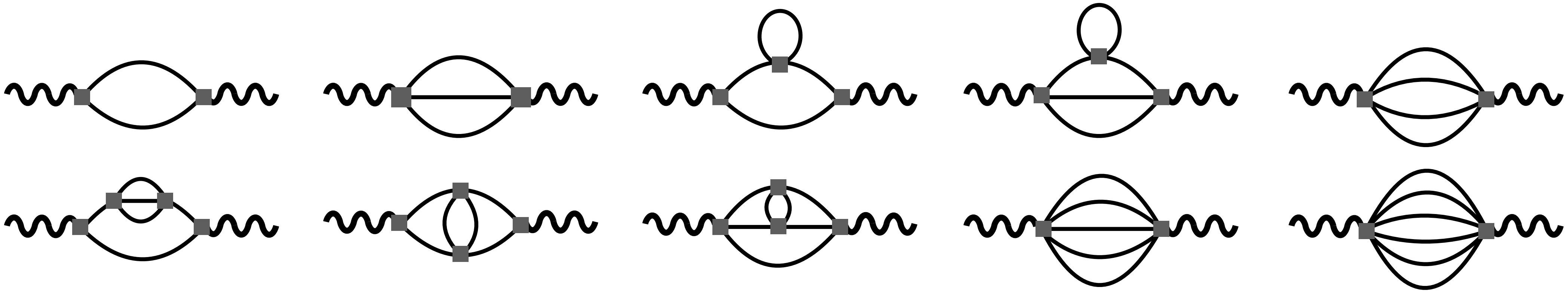

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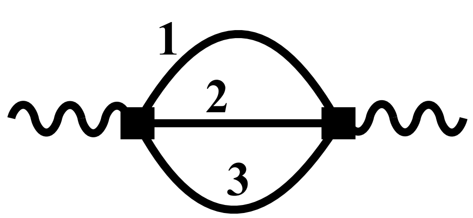
- Some subtleties in assigning ***Poisson modes***



$$\begin{aligned}\boldsymbol{n}_1 &= (0,0,1) \\ \boldsymbol{n}_2 &= (0,0,1)\end{aligned} \quad \equiv \quad \begin{aligned}\boldsymbol{n}_2 &= (0,0,0) \\ \boldsymbol{n}_3 &= (0,0,1)\end{aligned}$$

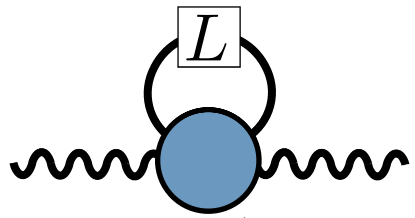
understood as a gauge redundancy → gauge fixing to catalogue effects

Lüscher (1986)

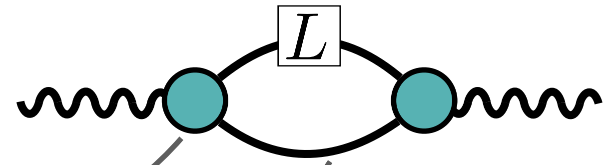
Extension (all orders diagrammatic)

- Leading contributions: *only one pion wraps the torus*

$$\Delta a_\mu(L) = \int_0^\infty dx_0\, \mathcal{K}(x_0) \Bigg[$$



$$+ \frac{1}{2}$$



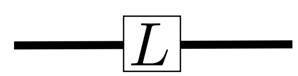
$$\Bigg] + \mathcal{O}(e^{-1.9 M_\pi L})$$

one-particle irreducible vertices

dressed propagators

leading 2-wrap contribution

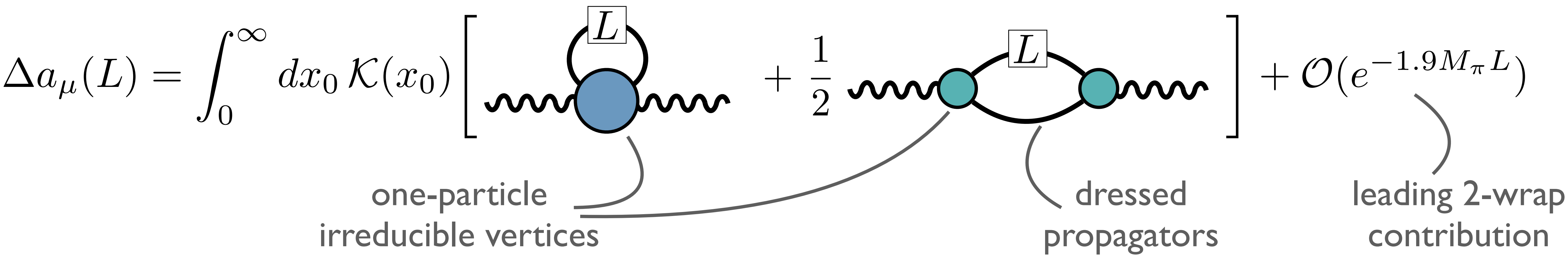
- All *L* dependence inside…



$$\equiv \sum_{\boldsymbol{n}} \frac{e^{iL\boldsymbol{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$$

Extension (all orders diagrammatic)

- Leading contributions: *only one pion wraps the torus*



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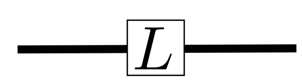
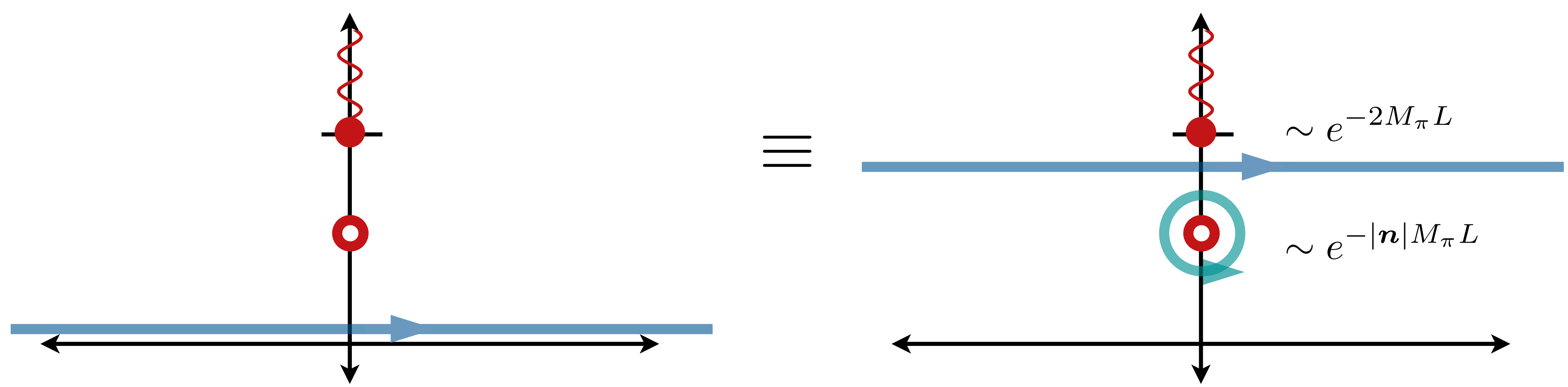


$$\equiv \sum_{\boldsymbol{n}} \frac{e^{iL\boldsymbol{n}\cdot\mathbf{p}}}{p^2 + M_\pi^2 + \Sigma(p^2)}$$

- Exponential decay?



pole contributions → on-shell pions → ***physical Compton amplitude***

Result

- Sum of ***all single-winding terms*** gives

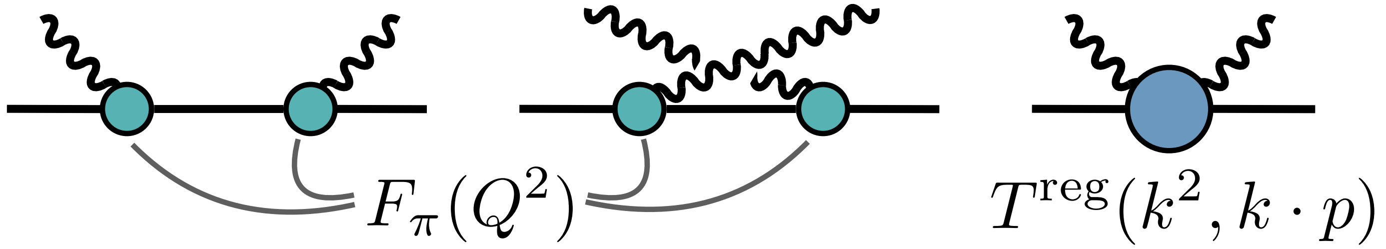
$$\Delta a_\mu(L) = -\frac{\alpha^2}{m_\mu^2} \sum_{\boldsymbol{n}\neq \boldsymbol{0}} \int \frac{dp_3}{2\pi} \frac{e^{-|\boldsymbol{n}|L\sqrt{M_\pi^2+p_3^2}}}{2\pi L|\boldsymbol{n}|} \int_0^\infty dx_0\, \widehat{\mathcal{K}}(m_\mu x_0) \int \frac{dk_3}{2\pi} \cos(x_0 k_3)\, \mathrm{Re}\, T(-k_3^2, -k_3 p_3)$$

volume dependence — kernel — Compton amplitude

reference to EFT has disappeared

requires only spacelike photons

$$T(k^2, k\cdot p) \equiv i \lim_{\boldsymbol{p}'\to\boldsymbol{p}} \sum_{q=0,\pm1} \int d^4x\, e^{ikx} \langle \boldsymbol{p}', q|\mathrm{T}\mathcal{J}_\rho(x)\mathcal{J}^\rho(0)|\boldsymbol{p}, q\rangle$$

$=$ 

Numerical estimates

$M_\pi = 137$ MeV $\quad m_\mu = 106$ MeV $\quad F_\pi(Q^2) = [1 + Q^2/M^2]^{-1} \quad M = 727$ MeV

$$-100 \times \Delta a_\mu(L)/(700 \times 10^{-10})$$

$M_\pi L$	$\lvert\mathbf{n}\rvert = 1$	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$2\sqrt{2}$	3	$\sum_\mathbf{n}$
4	1.26	1.16	0.317	0.104	0.193	0.0944	0.0128	0.0174	3.17
5	0.852	0.428	0.0813	0.0199	0.0287	0.0112	0.00102	0.00117	1.42
6	0.461	0.141	0.0189	0.00349	0.00394	0.00124	0.0000764	0.0000735	0.630
7	0.226	0.0433	0.00417	0.000582	0.000515	0.000130	5.46×10^{-6}	4.41×10^{-6}	0.274
8	0.104	0.0128	0.000883	0.0000936	0.0000652	0.0000132	3.79×10^{-7}	2.57×10^{-7}	0.118

- Slow convergence of $e^{-\alpha_n M_\pi L}$ series
- Last column estimates %-finite volume effect
- 1% precision goal for theoretical relevance

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} round one
  - ☑ Mass in $\lambda\phi^4$
  - ☑ Mass/matrix element in $g\phi^3$
- ☑ $2 \to 2$ formalism
  - ☑ Scattering basics
  - ☑ Derivation
  - ☑ Example application
  - ☑ Generalizations
- ☑ e^{-mL} round two
  - ☑ LO-HVP for $(g-2)_\mu$
  - ☑ Bethe-Salpeter kernel
- ☐ $(1+)\mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Example application
- ☐ $2 + \mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Conclusion and outlook