Excercises for MITP Summer School: Frontiers and Challenges in Lattice Gauge Theory

Scattering and Spectroscopy: Formalism Day 1

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1 Euclidean vs. Minkowski two-point function

1.1 Minkowski

Begin with the definition

$$\widetilde{G}_M(p_M) = \int d^4x_M \, e^{-ip_M\cdot x_M} \langle 0|\mathrm{T}\{\phi(x_M)\phi(0)\}|0\rangle, \tag{1}$$

where $p_M \cdot x_M = -p^0 x^0 + \mathbf{p}\cdot\mathbf{x}$ and

$$\langle 0|\mathrm{T}\{\phi(x_M)\phi(0)\}|0\rangle = \langle 0|\phi(0,\mathbf{x})e^{-i\hat{H}x^0-\epsilon x^0}\phi(0)|0\rangle, \qquad \text{for } x^0>0, \tag{2}$$

$$\langle 0|\mathrm{T}\{\phi(x_M)\phi(0)\}|0\rangle = \langle 0|\phi(0)e^{+i\hat{H}x^0+\epsilon x^0}\phi(0,\mathbf{x})|0\rangle, \qquad \text{for } x^0<0. \tag{3}$$

Insert the identity, defined as

$$\mathbb{I} = \sum_N \frac{1}{N!}\prod_{i=1}^{N}\int \frac{d^3\mathbf{p}_i}{(2\pi)^3 2\omega_{\mathbf{p}_i}}|\mathbf{p}_1,\cdots,\mathbf{p}_N\rangle\langle\mathbf{p}_1,\cdots,\mathbf{p}_N| \tag{4}$$

$$= \int d\alpha \int \frac{d^3\mathbf{P}}{(2\pi)^3 2\omega_{\mathbf{P},\alpha}}|\alpha,\mathbf{P}\rangle\langle\alpha,\mathbf{P}|, \tag{5}$$

where α is a collective index and

$$\hat{H}|\alpha,\mathbf{P}\rangle = |\alpha,\mathbf{P}\rangle\sqrt{M_\alpha^2+\mathbf{P}^2} = |\alpha,\mathbf{P}\rangle\omega_{\alpha,\mathbf{P}}, \tag{6}$$

$$\hat{\mathbf{P}}|\alpha,\mathbf{P}\rangle = |\alpha,\mathbf{P}\rangle\mathbf{P}. \tag{7}$$

Thereby demonstrate

$$\widetilde{G}_M(p_M) = \frac{1}{i}\int d\alpha \, \frac{|Z(\alpha,\mathbf{p})|^2}{p_M^2+M_\alpha^2-i\epsilon}, \tag{8}$$

and give the expression for $Z(\alpha,\mathbf{p})$ and M_α.

1.2 Euclidean

Now take

$$\widetilde{G}_E(p_E) = \int d^4x_E \, e^{-ip_E\cdot x_E} \langle 0|\mathrm{T}\{\phi(x_E)\phi(0)\}|0\rangle, \tag{9}$$

where $p_E \cdot x_E = p_4 x_4 + \mathbf{p}\cdot\mathbf{x}$ and

$$\langle 0|\mathrm{T}\{\phi(x_E)\phi(0)\}|0\rangle = \langle 0|\phi(0,\mathbf{x})e^{-\hat{H}x_4}\phi(0)|0\rangle, \qquad \text{for } x_4 > 0, \tag{10}$$

$$\langle 0|\mathrm{T}\{\phi(x_E)\phi(0)\}|0\rangle = \langle 0|\phi(0)e^{+\hat{H}x_4}\phi(0,\mathbf{x})|0\rangle, \qquad \text{for } x_4 < 0. \tag{11}$$

and derive the analogous result to that given in the previous subsection.

2 Twisted, asymmetric boundary conditions

Suppose we take the boundary conditions

$$\pi(\tau,\mathbf{x})e^{i\theta_i} = \pi(\tau,\mathbf{x}+L_i\mathbf{e}_i), \tag{12}$$

where $\mathbf{e}_i$ is a unit vector in the ith direction, $i \in \{x, y, z\}$. Derive the discrete set of momenta, $\mathbf{p}$, such that the Fourier transform respects the boundary conditions.

3 Volume dependence of the mass in $g\phi^3$

Begin with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi^2 - \frac{g_0}{3!}\phi_0^3\,, \tag{13}$$

where we are using bare fields and bare couplings.

- Give an expression for the $\mathcal{O}(g^2)$ correction in the relation between the physical finite-volume mass squared $m(L)^2$ and the bare mass squared m_0^2.
- Work out and simplify the expression for the mass shift

  $$\Delta m(L)^2 = m(L)^2 - m^2$$

  where m is the infinite-volume physical mass.
- Show that the finite-volume correction decays exponentially with L as

  $$\Delta m(L)^2 \sim e^{-\sqrt{3/4}\,mL}.$$

4 Single-particle matrix element in $g\phi^3$

4.1 Finite-volume decay constant

Define $j(x) = Z_j \phi(x)$ and define

$$f_\phi(L)^2 = |\langle 0|j(0)|\phi, \mathbf{p} = 0\rangle_L|^2, \tag{14}$$
$$f_\phi(\infty)^2 = |\langle 0|j(0)|\phi, \mathbf{p} = \mathbf{0}\rangle_\infty|^2. \tag{15}$$

Determine how to extract $f_\phi(L)^2$ from

$$\widetilde{G}_L^f(p^0) = Z_j^2 \int d^3x \int dx^0 \, e^{ip^0 x^0} \langle 0|T\{\phi(x^0, \mathbf{x})\phi(0)\}|0\rangle_L, \tag{16}$$

and give the analogous expression for $f_\phi(\infty)$.

4.2 Self energy summation

Show that the sum of all diagrams, to all orders can be written as

$$\widetilde{G}_L^f(p^0) = \frac{1}{i} \frac{Z_j^2}{-(p^0)^2 + m_0^2 + \Sigma_L(-(p^0)^2)}, \tag{17}$$

and define Σ_L.

4.3 Volume effects

Show that

$$\frac{f_\phi(L)^2}{f_\phi(\infty)^2} = 1 - \left[\Sigma_L'(-m^2) - \Sigma_\infty'(-m^2)\right] + \mathcal{O}(g^3), \tag{18}$$

and evaluate the g^2 result.

5 Two-particle phase space

Define the two-particle phase space as

$$\rho(s) = \frac{1}{2} \mathrm{Im} \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{k^2 + m^2 - i\epsilon} \frac{1}{(P-k)^2 + m^2 - i\epsilon} \right], \tag{19}$$

where $s = -P^2 = (P^0)^2 - \mathbf{P}^2$. Evaluate this to identify the square-root function that depends on s.