Spectroscopy and Scattering: Formalism (Lecture 1/3)

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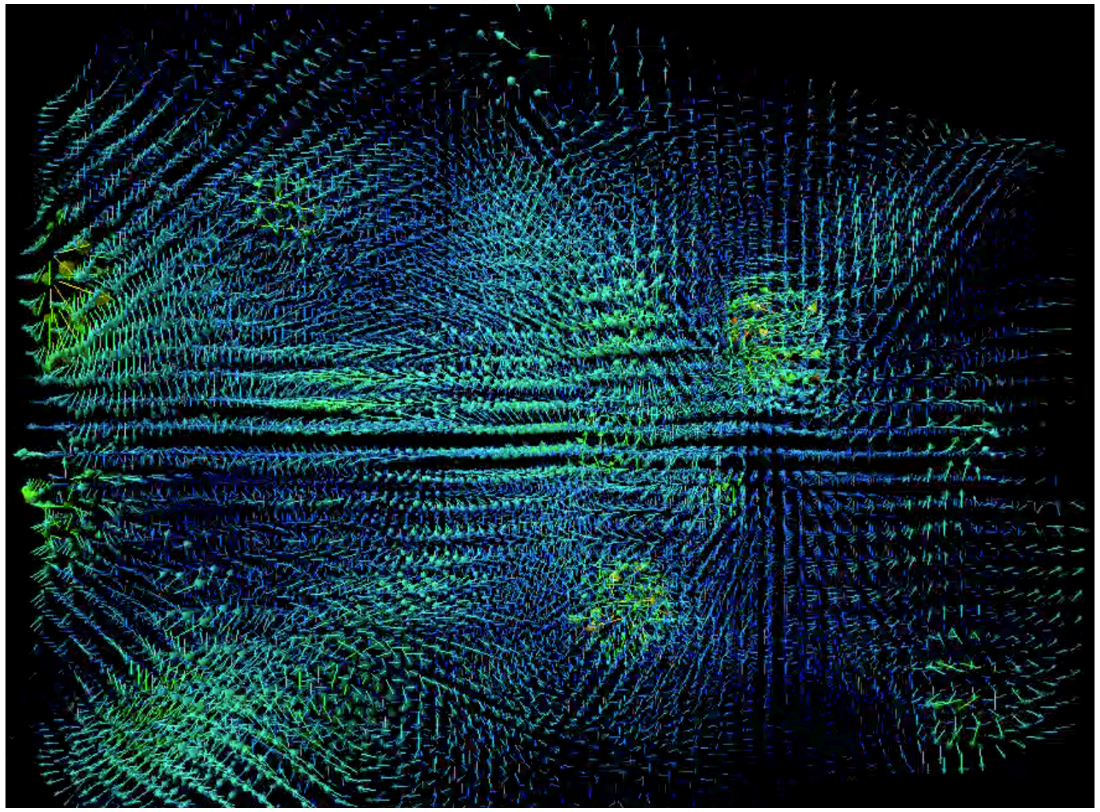
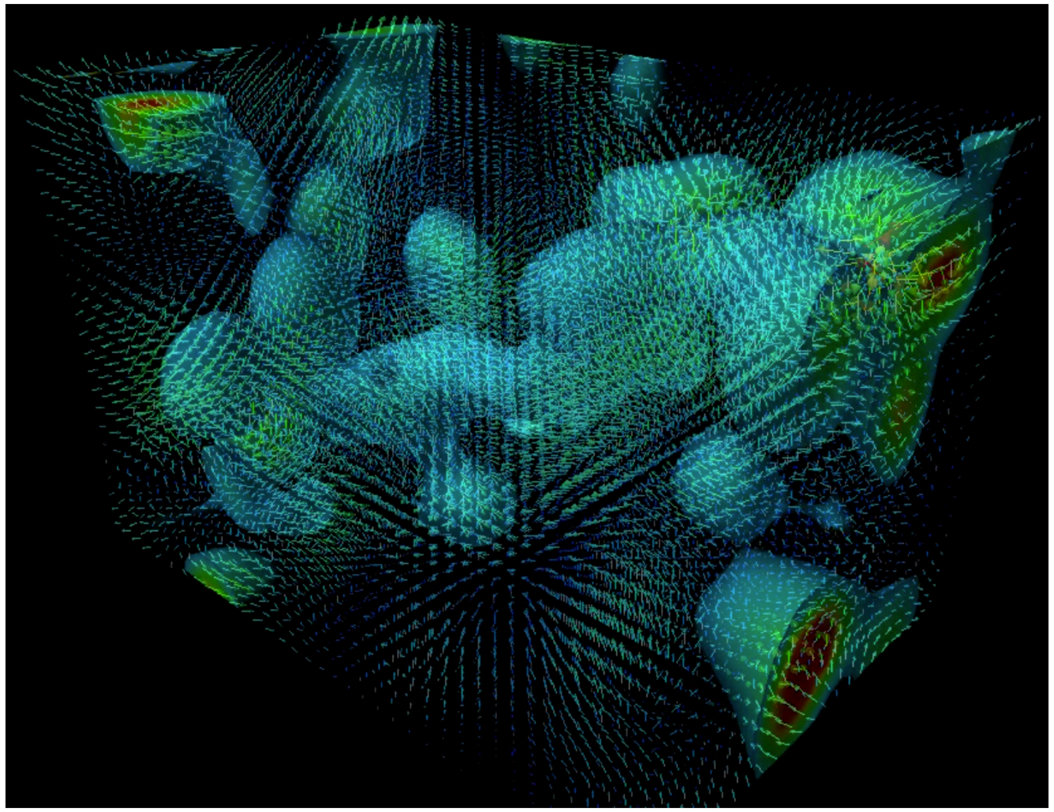
Mainz Institute for Theoretical Physics

THE UNIVERSITY of EDINBURGH

Lattice field theory

- Non-perturbative regulator of quantum field theory (QFT)
- Systematically improvable numerical method for extracting QFT's properties
- Exciting, vibrant, highly active research community
- Technical field that challenges all of us to be great communicators



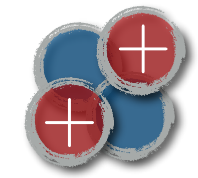


University of Adelaide, CSSM

Recipe for strong force predictions

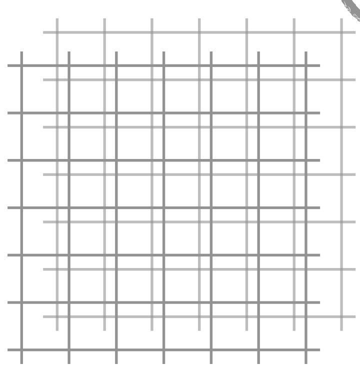
1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice field theory)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$



Wide range of precision pre-/post-dictions

$m_q^{\overline{\rm MS}}(\mu)$ $\quad m_p \quad$ $\alpha_{\overline{\rm MS}}(\mu)$ $\quad m_n \quad f_\pi \quad g_A \quad f_K \quad (g-2)_\mu^{\rm HVP} \quad f_D \quad f_B$



FLAG 2019 Flavour Lattice Averaging Group

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8)$$ lattice average

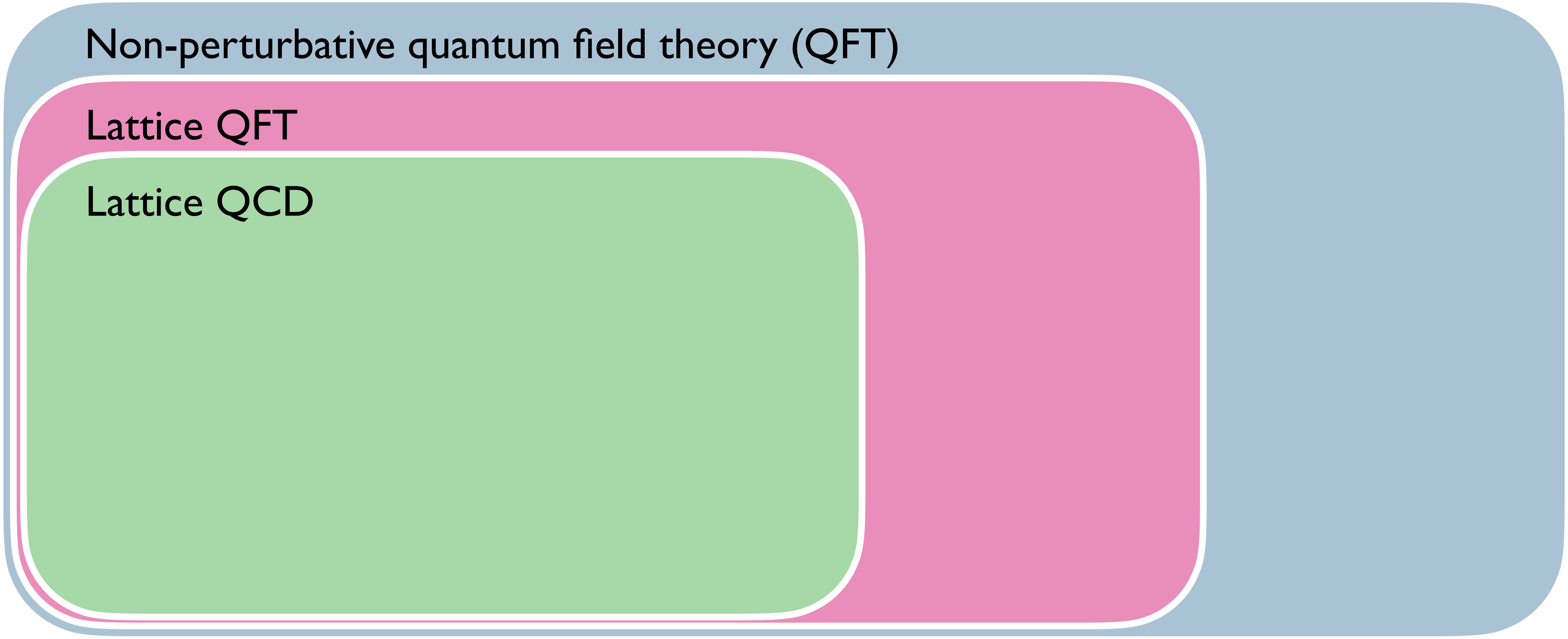
$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16)$$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Lattice QCD

- a non-perturbative regularization of QCD
- a definition that is well-suited to numerical evaluation

render the quantum path-integral finite-dimensional → *evaluate using Monte Carlo importance sampling*



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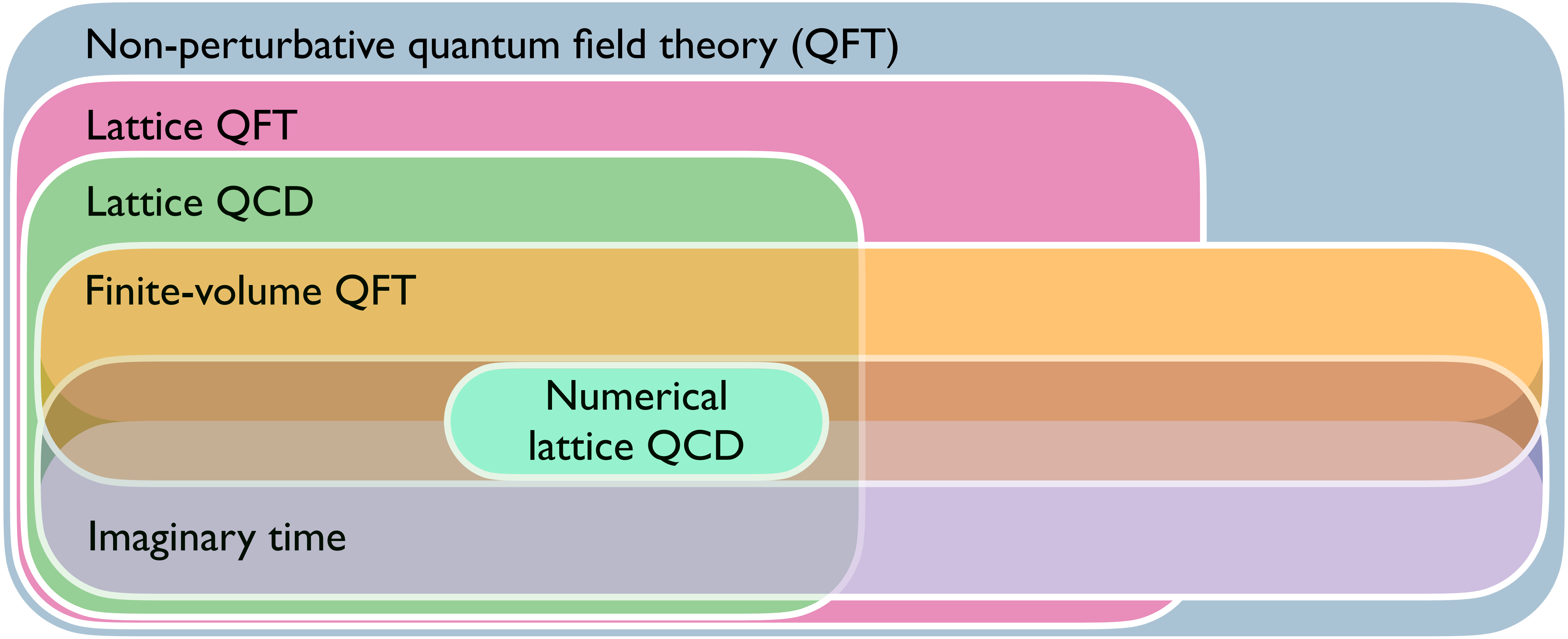


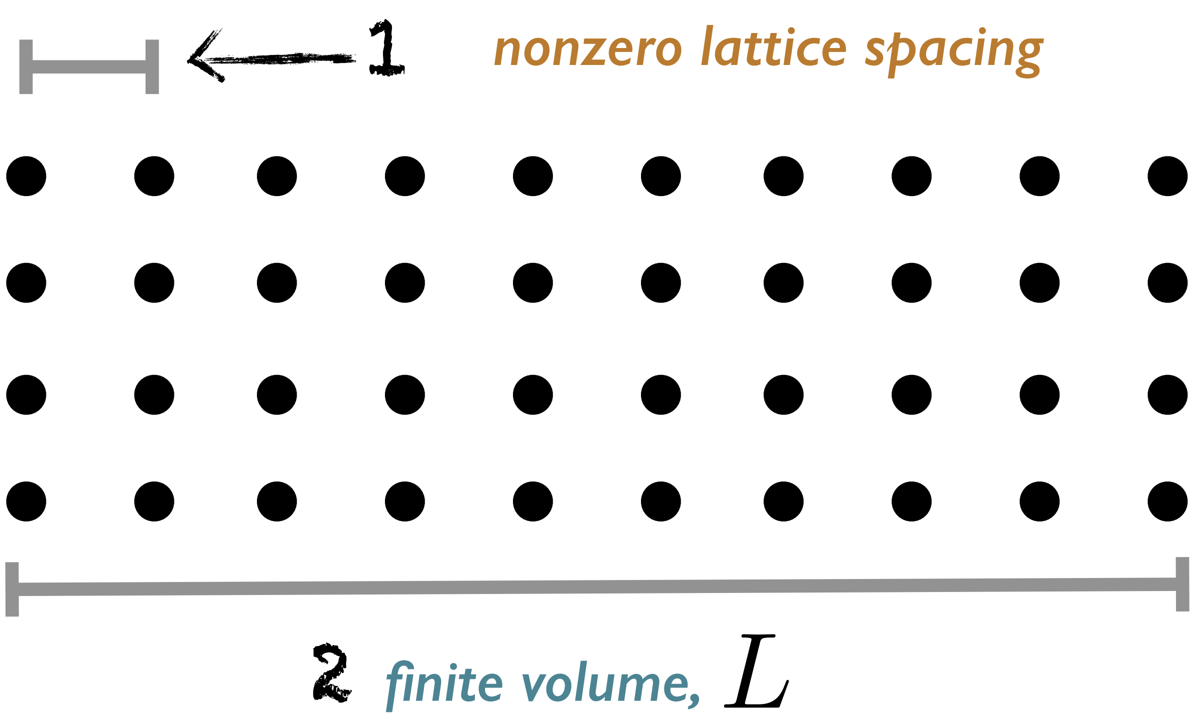
Limitations of lattice QCD

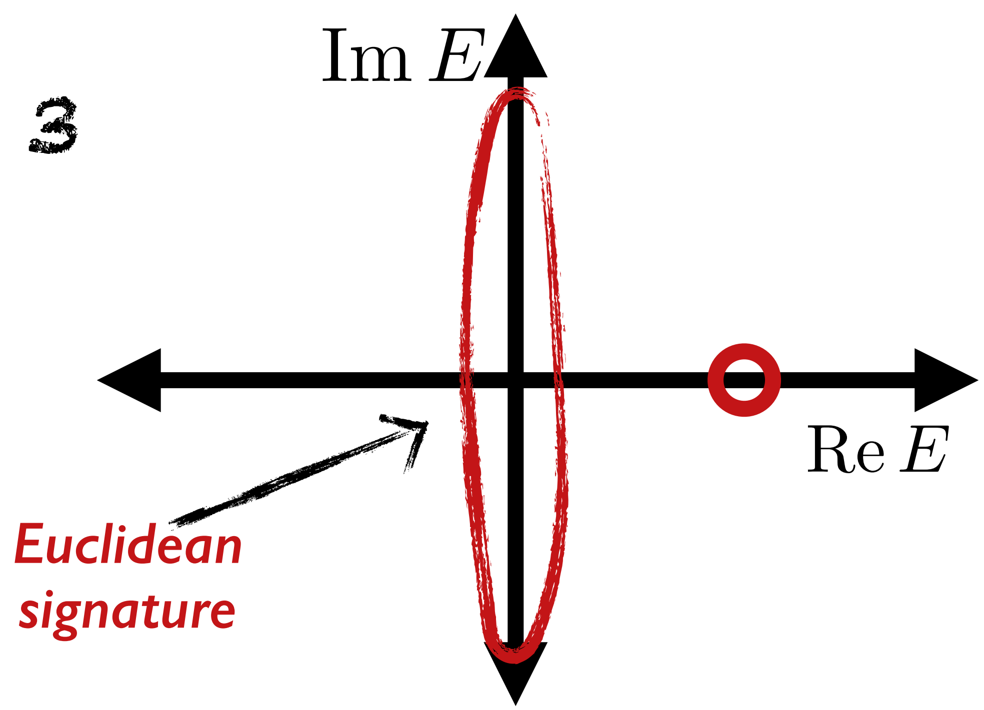
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Limitations of lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*

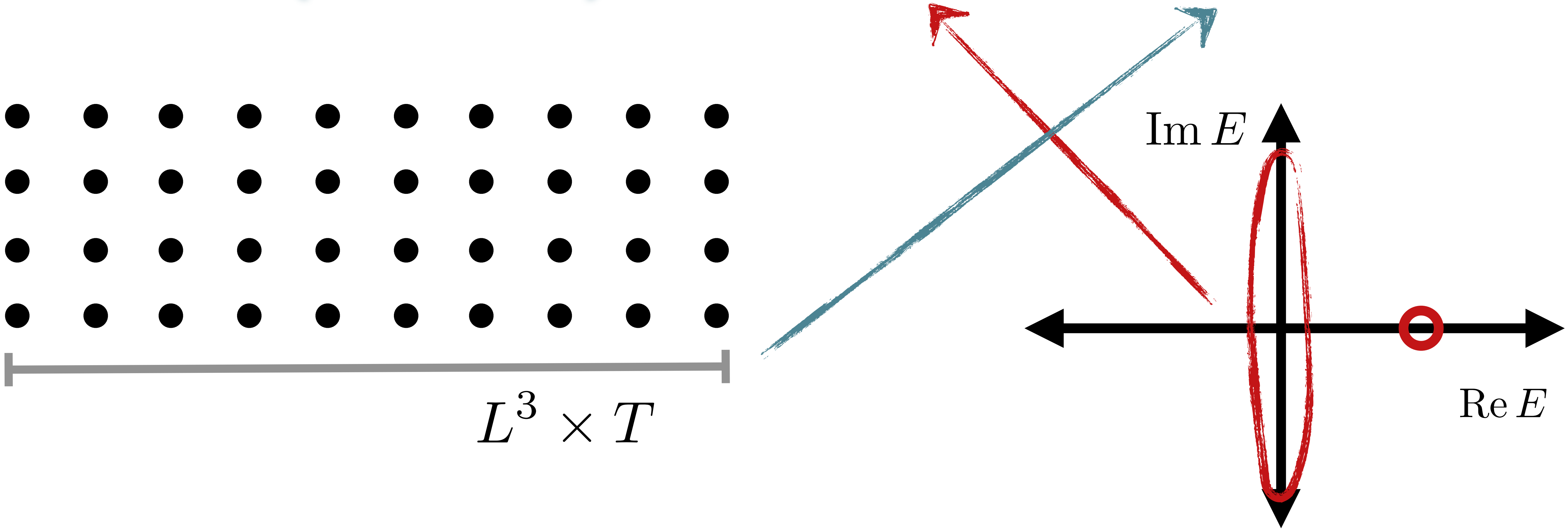




Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

QCD (and QED) in a Euclidean finite volume



Sometimes the **finite-volume** is an **unwanted artifact** (extrapolate $L \to \infty$)…

… but it can also be a **useful tool** (fit data to predicted L dependence)

Using the volume as a tool often resolves the issue of Euclidean signature

- ☐ Warm-up and definitions
  - ☐ Meaning of Euclidean
  - ☐ Finite-volume set-up
- ☐ e^{-mL} round one
  - ☐ Mass in $\lambda\phi^4$
  - ☐ Mass/matrix element in $g\phi^3$
- ☐ $2 \to 2$ formalism
  - ☐ Scattering basics
  - ☐ Derivation
  - ☐ Example application
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  - ☐ Testing the result
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  - ☐ Applications
- ☐ 3 → 3 formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
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Euclidean vs Minkowski

Four-vector basics

$$x_M^\alpha = (t, \vec{x})^\alpha, \quad p_M^\alpha = (E, \vec{p})^\alpha \quad \Big\} \quad x_M \cdot p_M = x_M^\alpha p_{M,\alpha} = -Et + \vec{p}\cdot\vec{x}$$

$$x_{E,\alpha} = (\vec{x}, it) = (\vec{x}, \tau)_\alpha \qquad \alpha = 1,2,3,4$$

$$x_E^2 = \tau^2 + \vec{x}^2$$

Minkowski two-point function (generic scalar theory)

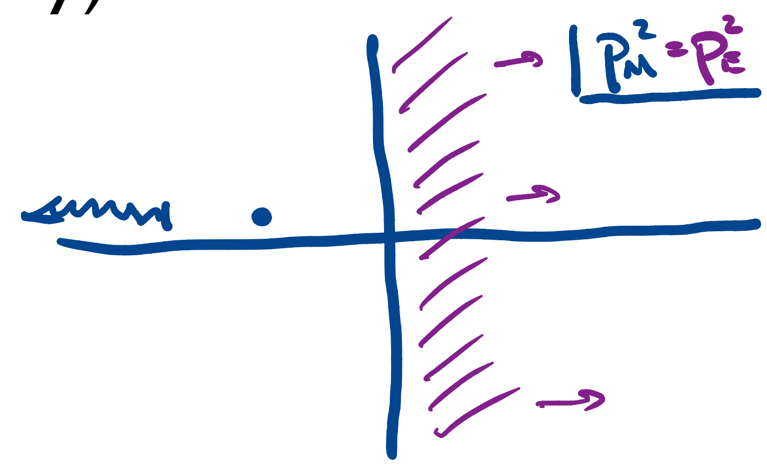
$$\tilde{G}_M(p_M) = \int d^4x_M\, e^{-ip_M\cdot x_M} \langle 0 | T\{\phi(x_M)\phi(0)\} | 0 \rangle$$

$$= \int_0^\infty dt\, e^{+iEt} \langle 0 | \tilde{\phi}(0,\vec{p})\, e^{-i\hat{H}t - \epsilon t}\, \phi(0) | 0 \rangle + \dots$$

$$= \int d\alpha \frac{1}{2\omega_{\alpha,\vec{p}}} \frac{1}{i} \frac{|Z_\alpha|^2}{-E + \omega_{\alpha,\vec{p}} - i\epsilon} + \dots$$

$$Z_\alpha = \langle 0 | \phi(0) | \alpha, \vec{p} \rangle, \qquad \omega_{\alpha,\vec{p}} = \sqrt{\vec{p}^2 + M_\alpha^2}$$

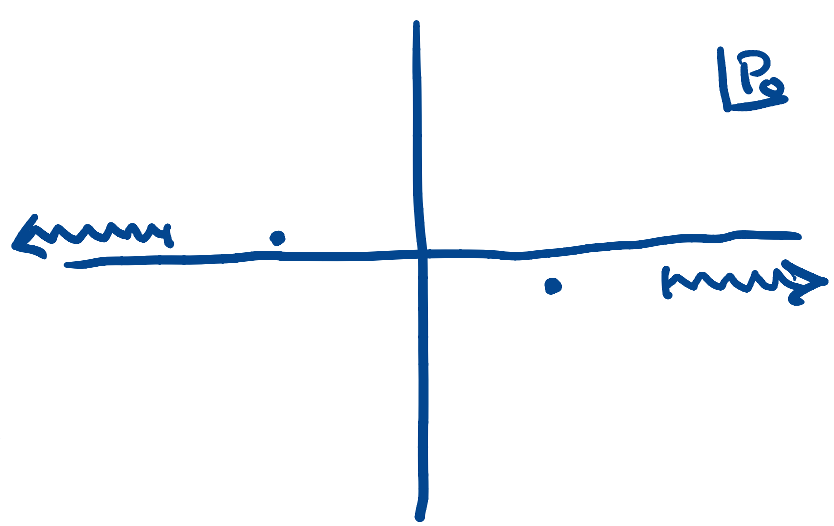
$$= \frac{1}{i} \int d\alpha \frac{|Z_\alpha|^2}{p_M^2 + M_\alpha^2 - i\epsilon}$$

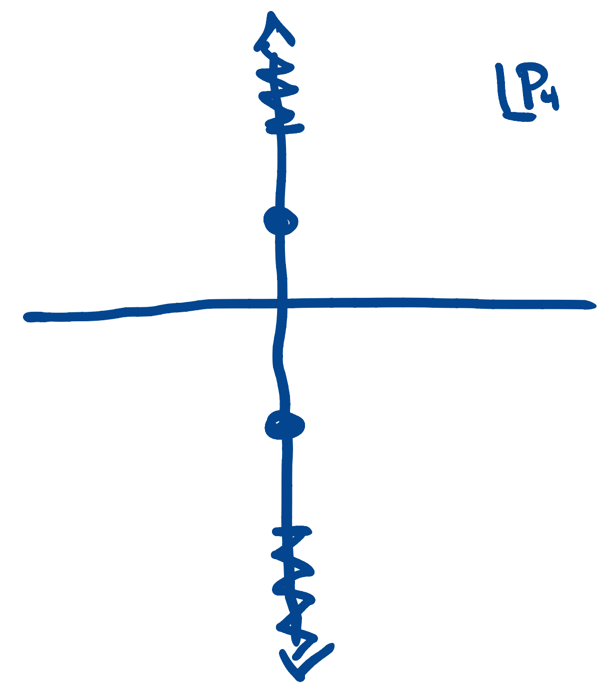


Euclidean two-point function

$$\tilde{G}_E(p_E) = \int d^4x_E\, e^{-ip_E\cdot x_E} \langle \dots \rangle = \int d\alpha \frac{|Z_\alpha|^2}{p_E^2 + M_\alpha^2}$$

$$e^{-\hat{H}\tau}$$





Meaning of Euclidean

When asked **"Is it Euclidean or Minkowski?"** first think:

"Is the question well-posed?"

Sometimes it is	Often it is not
❑ Correlation functions	❑ Masses
❑ Four-vectors	❑ Decay constants
❑ QFT path integral	❑ Finite-volume energies
	❑ Scattering amplitudes
	❑ Local matrix elements
	❑ Spectral densities

For the case of **correlation functions**, Minkowski vs Euclidean is…

Important **when we have limited knowledge**
(e.g. numerical estimate for real Euclidean momenta)

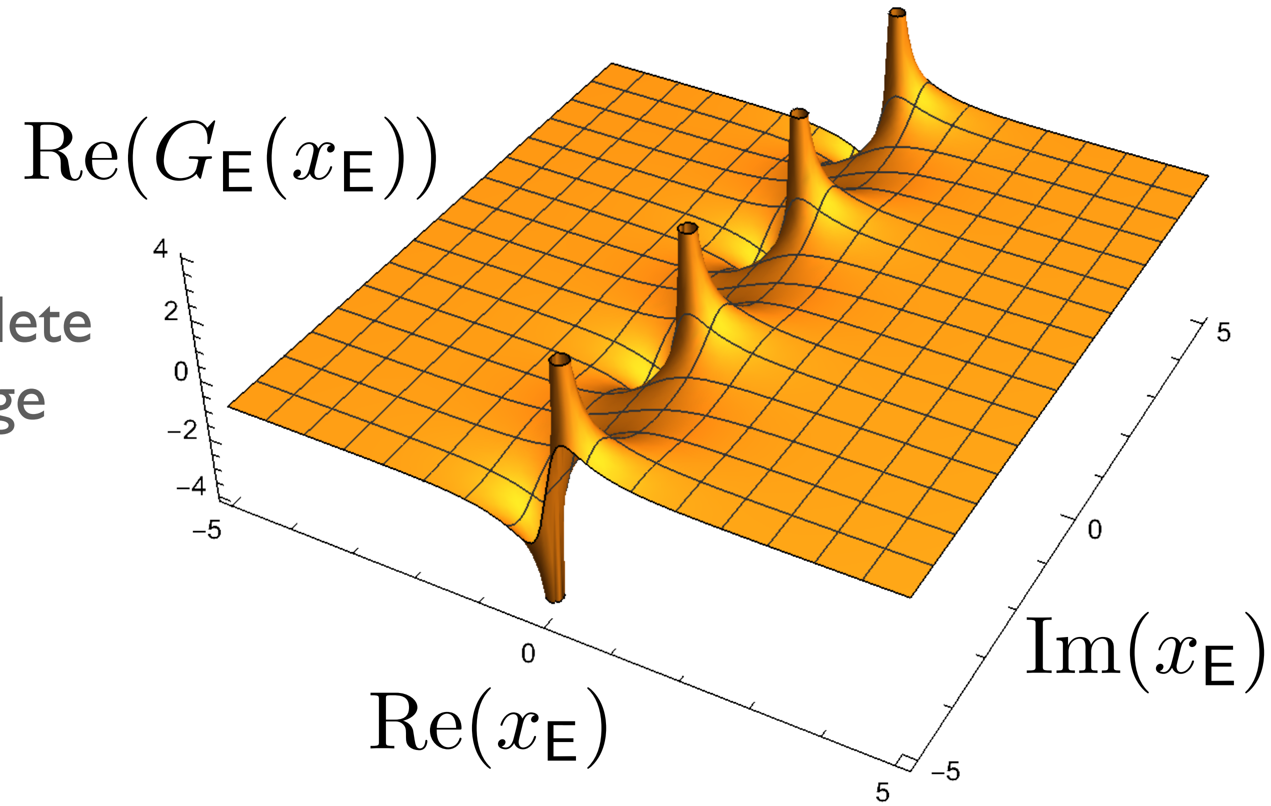
Irrelevant when we have **full analytic knowledge**

Analytic knowledge

Suppose the Euclidean correlator is

$$G_E(x_E) = \tanh(x_E)$$

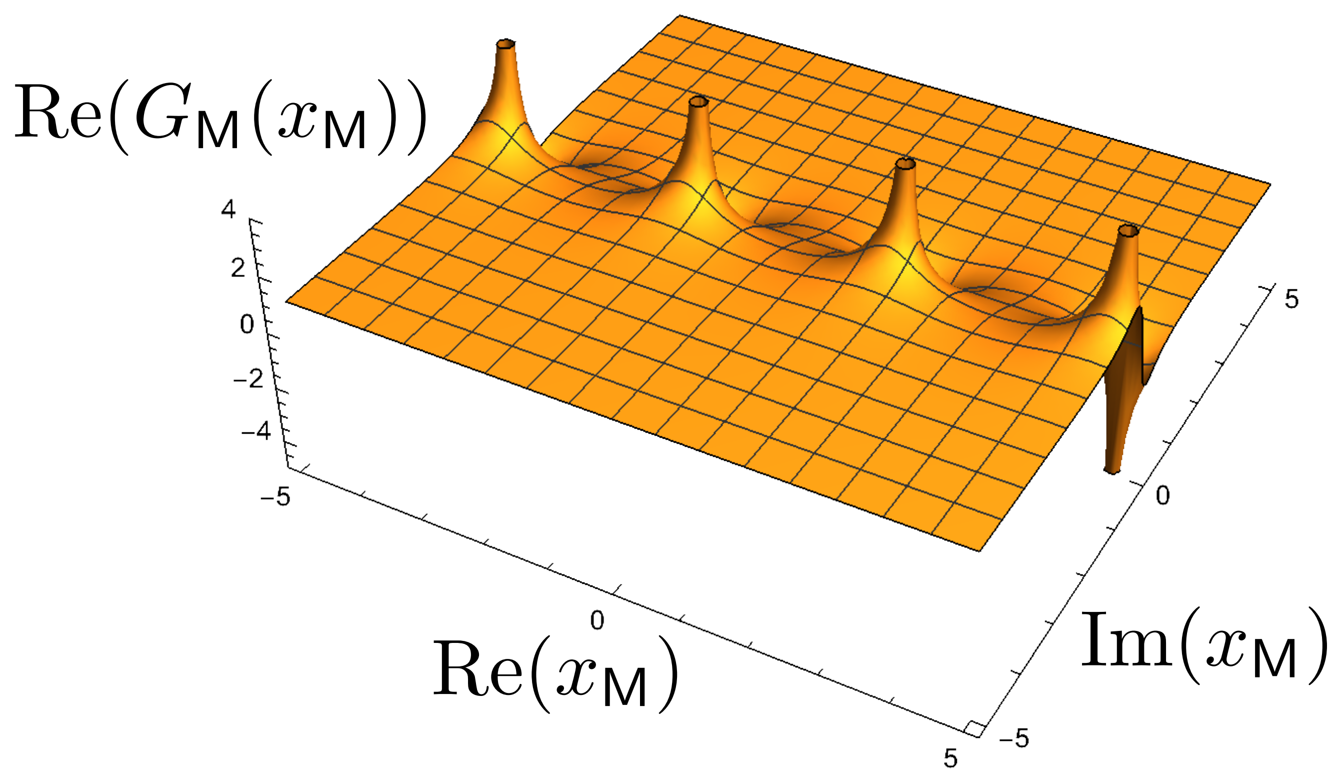
and we have complete analytic knowledge



… analytic continuation gives the Minkowski correlator

$$G_M(x_M) = i\tan(x_M)$$

but this is just a relabeling of the same information

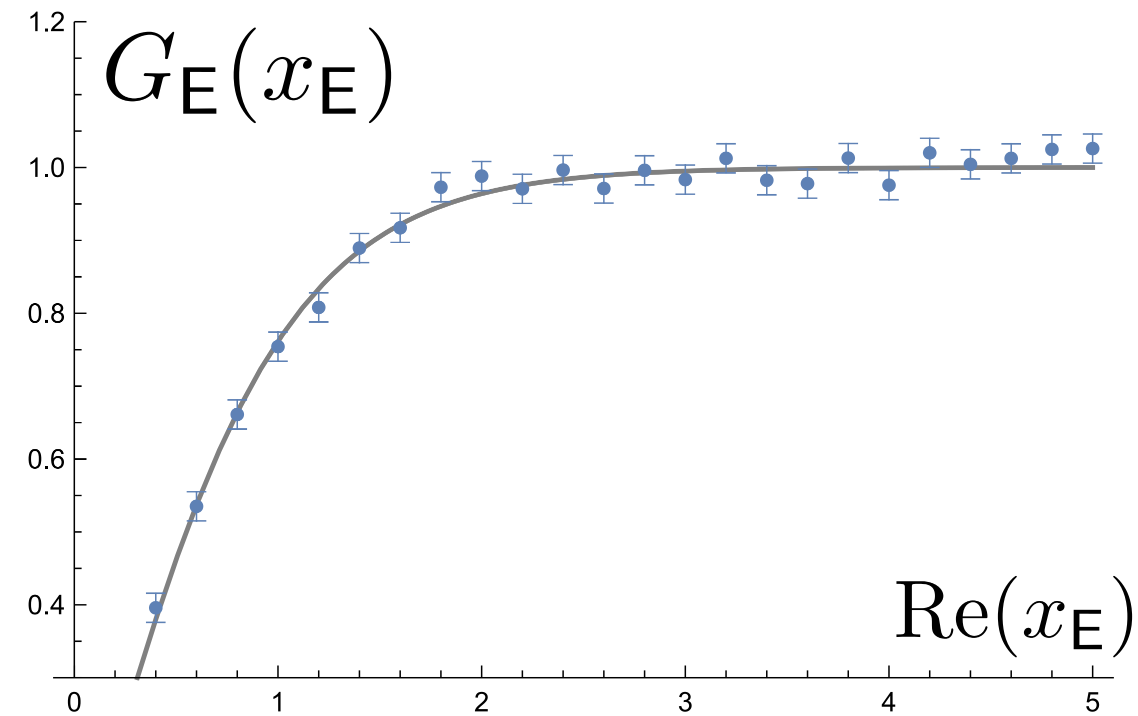


Limited numerical knowledge

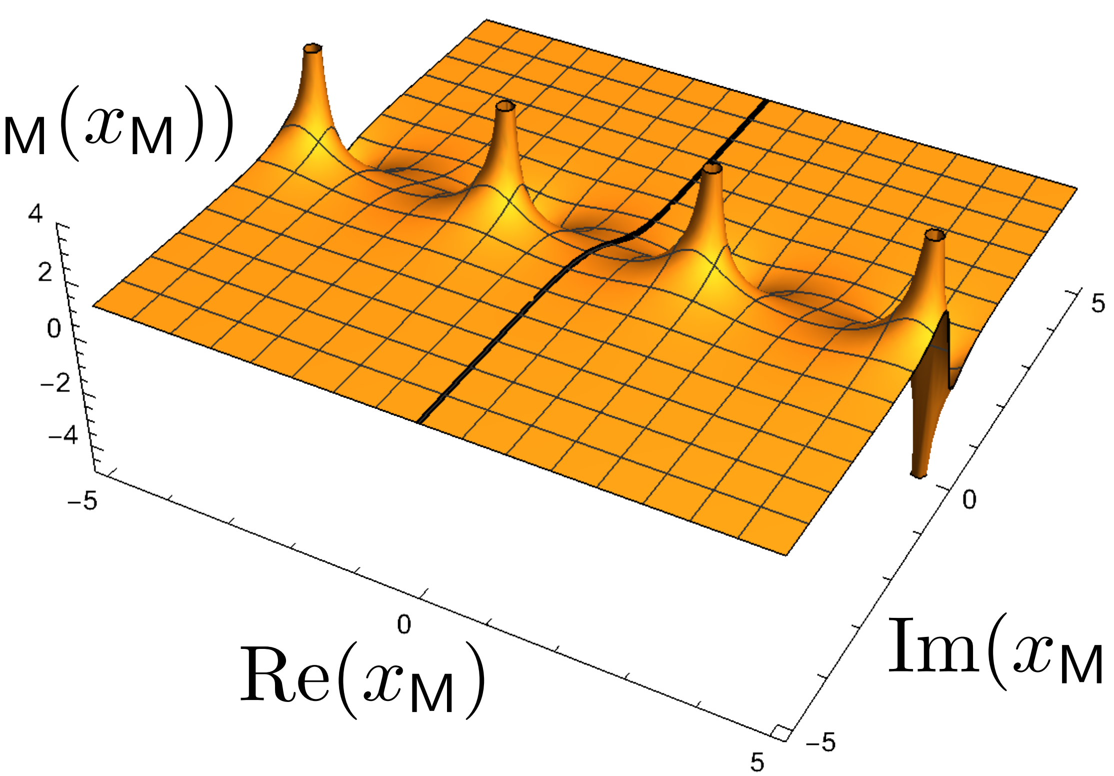
Suppose the Euclidean correlator is

$$G_\mathsf{E}(x_\mathsf{E}) = \tanh(x_\mathsf{E})$$

and we have a numerical estimate



Now the analytic continuation is ill-conditioned



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Mostly the lectures are concerned with these quantities

Correlators (where M vs. E is meaningful) are often used to access observables (where M vs. E is not meaningful)

Finite-volume setup

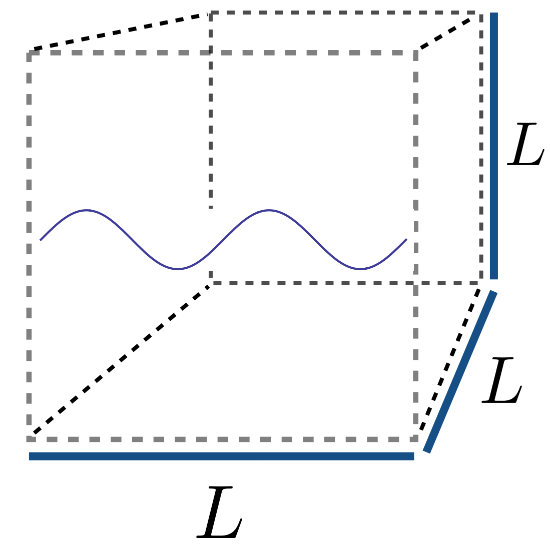
cubic, spatial volume (extent L)

periodic boundary conditions*

$$q(\tau, \mathbf{x}) = q(\tau, \mathbf{x} + L\mathbf{e}_i) \quad \Big| \quad \pi(\tau, \mathbf{x}) = \pi(\tau, \mathbf{x} + L\mathbf{e}_i)$$

time direction infinite*

continuum theory



*will also briefly consider finite T effects, alternative boundary conditions

$$\tilde{\pi}(\tau, \mathbf{p}) = \int_L d^3\mathbf{x}\ e^{-i\mathbf{p}\cdot\mathbf{x}}\, \pi(\tau, \mathbf{x})$$

$$= \int_L d^3\mathbf{x}\ e^{-i\mathbf{p}\cdot\mathbf{x}}\, \pi(\tau, \mathbf{x} + L\mathbf{e}_i)$$

$$= \int_L d^3\mathbf{x}\, e^{-i\mathbf{p}\cdot(\mathbf{x} - L\mathbf{e}_i)}\, \pi(\tau, \mathbf{x})$$

$$e^{-i\mathbf{p}\cdot L\mathbf{e}_i} = 1 \implies p_i L = 2\pi n_i$$

Quantization of momentum

$$\mathbf{p} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$$

Fourier transform conventions (here in one-dimension)

Begin in infinite volume

$$\mathcal{FT}^{-1}\Big[\mathcal{FT}[f]\Big] = f \quad \longrightarrow \quad \int \frac{dp}{N}\, e^{ipx} \left[\int \frac{dx'}{N'}\, e^{-ipx'} f(x') \right] \stackrel{!}{=} f(x)$$

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$$= 1$$

simplify via…

$$\int dp\, e^{ip(x-x')} = 2\pi\delta(x - x')$$

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Now repeat in a finite volume

$$\frac{1}{N} \sum_n e^{ix(2\pi n/L)} \left[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x') \right]$$

Rewrite this using $\displaystyle \sum_n e^{2\pi i z} = \sum_{n'} \delta(z + n')$

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$$\frac{1}{N}\sum_n e^{ix(2\pi n/L)}\left[\int_0^L dx'\, e^{-ix'(2\pi n/L)} f(x')\right] = \int_0^L dx' \sum_{n'} \frac{\delta\big[(x'-x)/L - n'\big]}{N} f(x') = \frac{L f(x)}{N}$$

$$= 1$$

Rewrite this using $\sum_n e^{2\pi i z} = \sum_{n'} \delta(z+n')$

argument can only vanish when $n' = 0$

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Now repeat in a finite volume

$$\mathcal{FT}_L^{-1}\Big[\mathcal{FT}_L[f]\Big] = f \quad \longrightarrow \quad \frac{1}{L}\sum_p e^{ipx}\left[\int_L dx'\, e^{-ipx'} f(x')\right] = f(x)$$

$p = 2\pi n/L$

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Take the infinite-volume limit as a sanity check

$$\lim_{L\to\infty}\frac{1}{L}\sum_p f(p) = \lim_{L\to\infty}\frac{1}{L}\sum_n f(2\pi n/L) = \lim_{L\to\infty}\frac{1}{L}\int dn\, f(2\pi n/L) = \lim_{L\to\infty}\frac{1}{L}\int dp\frac{L}{2\pi}\, f(p) = \int\frac{dp}{2\pi} f(p)$$

For a smooth function we can
replace the sum with an integral

Fourier transform conventions (here in one-dimension)

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$$\lim_{L\to\infty}\left[\frac{1}{L}\sum_p - \int\frac{dp}{2\pi}\right] f(p) = 0$$

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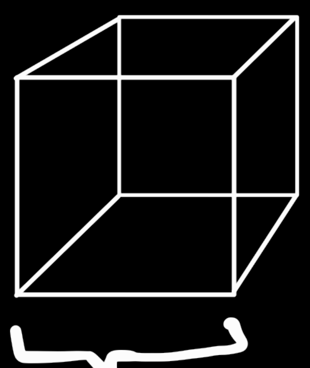
□ Finite-volume set-up



L

- 4-dimensional $(\mathbb{T}_3 \times \mathbb{R})$
- periodicity L
- $\underline{p} \in {}^{2\pi}/_{L}\,\mathbb{Z}^3$

- time-direction $T \to \infty$
- lattice spacing $a \to 0$

□ Exponentially suppressed volume effects

$M(L)^2 = M^2 +$  $+ \mathcal{O}(\lambda^2)$

( = propogator

× = vertex)

 $= \dfrac{(-i\lambda)}{2} \dfrac{1}{i} \displaystyle\int \frac{dk^0}{2\pi} \frac{1}{L^3} \sum_{\underline{k}} \frac{1}{-(k^0)^2 + \underline{k}^2 + M^2 - i\varepsilon}$

- Exponentially suppressed volume effects

$M(L)^2 = M^2 +$  $+ \mathcal{O}(\lambda^2)$

( = propagator ;  = vertex)

 $= \frac{(-i\lambda)}{2}\frac{1}{i}\int\frac{dk^0}{2\pi}\frac{1}{L^3}\sum_{\underline{k}}\frac{1}{-(k^0)^2+\underline{k}^2+M^2-i\epsilon}$

$$= \frac{\lambda}{2}\int\frac{dk_4}{2\pi}\frac{1}{L^3}\sum_{\underline{k}}\frac{1}{k^2+M^2} = \frac{\lambda}{2}\sum_{n}\int\frac{d^4k}{(2\pi)^4}e^{iLn\cdot k}\int_0^\infty d\alpha\, e^{-\alpha(k^2+M^2)}$$

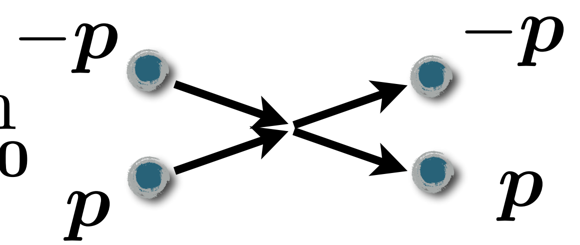
$=$  $+ M^2\frac{\lambda}{32\pi^2}\sum_{\underline{n}\neq\underline{0}}\int_0^\infty d\alpha\,\frac{1}{\alpha^2}e^{-\alpha-\frac{1}{\alpha}\left[\frac{M^2L^2\underline{n}^2}{4}\right]}$

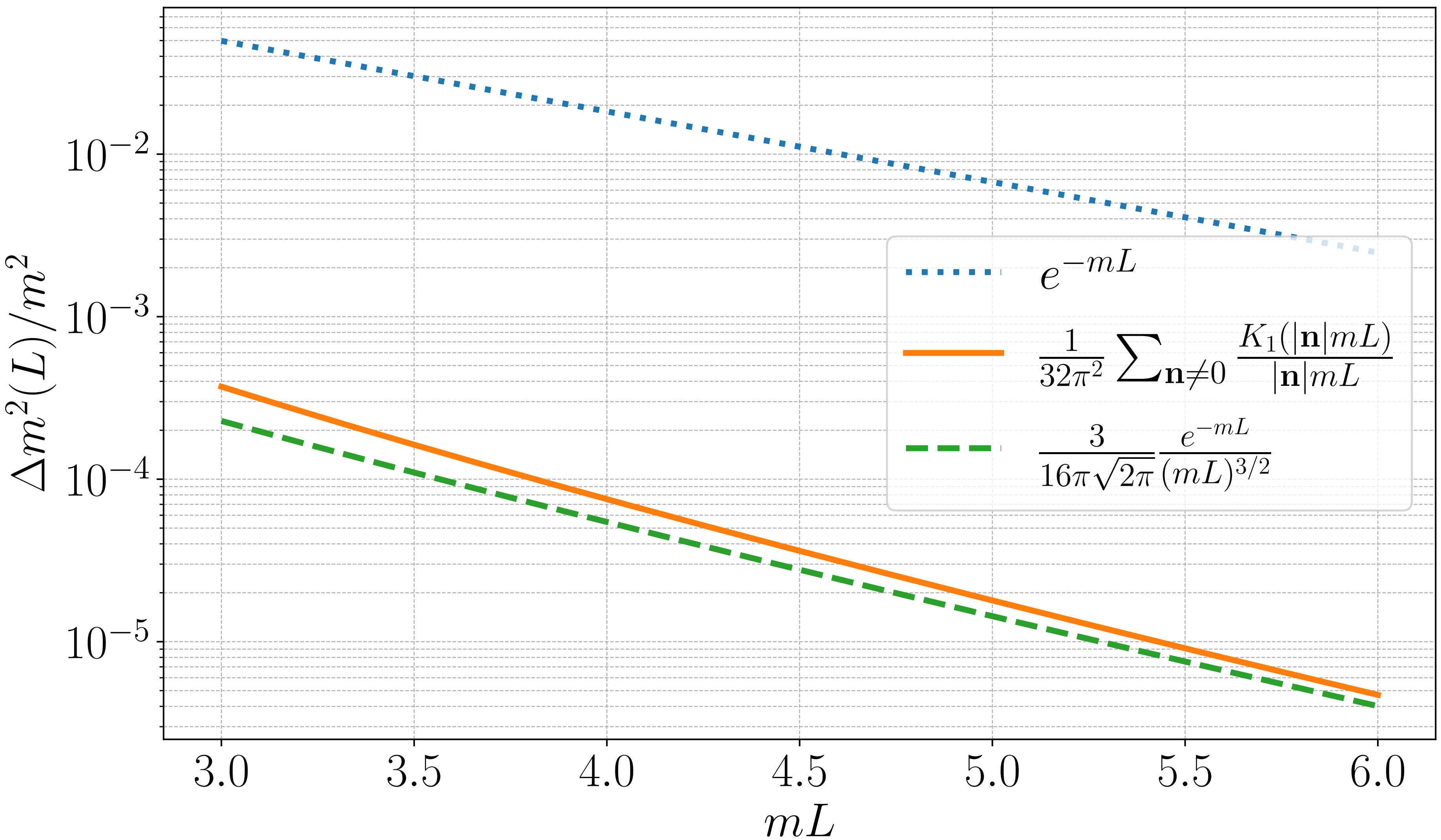
$$\frac{M(L)^2-M_{phys}^2}{M_{phys}^2} = \frac{\lambda}{32\pi^2}\sum_{\underline{n}\neq\underline{0}}\frac{K_1(ML|\underline{n}|)}{ML|\underline{n}|} = \frac{6\lambda}{32\pi^2}\frac{1}{ML}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{ML}}e^{-ML}$$

Particle mass in $\lambda\phi^4$

- Leading order volume correction ($\Delta m^2(L) = m(L)^2 - m^2 > 0$)

$$\frac{\Delta m^2(L)}{m^2} = \frac{\lambda_0}{32\pi^2}\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\frac{K_1(|\boldsymbol{n}|mL)}{|\boldsymbol{n}|mL} = \frac{3\lambda_0}{16\pi\sqrt{2\pi}}\frac{e^{-mL}}{(mL)^{3/2}}\left[1+\mathcal{O}(1/L, e^{-\beta mL})\right]$$

- To the order we work: coupling = threshold scattering amplitude $\lim_{\boldsymbol{p}\to\boldsymbol{0}}$ 



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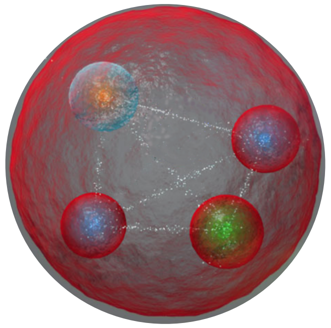
  - [ ] Integral equations ($\mathcal{K}_{\mathrm{df},3}$ to $\mathcal{M}_3$)
  - [ ] Testing the result
  - [ ] Numerical explorations/calculations
- [ ] Conclusion and outlook

Multi-hadron observables

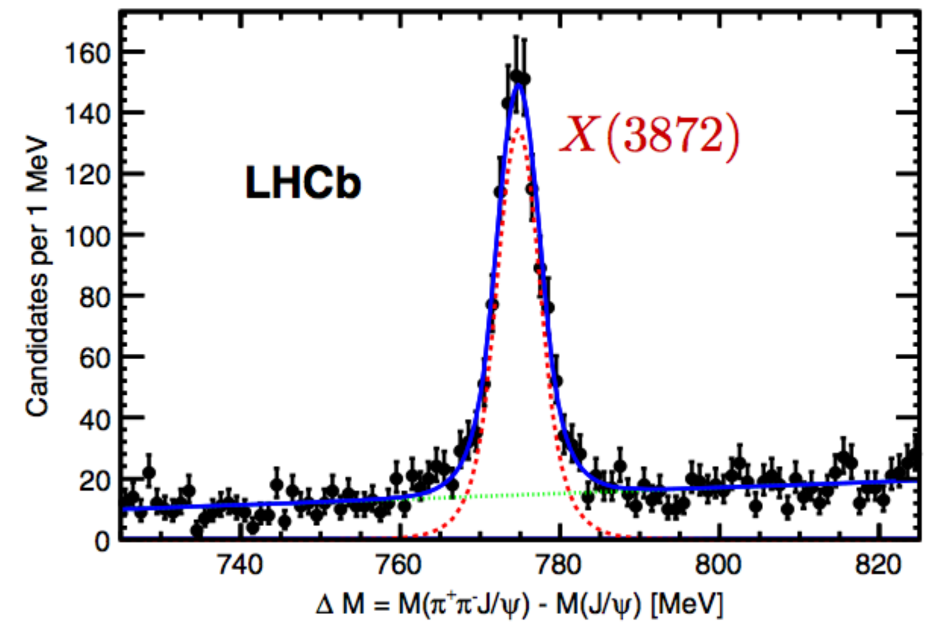
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

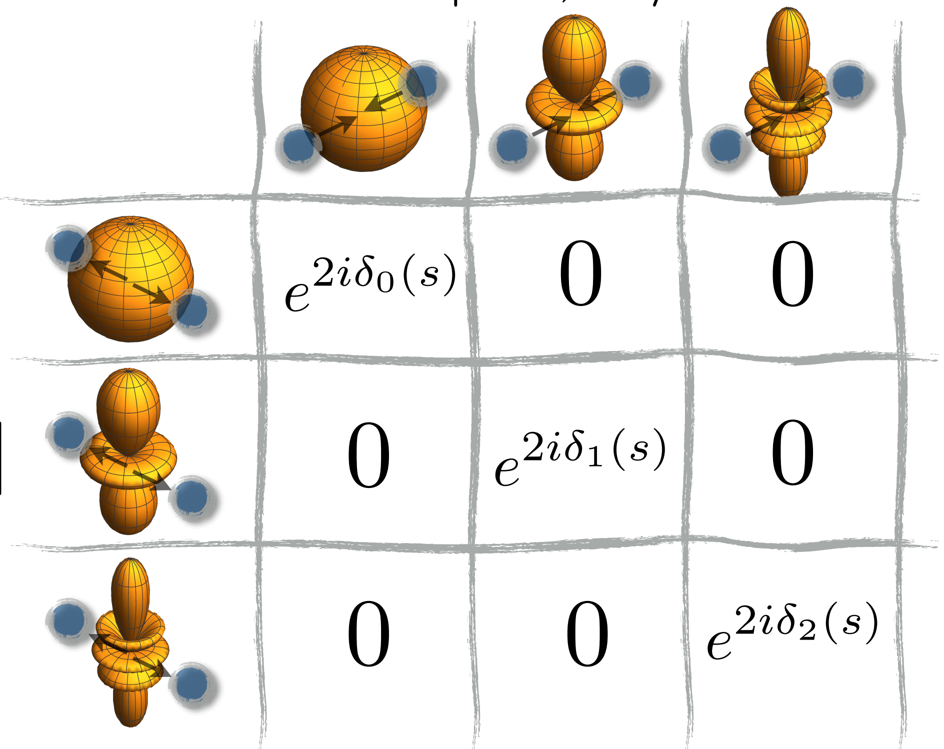
$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron *asymptotic states* → S matrix

$|\pi\pi, \mathrm{in}\rangle$

$S(s) \equiv \langle \pi\pi, \mathrm{out}|$



depends on $s = E_{\mathrm{cm}}^2$
and angular variables

diagonal in angular momentum

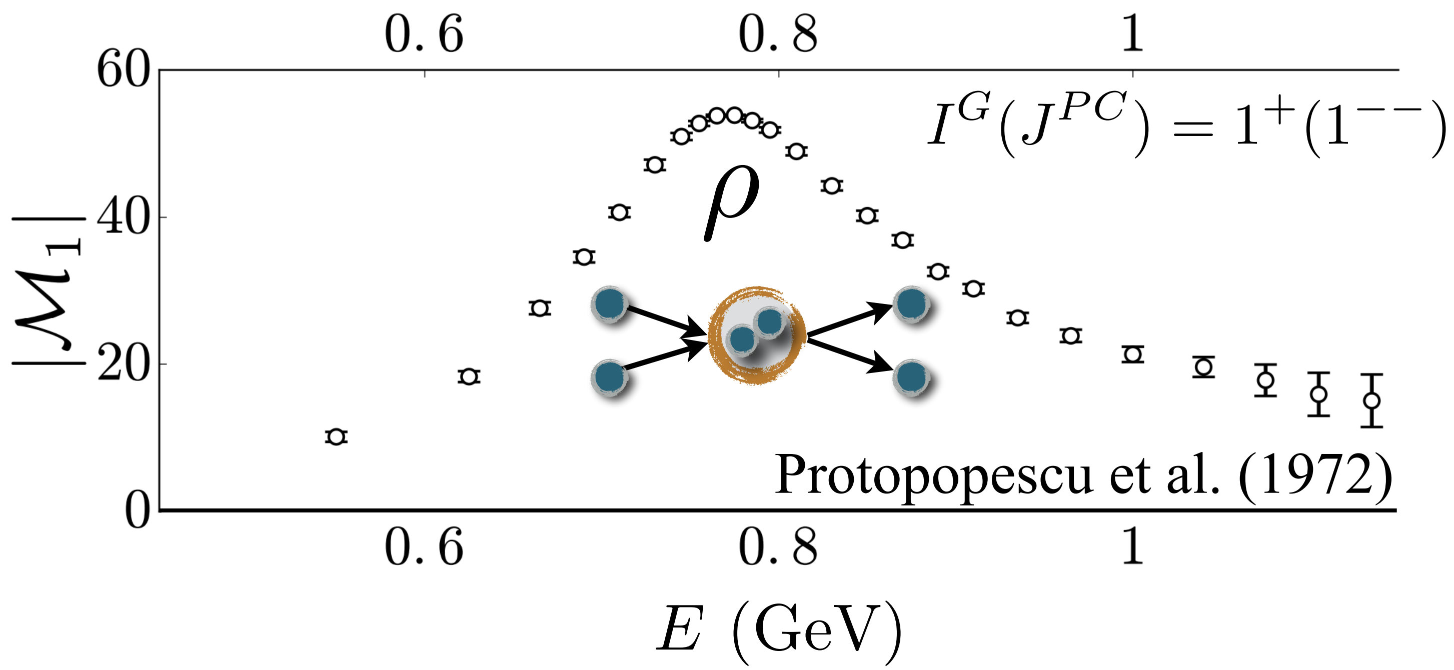
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

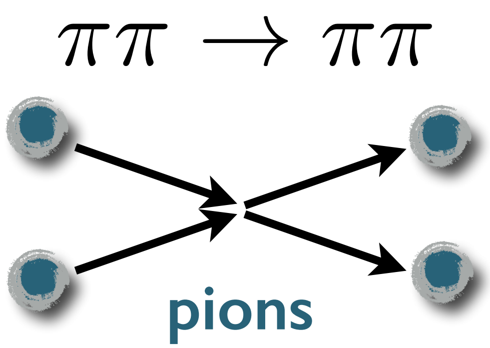
- ***An enormous space of information*** $|\pi\pi\pi\pi, \mathrm{in}\rangle \quad |K\overline{K}, \mathrm{in}\rangle \quad \cdots$

QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





QCD resonances

- ☐ Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

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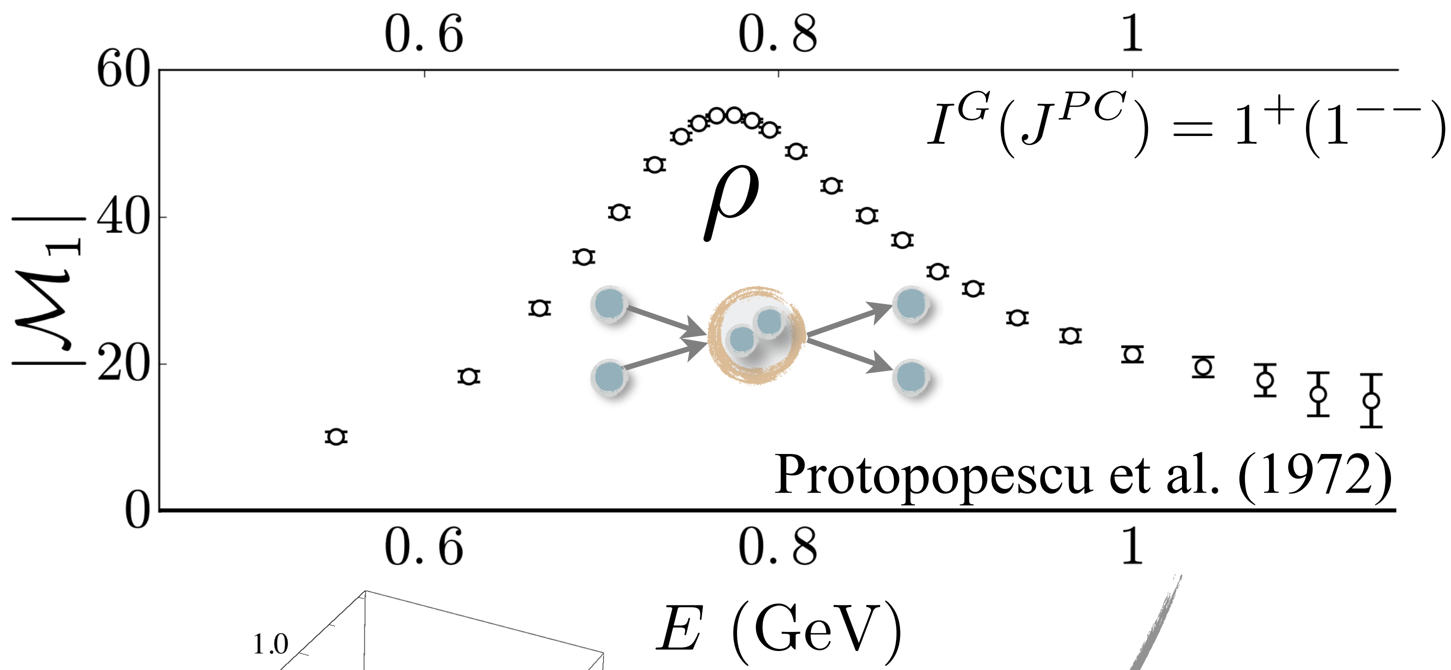
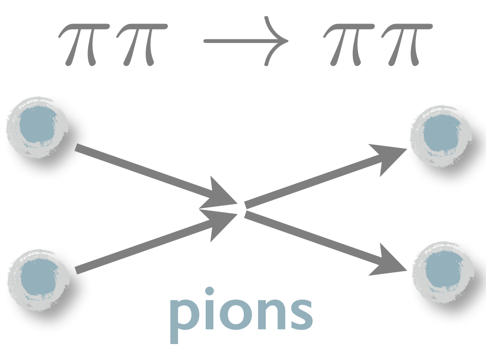
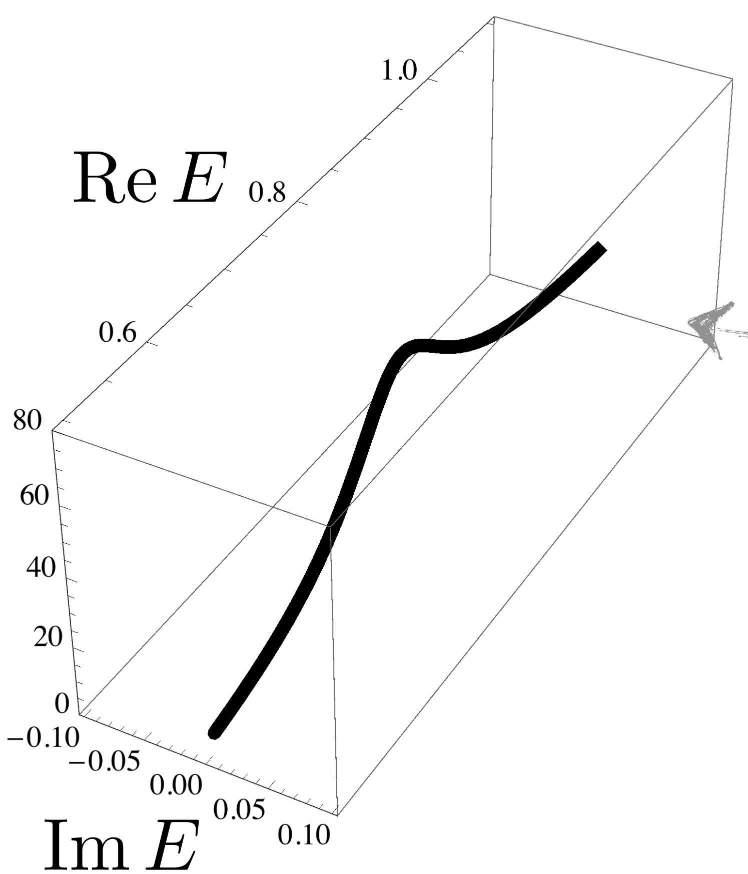




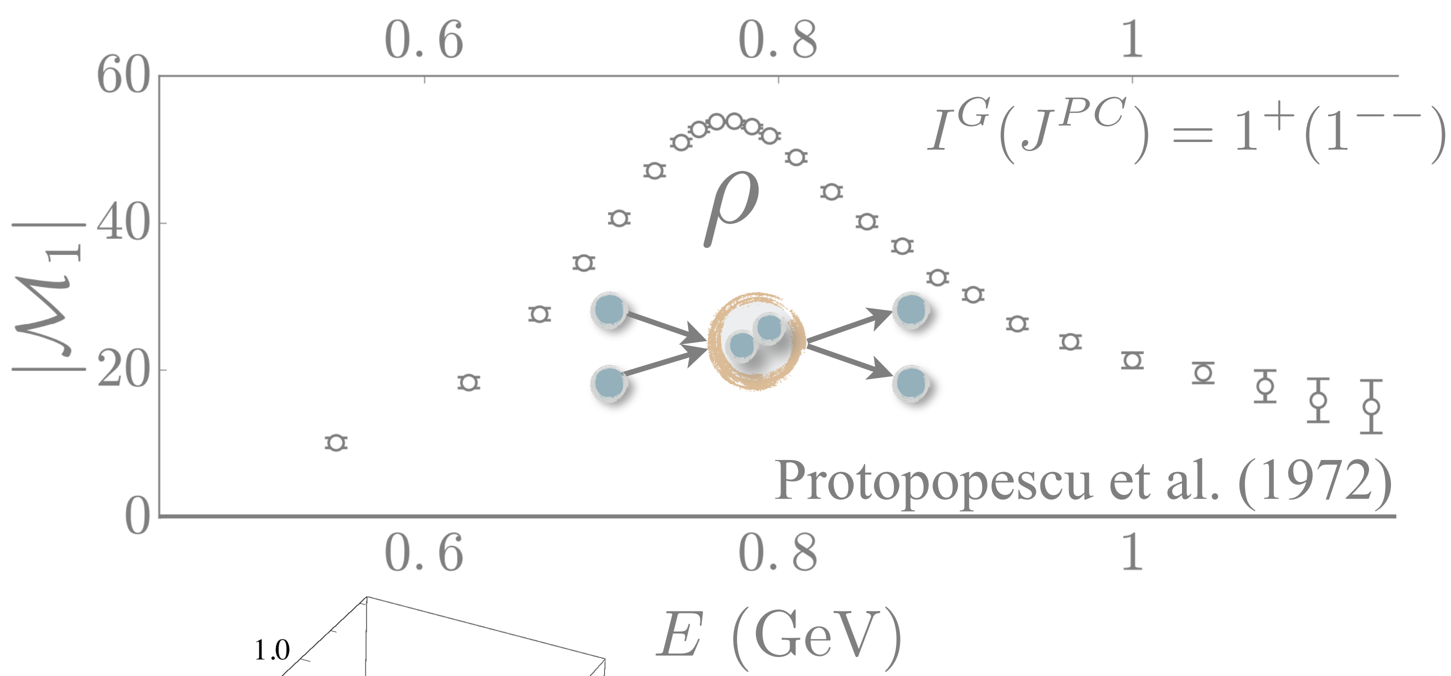


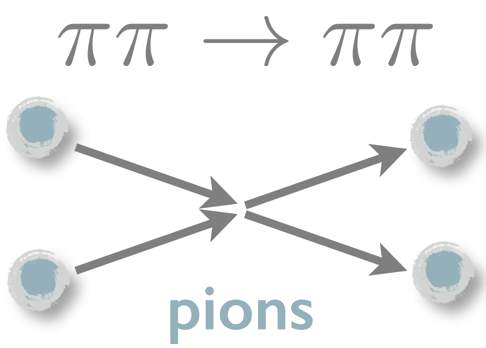
Extend to the **complex plane**

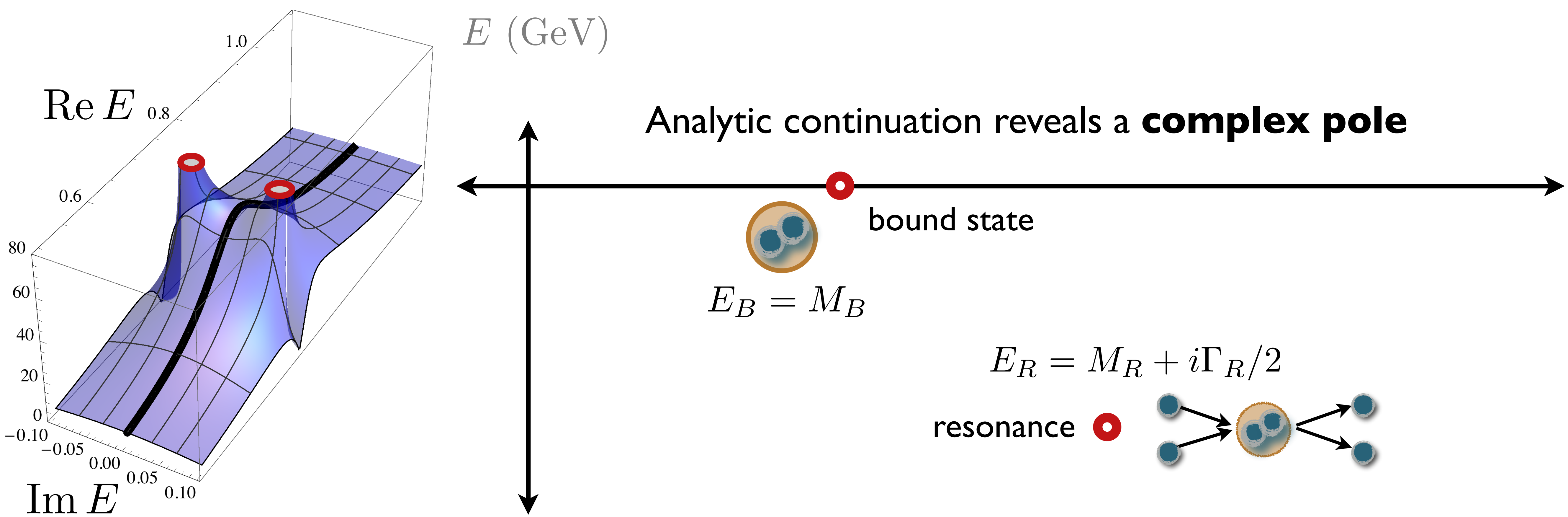
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate



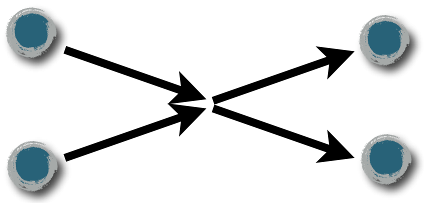




Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself

  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?



- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \mathrm{Im}\,\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

- Unique solution is…

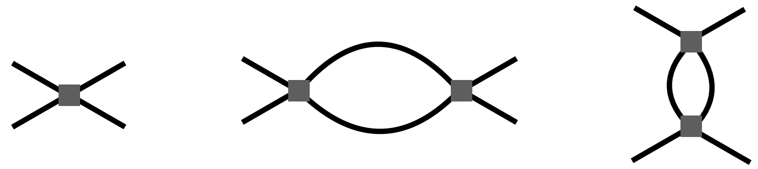
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance) phase-space cut (long distance)

Key message: *The scattering amplitude has a square-root branch cut*

Bethe Salpeter equation

- All orders diagrammatic expansion

$\mathcal{M} =$ 

Bethe Salpeter equation

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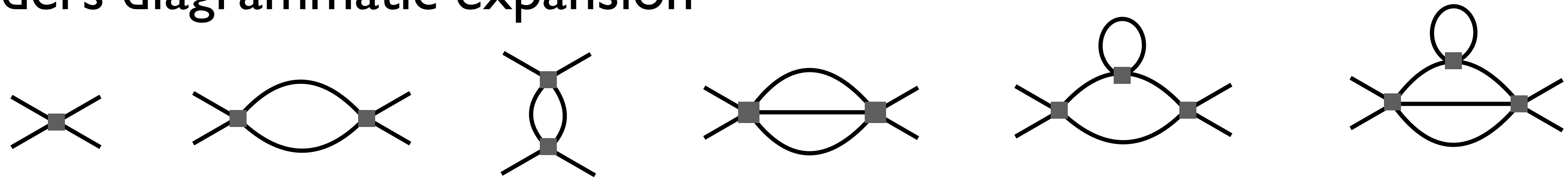
$\mathcal{M} =$ 

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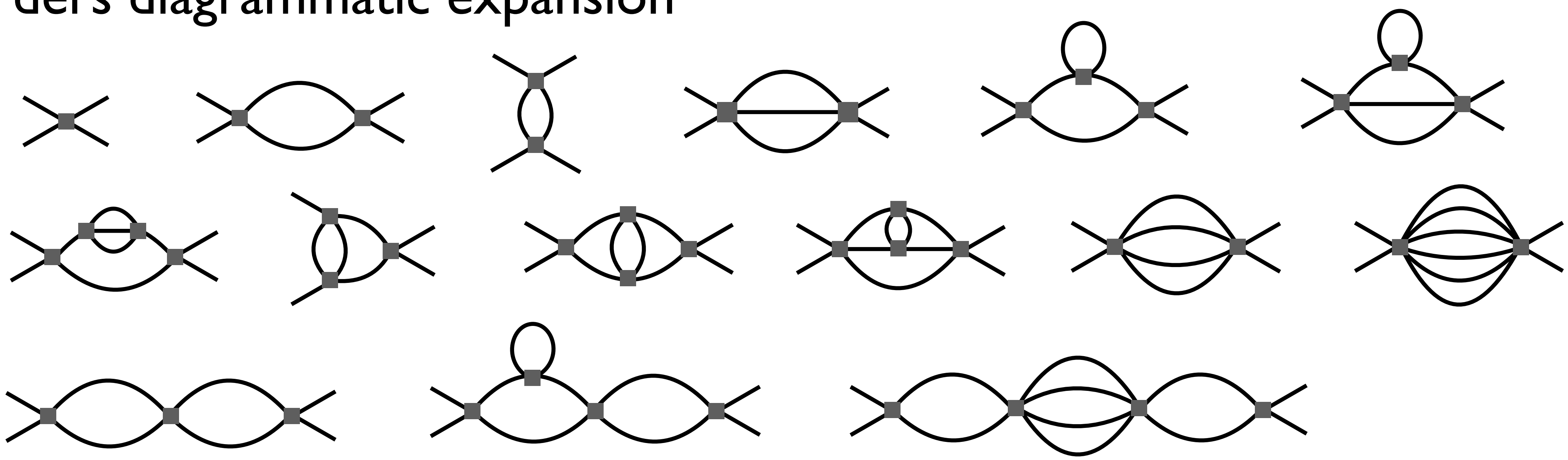
$\mathcal{M} =$ 

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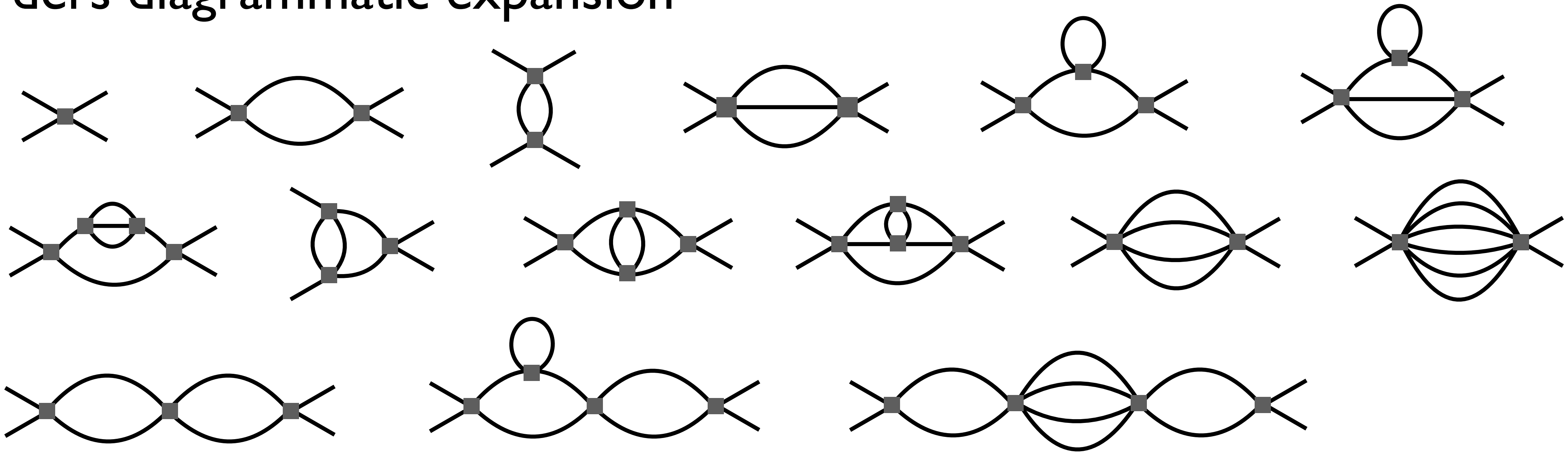
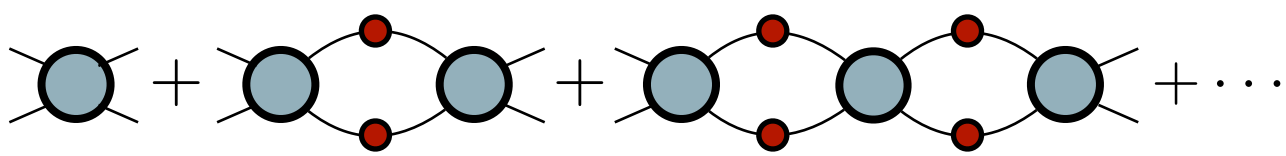
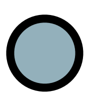
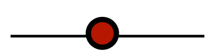
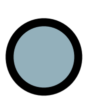
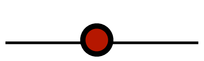
$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

 $=$

 $=$

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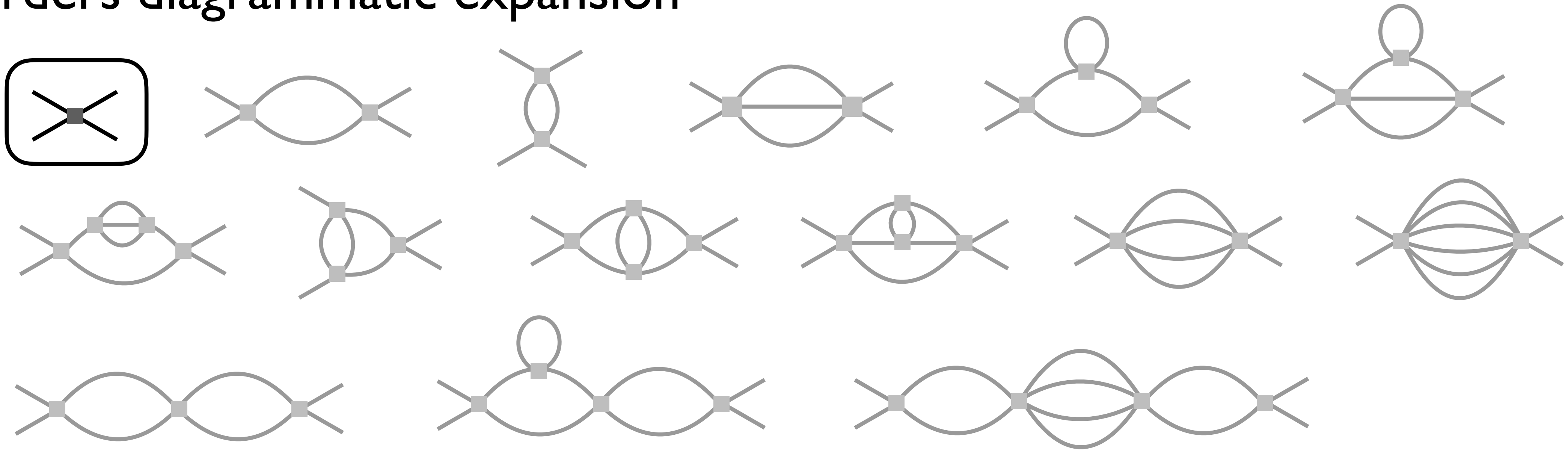
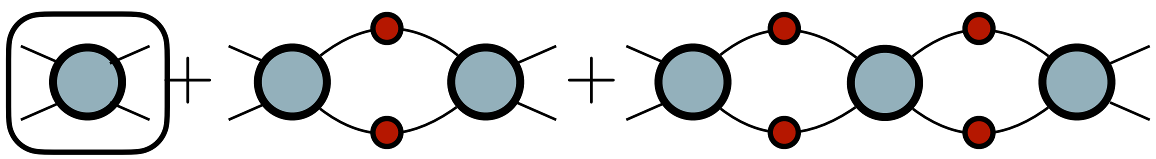
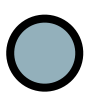
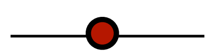
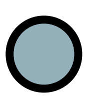
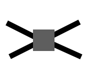
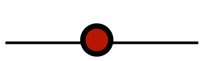
$\mathcal{M} =$ 

$=$  $+ \cdots$

- Construct  and  such that this is true

 $=$ 

 $=$

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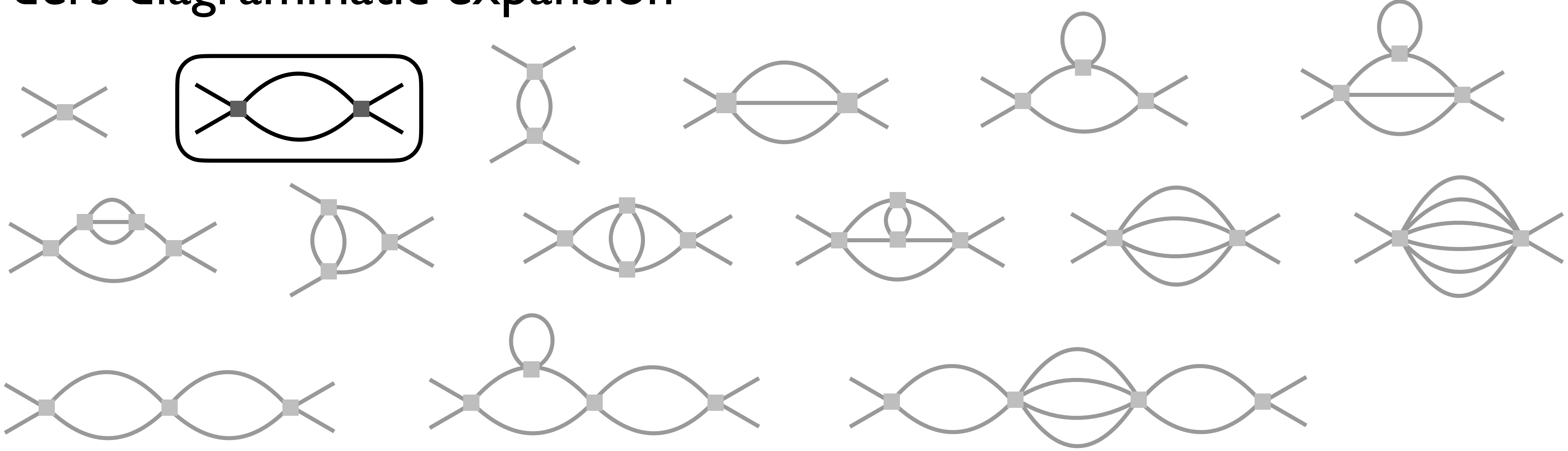
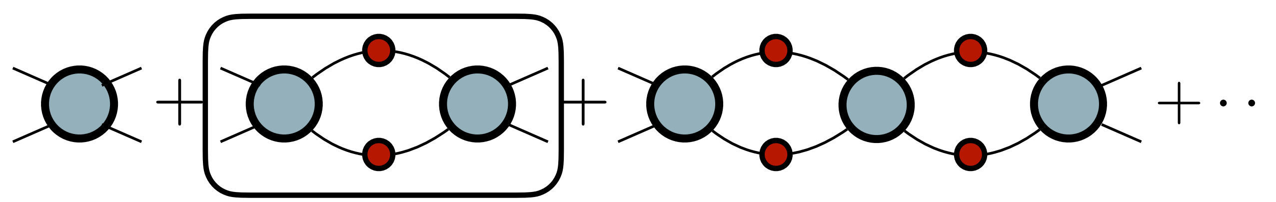
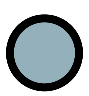
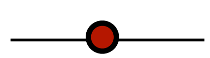
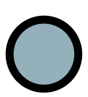
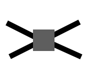
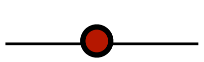
$\mathcal{M} =$ 

$=$ 

- Construct  and  such that this is true

 $=$ 

 $=$

Bethe Salpeter equation

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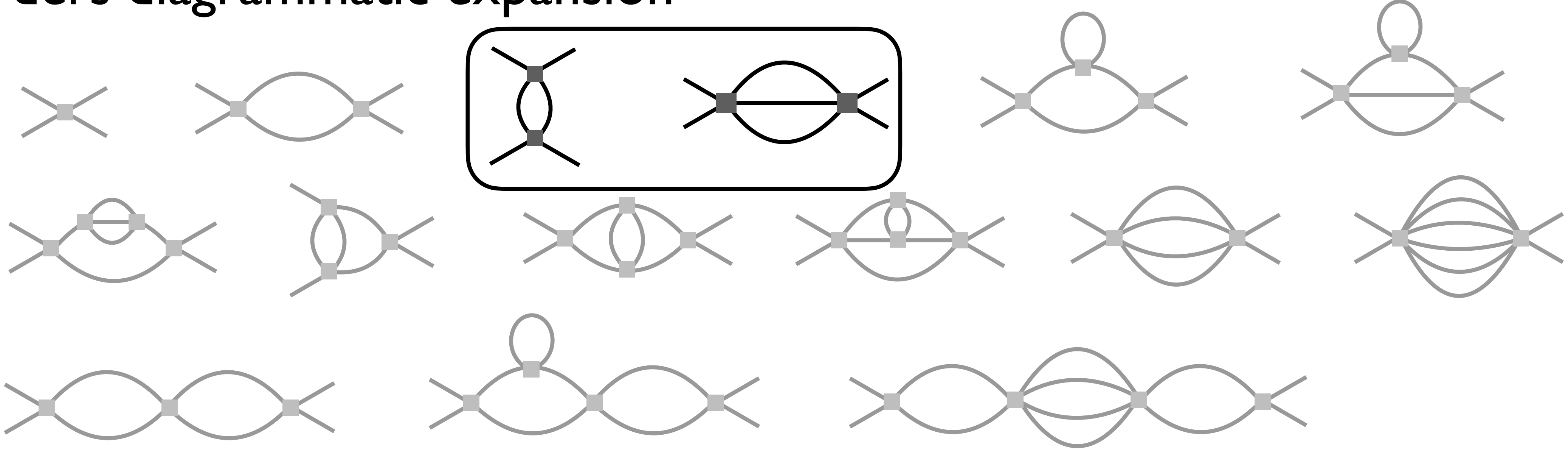
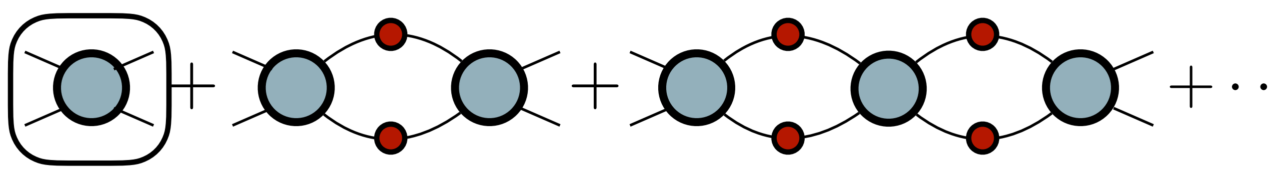
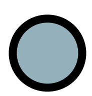
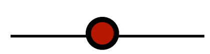
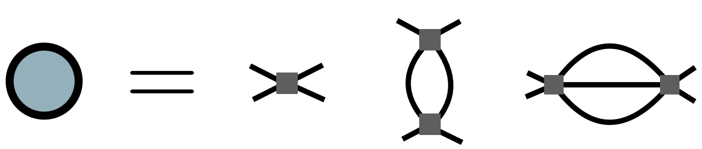
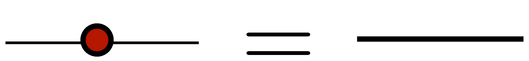
$\mathcal{M} =$ 

$=$ 

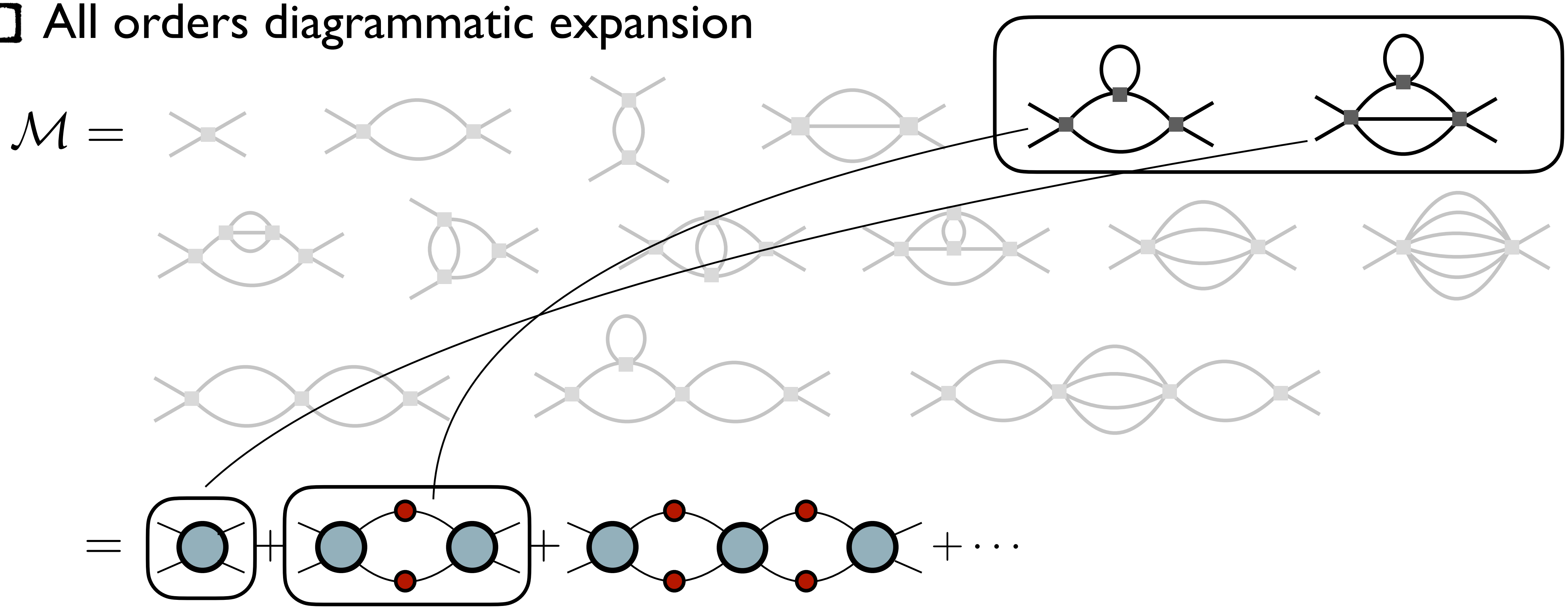
- Construct  and  such that this is true

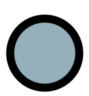
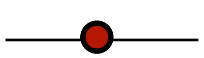


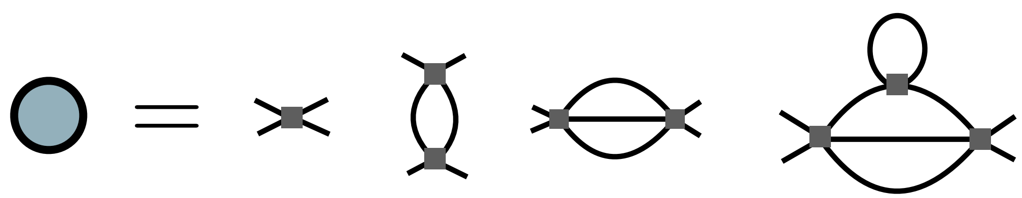


Bethe Salpeter equation

- All orders diagrammatic expansion

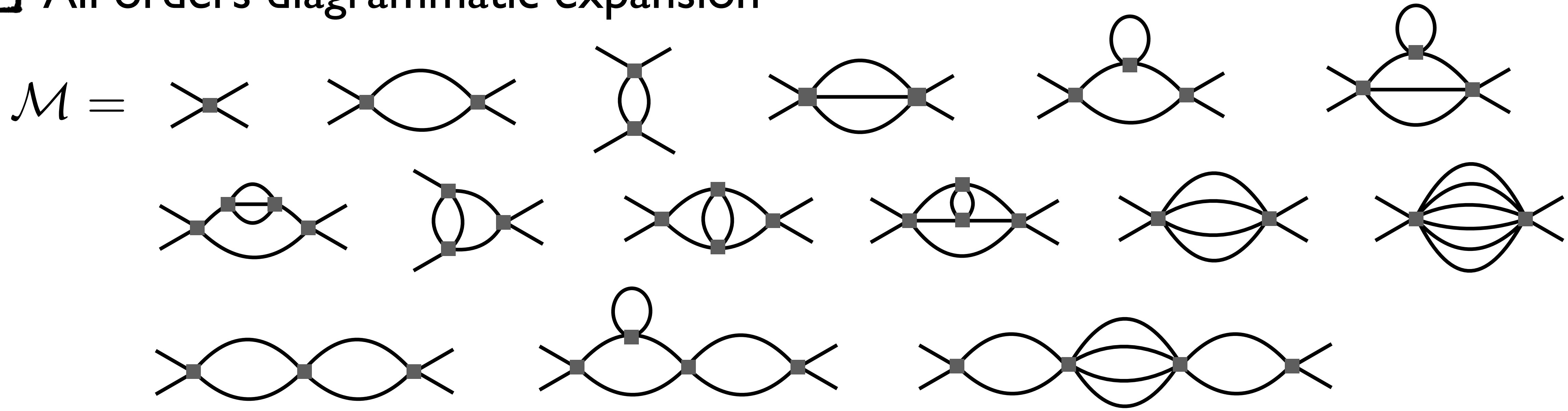


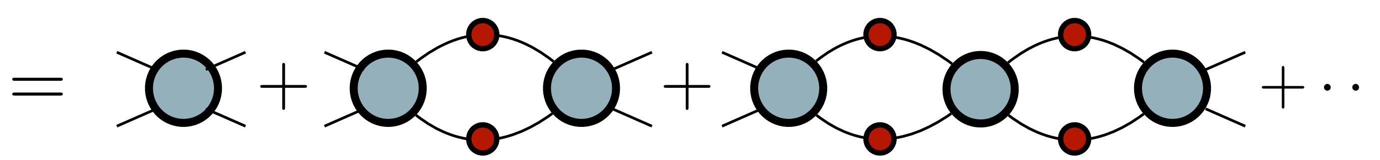
- Construct  and  such that this is true

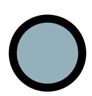
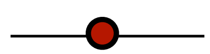


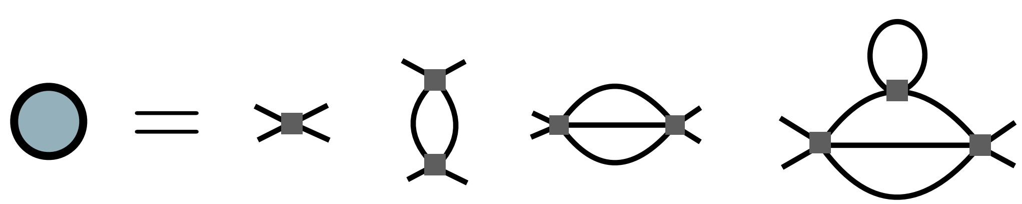
Bethe Salpeter equation

- All orders diagrammatic expansion

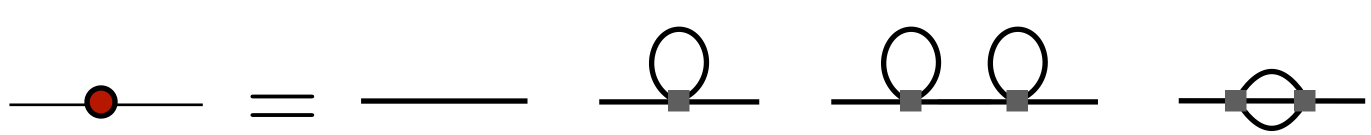




- Construct  and  such that this is true

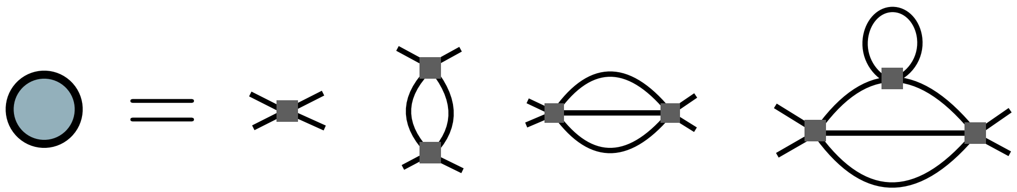


Bethe Salpeter kernel
(2PI in the s-channel)



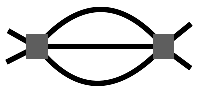
Fully dressed propagator

Bethe Salpeter kernel

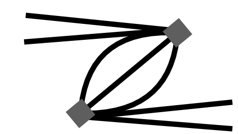
 $= B(s)$

- Analyticity of B.S. kernel

  time-ordered (old fashioned) perturbation theory



$$\int_{\boldsymbol{p}_1,\boldsymbol{p}_2} \frac{1}{E - \omega_{\boldsymbol{p}_1} - \omega_{\boldsymbol{p}_2} - \omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

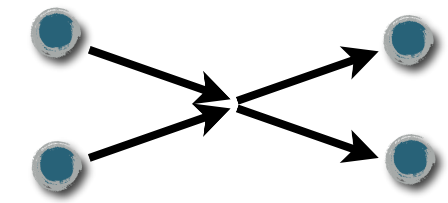


$$\int_{\boldsymbol{p}_1,\boldsymbol{p}_2} \frac{1}{-E - \omega_{\boldsymbol{p}_1} - \omega_{\boldsymbol{p}_2} - \omega_{\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2}}$$

- Real and analytic for $s < (3m)^2$

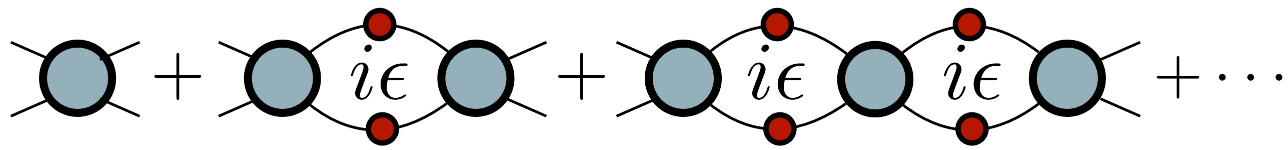
  no $i\epsilon$ needed if pole is not satisfied

Analyticity of $\mathscr{M}$ (from B.S. kernel)



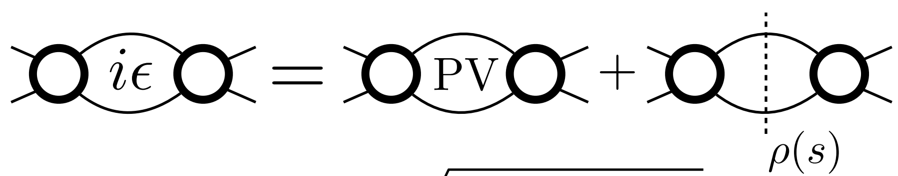
For two-particle energies $(2m)^2 < s < (3m)^2$, what is the analytic structure?

$\mathcal{M}(s) \equiv$



— propagating pion

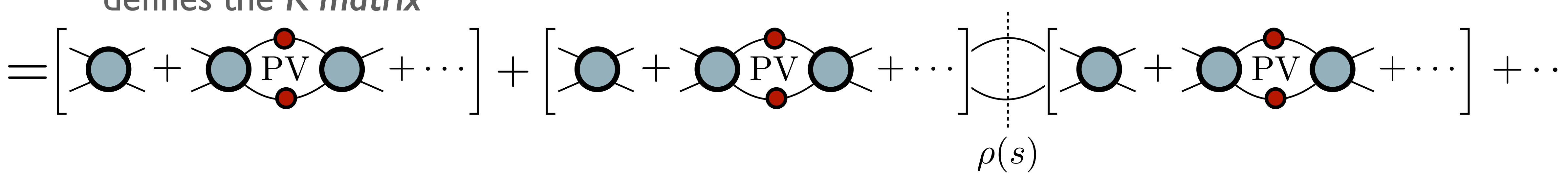
non-analytic:
on-shell particles = singularities



$\rho(s) \propto \sqrt{s-(2m)^2}$

cutting rule

defines the *K matrix*



$$= \mathcal{K}(s) + \mathcal{K}(s) i\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - i\rho(s)}$$

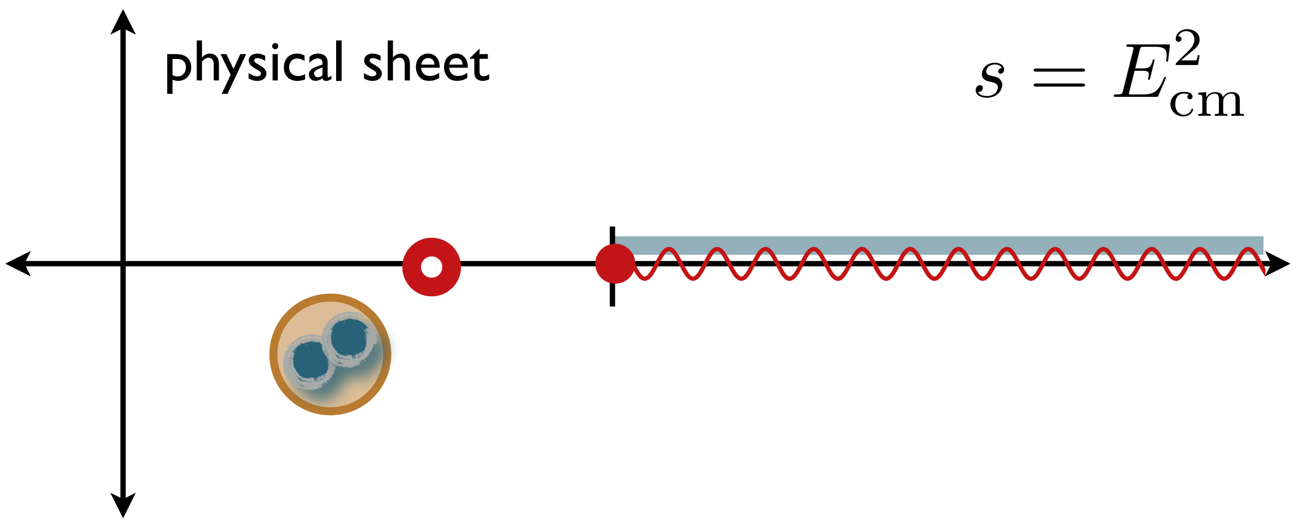
branch-cut singularity

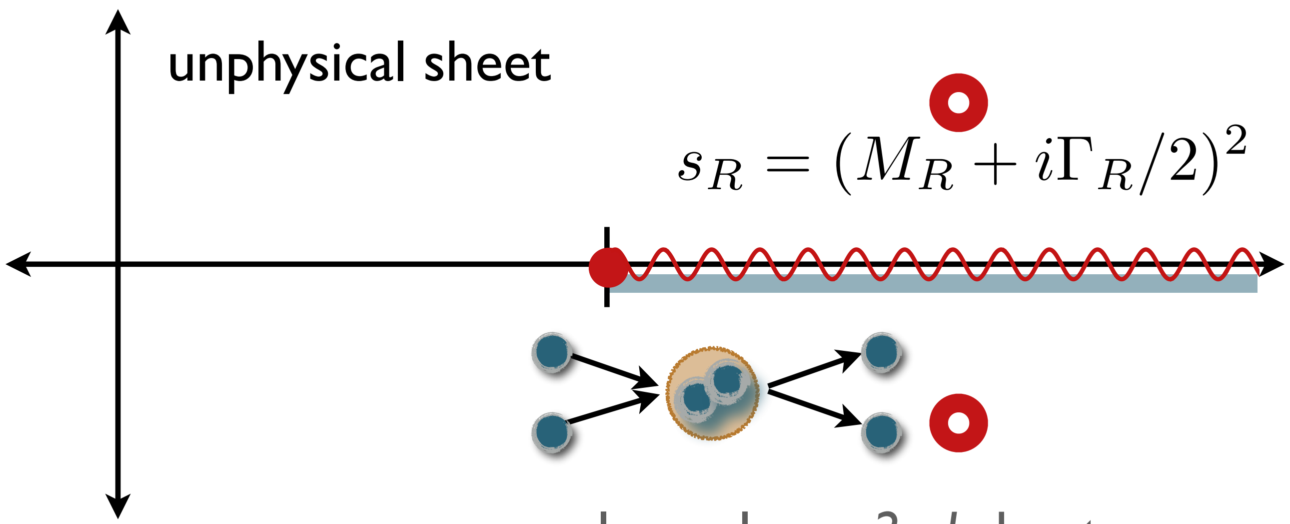
$\sqrt{s-(2m)^2}$

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto \sqrt{s-(2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets

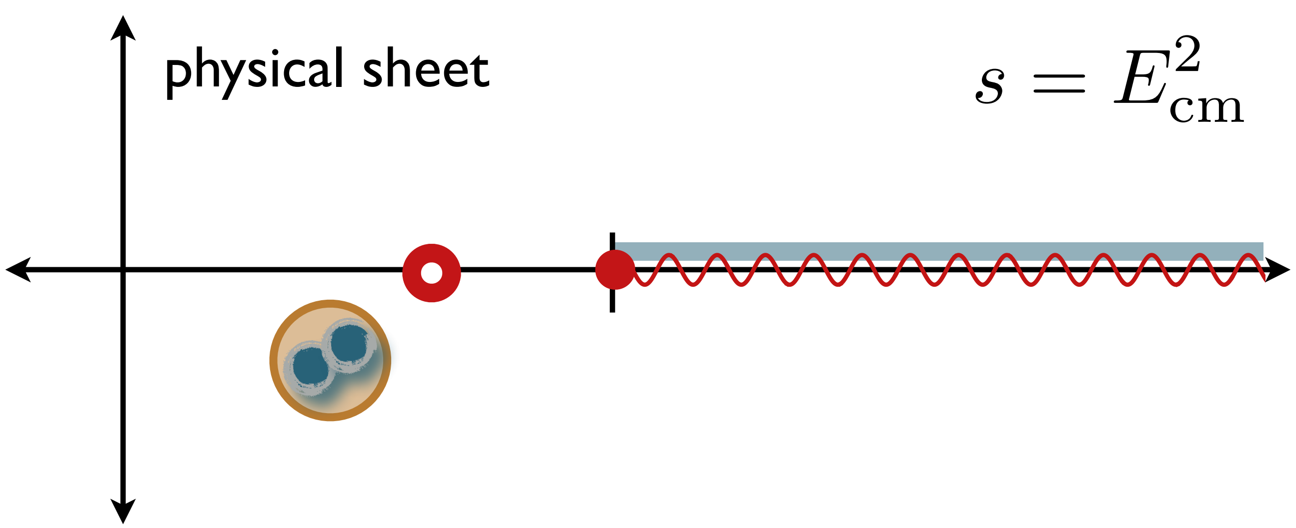


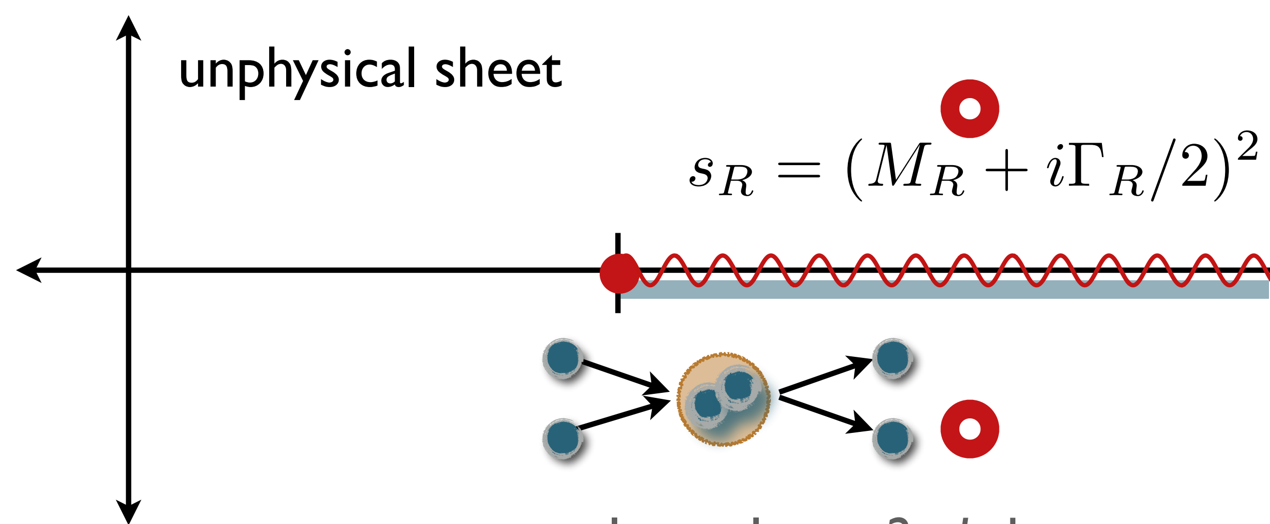


Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto \sqrt{s - (2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets





resonance poles only on *2nd* sheet

- Important lessons:

    Details of analyticity = important for quantitative understanding

    Possible to separate…
    (i) long-distance kinematic singularities
    (ii) short-distance/microscopic physics (depending on interaction details)

- ☑ Warm-up and definitions
  - ☑ Meaning of Euclidean
  - ☑ Finite-volume set-up
- ☑ e^{-mL} round one
  - ☑ Mass in $\lambda\phi^4$
  - ☑ Mass/matrix element in $g\phi^3$
- ☑ $2 \to 2$ formalism
  - ☑ Scattering basics
  - ☐ Derivation
  - ☐ Example application
  - ☐ Generalizations
- ☐ e^{-mL} round two
  - ☐ LO-HVP for $(g-2)_\mu$
  - ☐ Bethe-Salpeter kernel
- ☐ $(1+)\mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Example application
- ☐ $2 + \mathscr{J} \to 2$ formalism
  - ☐ Derivation
  - ☐ Testing the result
  - ☐ Numerical explorations
- ☐ Non-local matrix elements
  - ☐ Derivation
  - ☐ Applications
- ☐ $3 \to 3$ formalism
  - ☐ New complications
  - ☐ Derivation ($E_n(L)$ to $\mathscr{K}_{\text{df},3}$)
  - ☐ Integral equations ($\mathscr{K}_{\text{df},3}$ to $\mathscr{M}_3$)
  - ☐ Testing the result
  - ☐ Numerical explorations/calculations
- ☐ Conclusion and outlook