Gradient Flow:
perturbative perspective

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Perturbative approach

$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B \qquad\qquad \mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right) \quad \Rightarrow \quad \partial_t B_\mu = \mathscr{D}_\nu G_{\nu\mu}$$

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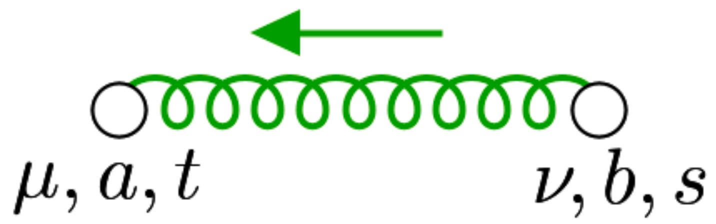
$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T\, B_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

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Perturbative approach

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B \qquad \mathscr{L}_B \sim \int_0^\infty dt\, {\color{green}L_\mu} \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right) \quad \Rightarrow \quad \partial_t B_\mu = \mathscr{D}_\nu G_{\nu\mu}$$

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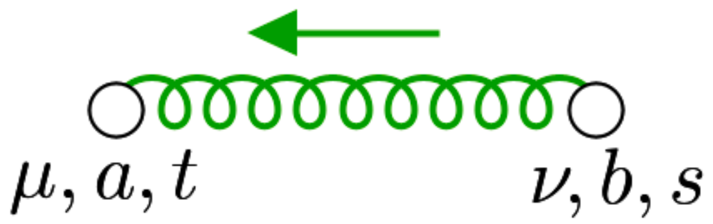

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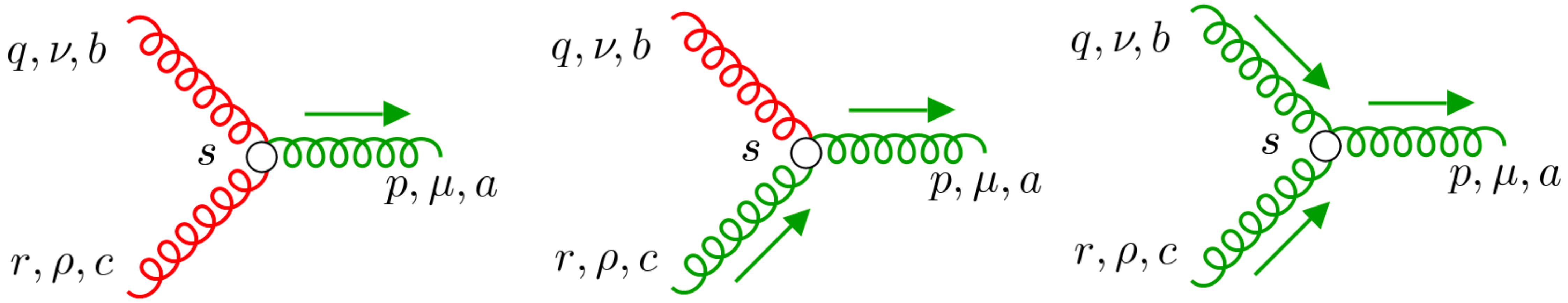
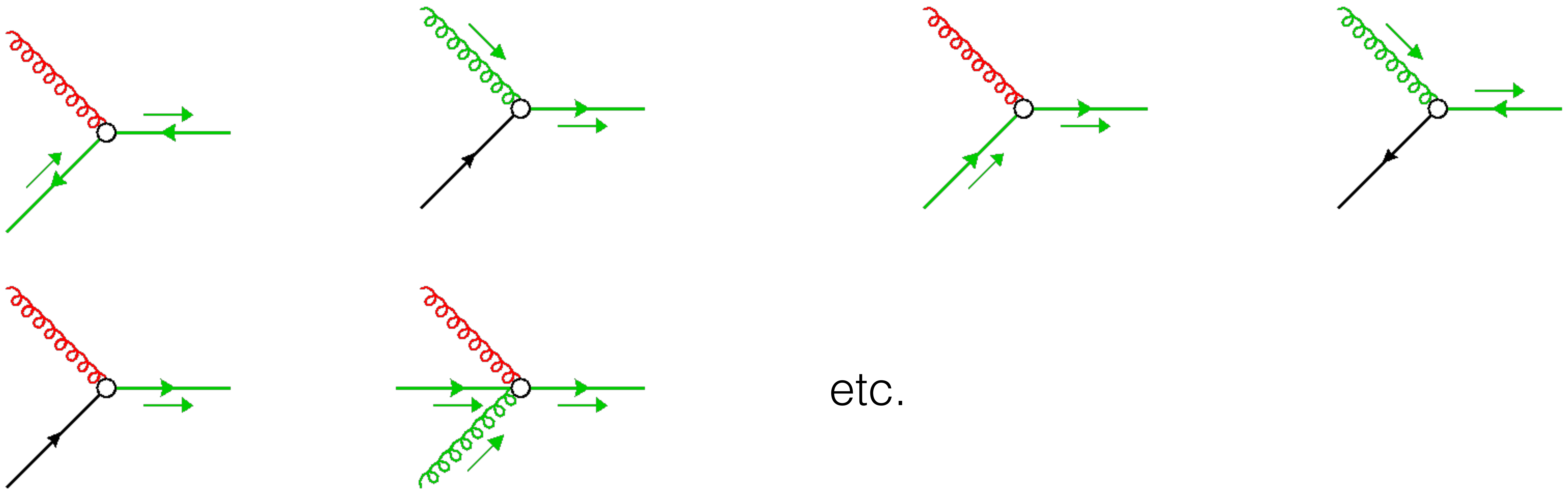


$$-\mathrm{i}g f^{abc} \int_0^\infty \mathrm{d}s\, \big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa - 1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

Perturbative approach

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \mathscr{D}^2\right) \chi + \mathrm{h.c.}$$



etc.

Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t) \rangle$$

$$G_{\mu\nu}(t) = \partial_\mu B_\nu(t) - \partial_\nu B_\mu(t) + [B_\mu(t), B_\nu(t)]$$

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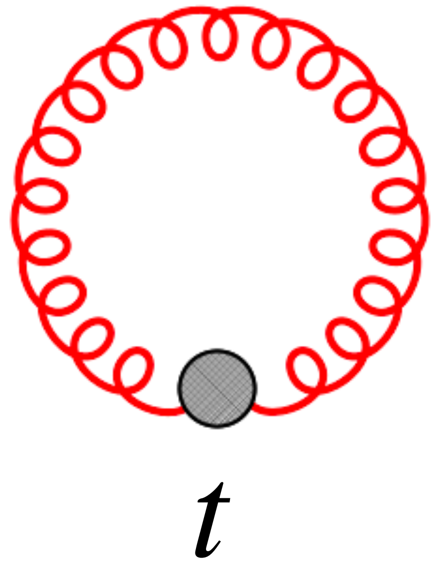
LO: 

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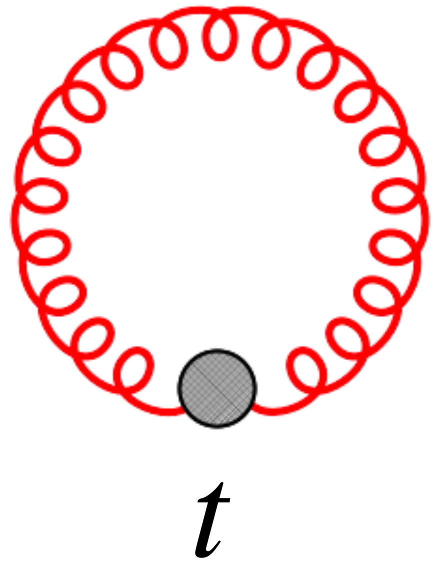
LO:  $\sim \int d^D p \, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

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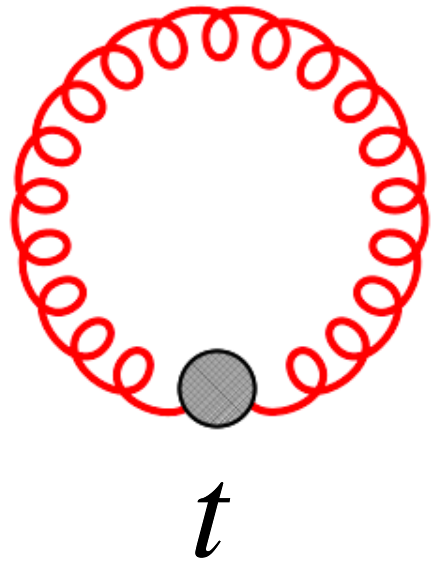
LO:  $\sim \int d^Dp\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

explicitly: $$E(t) = \frac{3\alpha_s}{4\pi t^2} \quad + \quad \mathcal{O}(\alpha_s^2)$$

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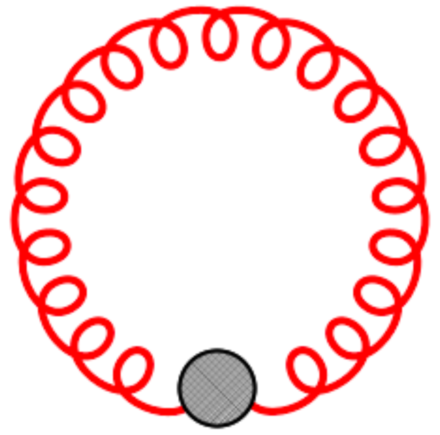
LO:  $\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$

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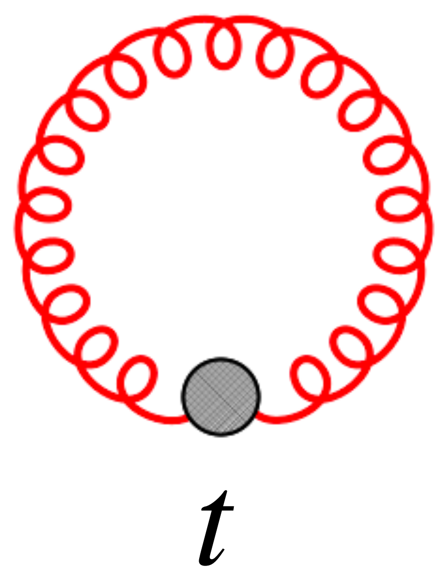
LO: 

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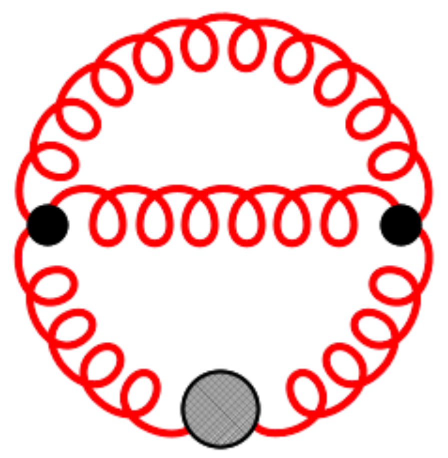
→ measure α_s on the lattice?

$$\alpha_s = \alpha_s(\mu)$$

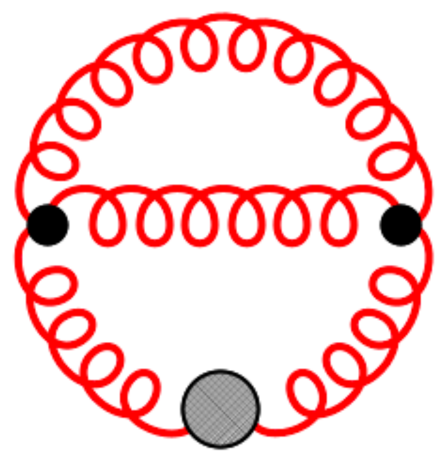
Higher orders



t

$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders



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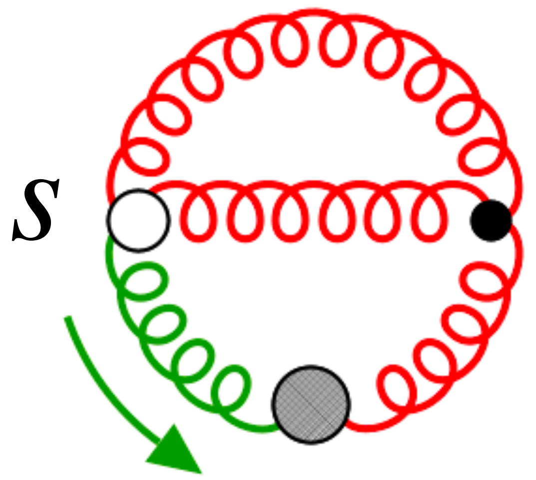
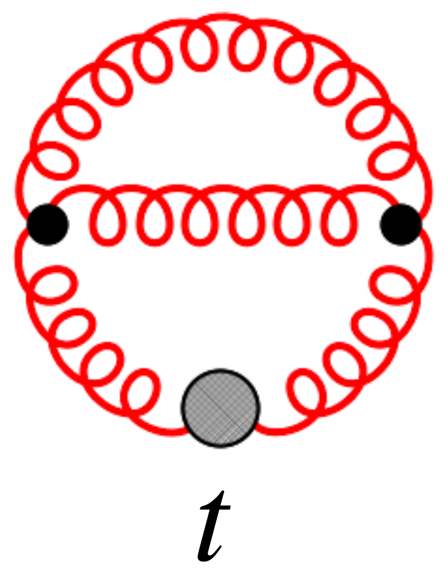
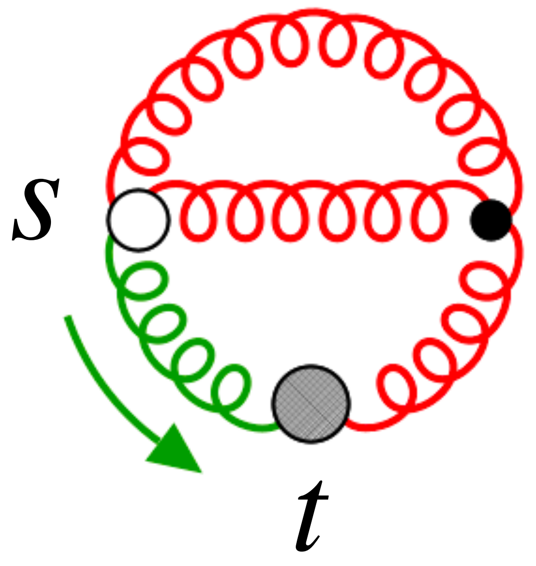


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\,k^2\,(p-k)^2}$$

Higher orders





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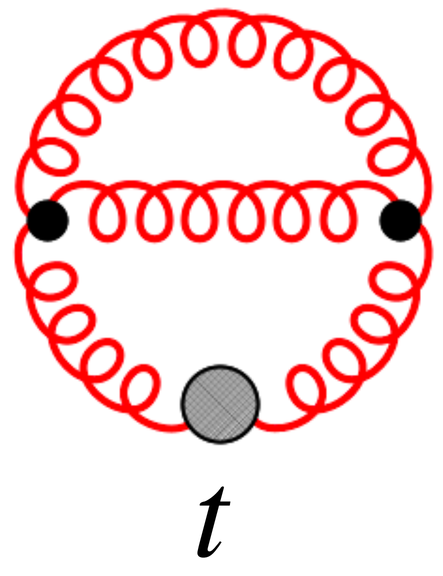
- generalized loop integrals

Higher orders



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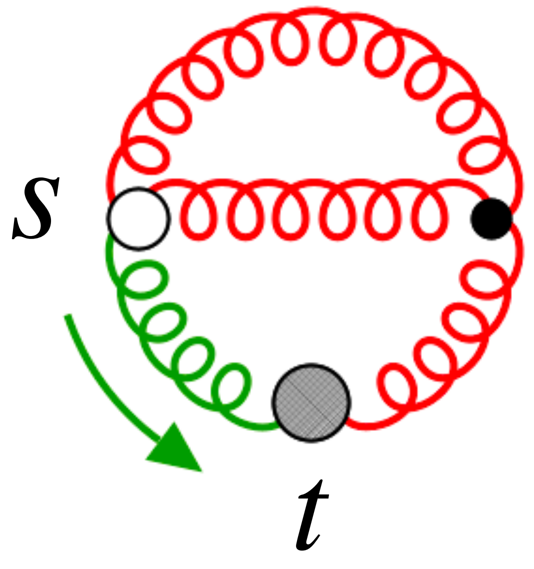
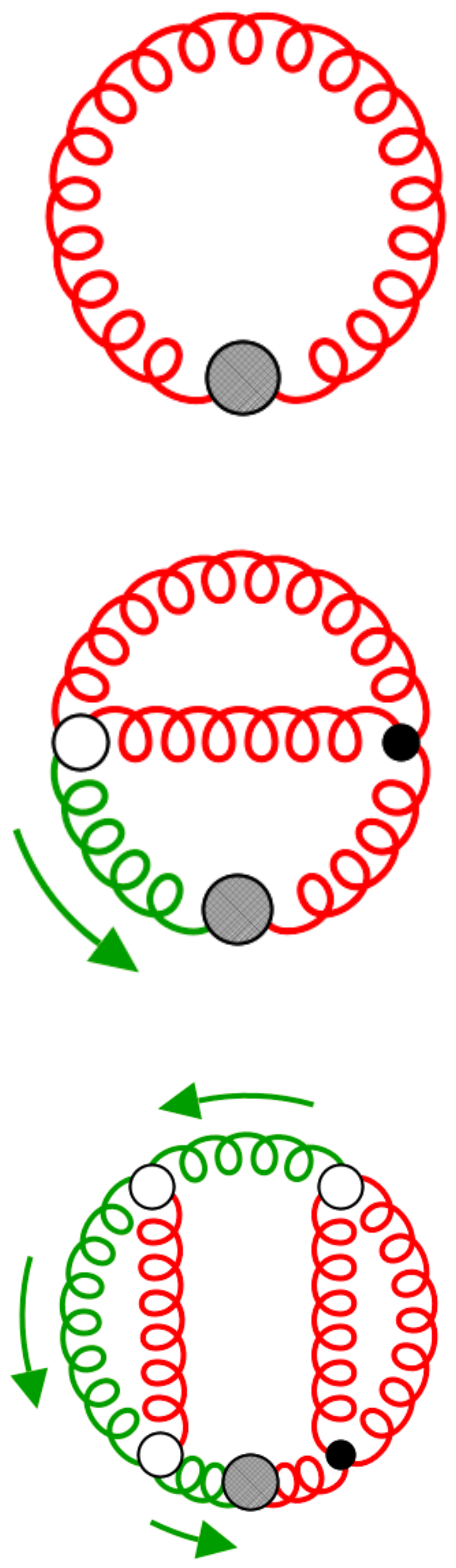


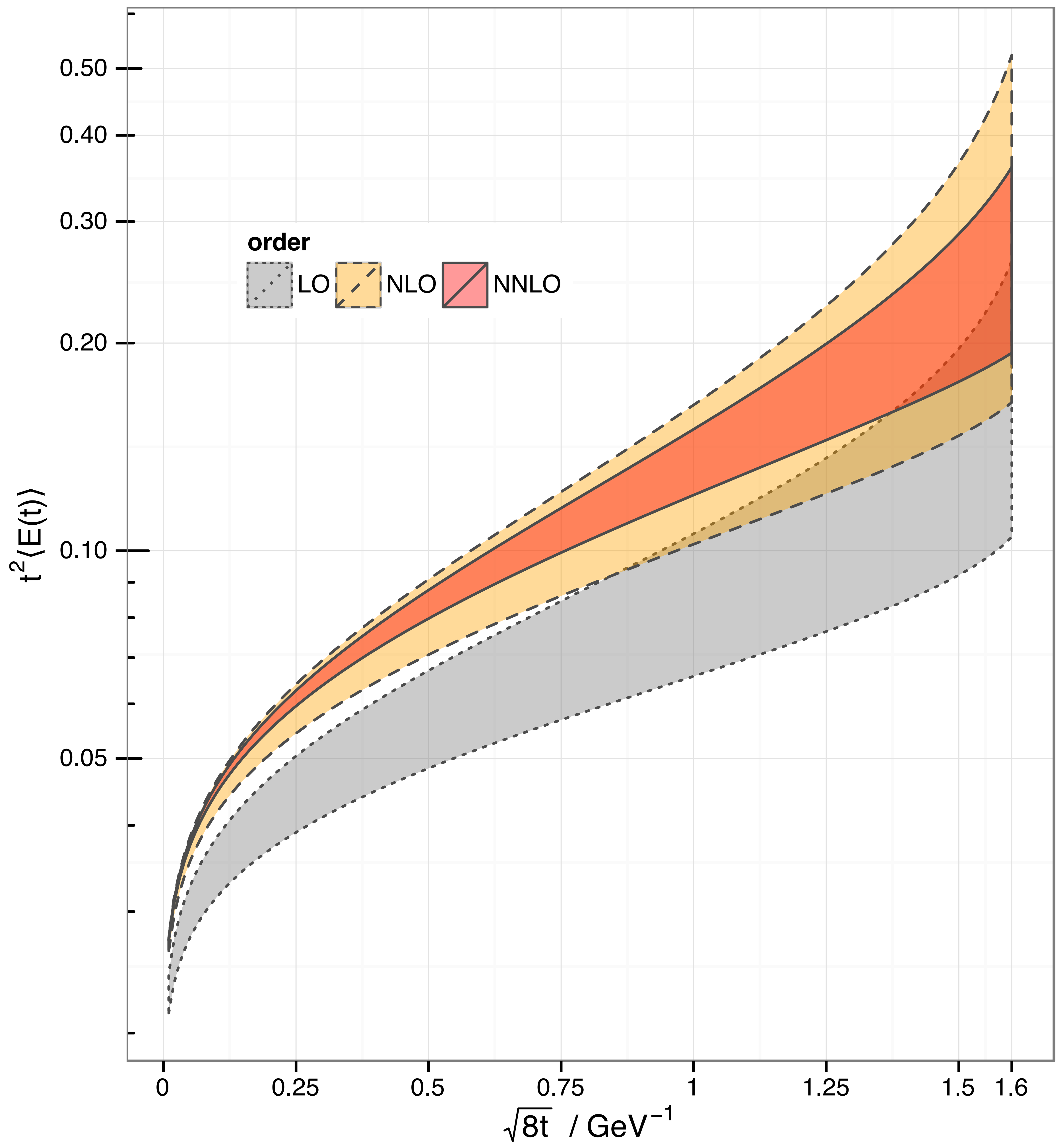
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

Lüscher 2010
RH, Neumann 2016





resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

Scale Setting and Strong Coupling Determination in the Gradient Flow Scheme for 2+1 Flavor Lattice QCD

Rasmus Larsen,[1] Swagato Mukherjee,[2] Peter Petreczky,[2] Hai-Tao Shu,[3,*] and Johannes Heinrich Weber[4,5]

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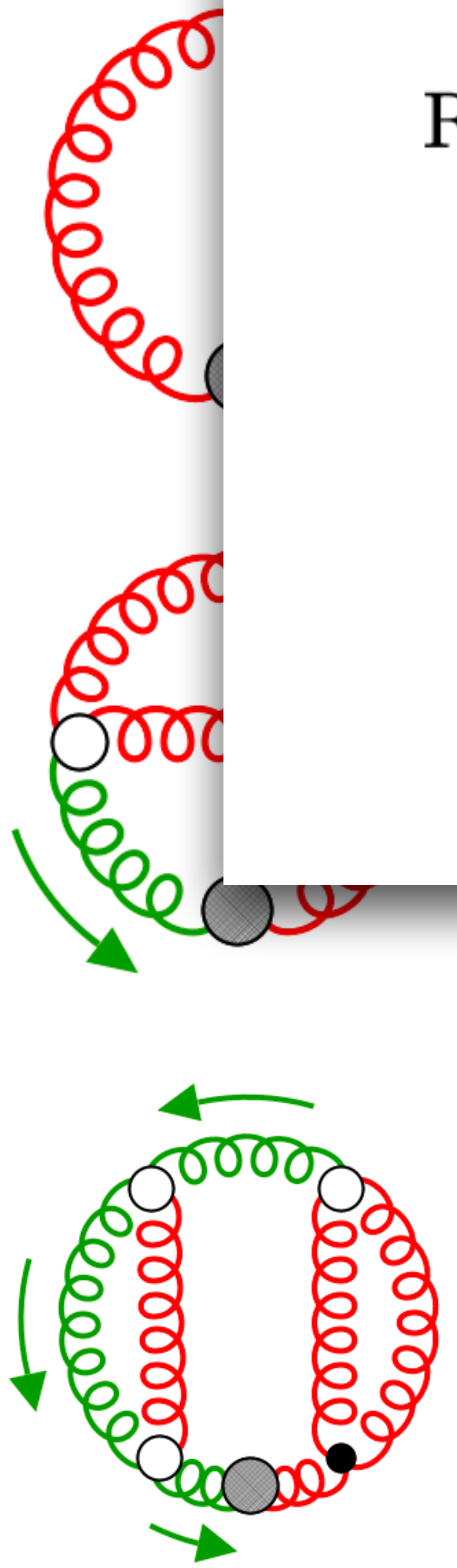
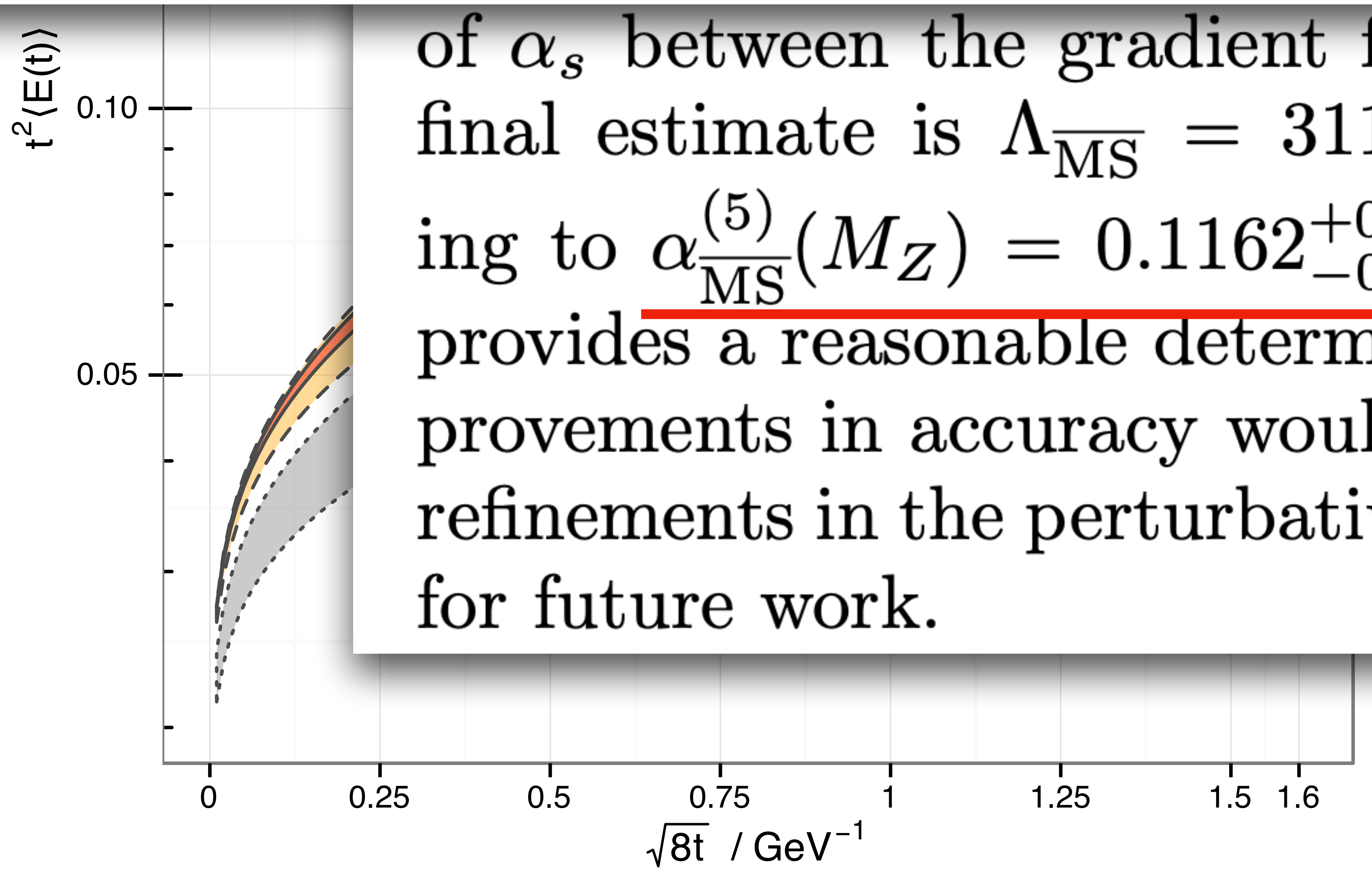
[2] *Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA*
[3] *Key Laboratory of Quark & Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan 430079, China*
[4] *Institut für Physik & IRIS Adlershof, Humboldt-Universität zu Berlin, D-12489 Berlin, Germany*
[5] *Institut für Kernphysik, Technische Universität Darmstadt, Schlossgartenstraße 2, D-64289 Darmstadt, Germany*

(Dated: February 13, 2025)





of α_s between the gradient flow and $\overline{\mathrm{MS}}$ schemes. Our final estimate is $\Lambda_{\overline{\mathrm{MS}}} = 311.0^{+0.7+4.8+34.0}_{-0.7-4.8-11.7}$ MeV, leading to $\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1162^{+0.0023}_{-0.0009}$. While this approach provides a reasonable determination of $\Lambda_{\overline{\mathrm{MS}}}$, further improvements in accuracy would benefit from higher-order refinements in the perturbative matching, which we leave for future work.

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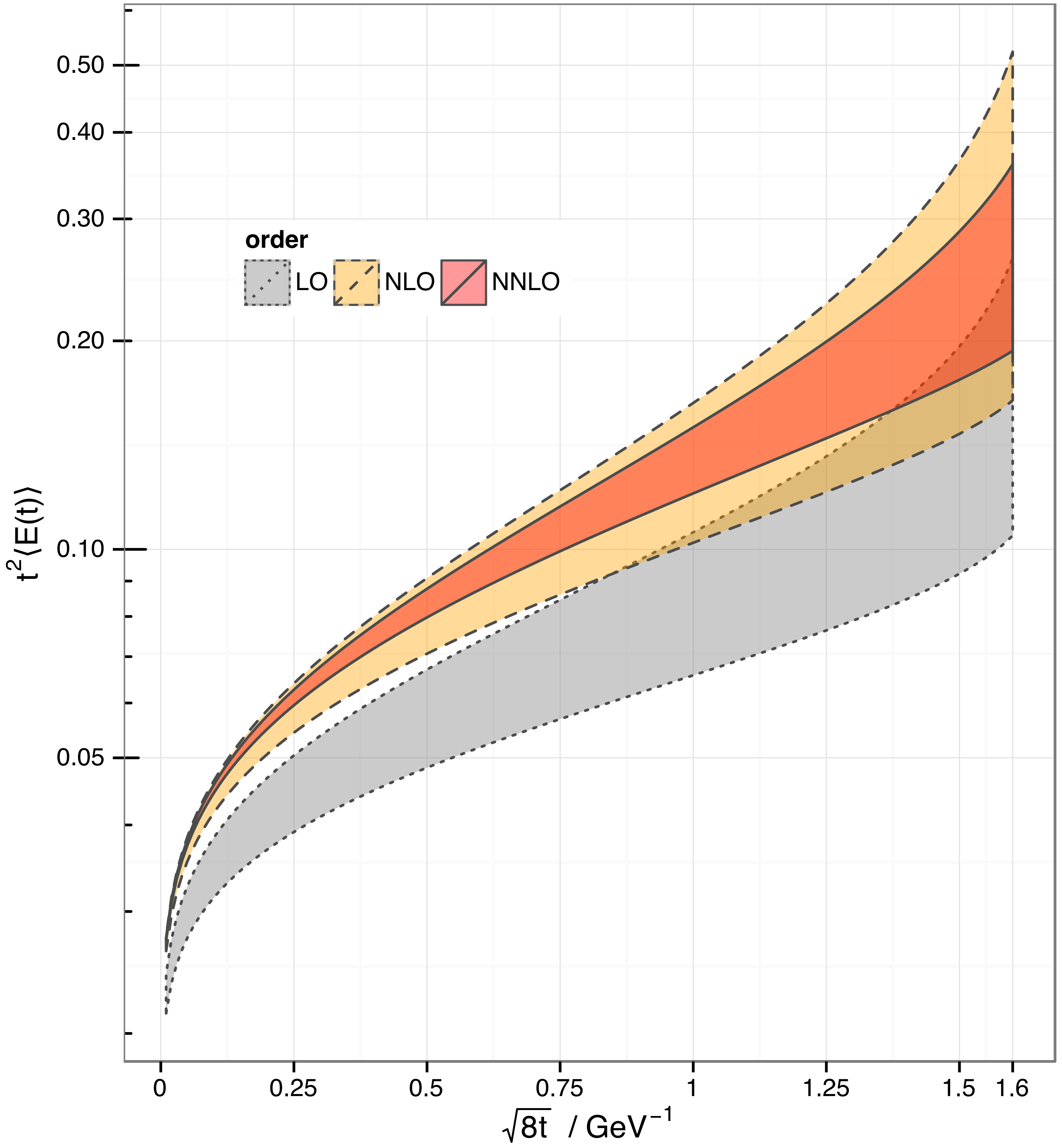

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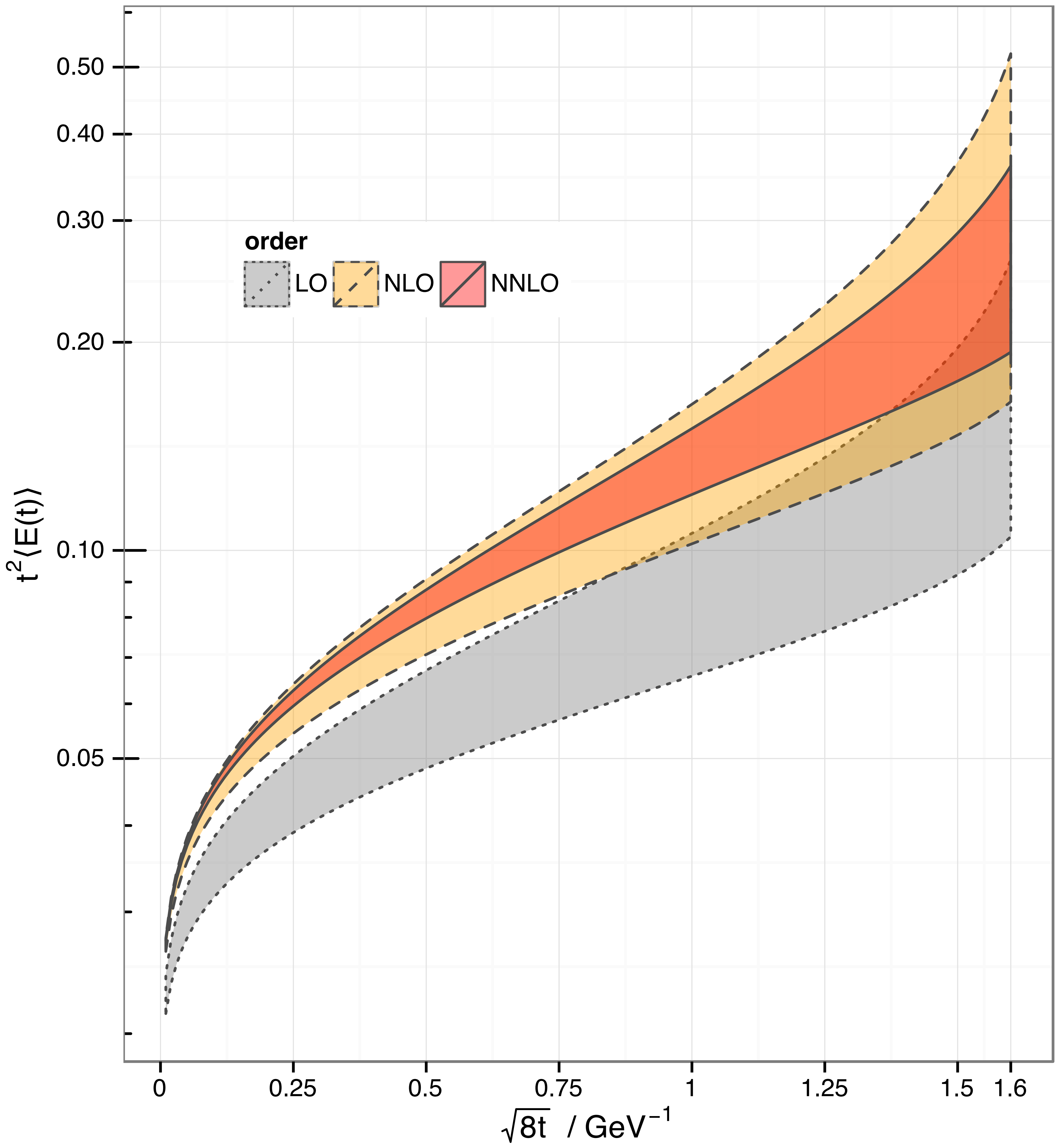

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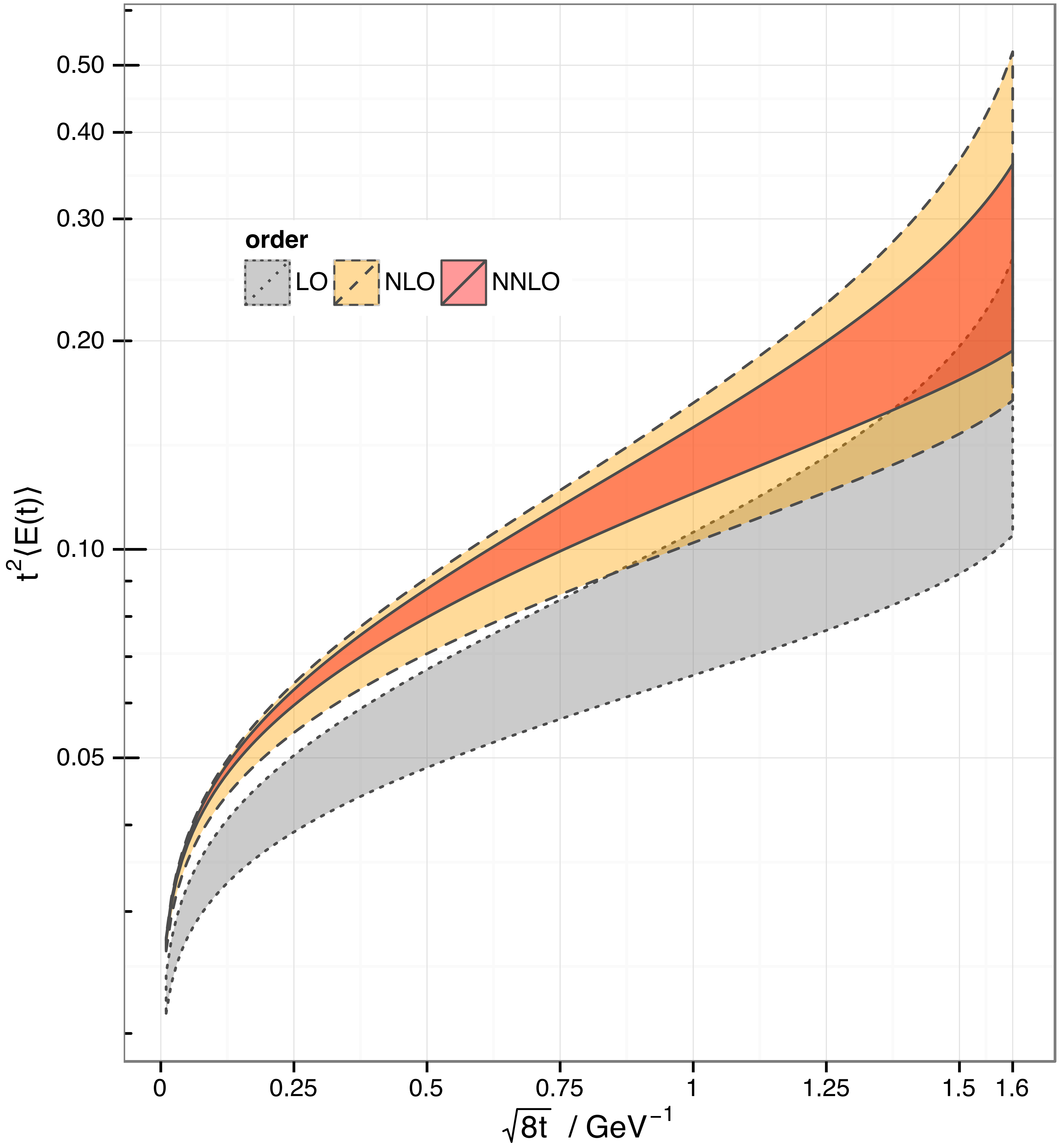

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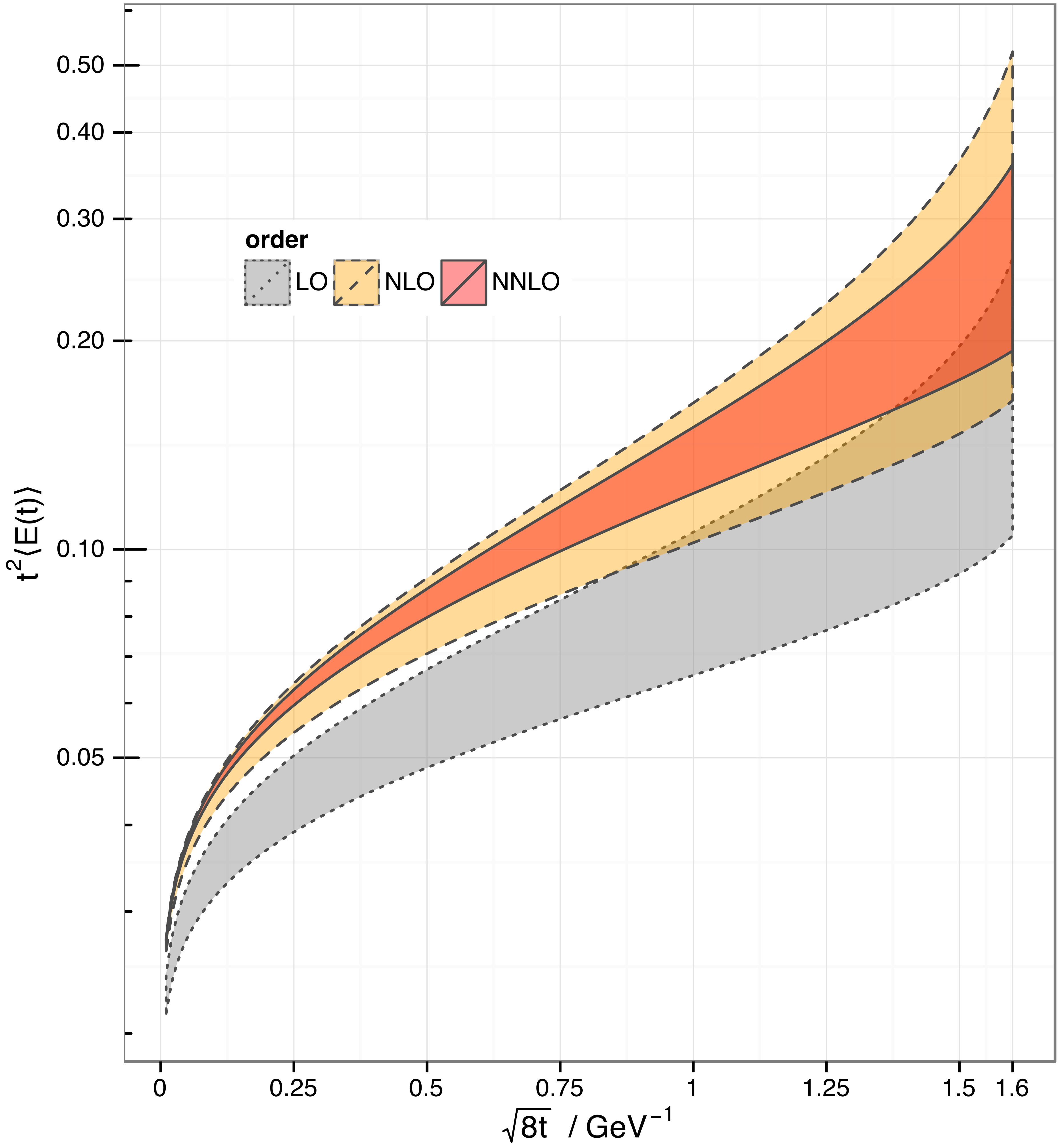

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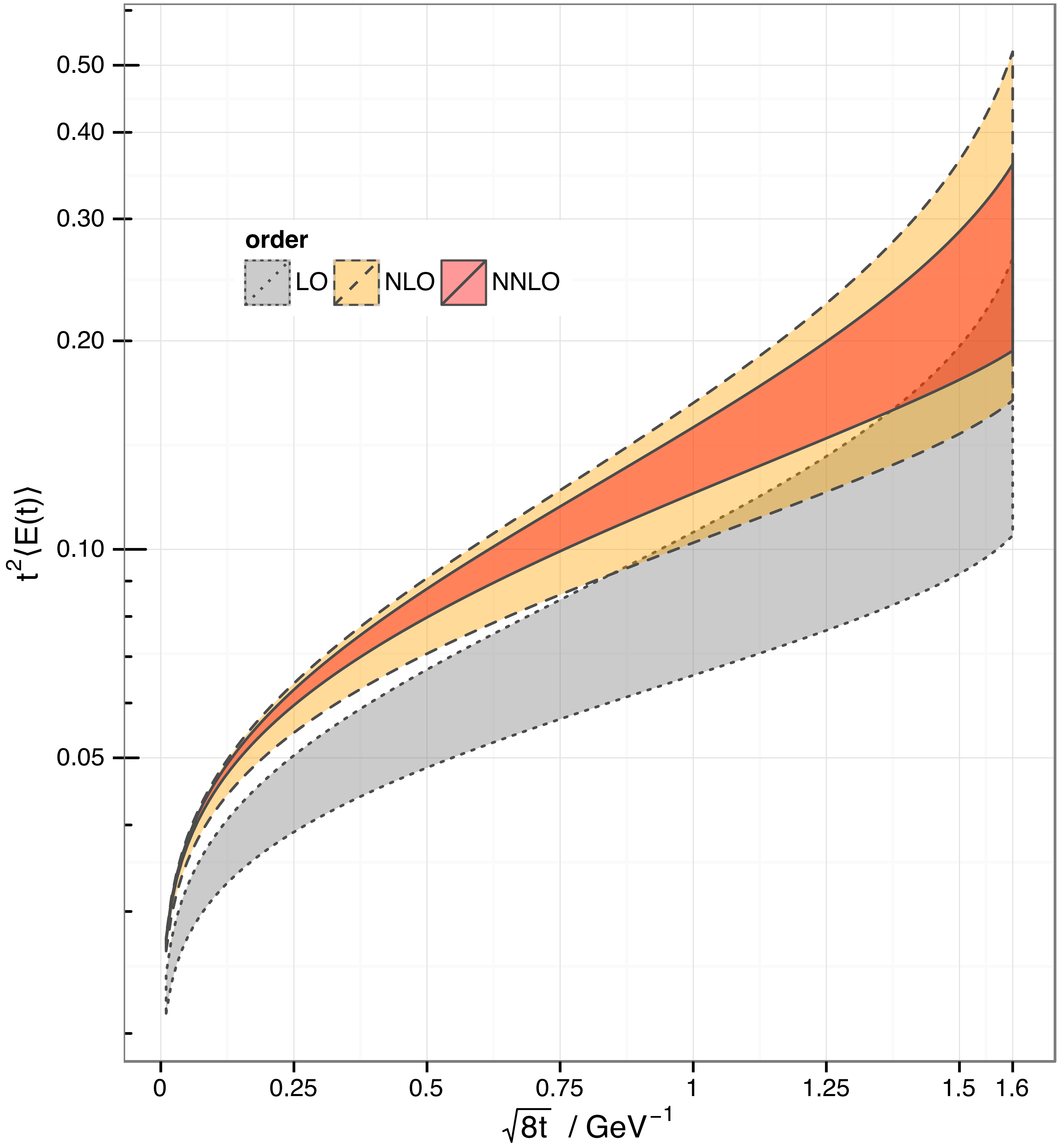

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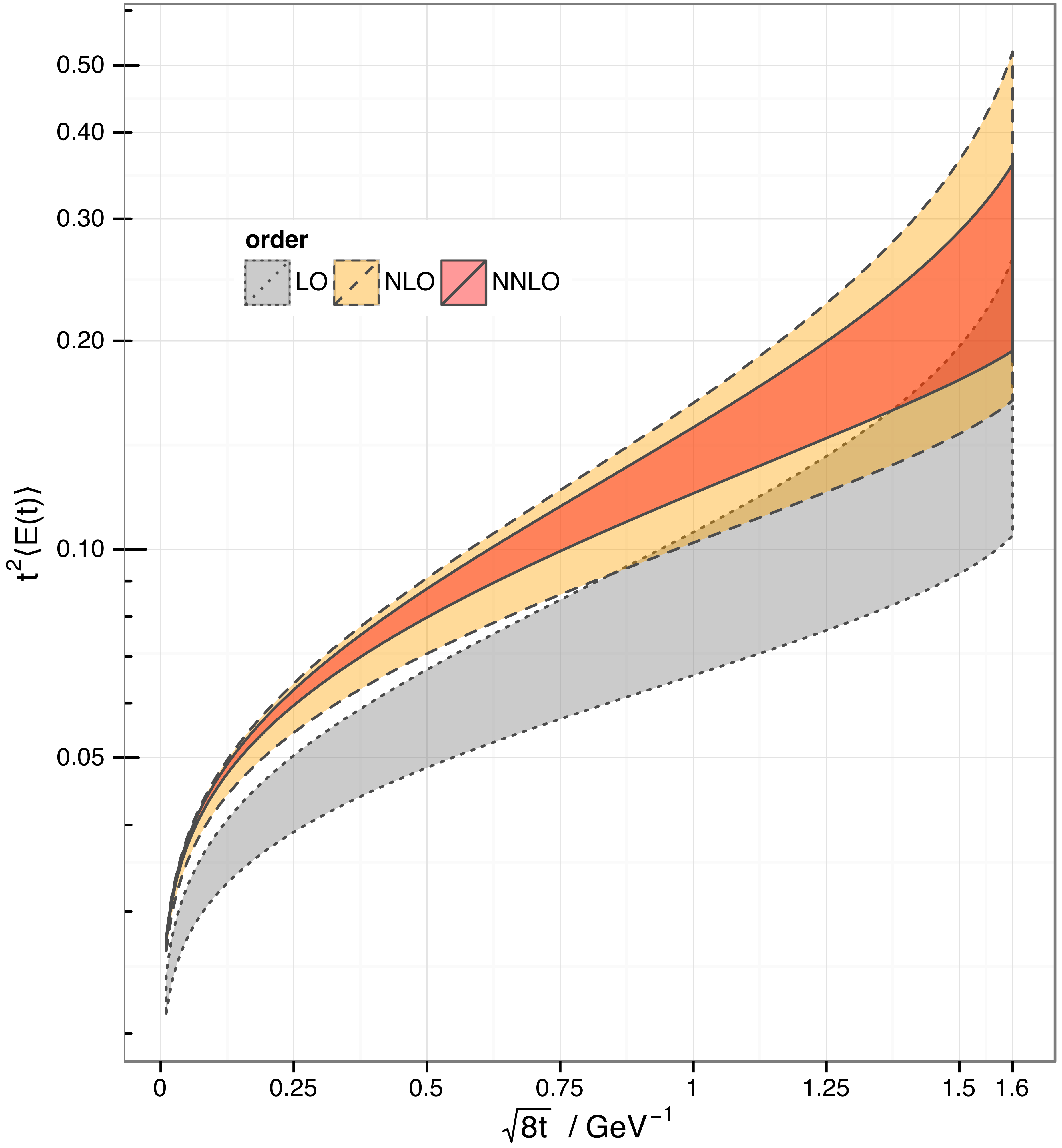

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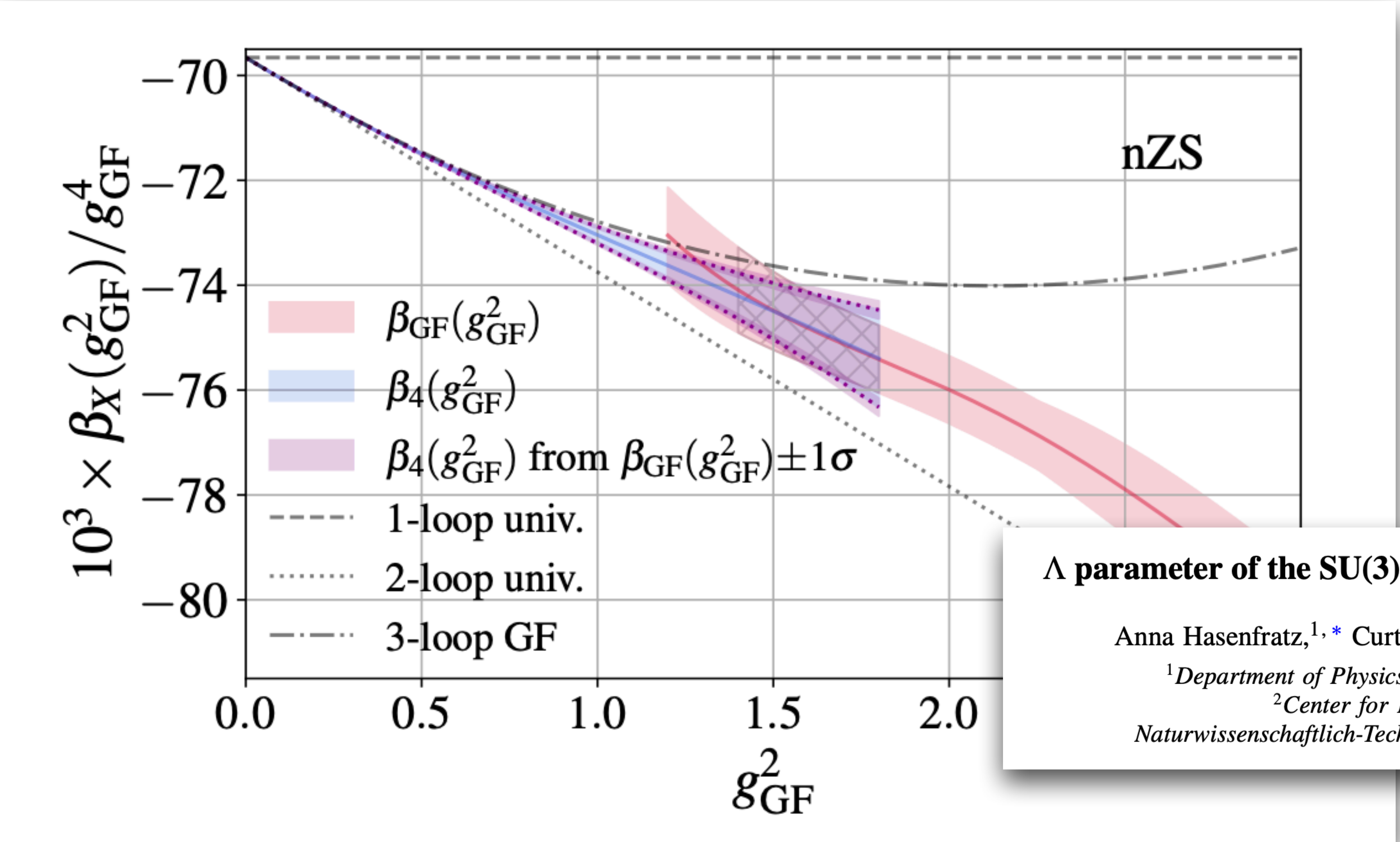

Λ parameter of the SU(3) Yang-Mills theory from the continuous β function

Anna Hasenfratz,[1,*] Curtis T. Peterson,[1,†] Jake van Sickle,[1] and Oliver Witzel[2,‡]

[1] *Department of Physics, University of Colorado, Boulder, Colorado 80309, USA*
[2] *Center for Particle Physics Siegen, Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät, Universität Siegen, 57068 Siegen, Germany*

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

match renormalization schemes?

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gradient flow renormalization

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gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim\limits_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

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→ need $\tilde{C}(t)$

Small flow-time expansion

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small flow-time expansion:

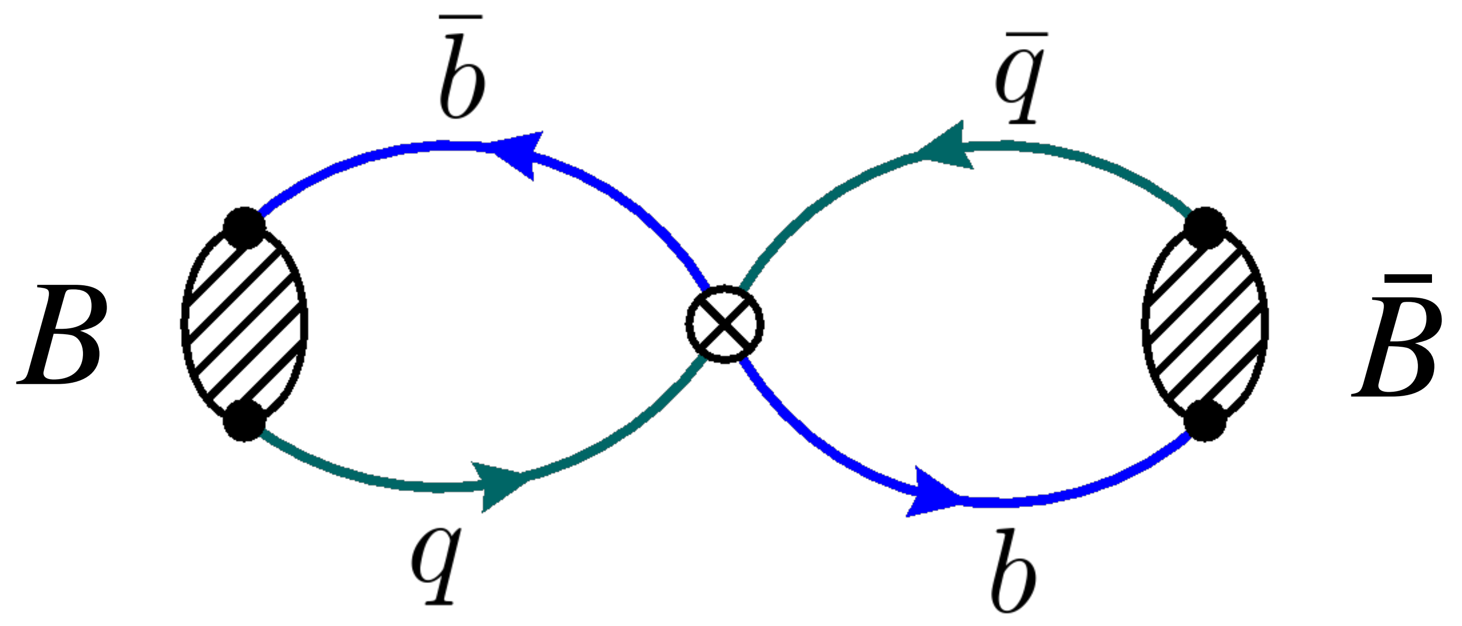
Lüscher, Weisz '11
Suzuki '13
Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

$$\tilde{C}_n(t) \overset{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

$\Rightarrow$ need $\zeta_{nm}(t)$ for small t $\Rightarrow$ perturbation theory

Example: Meson mixing



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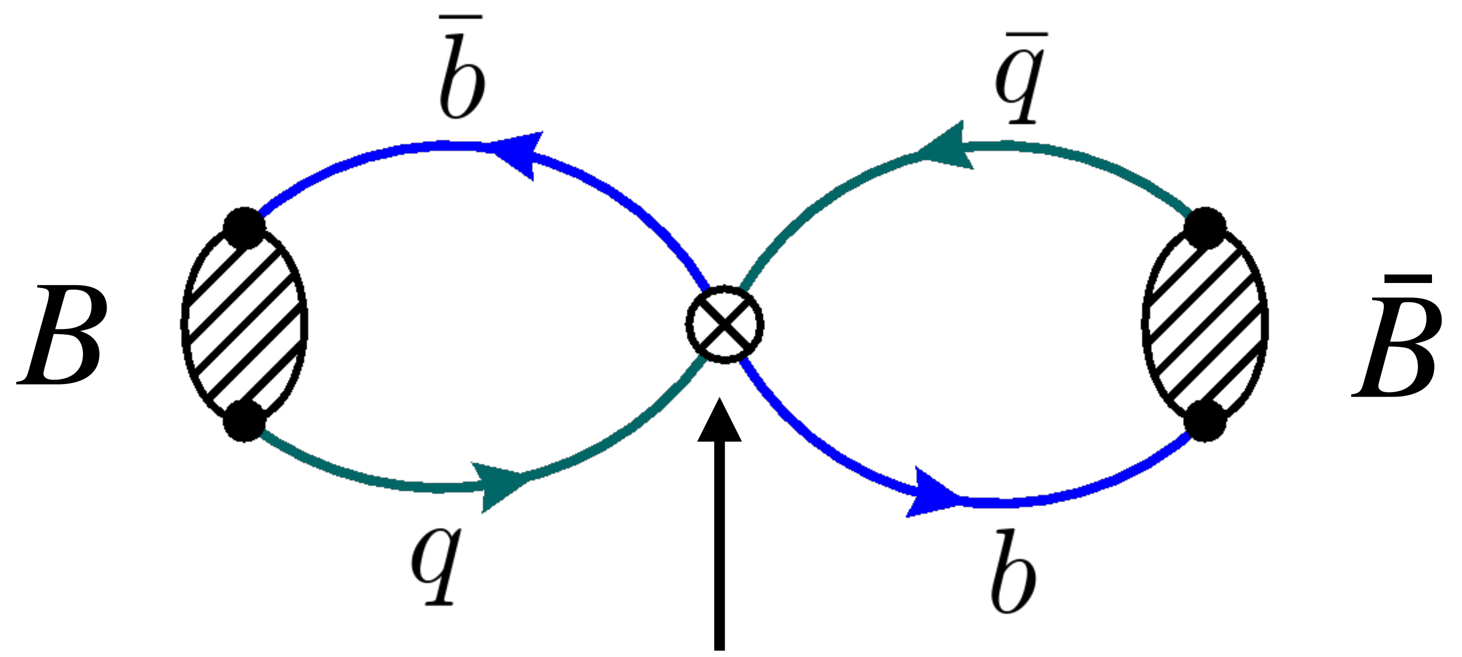

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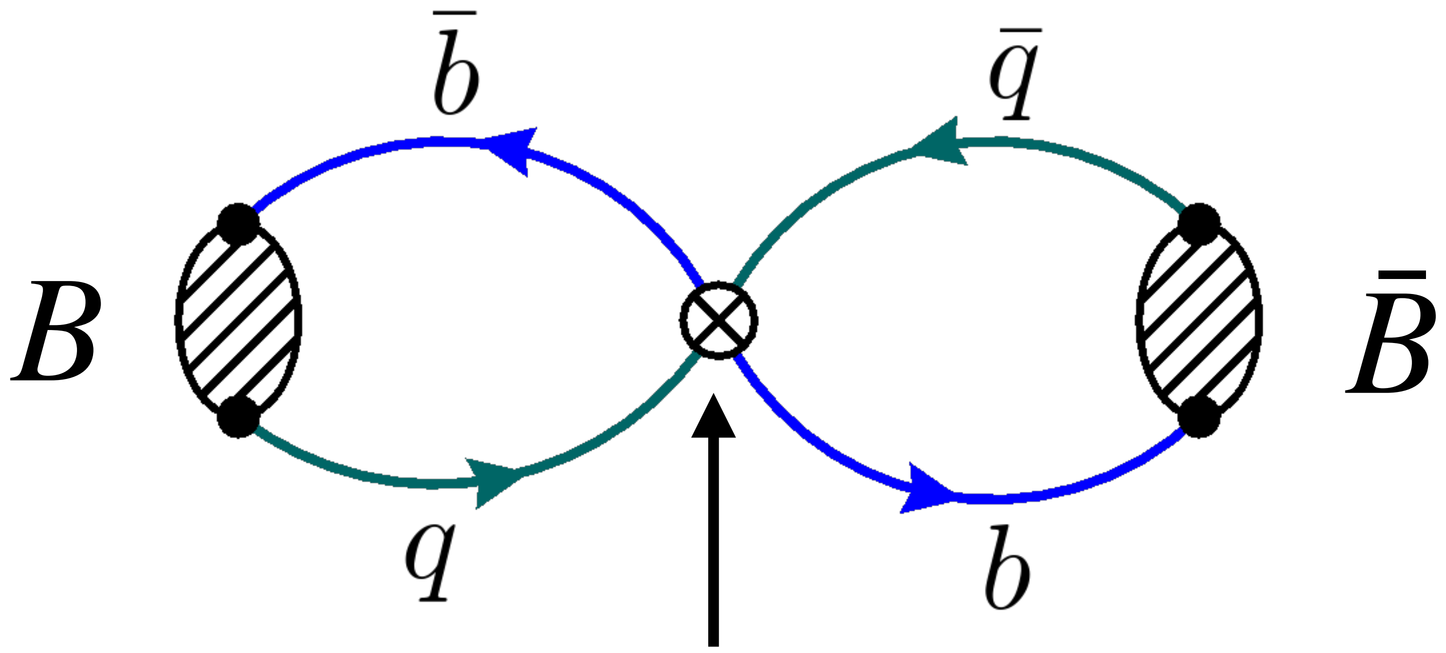

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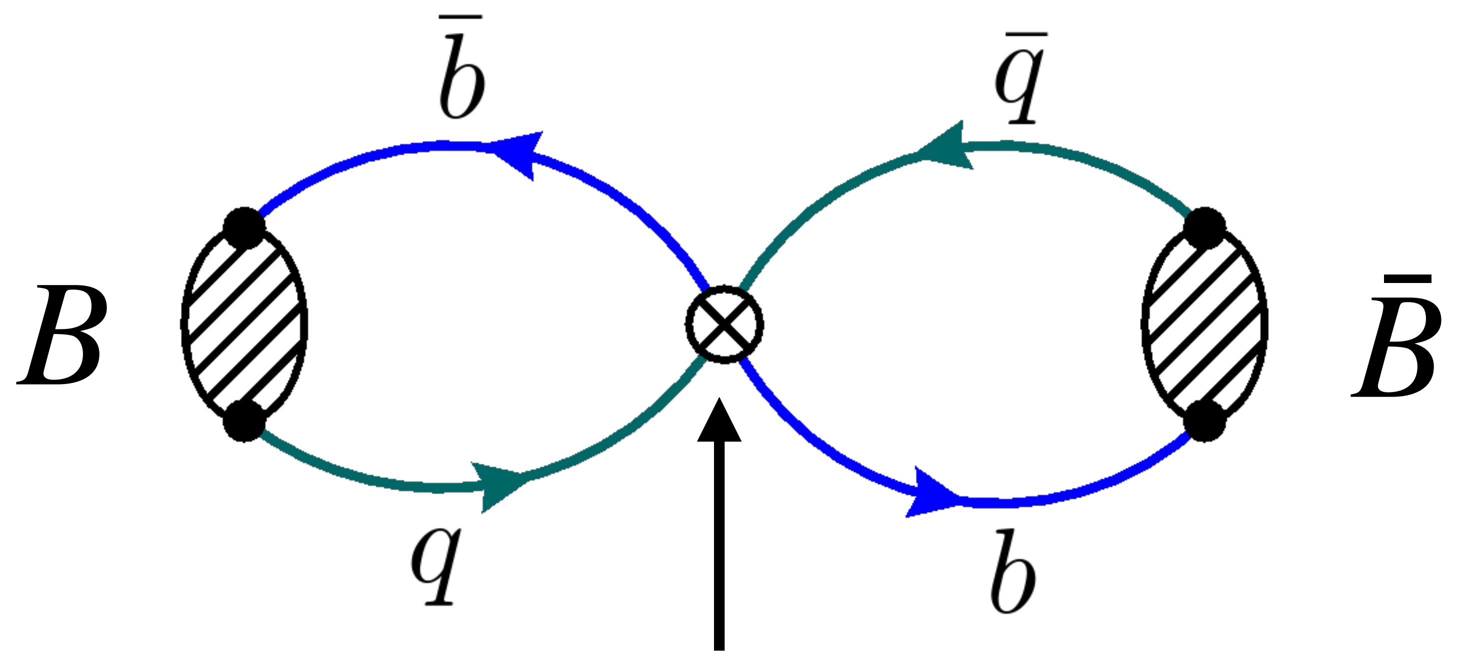

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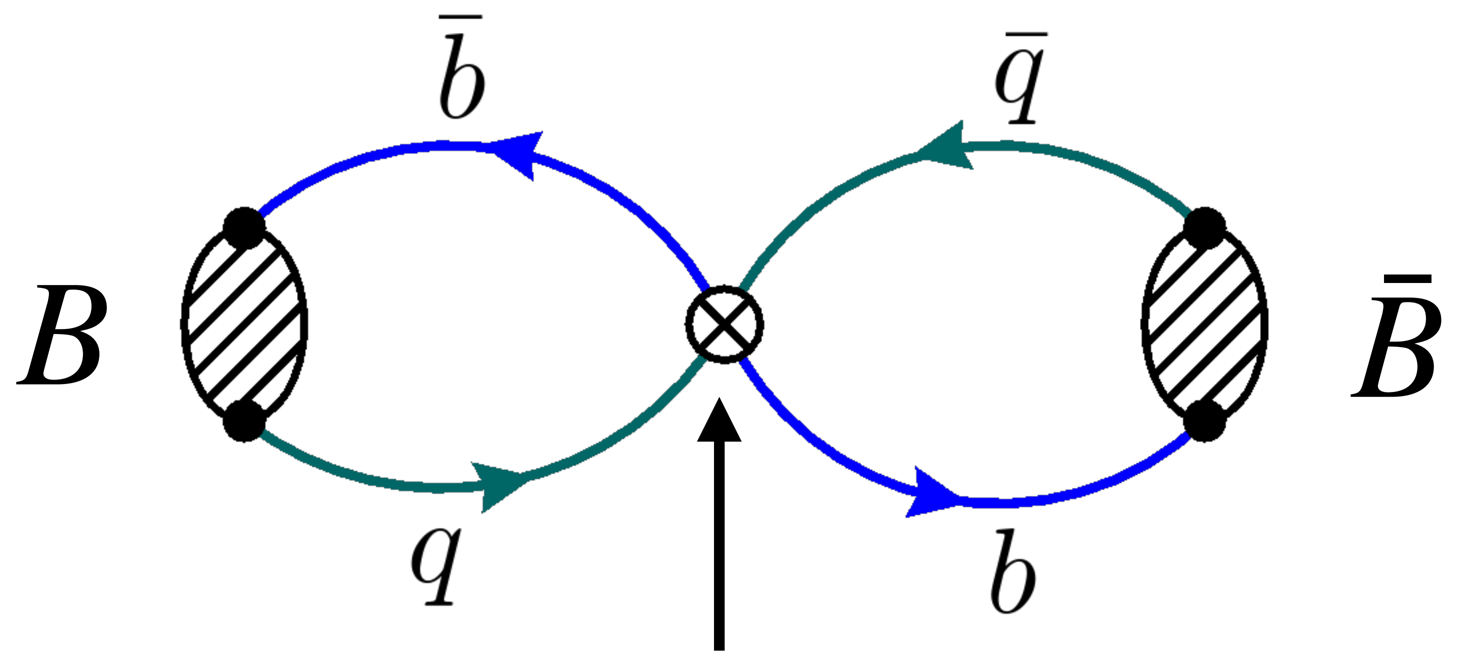

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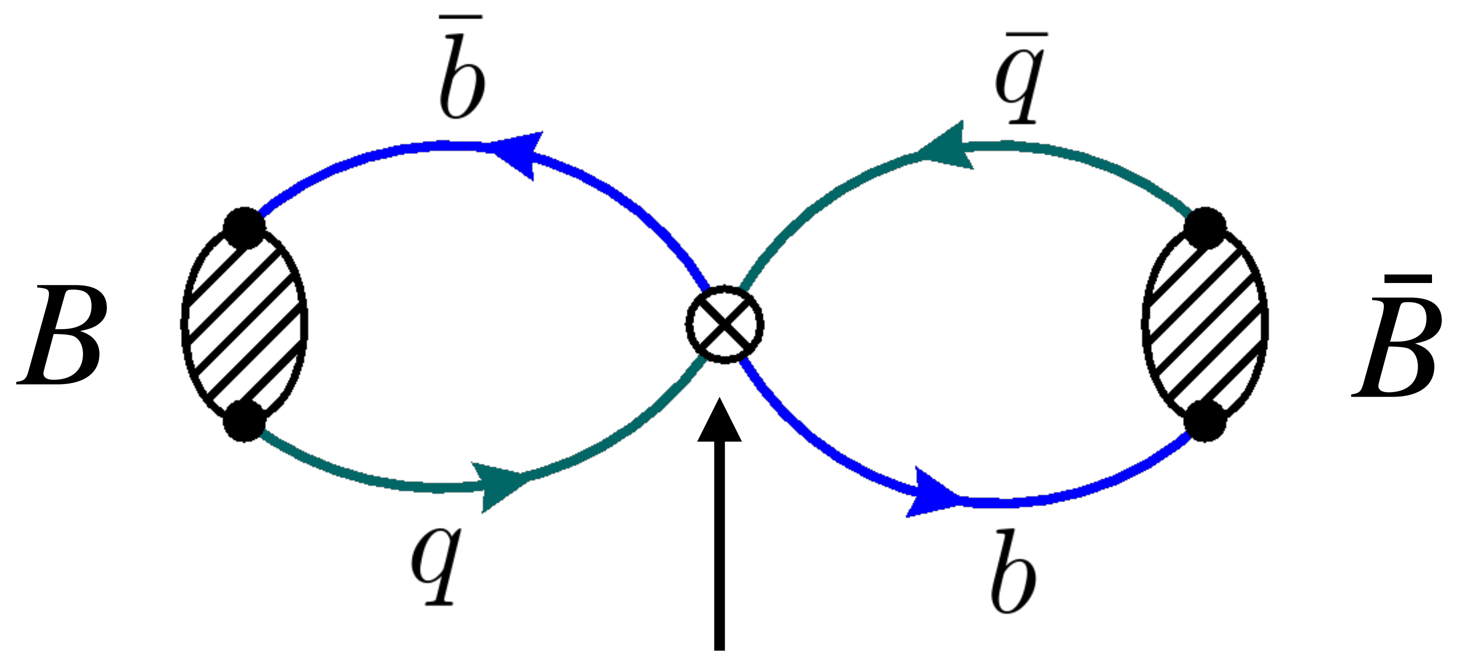
$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q) \quad \longrightarrow \quad B_1 \sim \langle B \,|\, \mathcal{O}_1 \,|\, B \rangle$$

bag parameter

Example: Meson mixing



$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}_n \equiv \sum_n \tilde{C}(t)_n \tilde{\mathcal{O}}(t)_n$$

$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q) \longrightarrow B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$$

bag parameter

$$= \zeta^{-1}(t) \langle B | \tilde{\mathcal{O}}_1(t) | B \rangle$$

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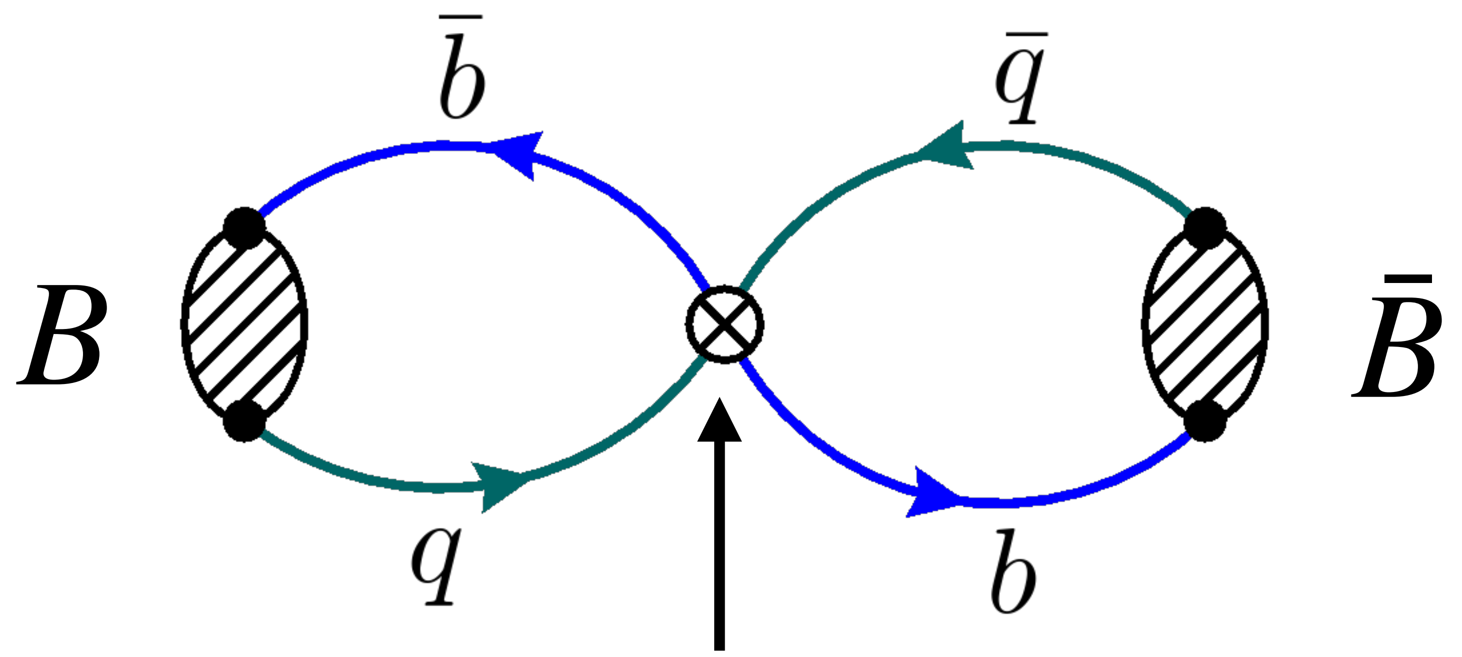

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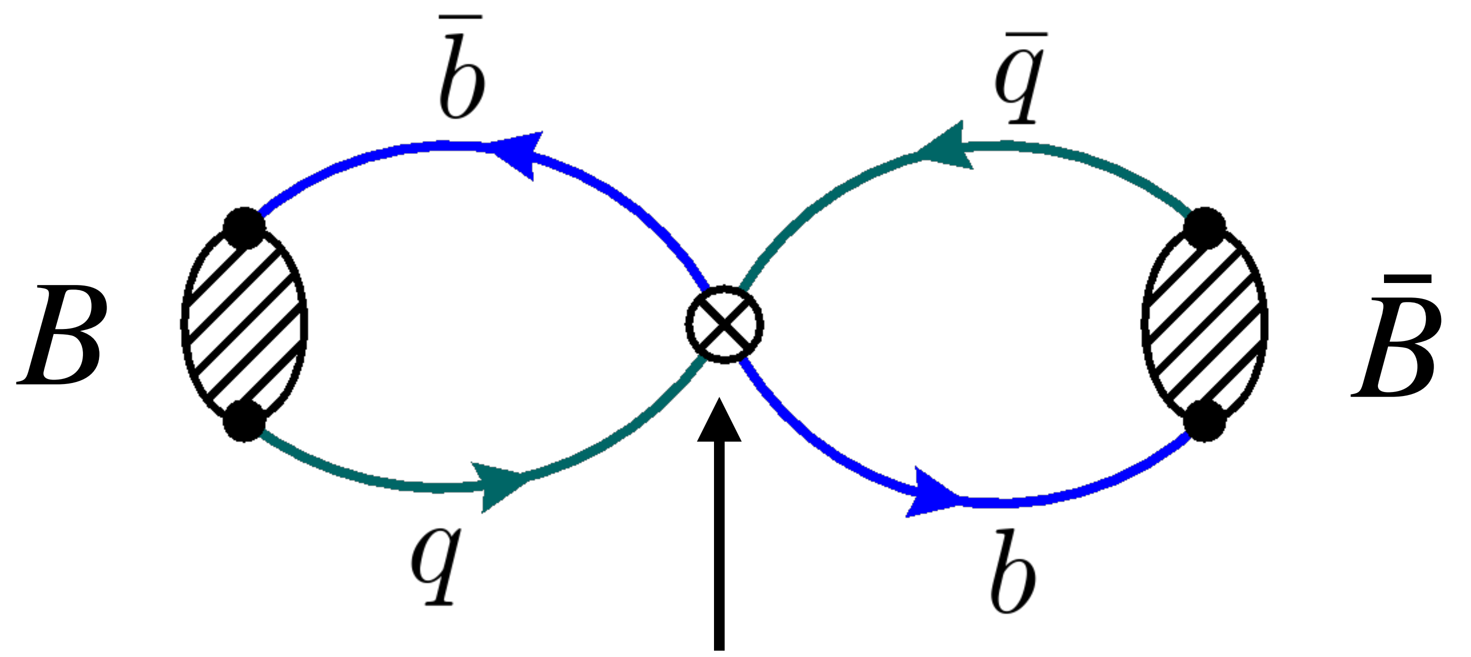

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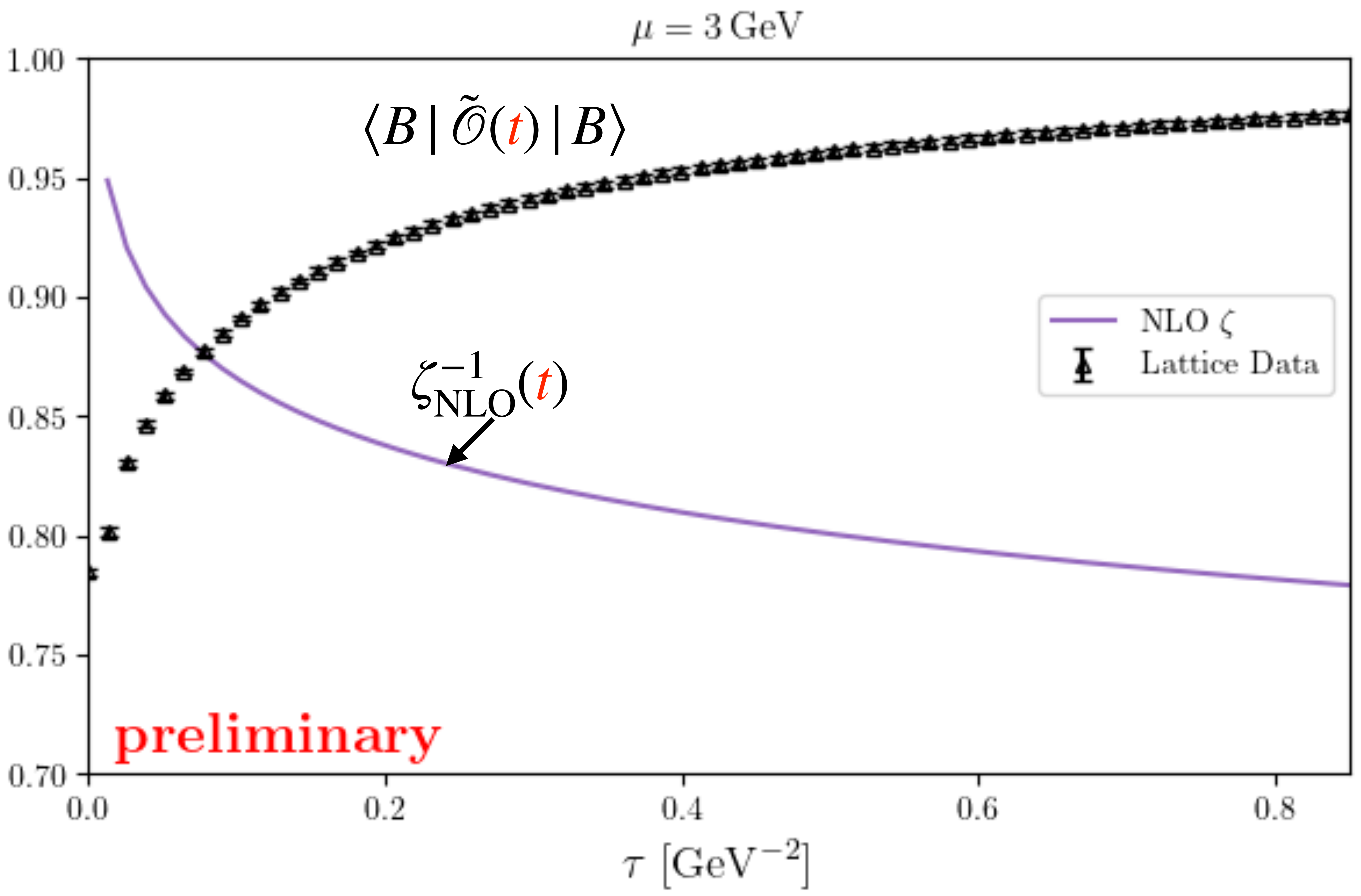

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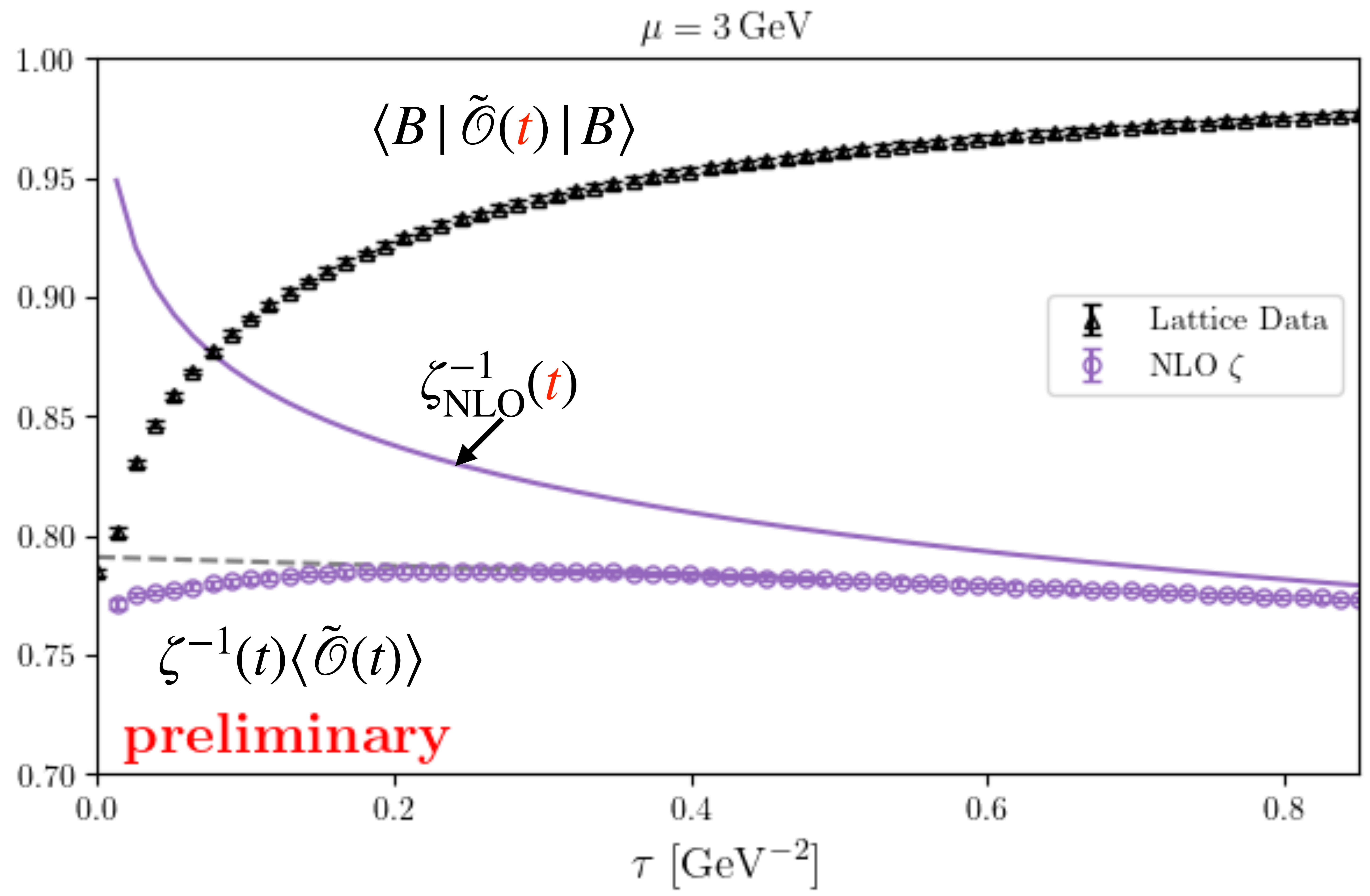

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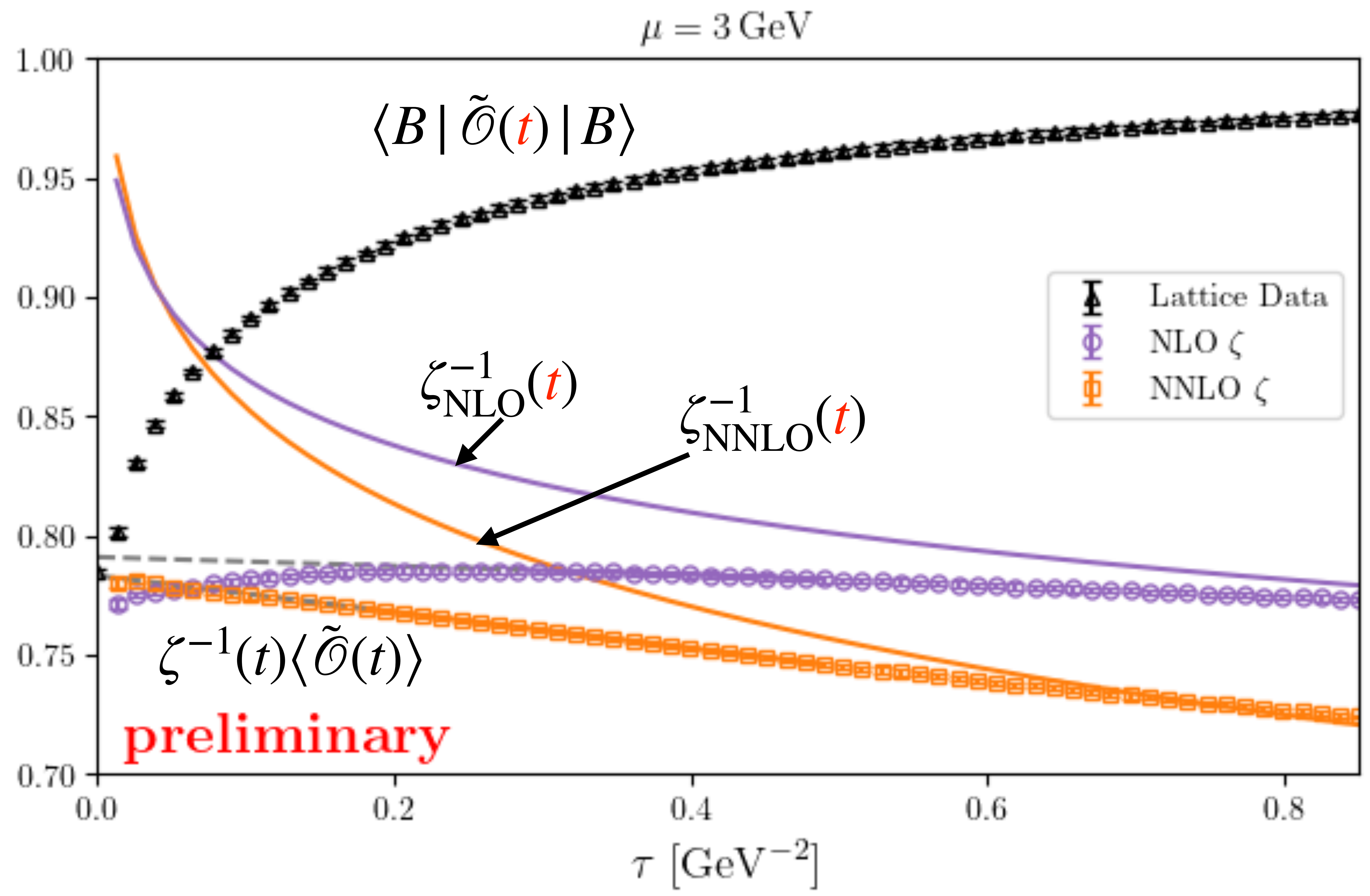

Black, RH, Lange, Rago, Shindler, Witzel (2023)

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Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

match renormalization schemes?

gradient flow renormalization

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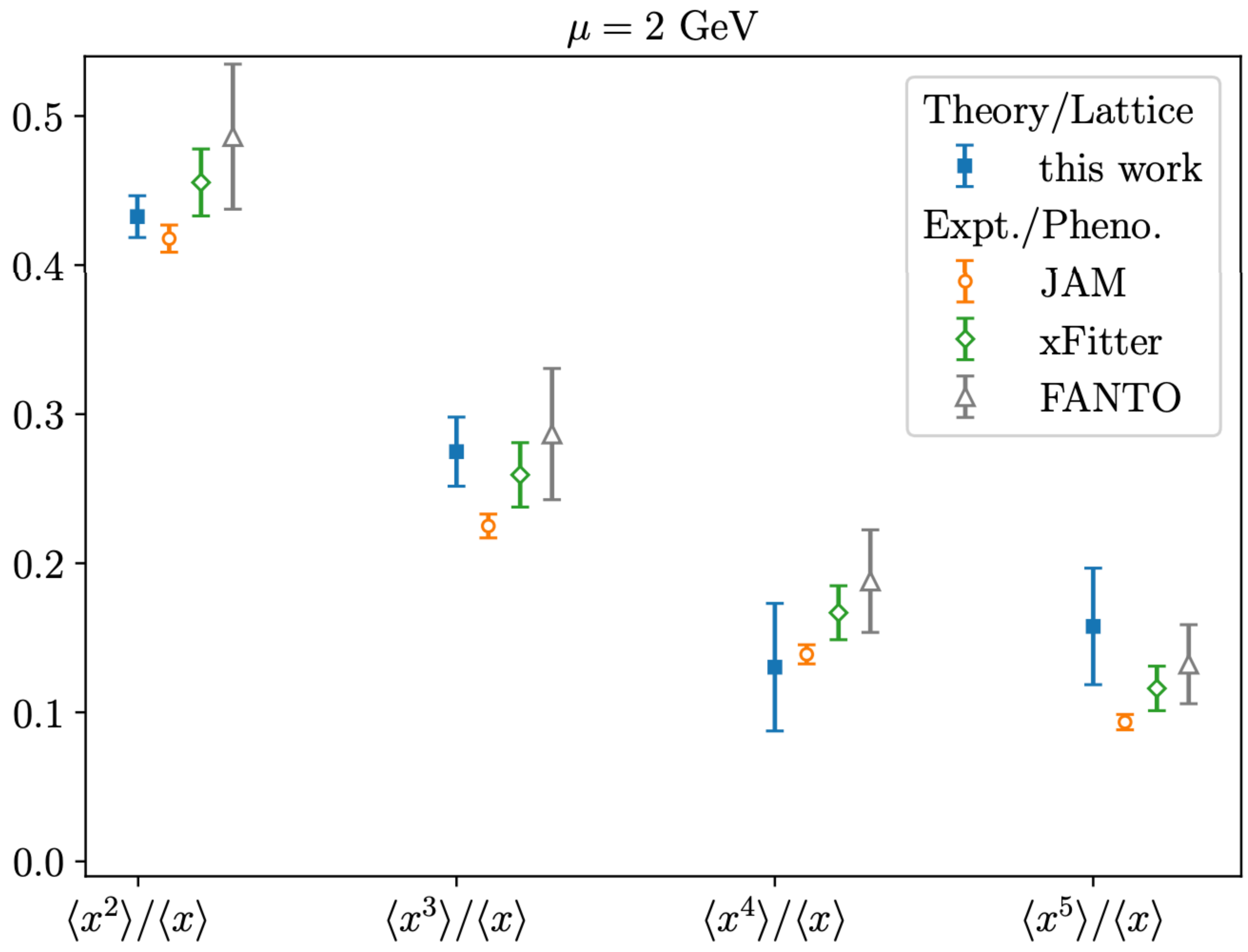
application: energy-momentum tensor Suzuki '13
parton density functions Shindler '24

Parton densities (pions)



Gradient flow for parton distribution functions: first application to the pion

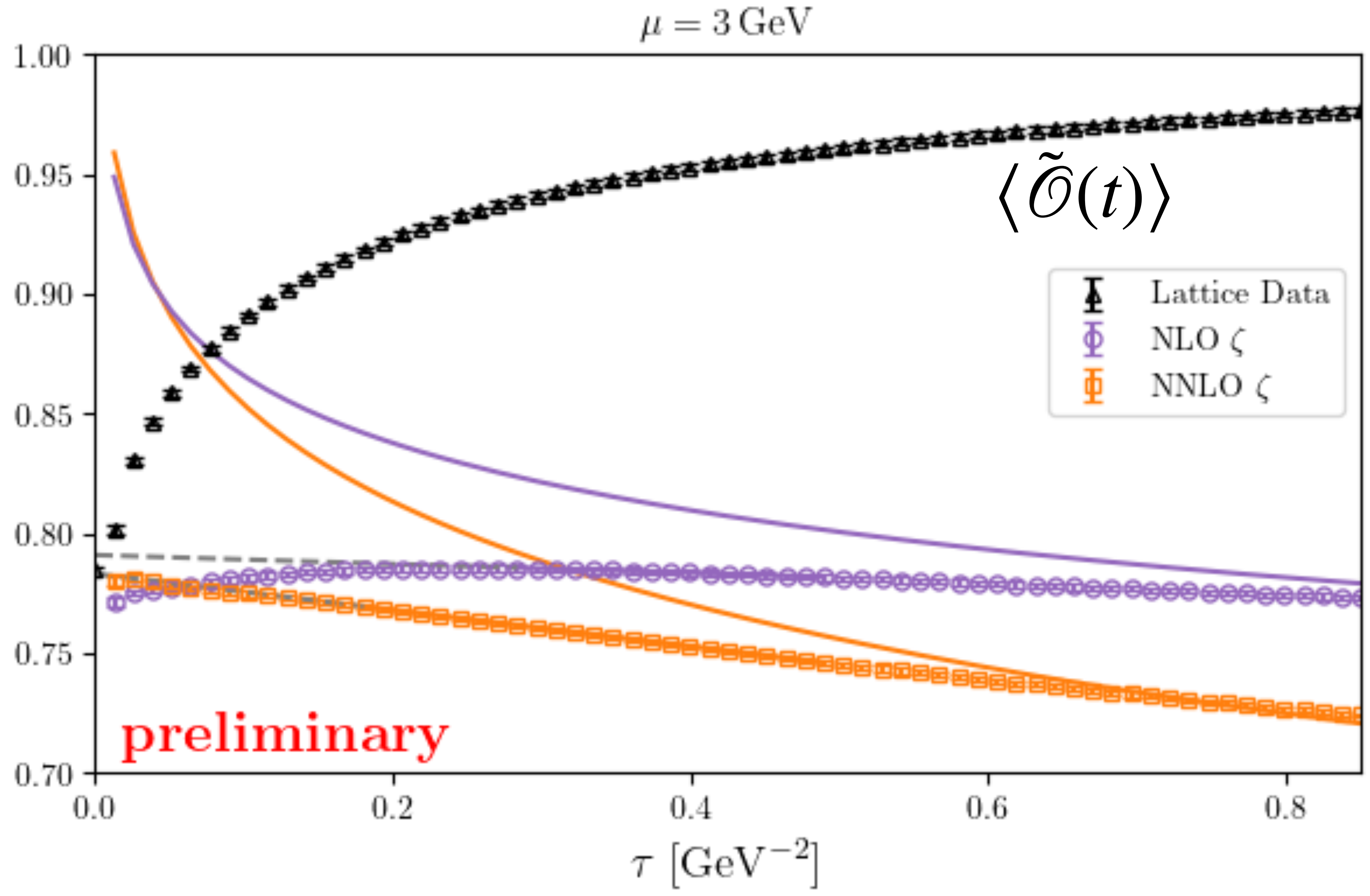
Anthony Francis,[1] Patrick Fritzsch,[2] Robert V. Harlander,[3] Rohith Karur,[4,5] Jangho Kim,[6] Jonas T. Kohnen,[3] Giovanni Pederiva,[7,8] Dimitra A. Pefkou,[4,5,*] Antonio Rago,[9] Andrea Shindler,[3,5,4,†] André Walker-Loud,[5,4] and Savvas Zafeiropoulos[10]

[1] *Institute of Physics, National Yang Ming Chiao Tung University, 30010 Hsinchu, Taiwan*
[2] *School of Mathematics, Trinity College, College Green, Dublin, Ireland*
[3] *Institute for Theoretical Particle Physics and Cosmology, TTK,*

2025

The gradient flow scheme

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$



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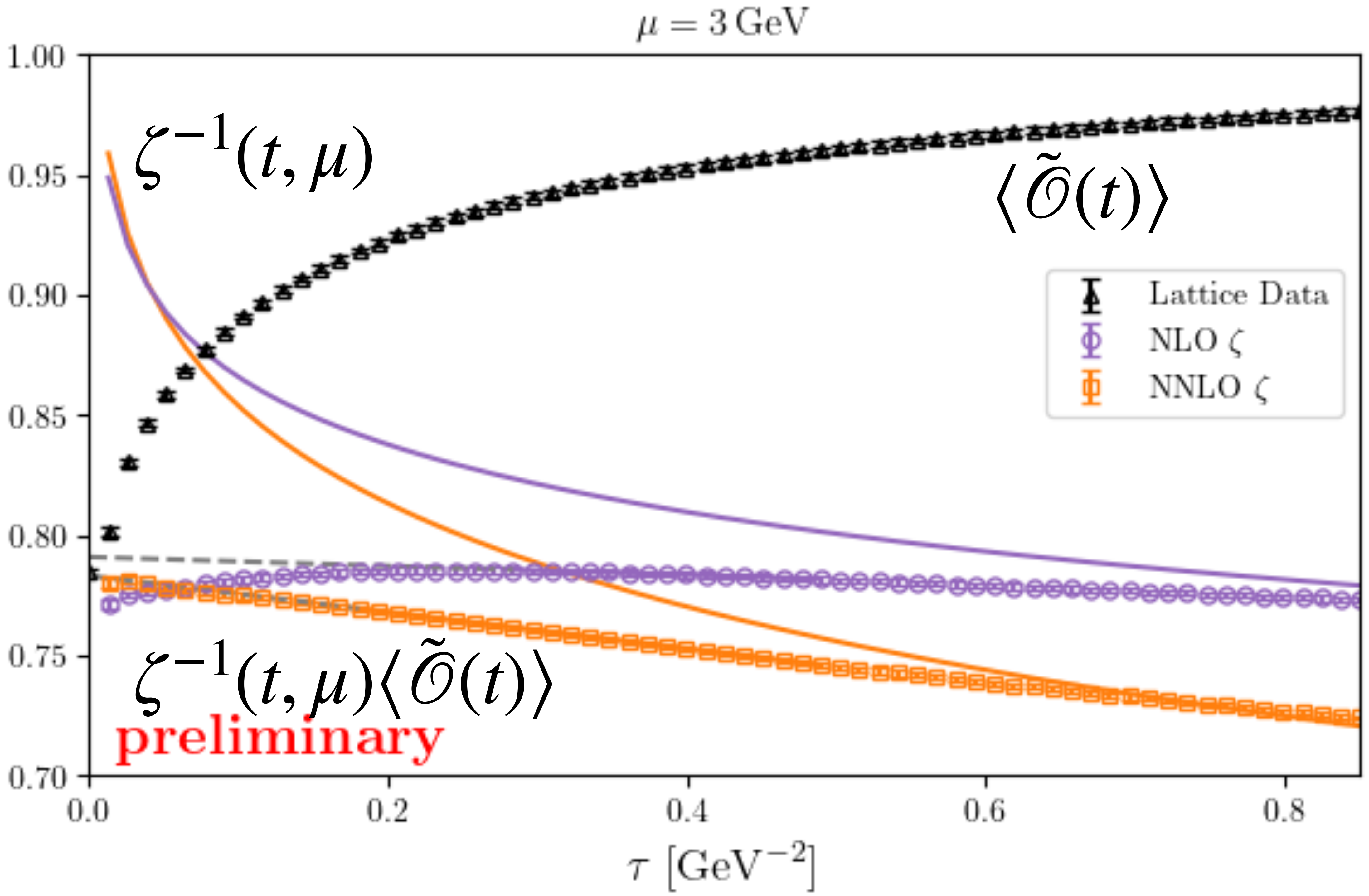

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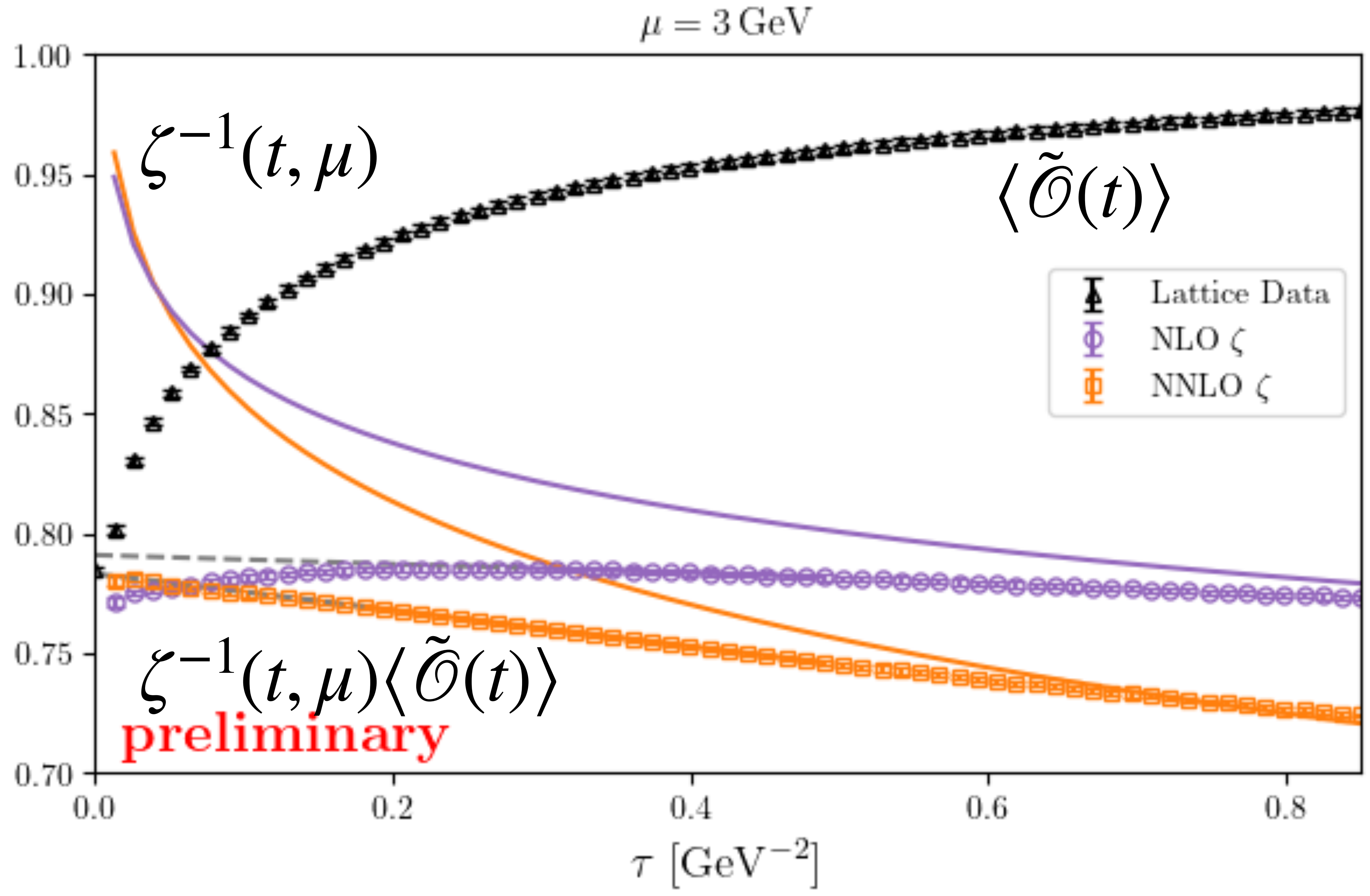

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RH, Lange, Neumann ‘20

Borgulat, Felten, RH, Kohnen ‘25

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