The calculation of the electron g-2 at 4 loops and the perspectives for 5 loops

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In Quantum Electrodynamics the *anomalous magnetic moment*, expressed in Bohr magnetons can be written as a power series in the small quantity $\left(\frac{\alpha}{\pi}\right)$ ($\alpha \approx 1/137$).

$$F_2(0) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

Another similar quantity, the so-called *slope* of the Dirac form factor, important for bound states calculation, can be expanded in power series

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

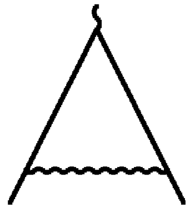
the coefficients C_i and A_i are pure numbers and can be extracted from the Feynman diagrams of the theory as linear combination of (a large number of) Feynman integrals.

These combinations are to be reduced to a linear combination of (irreducible) master integrals by solving of large systems of IBP identities, or, hopefully in near future, avoiding this step through intersection theory.

For a given n, both C_n and A_n are espressible in term of the same master integrals, and as a consequence have analytical expressions with similar structure.

Contributions at one loop

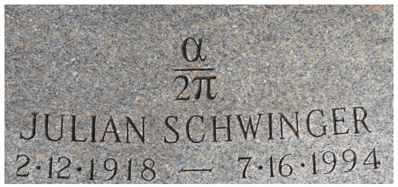
$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \ldots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \ldots$$



1 diagram $\rightarrow$ 1 master integral

$$C_1 = \frac{1}{2}$$

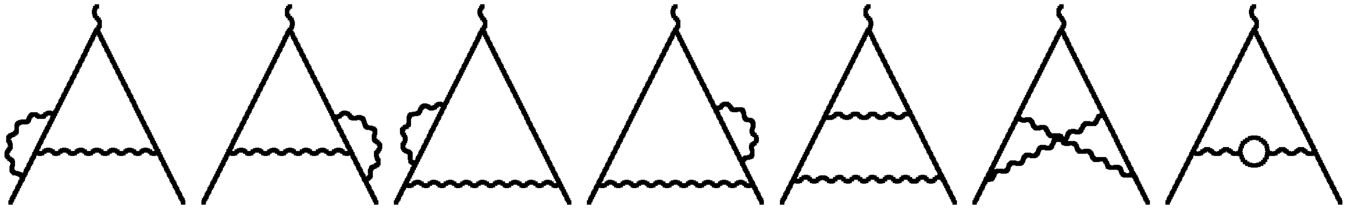
Obtained by Julian Schwinger in 1948



$$A_1 = -\tfrac{1}{8} - \tfrac{1}{3}\ln\tfrac{\Delta E}{m} + \tfrac{5}{18}$$ (Bethe 1947)

Contributions at two loops

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \dots$$

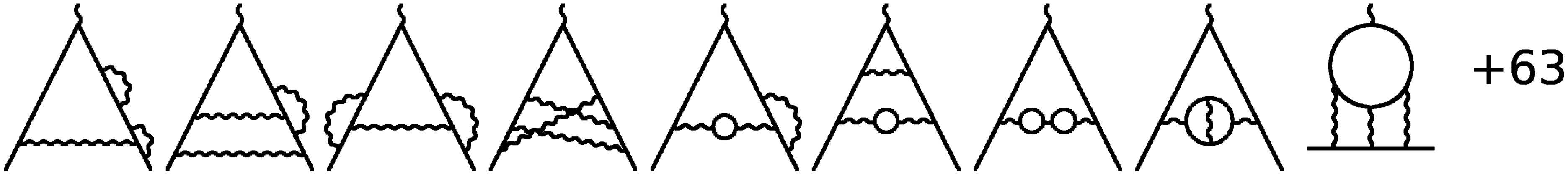


7 diagrams → 3 master integrals

$$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328\,478\,965\dots$$
$$A_2 = -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3) = 0.469\,941\,487\dots$$

- C_2 was computed by Petermann and Sommerfeld in 1957.
- A_2 was computed by R. Barbieri, J. A. Mignaco and E. Remiddi in 1972.

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \dots$$



72 diagrams → 17 master integrals

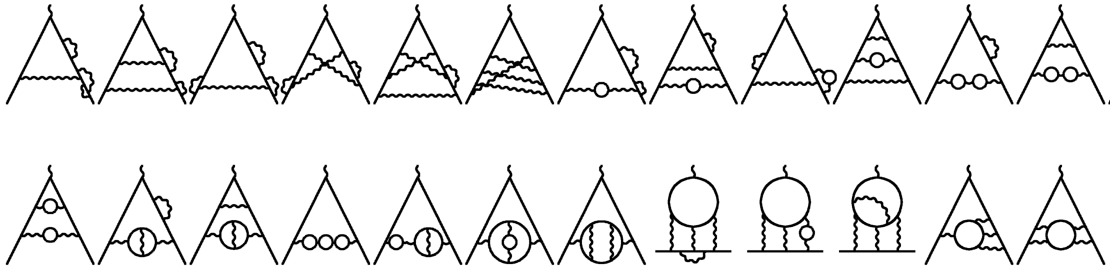
$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$
$$= 1.181\ 241\ 456\ \dots$$

$$A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2\ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{2160}\pi^2\ln 2 - \frac{454979}{38880}\pi^2$$
$$- \frac{77513}{186624} = 0.171\ 720\ 018\dots$$

- C_3 was obtained by **S.L.** and Ettore Remiddi in 1996.
- A_3 was obtained by Melnikov and Ritbergen in 1999.

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + C_5(\alpha/\pi)^5 + \ldots$$

$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + A_5(\alpha/\pi)^5 + \ldots$$



891 diagrams → 334 master integrals

$C_4 =$ `-1.9122457649264455741526471674398300540608733906587253451713298480060384439806517061427...`

$A_4 =$ `+0.8865456739464431458368217306103153593904240326600647453680559093208403164656289274548...`

- (**S.L** 2017, **S.L** 2019)
- Numerical values calculated with 1100 digits of precision
- Semi-analytical expressions fitted with the PSLQ algorithm with very high reliability. Components of analytical expressions known with at least 4800 digits.
- Jump in complexity: analytical fits contain ∼120 terms

$$
\begin{aligned}
C_4 =\, & \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
& + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
& + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
& - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
& + \sqrt{3}\Big[- \frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + 19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
& + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\Big] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
& + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \\
& + \sqrt{3}\Big[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\, f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
& + \pi\Big(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\, f_2(0,0,2) + \frac{5525}{162}\ln 2\, f_2(0,1,1) - \frac{5525}{243}\ln 2\, f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
& - \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\Big)\Big] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\Big(\frac{2541575}{82944}f_1(0,0,2) \\
& - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\Big) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}.
\end{aligned}
$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

analytical fit of A_4

$$A_4 = -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) + \frac{4572662443}{12247200}\ln^2 2\,\zeta(2) - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5)$$

$$+\frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 - \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2$$

$$-\frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7)$$

$$-\frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2$$

$$-\frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 + \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2$$

$$+\frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \sqrt{3}\Bigg[-\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right)$$

$$+\frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi$$

$$+\frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$

$$-\frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2$$

$$+\frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2\Bigg] + \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)$$

$$-507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)$$

$$-\frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)$$

$$-\frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)$$

$$-118\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) + \sqrt{3}\Bigg[\pi\left(+\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right)$$

$$-\frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\, f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1)$$

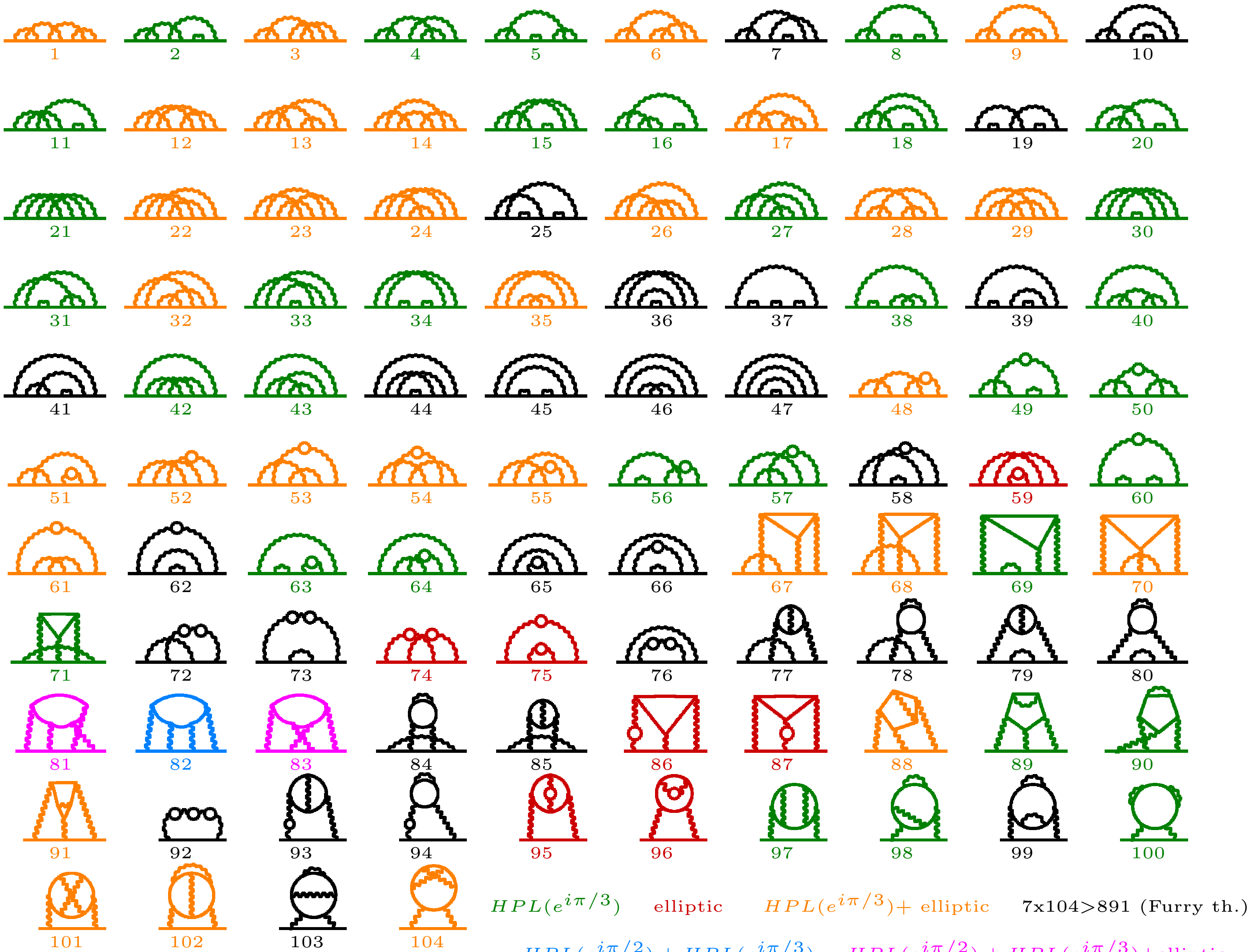
$$+\pi\left(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\, f_2(0,0,2) - \frac{2210}{81}\ln 2\, f_2(0,1,1) + \frac{4420}{243}\ln 2\, f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1)\right.$$

$$\left.+\frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\right)\Bigg] + \zeta(2)\left(-\frac{4629335}{165888}f_1(0,0,2)\right.$$

$$\left.+\frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\right) + \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

A coloured view of the 104 self-mass diagrams



$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

121 terms

$$\begin{aligned}
T =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) \\
&+ \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
&- \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) \\
&- \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 \\
&- \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) \\
&+ \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 \\
&+ \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 \\
&+ \frac{407771}{432}\zeta^2(3)\ln 2 - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 \\
&- \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2
\end{aligned}$$

$a_n = \mathrm{Li_n}(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_a =& -\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&-\frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi \\
&-\frac{223}{243}\zeta(4)\pi\ln 2 + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&-\frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

terms of weight 5 cancel out

$$\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{13487}{60}\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13487}{60}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) \\
&+\frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&-\frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi \\
&+\frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&+\frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+\frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)
\end{aligned}$$

$$\begin{aligned}
W_b =& -\frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\
&-2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg)
\end{aligned}$$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}E_a =&\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\ f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0)\right.\\&-\left.\frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) + \pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\ f_2(0,0,2) + \frac{5525}{162}\ln 2\ f_2(0,1,1)\right.\\&-\frac{5525}{243}\ln 2\ f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) - \frac{3710675}{1119744}f_2(0,3,0)\\&-\left.\frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\right)\end{aligned}$$

$$E_b = -\frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2) - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right).$$

$$f_1(i,j,k) = \int_1^9 ds\ D_1^2\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s) \qquad f_2(i,j,k) = \int_1^9 ds\ D_1(s)\sqrt{3}\mathrm{Re}D_m(s)\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s)$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \qquad D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)$$

$K(x)$ complete elliptic integral of the first kind

These constants have a hypergeometric expression (**S.L.** 2017, Y.Zhou 2018)

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1\text{-}x)}{\sqrt{1-x}} = \frac{\pi}{54}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right) + {}_4\tilde{F}_3\left({\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right)\right] \quad \text{sum}$$

$$B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right) - {}_4\tilde{F}_3\left({\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right)\right] \quad \text{difference}$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) - {}_4\tilde{F}_3\left({-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2} \atop -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}};1\right)\right] \quad \text{difference}$$

$$D_3 = \int_0^1 dx \frac{E_c(x)E_c(1\text{-}x)}{\sqrt{1-x}} = \frac{-\pi}{180}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) + {}_4\tilde{F}_3\left({-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2} \atop -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}};1\right)\right] + \frac{3}{10}A_3 + \frac{2}{9}\pi\sqrt{3} \quad \text{sum}$$

$${}_4\tilde{F}_3\left({a_1\ a_2\ a_3\ a_4 \atop b_1\ b_2\ b_3};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left({a_1\ a_2\ a_3\ a_4 \atop b_1\ b_2\ b_3};x\right) \text{regularized hypergeometric}$$

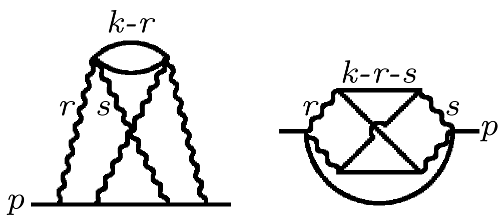
$$K_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left({\frac{1}{3}\ \frac{2}{3} \atop 1};x\right), \qquad E_c(x) = \frac{2\pi}{\sqrt{27}}\,{}_2F_1\left({\frac{1}{3}\ -\frac{1}{3} \atop 1};x\right).$$

A_3 cancels out in the diagram contributions; D_3 do not appear in 4-loop QED g-2 integrals.

$$C_4 = T + \sqrt{3}V_a + V_b + W_b + \sqrt{3}E_a + E_b + U$$

$$U = -\frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}\ .$$

C_{8xy} known only numerically ($\sim$ 4800 digits)



(f, f', f'') (g, g', g'')

C_{8xy} are the ϵ^0 coefficients of the $\epsilon-$expansion of six master integrals f, f', f'', g, g', g'': (f, f', f'') and (g, g', g'') have numerators respectively equal to $(1, p.k, (p.k)^2)$

Key points:

1. generation of the 891 4-loop Feynman diagrams as photon insertions on 104 self-mass diagram (C program)
2. extraction of the contribution to g-2 (or slope); one obtains up to $\sim 10^5$ integrals, with different powers of scalar products in the numerator and of factors in the denominator (FORM program).
3. A *large* system of analytic linear identities ($\sim 10^6$) between integrals is obtained integrating by parts in D dimensions. One gives different weight to the integrals, and the system is solved expressing the "more complicated" integrals in function of the "simplest ones". The integrals not reducible are the Master Integrals. (non canonical) M.I. choice maximizes cancellations of spurious poles for intermediate checks rather than speed. (`SYS`)
4. In order to find the unknown M.I., we need to known the systems of difference/differential equations that they satisfy. These systems are found generating and solving other systems of i.b.p. identities (`SYS`).
5. For topologies with several M.I., a single higher-order equation for the "scalar" M.I. is generated. (`SYS`).
6. High-precision solution of the systems of difference/differential equations (`SYS`)
7. High-precision analytical fit with PSLQ (parallel Fortran program by D.Bailey)

The program SYS

- C program, 30000+ lines.
- The program automatically determines the master integrals of a diagram, it builds and solves the systems of difference or differential equations.
- Input: description of the diagram, number of terms of the expansion in $D-4$.
- The program contains a simplified algebraic manipulator, used to solve systems of identities among integrals with this kind of coefficients: arbitrary precision integers, rationals, ratios of polynomials in one and two variables (for example D and x) with integer coefficients.
- Efficient management of systems of identities of size up to the limit of disk space (tested up to 500 million of identities).
- Numerical solution of systems of difference and differential equations up to 900 equations, using arbitrary precision floating point complex numbers and truncated series in ϵ.
- All the coefficients of the expansions in ϵ are worked out in numerical form, even those of divergent terms.
- Floating number precision: up to 9800 digits (essentially one sums expansions in *one* variable).
- Arithmetic libraries which deal with operations on arbitrary precision integers, polynomials, rationals, arbitrary precision floating point numbers and truncated series in ϵ were written on

purpose by the author. *Independent* of all other available libraries.

- Several Multicore/multinode parallel versions of the program were written on purpose.
- Sistematic protection of large buffers, I/O with crc/checksums. Found several subtle corruptions in the years, like marginal coupling of non-ECC RAM modules (*bit flipping*, 1 bit changed per week), failing RAID systems (corrupted blocks of 64KBytes), etc....)

$$
\begin{aligned}
G_7 = & -2342.20751410602307542352254059079270988532873205655947080 7\\
& 35948148357138469168064559169731859926148319489041973435698 6\\
& 64053648283918092773759937630697973782911060831170767176793 5\\
& 98313912596076691832992388387193058486849651607286872924318 3\\
& 31780051969475993991475176114128343581003079113683879370807 1\\
& 15734609978702030235752685241209543628733284644892624243050 3\\
& 23644954747440730758129112363792107858641867651754987797286 7\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
& = \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)\\
& - \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2\\
& + 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)
\end{aligned}
$$

black: ansatz (the input)

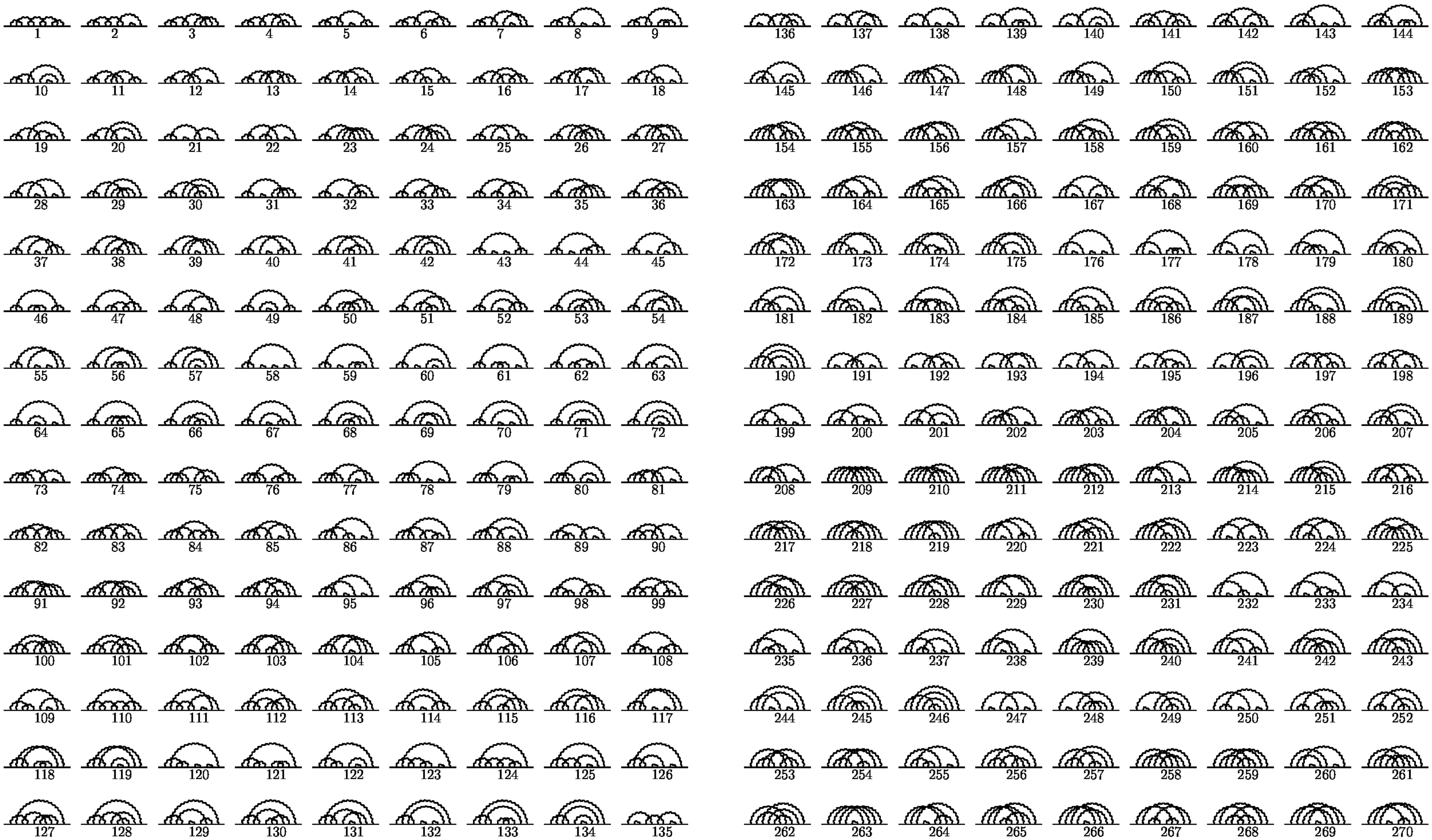
brown: coefficients found by PSLQ (the output)

This particular fit can be found using input data with a minimum precision of 415 digits ($> 7 \times 54$).

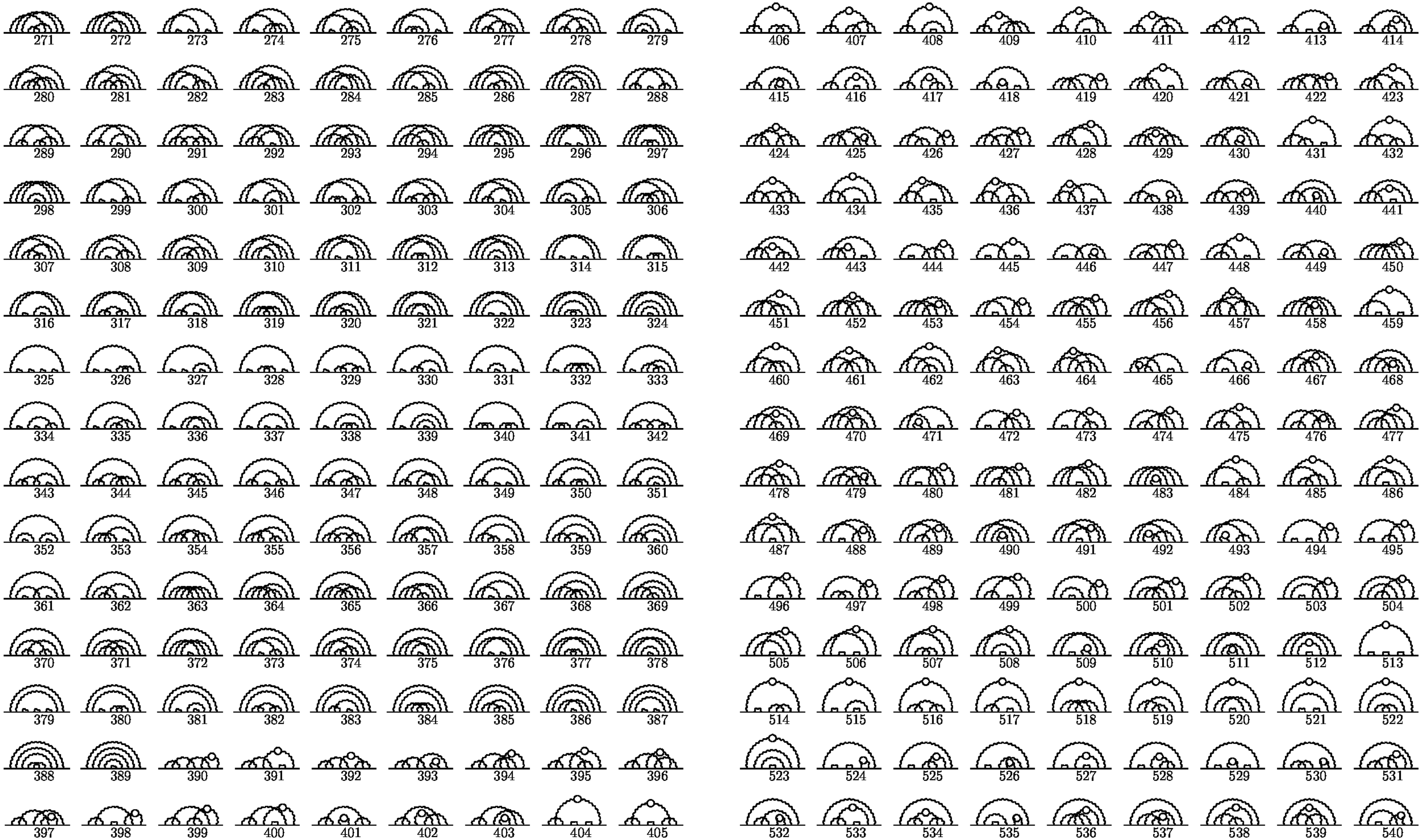
- system of i.b.p. identities solved for finding system of difference/differential equations satisfied by the M.I. item exact arithmetic with polynomials in 2 variables
- slower arithmetic, it requires more cpu
- “anti”-canonical choice of master integrals for difference/differential equations: it maximizes the cancellation of spurious poles in ϵ for the sake of internal checks. In the worst case spurious poles up ϵ^{-37} were generated. Drawback: (much) slower numerical solution. At 5 loop it could be not necessary.
- it needs fast cpu

- with ~ 120 analytical terms and ~ 10 digits in the numerical coefficients, minimum number of numerical precision is $\sim 10 * 120$; some analytical constants appear only in particular combinations. Using the combinations where possible, 1200 digits did suffice.
- Used a modified parallel version of PSLQ (Bailey,Broadhurst); tested up to ~ 9800 digits.

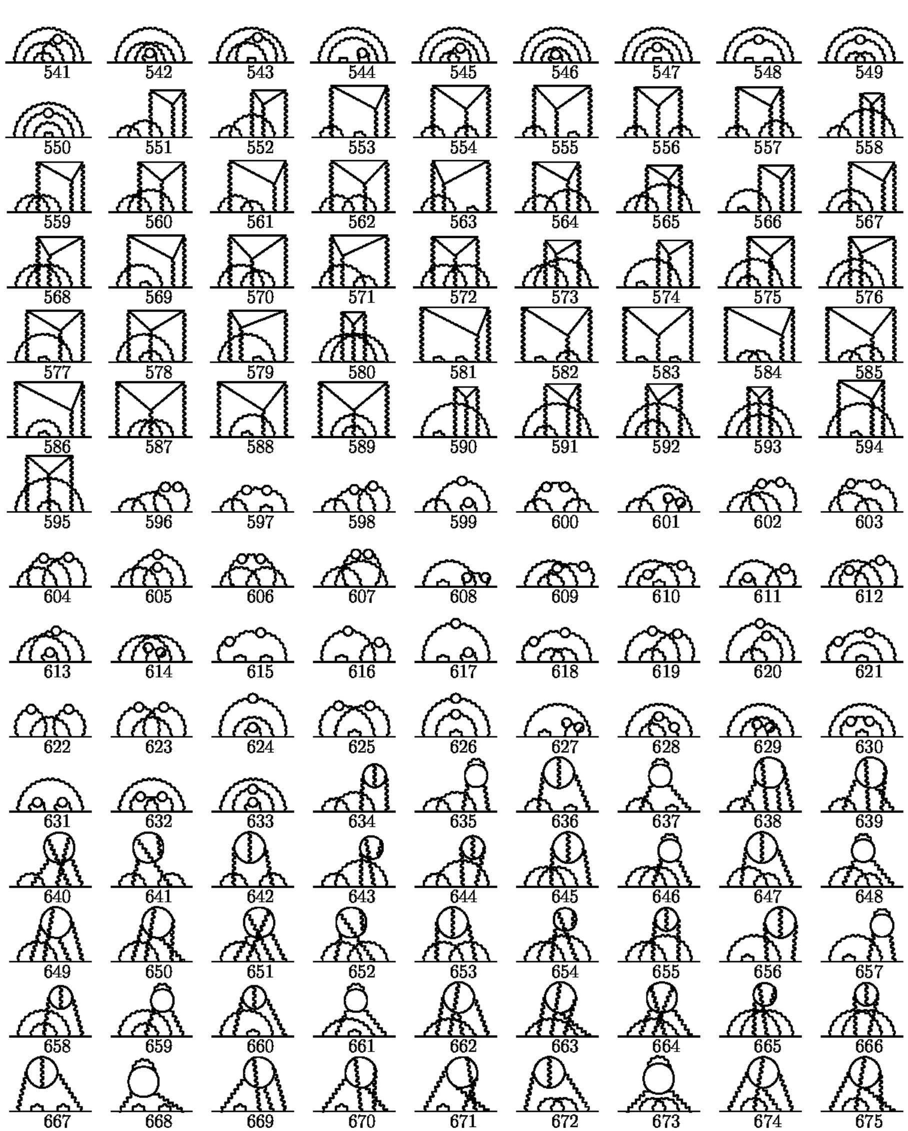
12672 Feynman diagrams generated by external photon insertions in 954 self-mass diagrams

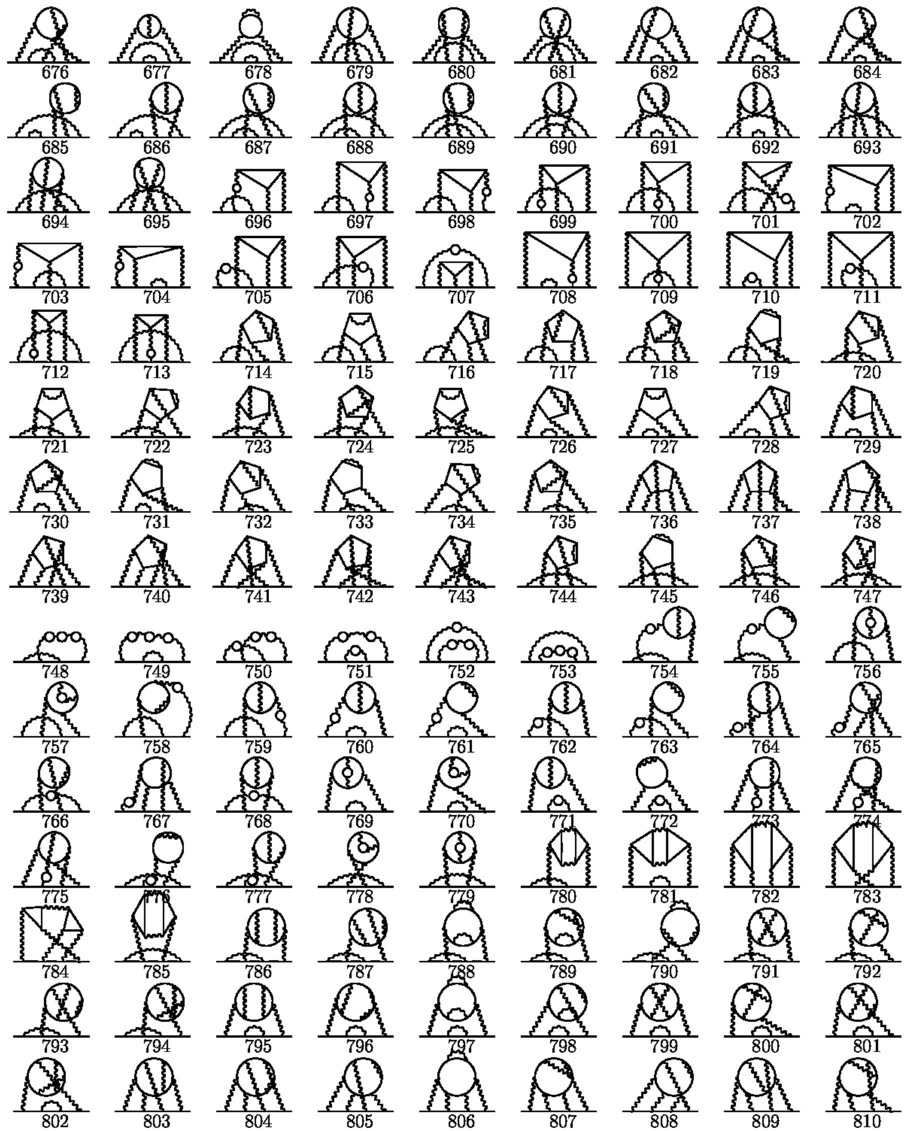


5 loops: the 954 self-masses

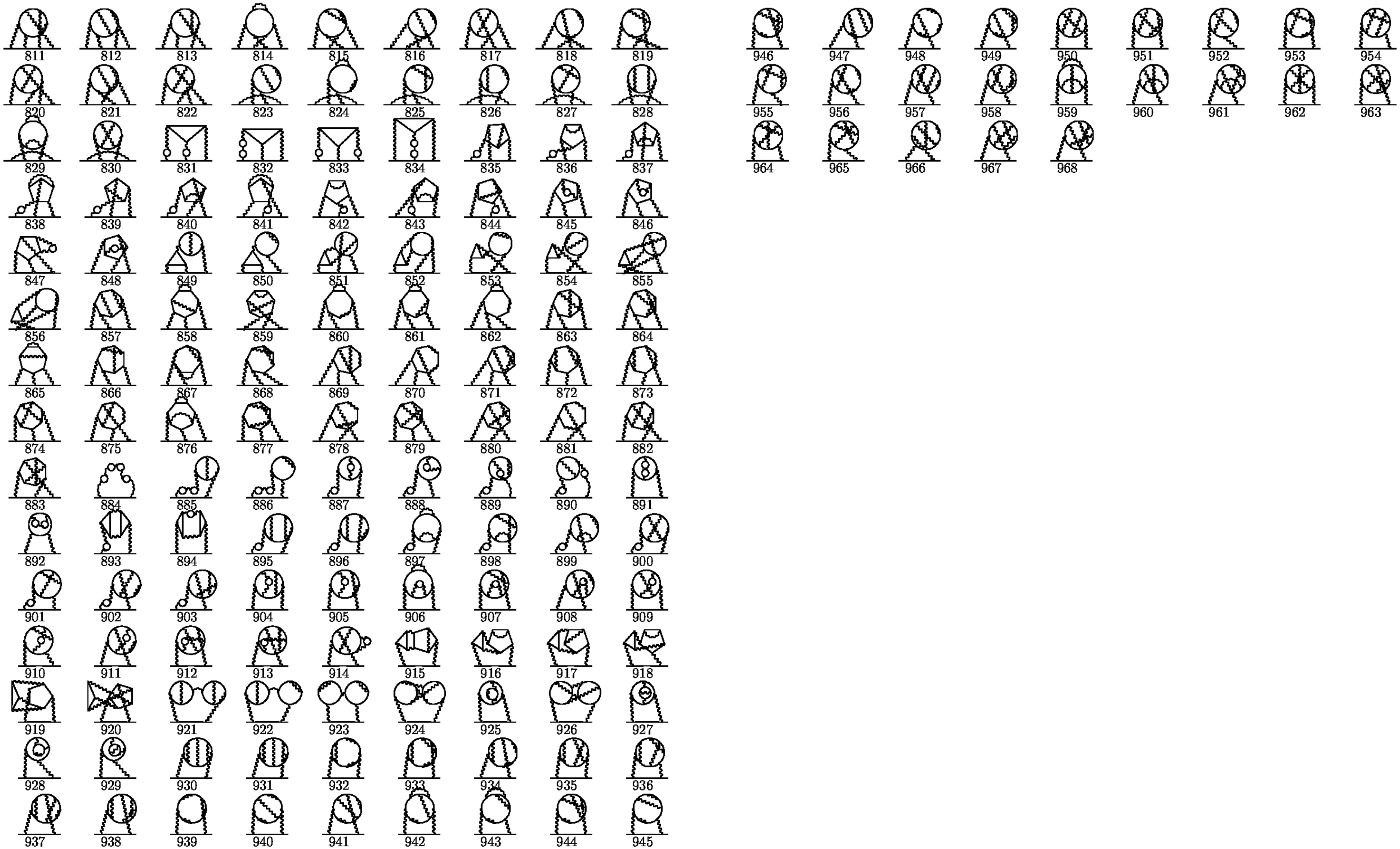


5 loops: the 954 self-masses





5 loops: the 954 self-masses



We include also the 14 non-contributing diagrams containing 2 electron loops with 3 or 5 vertices (Furry's theorem)

Initial 5σ discrepancy on the set "V" (diagrams without electron loops)

$$C_5(\text{Set V})\ \text{Volkov2024} = 6.828(60)$$

$$C_5(\text{Set V})\ \text{AHKN2019} = 7.604(140) \qquad \leftarrow \text{old}$$

$$C_5(\text{Set V})\ \text{AHKN2024} = 6.800(128) \qquad \leftarrow \text{new, higher statistics}$$

Discrepancy *solved.*

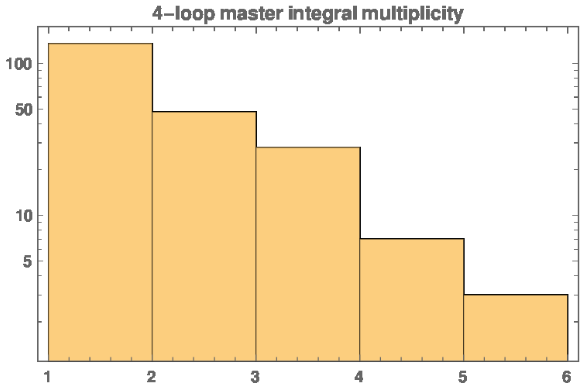
adding the contribution of diagrams with fermion loops, the total 5-loop contribution becomes

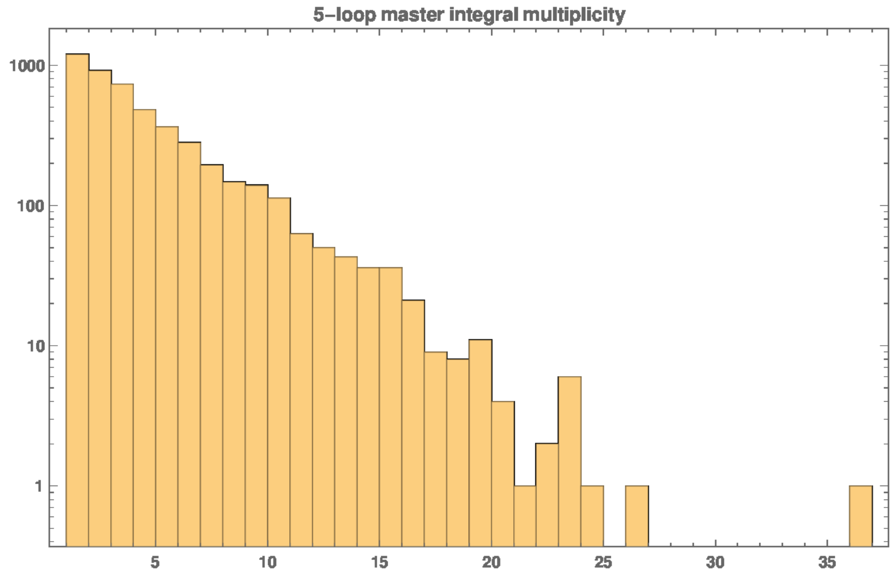
$$C_5(\text{all})\ \text{Volkov2024} = 5.891(61)$$

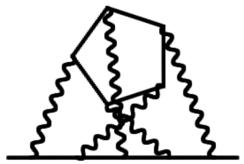
$$C_5(\text{all})\ \text{AHKN2024} = 5.870(128)$$

loop	# i.s.p.	# vertex	# self-mass	estimate # M.I.	actual # M.I.
1	0	1	1	1	1
2	0	7	3	3	3
3	1	72	15	18	17
4	3	891	104	358	325
5	6	12672	954	19670	?

The estimate of the number of M.I. has been done solving (limited) systems of i.b.p. identities on the maximal cut. It may slightly overestimates the real number. Number of master integrals at 5 loops is greater than expected; it is due to the high number of i.s.p. Worst example: the maximal cut of this diagram seems to have 36 M.I.







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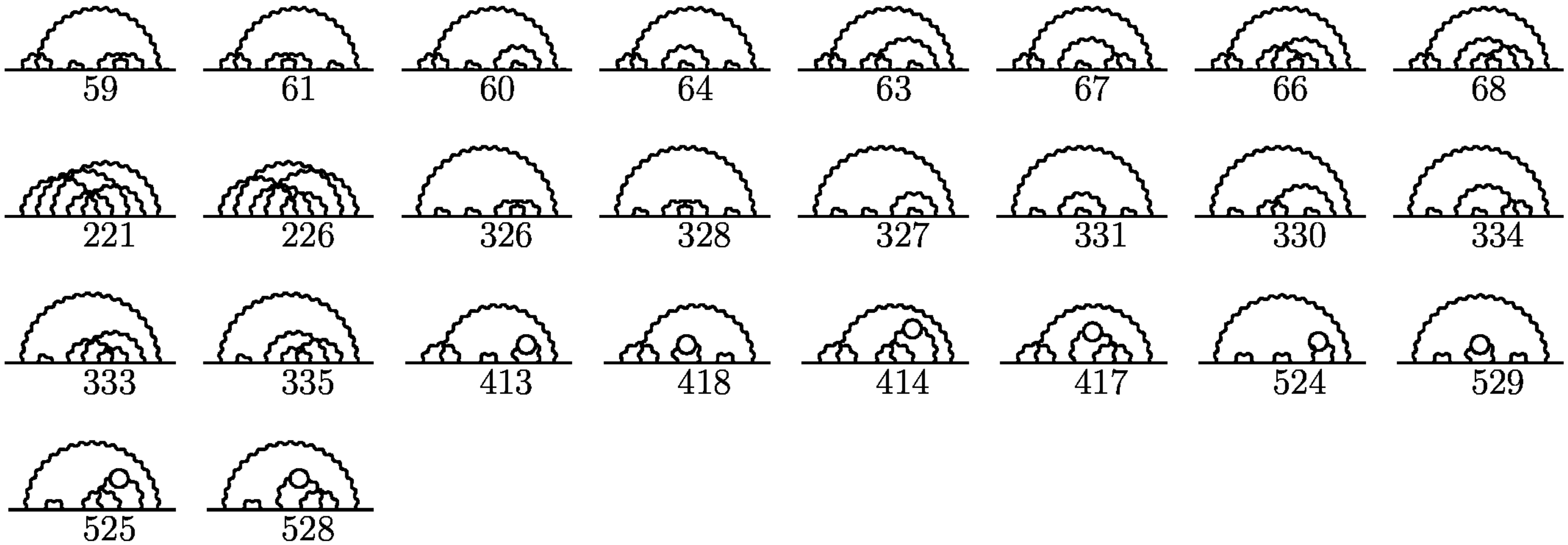
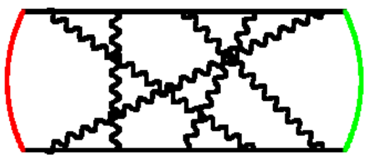
New at 5 loops:



The above pairs of self-mass diagrams have *all* the same master integrals, but the photon insertions generate different vertices which are not mirror images.

The pair 221-226 is a bit special  closing on the left-hand side (red), one gets 221; closing on the other side gives the self-mass 226. The main master integral is the same for both diagrams. There are some differences in the master integrals of the subdiagrams.

- systems of i.b.p. identities for reduction to master integrals
- systems of $\sim 10^9$ i.b.p. at 5 loops ($1000\times$ (4-loop)).
- already solved ~ 30 *small* 5-loop systems of 5×10^8 identities
- larger systems will need several TB memory and tens of TB of *fast* disk space

- numerical solution of systems of difference/differential equations
- "anti"-canonical choice of master integrals: (much) slower numerical solution. Not strictly necessary, at 5 loops it could be avoided.
- The high number of M.I. (> 10)for some topologies makes difficult to generate a single high-order equation for the "scalar" M.I, due to blow-up of the expression; numerical solution of the system of coupled first-order difference/differential equation is needed.
- 5-loop renormalization requires (re)calculations of some/all 4-loop M.I. with longer expansions in ϵ: currently work in progress, 71% done, enough terms for 5,6,7 loop, with a slightly higher precision of 1600 digits)

- PSLQ analytical fit
- at 5 loops, my guess is $\sim 5 \times 10^3$ terms and ~ 20 digits coefficient, so a numerical precision of $\sim 10^5$ digits could be necessary ($100\times$ (4-loop)).
- Unfortunately, runtime goes as the power $\sim 3-4$ of the number of digits so 10^5 digits seem to be unreachable.
- A moderate precision of a few hundreds of digits seem to be more realistic (but no analytical fit!)

Conclusions

- 1100-digits value of 4-loop C_4 and A_4 coefficients (1600 digits in progress) allows a successful analytical fit with (relatively) small coefficients
- the ability to fit analytical expression to the numerical value guarantees that all digits computed are correct
- the remaining QED error comes from the 5-loop coefficients
- 12672 diagrams obtained from 954 self-mass diagrams.
- $\sim$ 19500 master integrals.
- Extension to 5-loop of the 4-loop approach seems to be possible
- 10^5 digits needed for PSLQ analytical fit; not absolutely impossible, but extremely difficult. Hundreds of digits more realistic.
- Surely a gargantuan task, but with enough computer power…

The End