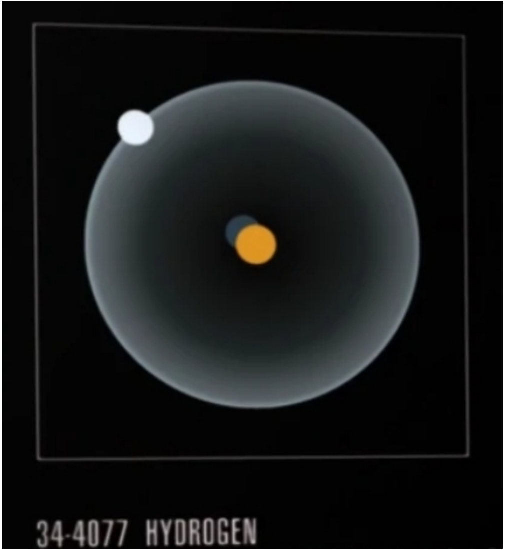
Improved bound-electron g-factor theory after completion of two-loop self-energy calculations

Bastian Sikora, Vladimir A. Yerokhin, Christoph H. Keitel, Zoltán Harman

Max Planck Institute for Nuclear Physics, Heidelberg, Germany

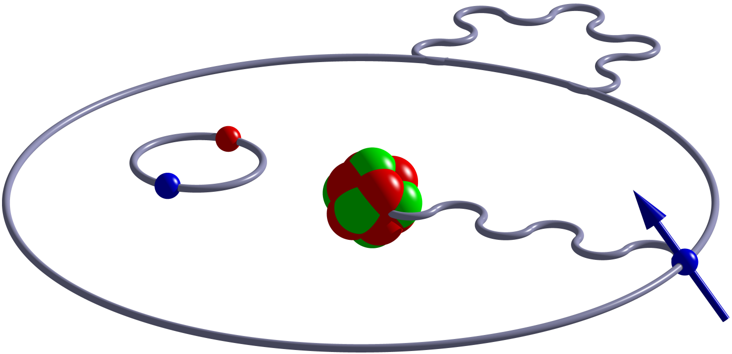
ALPHA2025: Precision Determinations of the Fine-structure Constant
October 29, 2025

The hydrogen atom



https://memory-alpha.fandom.com/wiki/Hydrogen

The hydrogenlike ion

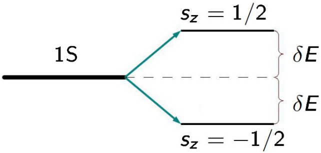


Zeeman effect

H-like ion with spinless nucleus in external magnetic field

$\Longrightarrow$ Splitting of ground-state energy level into two sublevels according to electron spin up or down

$$\delta E = -\langle 1s|\boldsymbol{\mu}_{\mathrm{e}} \boldsymbol{B}|1s\rangle = -g_e \frac{eB}{2m_e} s_z$$



δE: energy splitting due to external magnetic field $\boldsymbol{B}$

$$\boldsymbol{\mu}_{\mathrm{e}} = g_e \frac{e\boldsymbol{J}}{2m_e}$$

- $\boldsymbol{\mu}_{\mathrm{e}}$: electron magnetic dipole moment
- e, m_{e}: electron charge and mass
- $\boldsymbol{J}$: angular momentum (=spin, for s state)
- g_e: electron g-factor

QED test: g-factor of hydrogenlike tin (Z=50)

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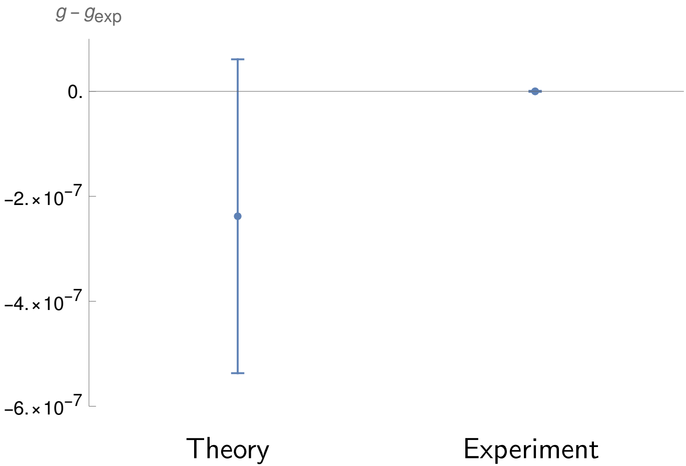

[J. Morgner, B. Tu, C. M. König, T. Sailer, F. Heiße, H. Bekker, **B. Sikora**, *et al.*, *Nature* **622**:53 (2023)]

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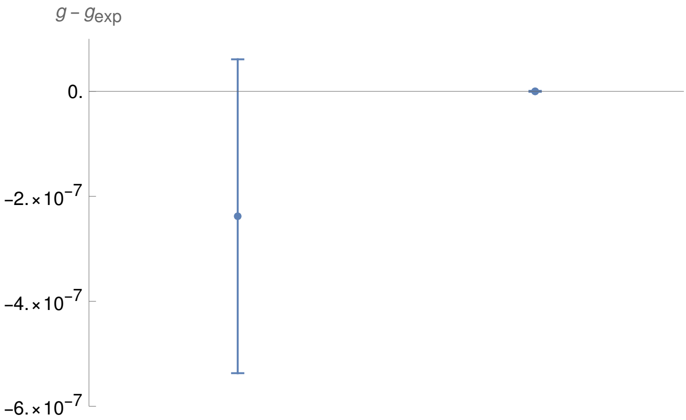

Theory $\delta g = 2.99 \times 10^{-7}$

Experiment $\delta g = 9.1 \times 10^{-10}$

[J. Morgner, B. Tu, C. M. König, T. Sailer, F. Heiße, H. Bekker, **B. Sikora**, *et al.*, *Nature* **622**:53 (2023)]

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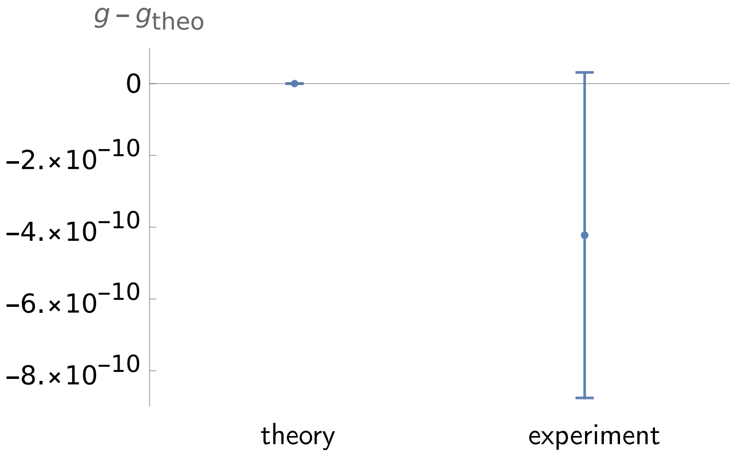

Uncertainty of theoretical g-factor limited by fine-structure constant α

[A. Schneider, **B. Sikora**, S. Dickopf, *et al*, *Nature*, **606**:878 (2022)]

Motivation

- Test of quantum electrodynamics in strong electric fields ALPHATRAP, with ions produced in Hyper-EBIT (MPIK)/ARTEMIS (GSI, Darmstadt)
  $\rightarrow$ measurement of heaviest hydrogenlike ions

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- Determination of the electron mass

  [S. Sturm, F. Köhler, J. Zatorski, *et al.* *Nature* **506**:467, 2014]

  [J. Zatorski, **B. Sikora**, S. G. Karshenboim, *et al.* *Phys. Rev. A*, 96, 012502 (2017)]

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- Competitive bounds for New Physics
  [T. Sailer, V. Debierre, Z. Harman, *et al., Nature* **606**:479, 2022]

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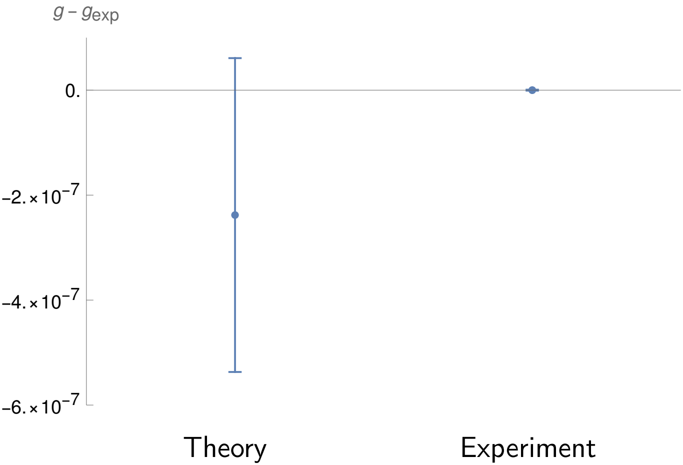

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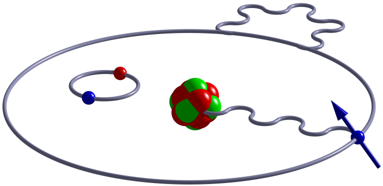
Outline

1. Bound-electron g-factor theory – status 2023
2. Bound-electron g-factor theory – update 2024/2025

Contributions to the bound-electron g-factor



- binding corrections
- nuclear corrections
- radiative corrections

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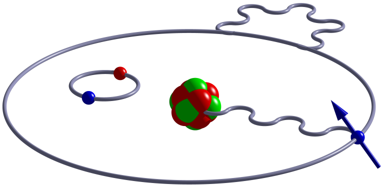

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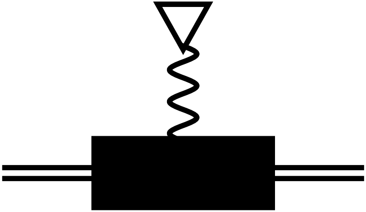

g-factor diagrams:

- 1 incoming electron line
- 1 outgoing electron line
- 1 photon line representing $\boldsymbol{B}$

Black box: Possible internal lines

Calculation of δE: E.g. Two-time Green's function method

[V.M. Shabaev, *Physics Reports* **356**:119, 2002]

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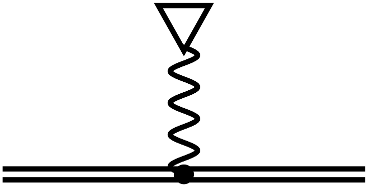

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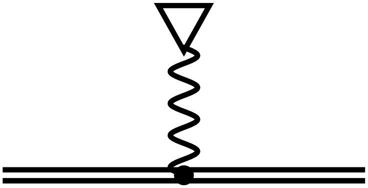

$$\Delta E_{\mathrm{mag}} = -\frac{2}{3} i \int \mathrm{d}r\, r^2\, Bier\, f(r) g(r)$$

Analytic result for point-nucleus model:

$$g_{\mathrm{D}} = \frac{2}{3} + \frac{4}{3}\sqrt{1 - (Z\alpha)^2}$$

[G. Breit. *Nature*, **122**:649, 1928]

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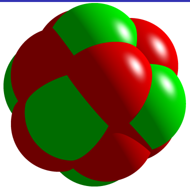
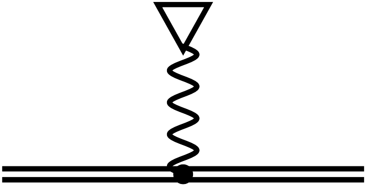
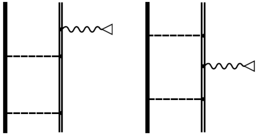




- Finite nuclear size (FS)
  [Y. Trouyet and **B. Sikora**, in preparation]
- Finite nuclear mass
- Nuclear deformation
- Nuclear polarization
  Impact of nuclear excited states



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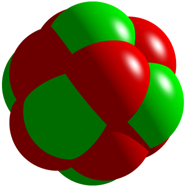
Finite size (FS) effect

Deviation of interaction potential between nucleus and bound electron from Coulomb potential assumed in Breit formula.



$$\Delta g_{\mathrm{FS}} \approx \frac{2(2\gamma+1)}{15}(Z\alpha)^2\left(1+\mathcal{O}\left((Z\alpha)^2\right)\right)(2Z\alpha m R)^{2\gamma}$$

$$R^2 \approx \frac{5}{3}\langle r^2\rangle$$

$$\langle r^n\rangle = \frac{\int \mathrm{d}^3\boldsymbol{r}\, r^n \rho(\boldsymbol{r})}{Ze}; \qquad \gamma = \sqrt{1-(Z\alpha)^2}$$

[J. Zatorski, N. S. Oreshkina, C.H. Keitel and Z. Harman, *Phys. Rev. Lett.*, **108**:063005, 2012]

Theoretical uncertainty of Δg_{FS} due to uncertainty of root-mean-square radius $r_{\mathrm{rms}} = \sqrt{\langle r^2\rangle}$ of the nucleus
(1.1×10^{-8} for $^{118}\mathrm{Sn}^{49+}$) ($\propto (Z\alpha)^4 \delta r_{\mathrm{rms}}$ in general)

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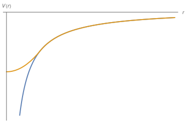

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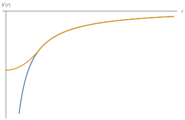

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(1.1×10^{-8} for $^{118}\mathrm{Sn}^{49+}$) ($\ll 2.99\times10^{-7}$ total theoretical uncertainty)
With rms radius from: [I. Angeli, K. P. Marinova, *At Data Nucl Data Tables* **99**:69, 2013]
$\Longrightarrow$ Possibility to measure rms radii from the bound-electron's g-factor...?!

Determining the fine-structure constant from the g-factor

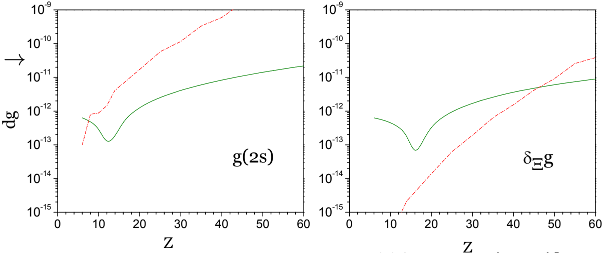
- **Problem:** nuclear parameters (e.g. $\langle r^2 \rangle$) are not known accurately
- **Solution:** weighted difference of H- and Li-ike ions (same Z):

$$\delta_\Xi g = g(2s) - \Xi g(1s)\,,$$

with Ξ theoretically optimized to suppress nuclear size effects

error due to $\delta\langle r^2\rangle$ + distr. $\rightarrow$

error due to present $\delta\alpha$ $\rightarrow$



[V. A. Yerokhin, E. Berseneva, Z. Harman, *et al.*, *Phys. Rev. Lett.* **116**:100801 (2016)]

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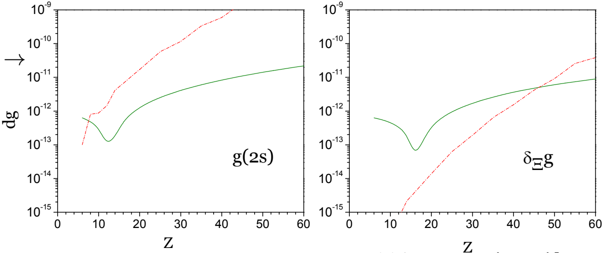
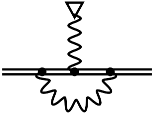


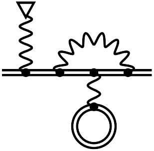
[V. A. Yerokhin, E. Berseneva, Z. Harman, *et al.*, *Phys. Rev. Lett.* **116**:100801 (2016)]

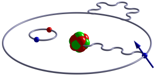
Previous idea: weighted difference of H- and B-ike ions

[V. M. Shabaev, D. A. Glazov, N. S. Oreshkina, *et al.* Phys. Rev. Lett., **96**:253002 (2006)]

Radiative corrections $\widehat{=}$ Feynman diagrams with closed loops.







$$g_e = 2\left(C^{(0)}(Z\alpha) + C^{(2)}(Z\alpha)\left(\frac{\alpha}{\pi}\right) + C^{(4)}(Z\alpha)\left(\frac{\alpha}{\pi}\right)^2 + \cdots \right)$$

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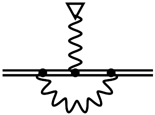

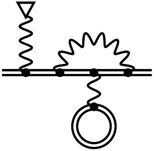

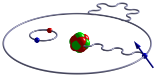

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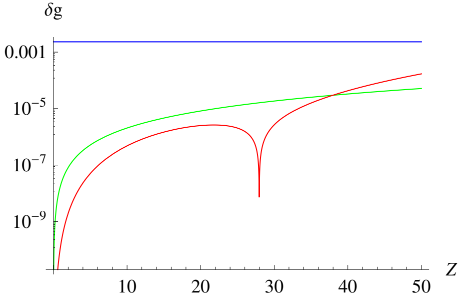
[T. Beier, *Phys. Rep.*, **339**:79, 2000]

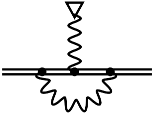


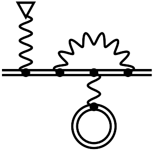
Absolute value of $\mathcal{O}\left((Z\alpha)^0\right)$ (blue), $\mathcal{O}\left((Z\alpha)^2\right)$ (green) and $\mathcal{O}\left((Z\alpha)^4\right)$ (red) contributions to the one-loop g-factor

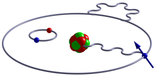
$Z\alpha$ expansion not a good approximation for high Z
$\Rightarrow$ Nuclear potential needs to be taken into account to all orders

Radiative corrections — introduction

Radiative corrections $\widehat{=}$ Feynman diagrams with closed loops.







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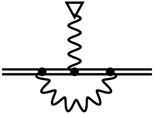

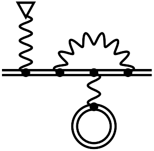

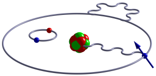

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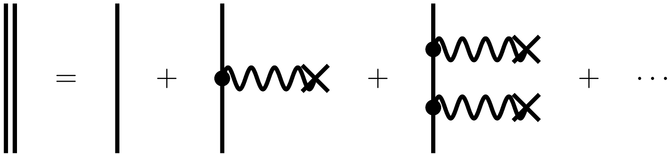
- Furry picture: $C^{(2n)}(Z\alpha)$

https://memory-alpha.fandom.com/wiki/B-4

"Why does the tall man have a furry face?" From Star Trek Nemesis

[T. Beier, *Phys. Rep.*, **339**:79, 2000]

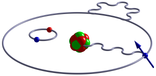
Furry picture



- Represent external fermion line by wave function $a_{1s}(\mathrm{x})$ of fermion bound in nuclear potential
- Represent internal fermion line by Green's function $G_{\mathrm{D}}(\mathrm{x}_1, \mathrm{x}_2)$ of fermion in nuclear potential

$$G_{\mathrm{D}}(\mathrm{x}_1, \mathrm{x}_2) = \sum_{\kappa, m_j} \begin{pmatrix} g_{11}(r_1, r_2)\chi_{\kappa m_j}\chi^{\dagger}_{\kappa m_j} & g_{12}(r_1, r_2)\chi_{\kappa m_j}\chi^{\dagger}_{-\kappa m_j} \\ g_{21}(r_1, r_2)\chi_{-\kappa m_j}\chi^{\dagger}_{\kappa m_j} & g_{22}(r_1, r_2)\chi_{-\kappa m_j}\chi^{\dagger}_{-\kappa m_j} \end{pmatrix}$$

Feynman diagrams with one loop

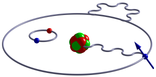


Self-energy (SE)

Vacuum polarization (VP)

- SE: e.g. [V. A. Yerokhin and Z. Harman, *Phys. Rev. A*, **95**:060501(R), 2017]
- VP: e.g. [T. Beier, *Phys. Rep.*, **339**:79, 2000]
- VP with virtual $\mu^+\mu^-$ pair: [N. A. Belov, **B. Sikora**, R. Weis, *et al.* arXiv:1610.01340v1 [physics.atom-ph]]

Feynman diagrams with one loop



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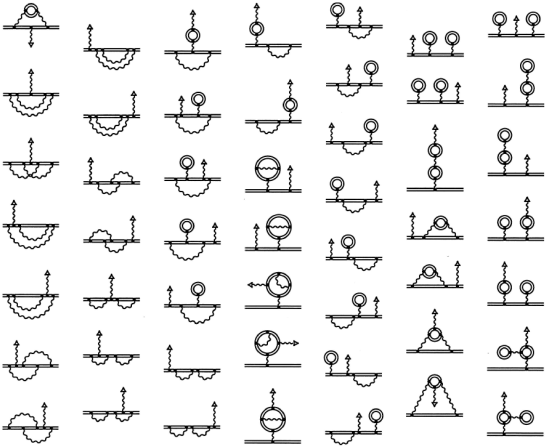
$^{118}\text{Sn}^{49+}$: (Numerical) Uncertainty of one-loop QED $= 5 \times 10^{-9}$

Two-loop QED corrections



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slide by V. Debierre

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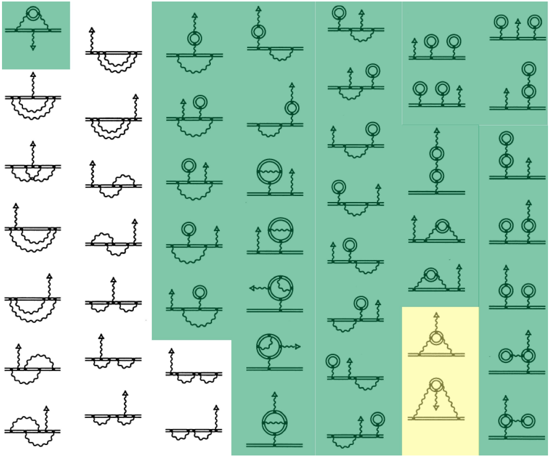

- Diagrams with 1 & 2 vacuum polarization loops (with free VP ($e^- e^+$) loops)

[V.A. Yerokhin, Z. Harman, *Phys. Rev. A*, 88:042502 (2013)]

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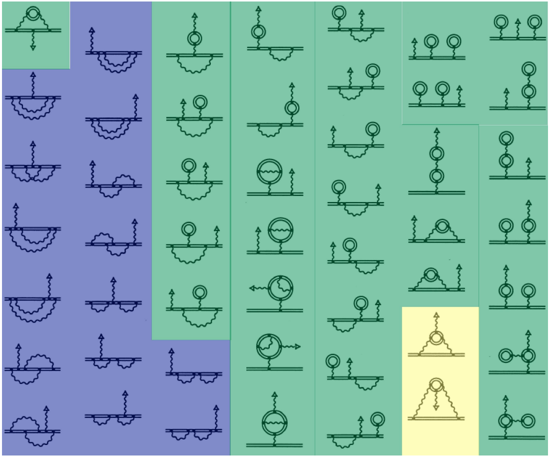

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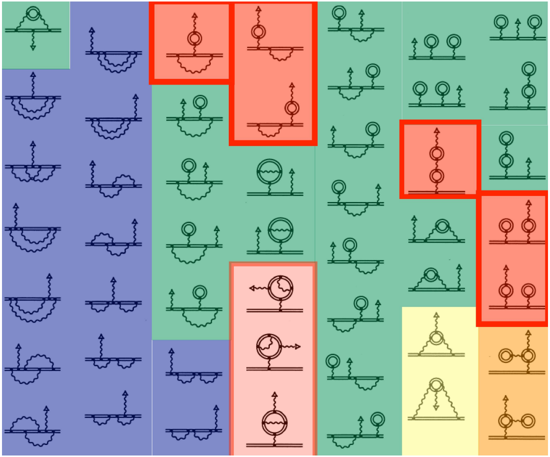

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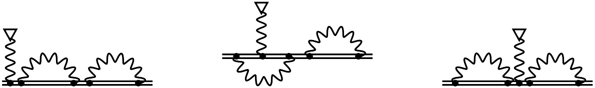
- Diagrams that vanished in the free VP loop approach

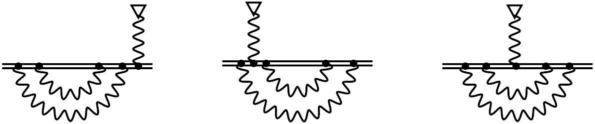
  [V. Debierre, **B. Sikora**, *et al.*, *Phys. Rev. A*, 103:L030802 (2021)]

Two-loop self-energy diagrams (SESE correction) - 2023

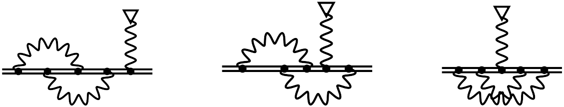
Loop after loop (LAL):



Nested loop (NL):



Overlapping loop (OL):



NRQED results up to order $(Z\alpha)^5$: e.g. [*Phys. Rev. Lett.* **120**:043203 (2018)]

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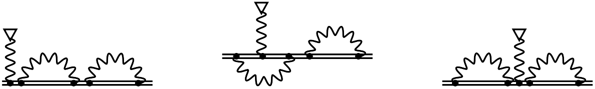

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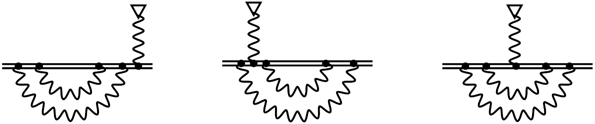

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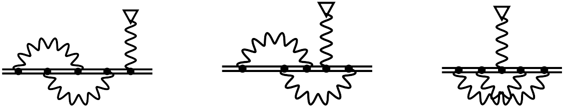

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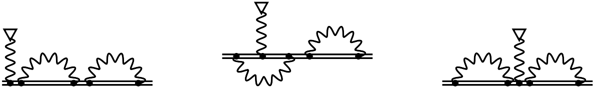

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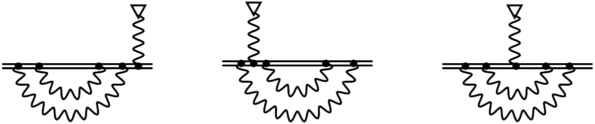

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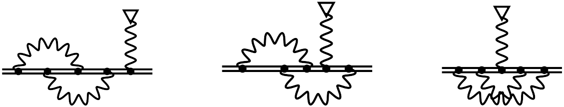

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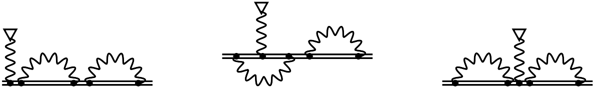

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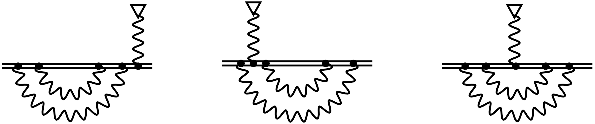

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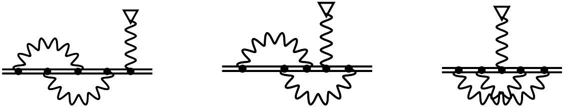

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- $^{118}\mathrm{Sn}^{49+}$: 2.97×10^{-7}

Outline

1. Bound-electron g-factor theory — status 2023
2. Bound-electron g-factor theory — update 2024/2025

Two-loop self-energy diagrams

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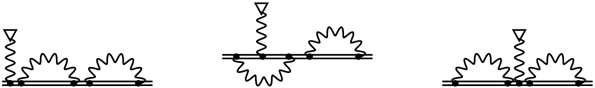

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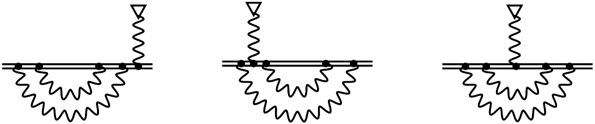

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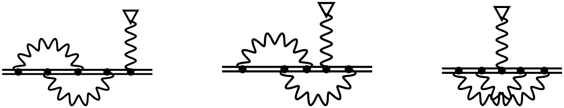
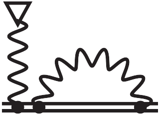


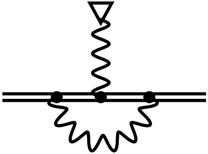
Furry picture calculation in analogy to SESE contribution to the Lamb Shift

[S. Mallampalli and J. Sapirstein, *Phys. Rev. A*, 57:1548, 1998]

[V. A. Yerokhin, *Phys. Rev. A*, 97:052509, 2018]

Loop after loop





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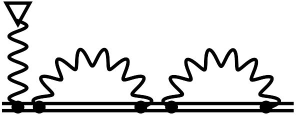

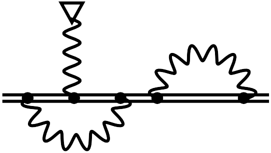

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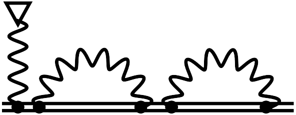
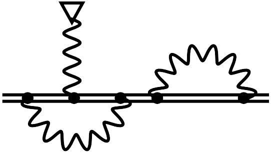




LAL diagrams can be computed as generalizations of one-loop QED calculations, with $|1s\rangle$ replaced by $|\delta_\Sigma 1s\rangle$

Define "self-energy perturbed wavefunction"

$$|\delta_\Sigma 1s\rangle = \sum_{n, n \neq 1s} \frac{|n\rangle\langle n|\Sigma|1s\rangle}{E_{1s} - E_n}$$

Tests of $|\delta_\Sigma 1s\rangle$: computation of one-loop g-factor and two-loop Lamb shift corrections.

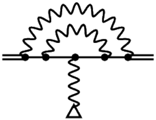
[N. S. Oreshkina, H. Cakir, **B. Sikora**, *et al.*, *Phys. Rev. A*, **101**:032511 (2020)]

Nested loop and overlapping loop, UV divergences

Separation of **all** nested and overlapping loop diagrams into three terms

F-term

- free internal electron lines
- UV divergences
- momentum representation



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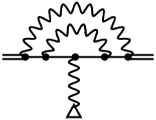

M-term

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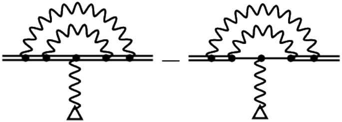
- coordinate representation



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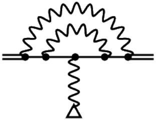

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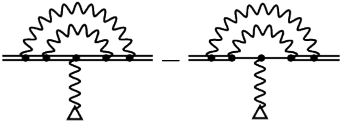
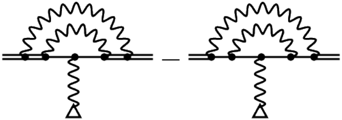


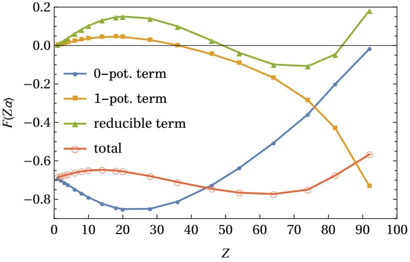
P-term

- bound internal electron lines
- UV divergent subdiagram
- mixed coordinate-momentum representation



F-term and LAL: numerical results

F-term



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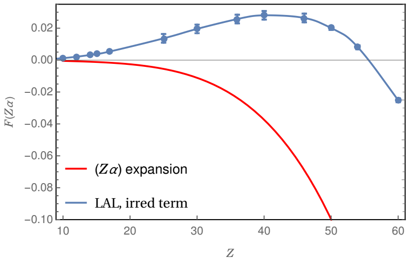

$$\Delta g = \left(\frac{\alpha}{\pi}\right)^2 F(Z\alpha)$$

Our results converge to the free-electron value $F(0) = -0.68833...$

[A. Peterman, *Helv. Phys. Act*, 30:407, 1957] [C. M. Sommerfield, *Ann. Phys.*, 5:26, 1958]

[**B. Sikora**, V. A. Yerokhin, N. S. Oreshkina, *et al.*, *Phys. Rev. Res.*, 2:012002(R) (2020)]

Update: M-term + P-term

IR divergences
M-term
P-term

- Summation of diagrams whose IR divergences cancel (g-factor, one-loop SE)
- Subtraction terms to cancel IR divergences (Lamb shift, two-loop SESE)

SESE correction to g-factor: combination of both methods

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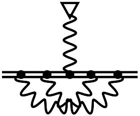
Numerical challenges



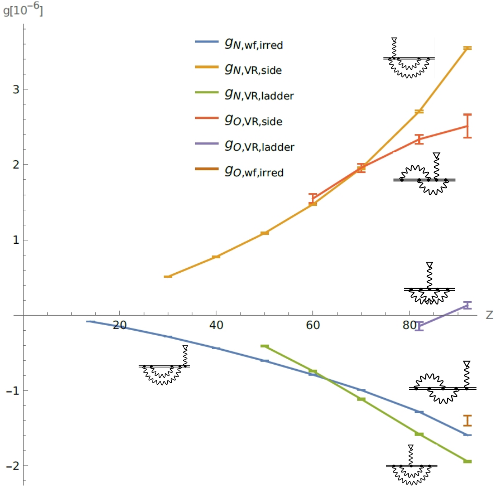
P-term
- Numerical Fourier transform
- $\sum_{\kappa_1} g_{\kappa_1}$ (infinite summation)

M-term
- $\int \mathrm{d}\omega_1 \int \mathrm{d}\omega_2 \int \mathrm{d}r_1 \cdots \int \mathrm{d}r_5 f(\cdots)$
- $\sum_{\kappa_1,\kappa_2} g_{\kappa_1,\kappa_2}$ (infinite summations)
- $\kappa_3 \in \{\kappa_2, -\kappa_2 - 1, -\kappa_2 + 1\}$, $|\kappa_2| \in \{|\kappa_1 - \kappa_4|, \cdots, |\kappa_1| + |\kappa_4|\}$

P-term: Numerical calculations performed by V. A. Yerokhin

[V. A. Yerokhin, **B. Sikora**, Z. Harman, and C. H. Keitel, submitted]

M-term: numerical results



[**B. Sikora**, V. A. Yerokhin, C. H. Keitel, Z. Harman, in preparation]

SESE correction — result for Sn

NRQED $\mathcal{O}\left((Z\alpha)^{\leq 5}\right)$ -4.25244×10^{-6}

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NRQED	$\mathcal{O}\left((Z\alpha)^{\leq 5}\right)$	-4.25244×10^{-6}
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Furry picture	all-order $Z\alpha$	$-4.099\,2(185) \times 10^{-6}$

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Experiment	$g_{\mathrm{exp}} =$	$1.910\ 562\ 058\ 962(73)_{\mathrm{stat}}(42)_{\mathrm{sys}}(910)_{m_{\mathrm{ion}}}$
Theory, previous	$g_{\mathrm{theo,prev}} =$	$1.910\ 561\ 821\ 0(2988)$

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Theory, update	$g_{\mathrm{theo}} =$	$1.910\ 561\ 974\ 2(392)$

[**B. Sikora**, V. A. Yerokhin, C. H. Keitel and Z. Harman. *Phys. Rev. Lett.* **134**:123001 (2025)]

QED test: g-factor of hydrogenlike tin (Z=50)

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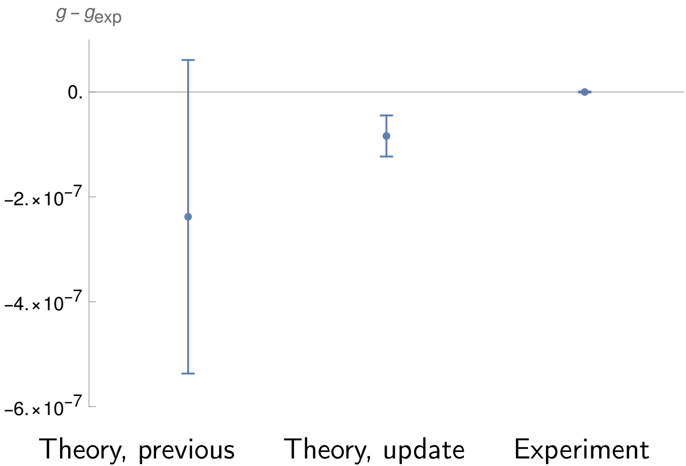

[**B. Sikora**, V. A. Yerokhin, C. H. Keitel and Z. Harman. *Phys. Rev. Lett.* **134**:123001 (2025)]

Outlook

For $Z \geq 50$:
SESE contribution with uncertainty according to

Z	NRQED	Furry picture
50	$-4.252(297) \times 10^{-6}$	$-4.099(18) \times 10^{-6}$
70	$-6.09(1.98) \times 10^{-6}$	$-5.326(31) \times 10^{-6}$
92	$-11.43(8.94) \times 10^{-6}$	$-9.650(30) \times 10^{-6}$

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Remaining uncertainties of bound-electron g-factor due to

Z	rms radius	uncalculated 2-loop QED	$\geq$ 3-loop QED
50	1.1×10^{-8}	2.2×10^{-8}	1.1×10^{-8}
70	2.5×10^{-7}	1.2×10^{-7}	4.5×10^{-8}
92	1.06×10^{-6}	4.6×10^{-7}	1.4×10^{-7}

$\Longrightarrow$ In high-Z regime, dominant uncertainty expected from nuclear effects

[**B. Sikora**, V. A. Yerokhin, C. H. Keitel, Z. Harman, in preparation]

Conclusions

- Experimental g-factor determination in high-Z regime orders of magnitude more precise than g-factor theory

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- Theoretical (2023) accuracy of g-factor limited by nuclear effects and uncalculated two-loop QED terms
- Significant improvement of total theoretical uncertainty through completion of SESE calculation in Furry picture approach
- After completion of two-loop QED calculations, accuracy of g-factor for high Z limited by nuclear effects

Conclusions

- Experimental g-factor determination in high-Z regime orders of magnitude more precise than g-factor theory
- Theoretical (2023) accuracy of g-factor limited by nuclear effects and uncalculated two-loop QED terms
- Significant improvement of total theoretical uncertainty through completion of SESE calculation in Furry picture approach
- After completion of two-loop QED calculations, accuracy of g-factor for high Z limited by nuclear effects
- Necessary step towards determination of α from the bound-electron g-factor

Thank you for your attention!