10-th order QED calculation of the electron g-2: special issues

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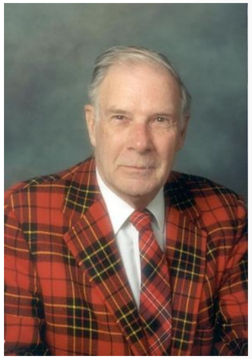
Mainz, 2025

To start...

The purpose of computing is insight, not numbers.

Richard Hamming

(a mathematician,
A.M. Turing award laureate)



Electron's g-2: the current status

Experiment:

a_e=0.00115965218073(28) [**2011**, D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, Phys. Rev. A 83, 052122]

a_e=0.00115965218059(13) [**2022**, X. Fan, T. G. Myers, B. A. D. Sukra, G. Gabrielse, arXiv:2209.13084]

Theory:

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geqslant 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.001159652181549(4)(12)(299)

α^{-1}=137.035999046(27) [**2018**, R. H. Parker et al., Science 360, 191-195]

a_e=0.001159652180197(4)(12)(94)

α^{-1}=137.035999206(11) [**2020**, L. Morel et al., Nature 588, 61-65]

Uncertainties come from: $A_1^{(10)}$, hadronic+electroweak, α

(recent hadronic data give a different value and are contradictive)

$A_1^{(10)}$=5.887(55) – S. Volkov & AHKN (T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio), **2025** consensus value

The history of the universal QED contributions $A_1^{(2n)}$ calculations

$$a_e(QED)=\sum_{n\geqslant 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}(m_e/m_\mu)+A_2^{(2n)}(m_e/m_\tau)+A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$$

- J.Schwinger [**1948**], analytically: $A_1^{(2)}=$**0.5**
- R. Karplus, N. Kroll [**1949**] – $A_1^{(4)}$ with a mistake
  A.Petermann [**1957**], C. Sommerfield [**1958**], analytically: $A_1^{(4)}=$**-0.328478966...**
- ~**1970**...~**1975**, $A_1^{(6)}$, numerically:
  1. M.Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.
     T. Kinoshita, P. Cvitanović [**1974**]: $A_1^{(6)}=$**1.195±0.026**
- E. Remiddi, S. Laporta et al., ~**1965**..**1996**, analytically: $A_1^{(6)}=$**1.181241456**...
- T. Kinoshita, M. Nio et al., numerically, **2015**: $A_1^{(8)}=$**-1.91298(84)**
  (first estimations in 1980-x)
- S. Laporta, semianalytically, **2017**: $A_1^{(8)}=$**-1.9122457649...**
- T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, numerically, **2012**: first numerical values of $A_1^{(10)}$
- S. Volkov, numerically **2019**: discrepancy of **4.8σ** in $A_1^{(10)}$
- **2025**: the discrepancy was resolved: $A_1^{(10)}=$**5.887(55)**

The history of the mass-dependent QED contributions calculations

$$a_e(QED)=\sum_{n\geqslant 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}(m_e/m_\mu)+A_2^{(2n)}(m_e/m_\tau)+A_3^{(2n)}(m_e/m_\mu,m_e/m_\tau)$$

- H.H.Elend [1966], analytically (**recently obtained masses are substituted into the analytical results**):
  for electron $A_2^{(4)}(m_e/m_\mu)=0.519738676(24)\cdot 10^{-6}$, $A_2^{(4)}(m_e/m_\tau)=0.183790(25)\cdot 10^{-8}$,
  for muon $A_2^{(4)}(m_\mu/m_e)=1.0942583093(76)$, $A_2^{(4)}(m_\mu/m_\tau)=0.000078076(11)$.
- ~**1970**...~**1990**, numerically, different research groups, increasing precision:
  J. Aldins, S. J. Brodsky, C. Chlouber, A. J. Dufner, T. Kinoshita, W. J. Marciano, B. Nizic, Y. Okamoto, M. A. Samuel...
  T. Kinoshita, W. J. Marciano [**1990**]: for muon $A_2^{(6)}(m_\mu/m_e)$=**22.8671(22)**.
- M. A. Samuel, G. Li, S. Laporta, E. Remiddi [**1991-1993**], analytically:
  for electron $A_2^{(6)}(m_e/m_\mu)$=**$-0.737394164(24)\cdot 10^{-5}$**, $A_2^{(6)}(m_e/m_\tau)$=**$-0.658273(79)\cdot 10^{-7}$**,
  for muon $A_2^{(6)}(m_\mu/m_e)$=**22.86837998(20)**, $A_2^{(6)}(m_\mu/m_\tau)$ =**0.000360671(94)**.
- S. Laporta [**1993**], semianalytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)$=**0.0005238(19)**.
- A. Czarnecki, M. Skrzypek, B. Krause [**1999**], analytically: for muon $A_3^{(6)}(m_\mu/m_e,m_\mu/m_\tau)$=**0.000527738(75)**.
- T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio [**2012**], numerically:
  for electron $A_2^{(8)}(m_e/m_\mu)$=**0.0009222(66)**, $A_2^{(8)}(m_e/m_\tau)$=**$8.24(12)\cdot 10^{-6}$**,
  for muon $A_2^{(8)}(m_\mu/m_e)$=**132.6852(60)**, $A_2^{(8)}(m_\mu/m_\tau)$=**0.04234(12)**, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)$=**0.06272(4)**,
  $A_2^{(10)}(m_\mu/m_e)$=**742.18(87)** [**NOT DOUBLE-CHECKED**], $A_2^{(10)}(m_\mu/m_\tau)$=**$-0.068(5)$**, $A_3^{(10)}(m_\mu/m_e,m_\mu/m_\tau)$=**2.011(10)**.
- A. Kurz, T. Liu, P. Marquard, M. Steinhauser [**2013**], semianalytically:
  for electron $A_2^{(8)}(m_e/m_\mu)$=**0.009161970703(372)**, $A_2^{(8)}(m_e/m_\tau)$=**$7.42924(118)\cdot 10^{-6}$**,
  for muon $A_2^{(8)}(m_\mu/m_\tau)$=**0.0424941(53)**.
- A. Kurz, T. Liu, P. Marquard, V. A. Smirnov, A. V. Smirnov, M. Steinhauser [**2016**], semianalytically:
  for muon $A_2^{(8)}(m_\mu/m_e)$=**132.86(48)**, $A_3^{(8)}(m_\mu/m_e,m_\mu/m_\tau)$=**0.0627220(100)**.

10-th order: my and AHKN results

- AHKN = T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio

$A_1^{(10)}$[AHKN] = 5.870(128) [**2025**]

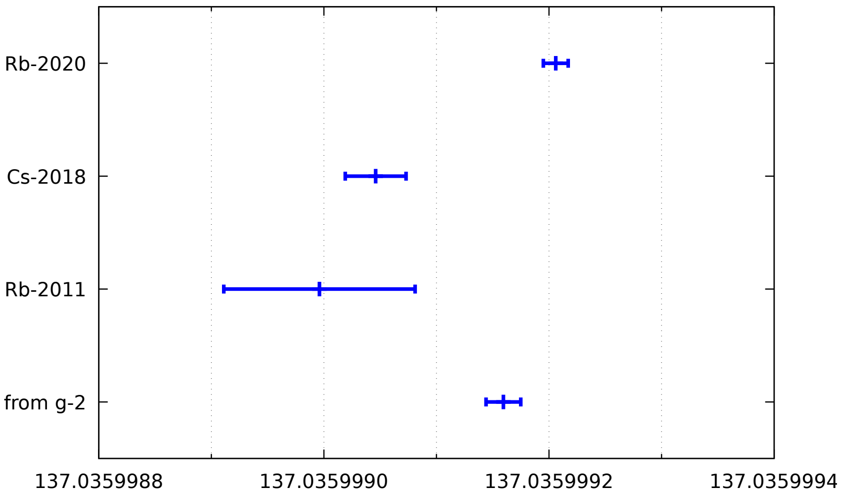
[T. Aoyama, M. Hayakawa, A. Hirayama, M. Nio, Phys. Rev. D 111, L031902]

- My result

$A_1^{(10)}$[Volkov] = 5.891(61) [**2024**]

[S. Volkov, Phys. Rev. D 110, 036001]

a_e and α



α^{-1} obtained by different methods

- $\alpha^{-1}[a_e]$ = 137.035999159(16)
- α^{-1}[Rb-2011] = 137.035998996(85)

  (R. Bouchendira, P. Cladé, S. Guellati-Khélifa, F. Nez, F. Biraben, PRL 106, 080801, 2011 + CODATA-2014)

  α^{-1}[Cs-2018] = 137.035999046(27)

  (R. H. Parker, C. Yu, W. Zhong, B. Estey, H. Müller, Science 360, 191, 2018)

  α^{-1}[Rb-2020] = 137.035999206(11)

  (L. Morel, Z. Yao, P. Cladé, S. Guellati-Khélifa, Nature 588, 61, 2020)

Outline

- Challenges in the calculation of $A_1^{(10)}$
- Prospects in higher-order calculations of $A_1^{(2n)}$

Challenges in $A_1^{(10)}$: how do calculations work?

Problem

Correctly defined mathematical object

Solving/Calculation

Challenges in $A_1^{(10)}$: foundations of quantum field theory (logical inconsistency)

The foundations of QFT are **inconsistent**!
(in **all** possible formulations)

There's no sense in being precise when you don't even know what you're talking about.

John von Neumann
(a mathematician, physicist,
founder of computer science)

~~Reliance on foundations~~

↓

Reliance on other successful calculations

Challenges in $A_1^{(10)}$: foundations of quantum field theory (infinities)

- Infinities arise in intermediate values
- Example from classical electrodynamics:

  the energy of the electric field of a point-like charge = ∞
- One has to be very careful with infinities:

  1-1+1-1+…. = (1-1)+(1-1)+…. = 0+0+… = **0**

  1-1+1-1+… = 1+(-1+1)+(-1+1)+… = 1+0+0+… = **1**
- In quantum electrodynamics: the infinities are absorbed by renormalization of the electron **mass** and **charge**

  (the constants in the Lagrangian are infinite, but physically observable values are finite)

Challenges in $A_1^{(10)}$: foundations of quantum field theory (regularizations)

- In mathematics, infinities are functions, not numbers.
- A **regularization** is needed:

  f is infinite means $f(\varepsilon) \to \infty$, $\varepsilon \to 0$.

  A natural example:

  ε is the particle radius or interaction radius.
- Unfortunately, natural (physical) regularizations do not work at all!
- Dimensional regularization (it somehow works):

  space-time dimensionality = $4-\varepsilon$

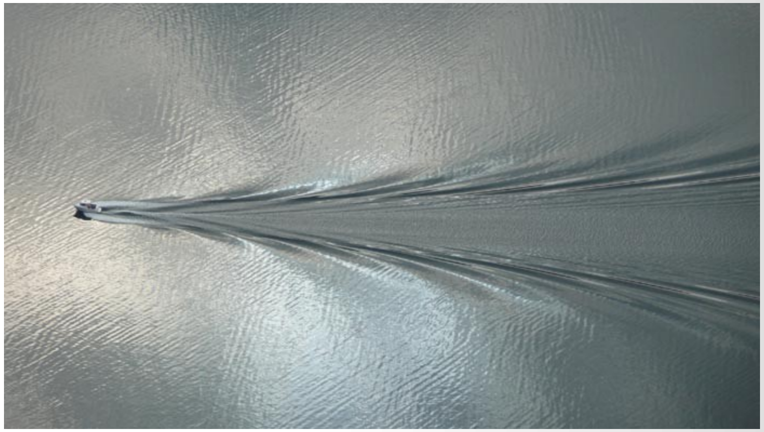
  Requires analytical continuation to be correctly defined!
- Dreams and hype: the final result does not depend on the regularization.

  Reality: there is no formal definition of regularization and no theorems like this.
- Dreams and hype: ε is a cut-off of our unknowledge.

  Reality: quantum field theory has never been formulated in the form of unknowledge cut-off.

Challenges in $A_1^{(10)}$: foundations of quantum field theory (not-only-mass-and-charge renormalization)

- Every particle is *dressed*, because of the interaction between fields.
- Particle **wave function renormalization**: the corresponding constant *Z* relates to the probability that the dressed particle is a free particle.



Every particle is inseparably surrounded by its own field

- The electron is surrounded by infinite number of photons (on average), because of the classical electromagnetic field.
- Therefore, the probability, that the dressed electron is a free electron, is **0**.
- The zero-probability leads to an **infrared divergence** in *Z*.
- Dreams and hype: renormalization accounts for the cancellation of only ultraviolet divergences.
  Reality: renormalization accounts for the cancellation of ultraviolet, infrared, and mixed ultraviolet-infrared divergences.

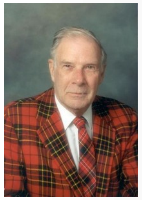
Challenges in $A_1^{(10)}$: foundations of quantum field theory (summary)

- The Lagrangian of QED is simple, but it is not possible to derive a calculation scheme exactly from it.
- The only way is to rely on other calculation schemes that have already demonstrated their reliability.
- These schemes are:
  - derived from foundations, but not fully logically correctly;
  - cumbersome (complicated).
- The path of the usage of such a scheme is **very narrow**.
  It is very difficult to change something.
- It is well known how much computer resources this path takes (too much for $A_1^{(10)}$).
- If we want to go further, we have to invent something completely different, but equivalent.

In science, if you know what you are doing, you should not be doing it.

In engineering, if you do not know what you are doing, you should not be doing it.

Richard Hamming
(a mathematician,
A. M. Turing award laureate

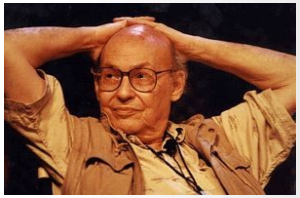


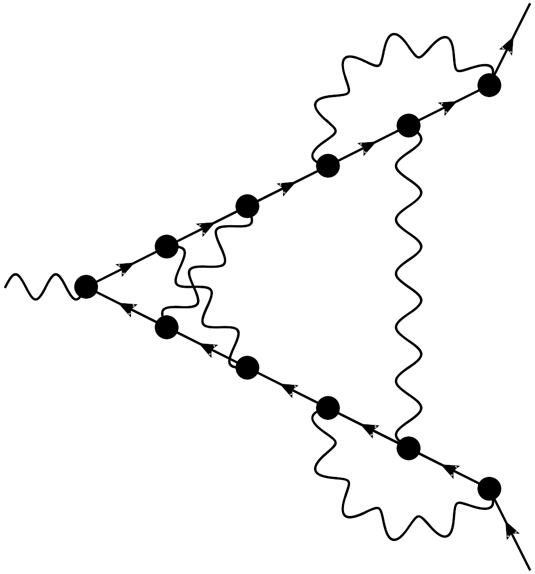
Challenges in $A_1^{(10)}$: handling infinities (Feynman diagrams and parameters)

You don't understand anything until you learn it more than one way.

Marwin Minsky

(a mathematician, computer scientist, cognitive scientist, father of AI)



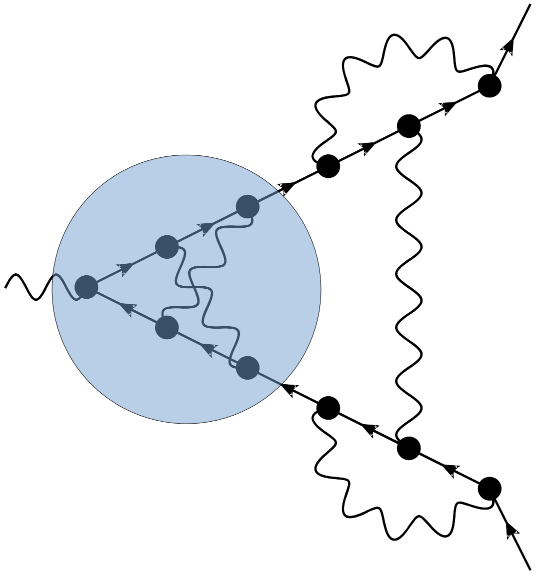


an example of a diagram of $A_1^{(10)}$

Diagrams: 12672
(undirected: 5536)

- We construct an integral in Feynman-parameteric space for each diagram:
  $$\int_{z\geq 0} f(z_1,z_2,\ldots,z_n)\,\delta(z_1+\ldots+z_n-1)\,d^n z$$
  A Feynman parameter z_j is assigned to each line ($n=15$).
- Divergences (UV, IR, mixed) are removed under the integral sign (before integration).
- Things like dimensional regularization or expansions of infinities are never used.

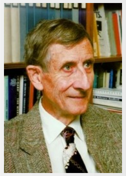
Challenges in $A_1^{(10)}$: handling infinities (UV divergences in Feynman diagrams)



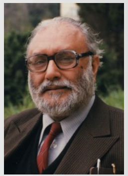
To obtain subtraction counterterms for removing UV divergences, we modify the Feynman amplitudes of subdiagrams.

The procedure is equivalent to the renormalization (of the **mass**, **charge**, **wave function**).

General idea how to work at any order:
F. J. Dyson, Phys. Rev. 75, 1736 (1949)
A. Salam, Phys. Rev. 82, 217 (1951)

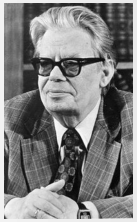


F. Dyson



A. Salam

First attempt of rigorous proof, the hope that QFT can be treated rigorously, inspiration:
N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957)



N. Bogoliubov

O. Parasiuk

Rigorous proof of finiteness (UV only), Schwinger parameters:
K. Hepp, Commun. Math. Phys. 2, 301 (1966)

Rigorous proof of finiteness (UV only), momentum space:
W. Zimmermann, Commun. Math. Phys. 15, 208 (1969)

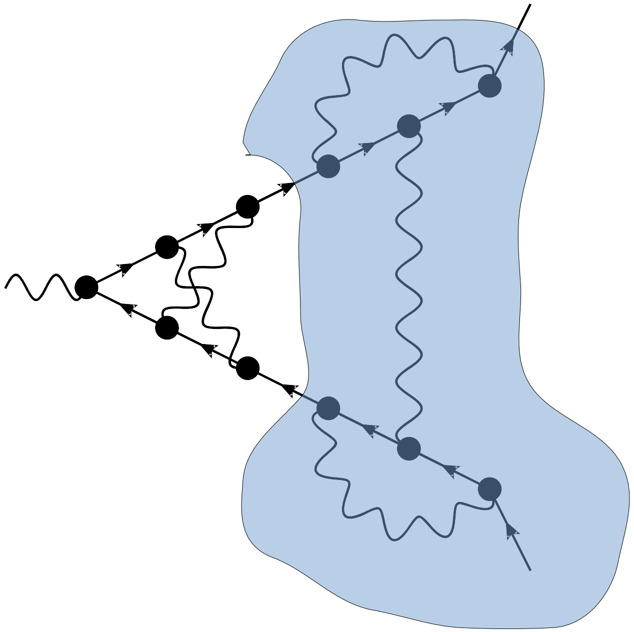
K. Hepp

W. Zimmermann

Handling UV divergences is a **beautiful story**:

- rigorous proofs;
- It is universal.

Challenges in $A_1^{(10)}$: handling infinities (IR divergences in Feynman diagrams)

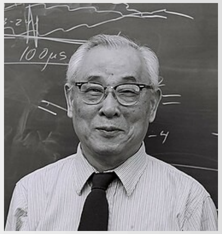


To obtain subtraction counterterms for removing IR divergences, we typically modify the expression part related to the part outside of a subdiagram.

Different approaches to the removal of IR divergences (together with the UV and mixed ones):

- M. J. Levine, J. Wright, Phys. Rev. D 8, 3171 (1973);
- R. Carroll, Y.-P. Yao, Phys. Lett. 48B, 125 (1974);
- P. Cvitanović, T. Kinoshita, Phys. Rev. D 10, 3991 (1974);
- T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Nucl. Phys. B 796, 184 (2008);
- S. Volkov, J. Exp. Theor. Phys. 122, 1008 (2016);
- S. Volkov, Phys. Part. Nuclei 53, 805 (2022).

simplest



Toichiro Kinoshita
first went beyond $A_1^{(6)}$

Handling IR divergences is **not a beautiful story at all**:

- not universal (for specific observables only, like the anomalous magnetic moment);
- no rigorous proofs.

Challenges in $A_1^{(10)}$: integration (**difficult integrands**)

- We have to integrate
  $$\int_{z\geq 0} f(z_1,\ldots,z_n)\,\delta(z_1+\ldots+z_n-1)\,d^n z$$
  with $n=15$ for each of 5536 Feynman diagrams.
- Many variables: Monte Carlo is the only feasible solution.
- Uniform probability distribution for random samples works extremely bad.
- Importance sampling: more samples for regions with large $|f(z)|$.
- $f(z)$ is singular at the boundary, has acute peaks.
- Adaptive algorithms work not so well, because the number of qualitatively different regions in the integration domain grows exponentially (or even faster) with n.

Challenges in $A_1^{(10)}$: integration (Monte Carlo probability density from combinatorics)

- We employ a **nonadaptive** approach for the Monte Carlo integration.
- The ideas come from the theory of renormalization:
  E. Speer, J. Math. Phys. 9, 1404 (1968)
  (the asymptotic behaviour of the integrand can be explained by ultraviolet degrees of divergence of **all** subdiagrams).
- If there is no infrared cut-off, these ideas are not enough. We need more:
  S. Volkov, Nucl. Phys. B 961, 115232 (2020).
- Not every probability density function can be used for random sample generation. We developed an algorithm for the needed class of functions:
  S. Volkov, Phys. Rev. D 96, 096018 (2017).
- The algorithm is still partially adaptive:
  - all diagrams are evaluated simultaneously, the time distribution is being adjusted adaptively;
  - several parameters are preadjusted (by lower-order experiments).

Challenges in $A_1^{(10)}$: integration (huge computer resources)

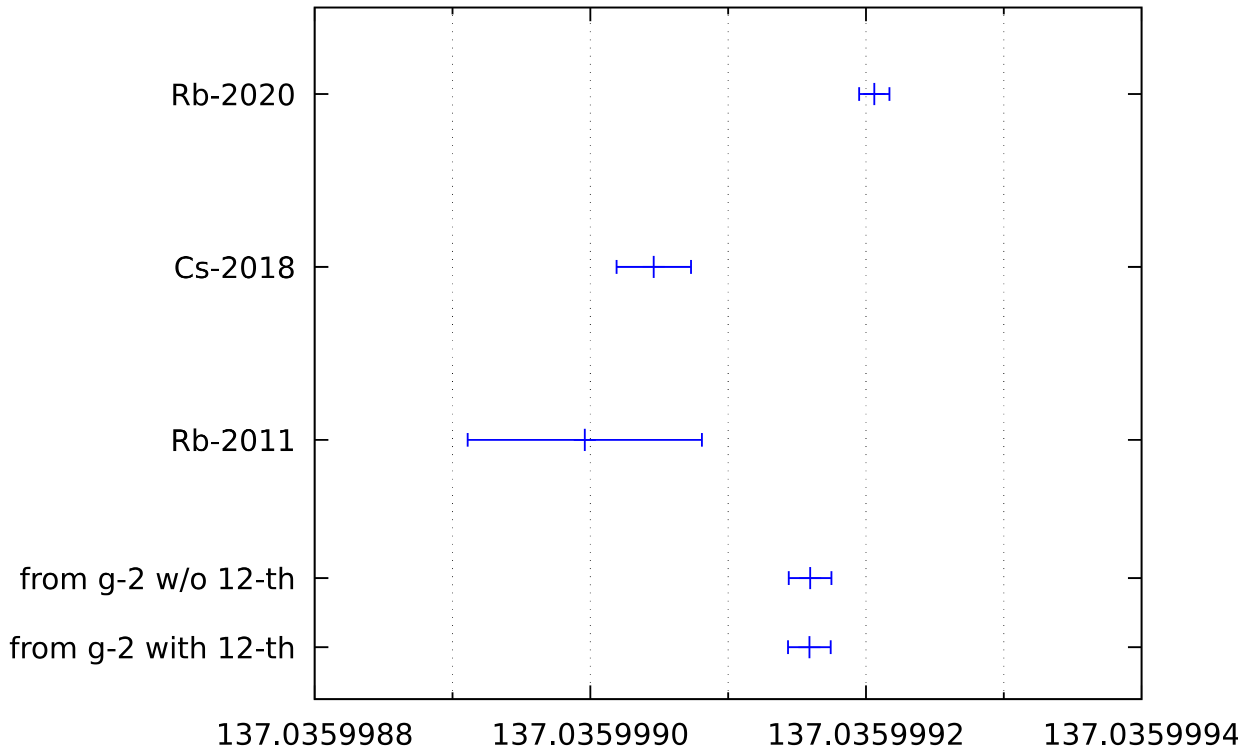
- The integrand code size: **1 TB** (the compilation required **1 CPU core-month**)
- Nvidia GPUs were used for integration.
- Two calculations:
  - [2019] a part of $A_1^{(10)}$, supercomputer “Govorun” (Dubna), Nvidia V100, **4.5 GPU-years**;
  - [2024] full $A_1^{(10)}$, supercomputer “HoreKA” (Karlsruhe), Nvidia A100, **11.5 GPU-years**.
- Total number of Monte Carlo samples $\approx 10^{15}$.

Challenges in $A_1^{(10)}$: integration (numerical cancellations)

Types of numerical cancellations:

- between different terms contributing to the integrand value at a point;
  Reason: cancellation of divergences under the integral sign.
  Problem: round-off errors.
  Solution: interval arithmetic, high-precision arithmetic.
- between different integrals;
  The sum of the absolute values is
  2500 times more
  than the absolute value of the sum!
  Reason: not fully understood.
  Solution: obey and suffer.
- inside one integral;
  not significant

Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: is it really needed?



α^{-1} obtained by different methods

Geometric progression estimation:

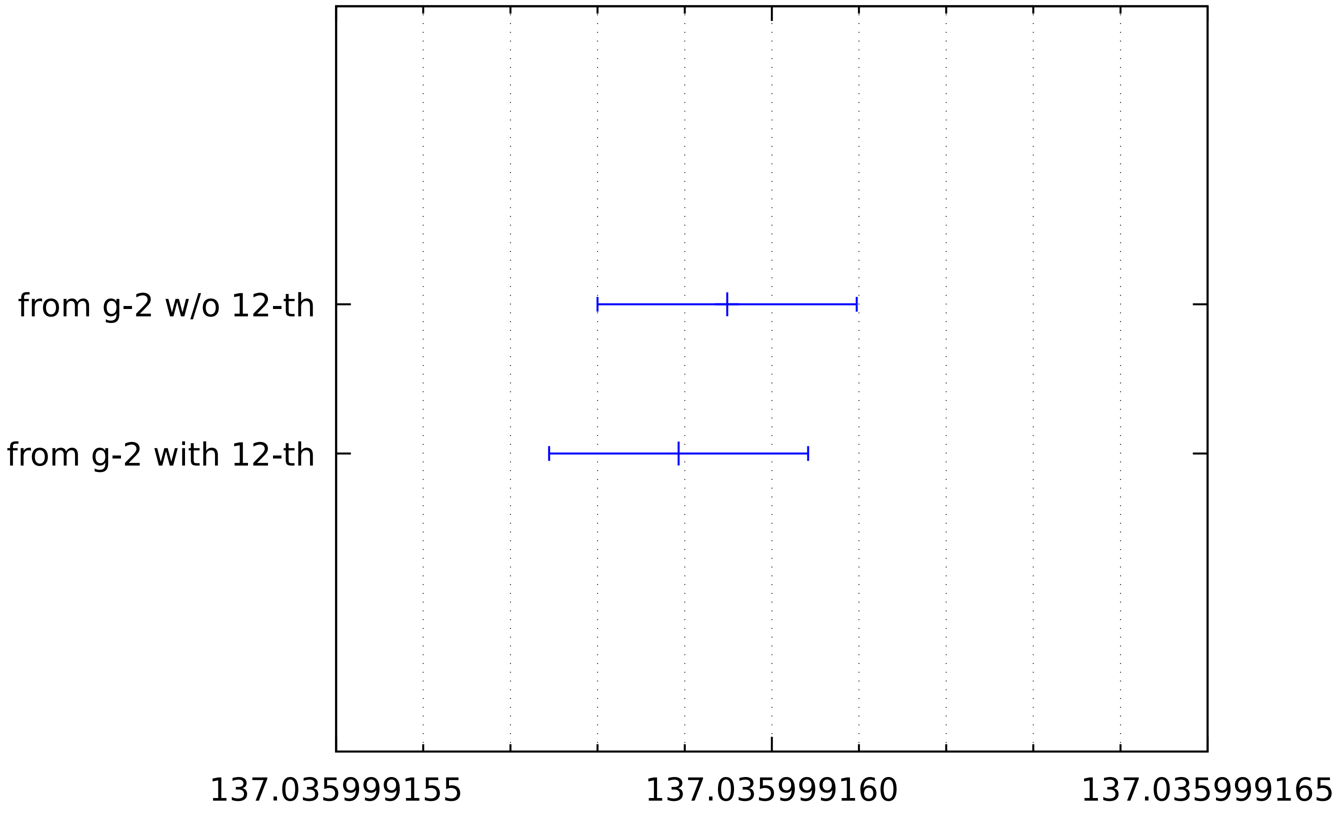
$A_1^{(2)} = 0.5$,
$A_1^{(4)} = -0.328478966...$,
$A_1^{(6)} = 1.181241456...$,
$A_1^{(8)} = -1.9122457649...$,
$A_1^{(10)} = 5.887(55)$,
$\mathbf{A_1^{(12)} \approx -18}$.

Let us take
$\mathbf{A_1^{(12)} = -30}$.

No difference!

Prospects in the calculation of $A_1^{(2n)}$, $n\geq 6$: is it really needed?

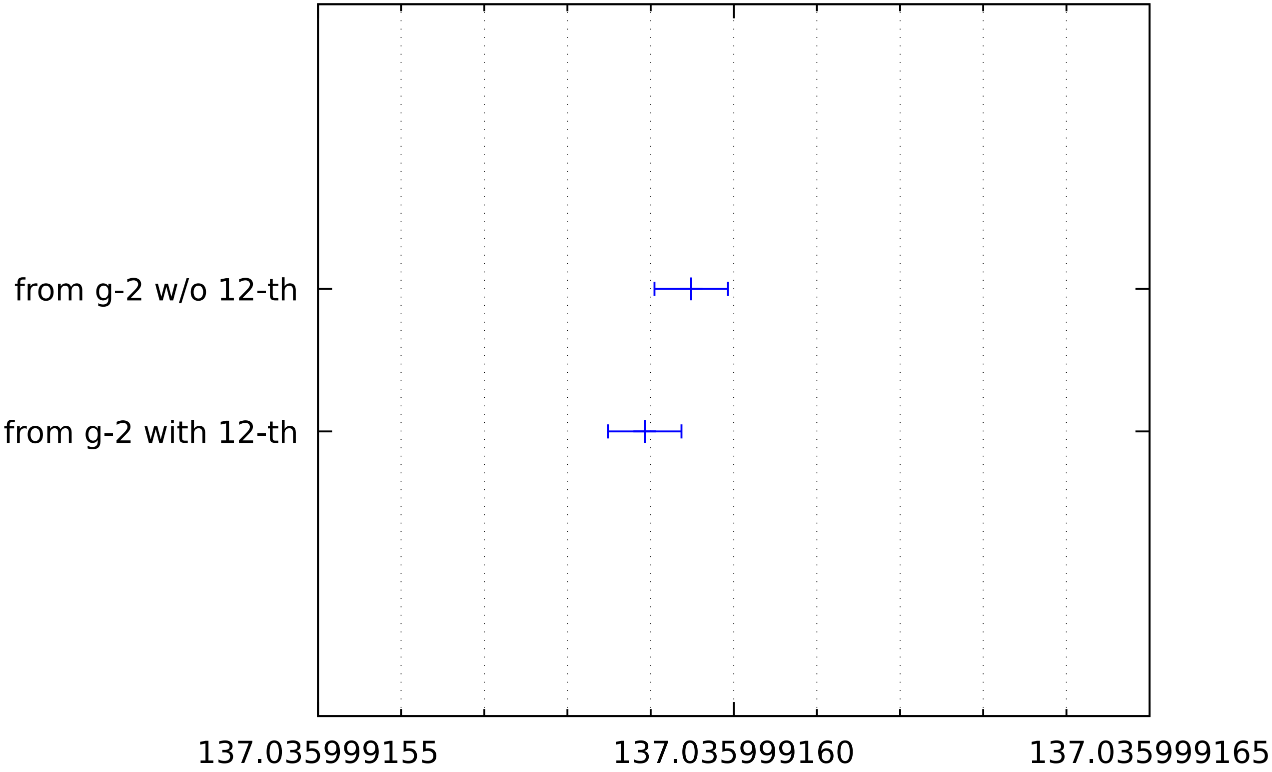
Suppose $A_1^{(12)}=-30$ and we resolved the experimental uncertainty of a_e.



More noticeable (at least if we underestimated $A_1^{(12)}$)

Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: is it really needed?

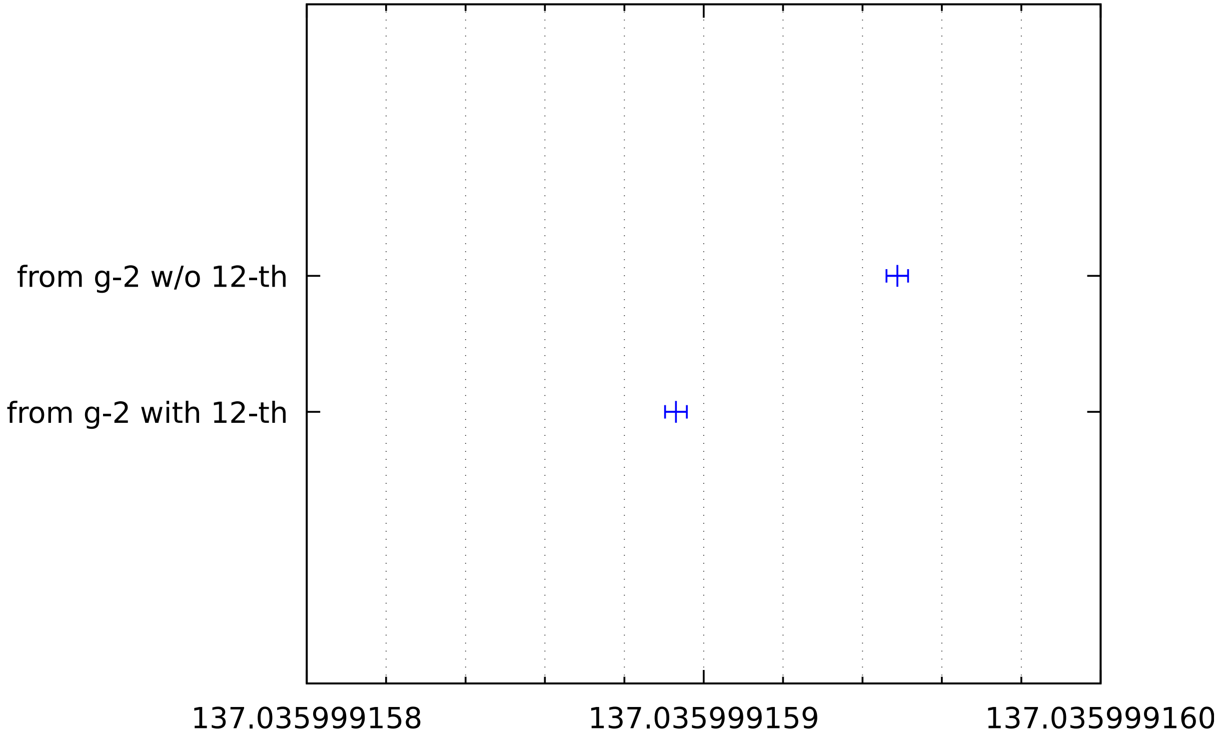
Suppose $A_1^{(12)}=-30$ and we resolved:
- the experimental uncertainty of a_e;
- the hadronic uncertainty.



Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: is it really needed?

<u>Suppose</u> $A_1^{(12)}=-30$ and we resolved:

- the experimental uncertainty of a_e;
- the hadronic uncertainty;
- the Monte Carlo uncertainty in $A_1^{(10)}$.



$A_1^{(12)}$ becomes crucially important!

Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: bottlenecks

	Diagrams (directed)	Integrand code size	GPU time to reach **10% uncertainty**
$A_1^{(8)}$	891	10 GB	5 min
$A_1^{(10)}$	12672	1 TB	1 month
$A_1^{(12)}$	202770	100 TB (estimated)	700 years (estimated)
$A_1^{(14)}$	3602880		
$A_1^{(16)}$	70425747		

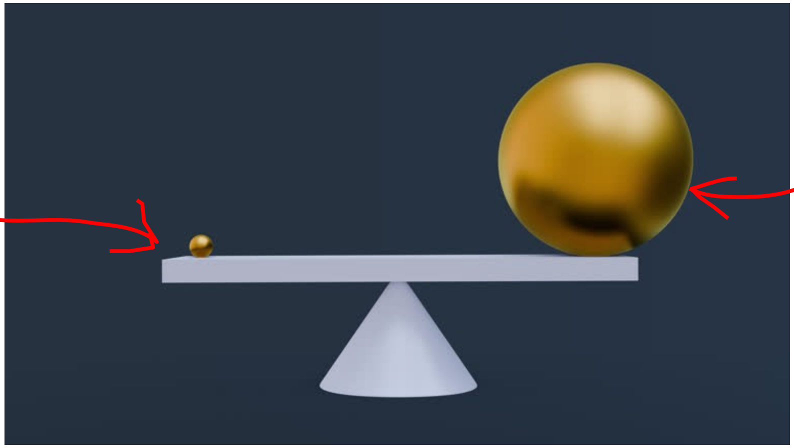
100 terabytes: doable on a supercomputer

700 GPU-years: ?????

Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: a supercomputer for the Monte Carlo

- The Monte Carlo integration time is proportional to $1/\sigma^2$, where σ is the uncertainty.
- Reaching 10% uncertainty in $A_1^{(12)}$ takes ≈ **700 years** on the GPU Nvidia A100.
- Jupiter Booster is the fastest supercomputer in Germany (in the world top-5 list)
  24000 GPUs Nvidia GH200 Grace Hopper
  Reaching 10% uncertainty in $A_1^{(12)}$ ≈ **3 days of full occupation**.

The supercomputer time is distributed according to the problem importance



Calculation of $A_1^{(12)}$

Biological immortality for humans

and other more important computational problems

Prospects in the calculation of $A_1^{(2n)}$, $n \geq 6$: environmental aspects

- Reaching 10% uncertainty $A_1^{(12)}$ = **3 days of full occupation** of the supercomputer “Jupiter Booster”.
- The energy consumption of “Jupiter Booster” is ≈ **13 MW**.
- The average electricity consumption of a German person is ≈ **700 W** (20% comes from renewable sources).
- Reaching 10% uncertainty in $A_1^{(12)}$ =
  - 150 years of an average German person’s life;
  - 6 hours of the consumption of Mainz;
  - 1 min of the consumption of Germany;
  - 1 sec of the total world’s consumption;
  - 8 hours of Large Hadron Collider.

Prospects in the calculation of $A_1^{(2n)}$, $n\geq 6$: ways to improve the method

- More efficient Feynman-parametric integrands (less number of diagrams, AHKN-like?)
  100 TB → several terabytes?
- More integration variables. For example, combine Schwinger parameters with momentum integration.
  14 variables → 35 variables (not a problem for Monte Carlo)
  100 TB → several **kilobytes**???
  How to integrate? Cancellations inside integrals?
  It requires a complete reconstruction of the programs and a new large investigation.
- Remove the inter-diagram cancellations by applying the Ward-Takahashi identities to subdiagrams of any type (expressing the problematic form factors from less problematic ones).
  better Monte Carlo convergence rate
  It does not require too large modifications.
  No one knows how it works…
- More efficient integrand approximation for Monte Carlo.
  Hepp's sectors → more natural sectors for this problem
  better convergence rate
  a new large investigation

Summary

- $A_1^{(10)}$ has been calculated and double-checked (1.5% uncertainty).
- From the phenomenological point of view, the calculation of $A_1^{(12)}$ makes sense only after a significant improvement of the measurement uncertainty in a_e and α, and only simultaneously with the improving the precision of $A_1^{(10)}$ and resolving the hadronic uncertainty.
- From the theoretical point of view, the calculation of $A_1^{(12)}$ makes a great sense, because it is a very challenging problem that requires a very deep understanding of quantum field theory.
- The usage of the current methods for $A_1^{(12)}$ is not recommended.
- Promising approaches for improving the method exist, but no one knows how it will work.

Sometimes a problem will seem completely insurmountable. Then someone comes up with a simple new idea, or just a rearrangement of old ideas, that completely eliminates it.

Marwin Minsky

(a mathematician, computer scientist, cognitive scientist, father of AI)